Liquidity and Capital Structure *

Ronald W. Anderson †and Andrew Carverhill ‡

August 1, 2011

*We would like to thank an anonymous referee and the editor (Pietro Veronesi) for making many useful suggestions. We have benefitted from discussions with Christopher Hennessy, Christian Laux, and Jean-Charles Rochet and from the comments of seminar participants at the LSE Financial Markets Group, Warwick University, Toulouse University, the Goethe University of Frankfurt, HEC Lausanne, University of Zurich, the Symposium on Dynamic Corporate Finance and Incentives at Copenhagen Business School, and the Skinance Conference in Hemsedal 2006. This research has benefitted from the support of the Dr Frederik Paulsen Foundation. Responsibility for all views expressed and for any errors is our own.

†London School of Economics, Department of Finance, Houghton Street, London WC2A 2AE, UK, r.w.anderson@lse.ac.uk

‡Hong Kong University, carverhill@business.hku.hk

Abstract

We solve for a firm's optimal cash holding policy within a continuous time, contingent claims framework using dividends, short-term borrowing and equity issues as controls under the assumption of mean reversion of earnings. Optimal cash holdings are a non-monotonic function of business conditions and an increasing function of the level of long-term debt. The model matches closely a wide range of empirical benchmarks and predicts cash and leverage dynamics in line with the empirical literature. Firm value is quite insensitive to changes in the level of long-term debt. The model has interesting implications for the asset substitution hypothesis, corporate hedging, and the pecking order hypothesis. We find that growth opportunities do not greatly affect the cash holding policy.

Liquidity and Capital Structure

> "Kerkorian's numbers just don't add up," said Nicholas Lobaccaro, an auto analyst with S G Warburg. "Ford says it needs double-digit billions of cash to survive the next downturn in the market. General Motors says it wants to put aside \$13-15 billion. How can anyone believe Kerkorian when he says \$2 billion is enough?"[for Chrysler] [1]

Introduction

The quotation above illustrates the range of opinions that can be found among practitioners about the levels of liquid assets that are appropriate for firms. This observation is not an isolated case - it is often remarked that many large corporations carry surprisingly large amounts of cash on their balance sheets. However, finance theory has given very little precise guidance as to how much cash is enough.

In this paper we attempt to fill this gap by directly asking what is the optimal policy toward holding liquid assets in the firm. Specifically we determine the dynamically optimal cash holding and leverage policy in a firm with given assets in place and long-term debt outstanding. The controls are the firm's dividend, short-term borrowing, and share issuance choices. This analysis is important because in standard finance teaching cash holding is treated as a short-term, operational problem separate from the firm's capital structure choice. In fact, for a firm choosing cash holding with foresight, this choice is inherently related to the long-term capital structure. Under the optimal policy the level of cash is adjusted dynamically as the firm is exposed to changing business conditions. In addition, because our model employs a rich continuous time formulation of the firm's cash flow process, we can calibrate the model to several empirical benchmarks and thereby gain some insight into the quantitative magnitudes of optimal cash holding in a variety of circumstances.

The core qualitative finding is that the optimal cash holding of the firm is a function of the firm's expected cash flow (earnings) which displays a hump peaking in the neighborhood of the firm's long-term average rate of cash flow. As business conditions improve beyond the long-term average, the firm targets a lower level of cash and pays dividends that maintain cash at the target level. Depending on the firm's level of long-term debt, it may be optimal for the firm to borrow short-term in order to increase leverage. When business conditions fall below the long-term rate, the firm will target a higher level of cash holdings and will suspend dividends in an attempt to achieve this. However, with the decline in revenues, the firm will see average cash holdings fall well below the target level. When there is a sustained or deep decline in revenues, cash will drop to a minimum level at which point the firm will issue equity. However, at a still lower threshold of expected cash flow, the firm's equity value drops to zero at which point the firm is bankrupt and the remaining assets are transferred to creditors.

A rich variety of consequences follow from this optimal policy. First, since the firm is able to control its net leverage through its dynamic cash policy, firm value is quite insensitive to its long-term debt level structure. If the long-term debt level is lower, then the firm can compensate in terms of tax shield benefits, by holding less liquidity or by short-term borrowing. A second implication is that there is a hierarchy or "pecking order" among sources of funds but this hierarchy is a function of business conditions. If business conditions are very good the hierarchy is internal funds first, short-term debt second, and equity third. However, in poorer business conditions the firm will avoid short-term debt and will turn to equity issuance if internal funds are not available. A third implication is that for our firm, and in many situations that we explain, the interests of debt holders and shareholders are not in conflict with respect to the choice of asset volatility. Thus, the the basic corporate finance teaching about the "asset substitution" problem must be refined for our firm. Fourth, our model yields insights into corporate hedging, and its limitations.

We explore the empirical implications of the model in more detail by simulating it in the baseline case calibrated to match simultaneously benchmarks for cash holdings, leverage, equity volatility, yield spreads, default probabilities and recovery rates. Using sample paths generated from this baseline model, we are able obtain results quite similar to the cross-sectional and time series results reported in recent empirical studies of cash holding.

We finally extend the model to allow for an increase in the scale of the firm's risky technology. We find that the growth opportunity does not greatly impact cash holding policy, unless the firm expects to grow at an unusually large rate and in that case only shortly before the investment is undertaken.

Papers closely related to ours include Mello and Parsons (2000), Rochet and Villeneuve (2004), Hennessy and Whited (2005), Riddick and Whited (2009), and Bolton, Chen and Wang (2011). All of these papers present theoretical treatments of the firm's liquidity policies that share some but not all of the features of the present paper. Mello and Parsons (2000) and Rochet and Villeneuve (2004) are concerned with the benefits of hedging and show that that hedging reduces the necessity of inefficient cash holdings. Hennessy and Whited (2005) consider a discrete time model with cash and short-term debt but no long-term debt. They explore the close dynamic linkage among debt, cash, and investment.

Riddick and Whited (2009) address the question of "Why do corporations save?", with 'saving' defined as increasing the firm's cash reserves. The firm "dis-saves" when its profitability increases, because this is when it is optimal to expand real capital. There is a negative"cash flow sensitivity of cash", in the sense of Almeida, Campello and Weisbach (2004). Our model addresses the more basic question of the determinants and levels of cash reserves themselves, rather than just their time increments. In our model, firms have a negative marginal propensity to save when the profitability is high, independent of their real investment plans. At lower profitability, this propensity is highly path dependent.

Bolton, Chen and Wang (2011) present a model which is similar to ours in many ways and which derives the firm's optimal cash management policy in the face of costly external finance. The main difference is that they assume the firm is subject to i.i.d. productivity shocks which allows them to solve the associated PDE analytically. In contrast, we assume a mean reversion of profitability which allows us to explore cyclical aspects of the firm's optimal cash management. One of their key findings is that investment of a financially constrained firm is determined by the ratio of marginal q to the marginal cost of financing. Increases in cash decrease the marginal cost of financing and thus induce greater investment. This differs from our model extended for growth because in our model investment tends to occur only when earnings are high and in such states cash reserves need not be high. This comparison is discussed in detail in Section 3 below.

The remainder of the paper is organized as follows. In Section 1 we introduce the model and the dynamic programming technique we use to solve it. We explore in some detail a benchmark solution of the model and discuss its relation to previous empirical findings. In Section 2 we draw out the implications of our model for the firm's choice of capital structure, emphasizing the role of mean reversion. In section 3 we extend our model first to allow for investment opportunities. Finally, in Section 4 we summarize our results and conclusions. In an online Appendix we present technical details of our numerical procedures and give derivations and further details of results reported in the text.

1 The Model

1.1 Overview

Before presenting our model formally, it is useful to set out the main ideas in informal terms. We consider a firm with a fixed asset in place which has been financed by equity, variable short-term debt and fixed long-term debt. The asset generates a random cash flow according to a stochastic process whose drift is itself random and follows a mean-reverting process. Any cash flow in excess of contractual debt service and fixed operating costs is subject to proportional corporate income tax, and the after-tax residual may either be paid out as dividends, used to reduce short term debt, or retained as liquid assets within the firm. Debt is assumed to be a hard claim, and any failure to meet contractual debt service results in bankruptcy. We assume that strict priority is observed in bankruptcy, with the firm's assets in excess of bankruptcy costs being awarded to the firm's creditors. Shareholders lose all. When cash flows fall short of debt service, the firm may draw-down its liquid assets or issue short term debt. It may also issue new equity; however, this external finance is costly so that the firm receives less than the full value of the shares it issues. The asset in place is indivisible so that partial sales of the risky asset are not allowed.

In this setting, firm faces two decisions. How much of the firm's earnings should be paid out as dividends? And how many new shares should be issued? Jointly, the two decisions will determine the firm's policy toward holding liquid assets and short term debt issuance. In our framework there is no reason to hold cash and borrow short term simultaneously. We assume that these decisions are under the control of shareholders, who maximize the value of equity, calculated as the present discounted value of the future stream of dividends.

The firm's decision will depend upon two state variables: the current rate of revenue cash flow and the current level of liquid assets (which is interpreted as borrowing when negative). Since all the other features of the environment are constant, this is a stationary problem. The solution of the model involves solving for the optimal policy as a function of the two state variables.

Shareholders face various costs for capital market operations. Firm insiders will to some degree extract rents from liquid assets inside the firm, so that inside cash will grow at something less than the money market rate. On the other hand, the firm borrows short-term at a rate greater than the money market rate reflecting informational rents conceded as part of its banking relationship. Finally, issuing equity will incur floatation costs. In this context, the optimal dividend and share issuance strategy will be of the 'bang-bang' type, under which the state space is divided into 3 regions: in the 'save' region zero dividend is paid and earnings are accumulated in the reserve of liquid assets or used to pay down short term debt; in the 'dividend' region, the liquid reserve is immediately paid out, until it is brought back to the 'save' region, or to abandonment or bankruptcy; and in the 'issue' region, the firm immediately issues equity until liquid reserve is brought back into the 'save' region. The solution to the problem is studied by characterizing the boundaries between these regions as free boundaries, in a dynamic program.

1.2 Model Assumptions and Detailed Specification

The firm has fixed assets in place[2], which incur operating costs at a constant rate f, and which generate operating revenues at a rate dS_t according to the Ito equation

$$dS_t = \rho_t dt + \sigma dW_t^\sigma. \tag{1}$$

Here expected revenue ρ_t at time t itself obeys the Ito equation

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \sqrt{\rho_t}\eta dW_t^\rho. \tag{2}$$

In these equations, dW_t^σ and dW_t^ρ are infinitesimal increments of independent, standard Brownian motions, and $\kappa, \bar{\rho}, \sigma$ and η are positive constants. Equation (2) reflects our assumption that the expected rate of revenue is positive and mean reverting, representing variation of business conditions over the business cycle[3]. Mean reversion of ρ_t is discussed further in Section 2.3 below. Equation (1) adds an unpredictable random element to the operating income. Note that after deducting the operating costs the profitability of the fixed assets is given by $dS_t - f dt$.

The firm is financed by equity, variable short-term debt, and fixed long-term debt.[4] We assume that the firm's short-term debt is instantaneously maturing, and long-term debt takes the form of a perpetual bond promising a continuous payment at rate q. In practice, a typical financial structure will involve short-term secured bank loans and long-term debentures which may be protected against dilution through restrictions on the amount of debt issuance (see, Petersen & Rajan, 1994, and Bolton & Freixas, 2000). We capture this by assuming that given a fixed q the firm can instantaneously borrow up to its debt capacity, determined in our model by the value of the firm in bankruptcy. This will have the effect of making short-term debt default risk-free. We assume that the firm pays an interest rate on short-term borrowing, r_{bank}, which is in excess of the risk-free money market rate, r. The amount $r_{bank} - r$ reflects the informational rents ceded by the firm as part of its banking relationship.

The firm also may issue new equity to cover interest payments and operating losses. Such equity issues will be costly, in that the firm will be able to sell new shares at a fraction $\theta < 1$ of their fair value. The parameter θ will reflect fees and pricing concessions associated with primary equity market operations and may vary systematically with the efficiency of the capital markets where the firm operates. In a highly efficient market θ will be close to unity; whereas in a very underdeveloped capital market θ may be close to zero.

In addition to its fixed asset, the firm may hold a variable amount of liquid reserves. At any time t, the value of these will be denoted by C_t. Liquid reserves held within the firm will earn an 'internal' return at rate r_{in}, which will be less than the riskless rate r earned on outside funds. This wedge $r - r_{in}$between r_{in} and r reflects the moral hazard faced by the shareholders, as discussed by Myers and Rajan (1998). As already mentioned, in our framework there will be no incentive to hold cash and issue short-term debt simultaneously. Thus $C_t < 0$ will correspond to a situation where the firm is borrowing short-term.

Under these assumptions, and for the time-being ignoring the possibility of equity issues, the liquid reserve is the accumulation of total earnings net of dividends, fixed costs, and interest payments on debt, and we can write

$$dC_t = (1-\tau)(dS_t - ((f+q) - r_{C_t}C_t)dt) - dD_t, \tag{3}$$

where τ is the corporate tax rate, and D_t is the accumulated payments of dividends, r_C is equal to r_{in} when $C > 0$, and equal to r_{bank} when $C \leq 0$. This equation recognizes that tax is paid on the operating income and interest on the cash reserve, net of interest payments and fixed costs, i.e. $dS_t - ((f+q) - r_{C_t}C_t)dt$. Applying this to negative earnings is an analytically tractable way to model the loss carry-back and carry-forward provisions of many tax regimes.[5]

The firm becomes bankrupt if it does not meet its debt obligations and fixed costs, either from its operating revenue, its cash reserve, or by issuing new shares. On bankruptcy, the creditors are awarded the firm's fixed assets. We assume their value equals the value of the unlevered firm less bankruptcy costs, which we take to be a fraction α of this value. We denote the equity value with long-term debt coupon, q, by $J^q(\rho, C)$. Then the debt holders get value $(1-\alpha)J^0(\rho, C)$ upon bankruptcy. We assume that the firm can borrow short-term up to the value of this collateral less the face value of long-term debt, q/r. Thus for given ρ, the short-term borrowing limit will be a negative cash holding denoted $\underline{\mathrm{C}}(\rho)$, given by $\underline{\mathrm{C}}(\rho) = \min\{0, (1-\alpha)J^0(\rho, C) - q/r\}$

Finally, we assume the firm chooses the dividend and capital market policy so as to maximize equity value, which is taken to be the present value of expected dividends discounted at the risk-free rate r. The debt is also valued by discounting at the risk free rate the coupon payments until bankruptcy, and then the bankruptcy value. This is consistent with Equations (1), (2) referring to the risk neutral probability measure.[6]

1.3 PDEs for the Solution

Under the above assumptions, and ignoring for the moment the possibility of new equity issues, the value of the firm's equity is determined at any time t by the current values of profitability ρ and cash reserve C, which represents short term borrowing, if it is negative. Denoting this value by $J_t^q(\rho, C)$, then we can write the HJB equation

$$J_t^q(\rho, C) = \max_{dD_t} \left\{ dD_t + e^{-rdt} E_t^{(\rho,C)} \left[J_{t+dt}^q(\rho_{t+dt}, C_{t+dt}) \right] \right\}, \tag{4}$$

in which dt is an infinitesimally short time step and dD_t is the optimal dividend payment over this time step, which must be non-negative. Also, $E_t^{(\rho,C)}$ means the expectation at time t, given that $(\rho_t, C_t) = (\rho, C)$. If the liquid reserve becomes low, then the firm can increase it by issuing more equity, if this is feasible in terms of the share price.

Expanding $J^q_{t+dt}(\rho_{t+dt}, C_{t+dt})$ in Equation (4), using the Ito Formula, $E[dW_t^\sigma] = 0$, and following standard manipulations we obtain the Ito Equation

$$J_t^q(\rho, C)(1 - e^{-rdt}) = \max_{dD_t \geq 0} \left\{ dD_t + \left(\frac{\partial}{\partial t} J_t^q + \kappa(\bar{\rho} - \rho) \frac{\partial}{\partial \rho} J_t^q + \frac{1}{2} \rho \eta^2 \frac{\partial^2}{\partial \rho^2} J_t^q \right) dt \right.$$

$$\left. + \left[(1-\tau)(\rho - (f+q) + r_C C) dt - dD_t \right] \frac{\partial}{\partial C} J_t^q + \frac{1}{2} \sigma^2 (1-\tau)^2 \frac{\partial^2}{\partial C^2} J_t^q dt \right\} \qquad (5)$$

We emphasize that this equation holds only for the optimal choice of dD_t, which depends on (ρ, C).

The optimal choice of dD_t here is singular: if $\frac{\partial}{\partial C} J_t^q < 1$, then it is optimal to pay dividends as quickly as possible, reducing the cash holding until either $\frac{\partial}{\partial C} J_t^q \geq 1$, or until the firm becomes bankrupt. If $\frac{\partial}{\partial C} J_t^q \geq 1$, then the firm will not pay dividends.

The optimal decision can thus be characterized in terms a "save" region $\mathcal{S}$ and a "dividend" region $\mathcal{D}$ in the state space $\{(\rho, C) : C \geq \underline{C}(\rho)\}$. In $\mathcal{S}$ we have $\frac{\partial}{\partial C} J_t^q > 1$, and also Equation (5) holds, with $dD_t/dt = 0$, i.e.

$$\frac{\partial}{\partial t} J_t^q - r J_t^q + \kappa(\bar{\rho} - \rho) \frac{\partial}{\partial \rho} J_t^q + \frac{1}{2} \rho \eta^2 \frac{\partial^2}{\partial \rho^2} J_t^q$$

$$+ (1-\tau) \left[\rho - (f+q) + r_C C \right] \frac{\partial}{\partial C} J_t^q + \frac{1}{2} \sigma^2 (1-\tau)^2 \frac{\partial^2}{\partial C^2} J_t^q = 0. \qquad (6)$$

In $\mathcal{D}$ we have

$$\frac{\partial}{\partial C} J_t^q = 1, \qquad (7)$$

and Equation (6) does not apply, since the value of an extra dollar in the liquidity reserve is just its value if immediately paid as a dividend. If the liquid reserve C_t becomes so high that $(\rho_t, C_t) \varepsilon \mathcal{D}$, then a dividend should immediately be paid, to take (ρ_t, C_t) back into the region S , or to bankruptcy.

If the liquid reserve becomes low, then it may be optimal for the firm to issue new equity. We have not included this possibility in the above formulation. In fact it is optimal to issue more equity if [7] $\frac{\partial}{\partial C} J_t^q > \frac{1}{\theta}$. This possibility leads to there being a third, 'issue' region, which we will denote by $\mathcal{I}$, lying below $\mathcal{S}$, and in which

$$\frac{\partial}{\partial C} J_t^q = \frac{1}{\theta}. \tag{8}$$

If the liquid reserve C_t becomes low, so that $(\rho_t, C_t) \varepsilon \mathcal{I}$, then new equity should immediately be issued, to take (ρ_t, C_t) back into the region S . Note that until bankruptcy occurs, the process (ρ_t, C_t) will always remain in the save region $\mathcal{S}$, since it is immediately pushed away, whenever it enters the region $\mathcal{D}$ or $\mathcal{I}$.

These regions must be chosen to maximize $J_t^q(\rho, C)$ and the boundaries between $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{I}$. are 'free', in that they are determined as part of the solution to Equations (6), (7), (8) [8] The boundary condition at the lower boundary $\{\underline{\mathrm{C}}(\rho)\}_\rho$ is just $J^q(\rho, \underline{\mathrm{C}}(\rho)) \geq 0$. Our solution technique will be described in detail in our online Appendix A. The approach is to develop the solution to Equation (6) numerically from a distant horizon $t = T$ to $t = 0$, and to test at every point whether the value is increased by applying Equation (7) or (8), instead of Equation (6). The horizon T is taken to be sufficiently far away, that the solution is independent of T for t near zero.

The debt value $P_t^q(\rho, C)$ can be calculated by solving a PDE, in a similar way to the equity value $J_t^q(\rho, C)$ above. In fact, the calculation is simpler because the boundaries of the region in which the debt is defined, i.e. $\mathcal{S}$, have already been determined in the equity valuation. The PDE for the debt, in the region $\mathcal{S}$, is

$$q + \frac{\partial}{\partial t} P_t^q - r P_t^q + \kappa(\bar{\rho} - \rho) \frac{\partial}{\partial \rho} P_t^q + \frac{1}{2} \rho \eta^2 \frac{\partial}{\partial \rho^2} P_t^q +$$

$$[(1-\tau)(\rho - (f+q) + r_C C)] \frac{\partial}{\partial C} P_t^q + \frac{1}{2} \sigma^2 \frac{\partial}{\partial C^2} P_t^q = 0. \tag{9}$$

Again, we evolve the solution backwards from a distant horizon $t = T$. The boundary conditions for the debt valuation are as follows: First $\frac{\partial}{\partial C} P_t^q = 0$, where $\mathcal{S}$ meets $\mathcal{I}$ or where $\mathcal{S}$ meets $\mathcal{D}$ and $\mathcal{D}$ is above $\mathcal{S}$. This corresponds to the reflection of the process C_t at these boundaries[9]; Second $P_t^q = (1-\alpha) J_0^0(\rho, 0)$ where $\mathcal{S}$ meets $\mathcal{D}$ and $\mathcal{D}$ is below $\mathcal{S}$, or when $C = \underline{\mathrm{C}}(\rho)$ and ρ is not high enough to induce the equity holders to maintain payments to debt and fixed costs. This corresponds to bankruptcy, under which the debt holders receive the unlevered value of the firm, net of bankruptcy costs. (Note that the restrictions on short term debt mean that the firm will not go bankrupt if it has any short term debt, and so we must have $C = \underline{C}(\rho) = 0$ at bankruptcy.)

1.4 Benchmark Solutions to the PDEs

Our benchmark parameters are set as $r_{in} = 5\%, r = 6\%, r_{bank} = 7\%, f = 1.0, \bar{\rho} = 1.09, \eta = 0.16, \kappa = 0.4, \sigma = 0.0, \tau = 30\%, \theta = 0.8, \alpha = 0.3$, and $q = 0.04$. Perhaps the most important parameter in our model is κ which determines the degree of mean reversion of profitability, a feature that has been recognized as important in capturing business cycle effects in capital structure dynamics (see, Gomes (2001), Hennessy and Whited (2005), and Pastor and Veronesi (2003)) The value of $\kappa = 0.4$ is taken from Pastor and Veronesi and reflects a moderate rate of mean reversion.[10]

Other important parameters are $\bar{\rho}, \eta, f$, and q which are all expressed in annual rates of monetary flows. The long run objectively realized average $\tilde{\bar{\rho}}$ of ρ_t will be higher than $\bar{\rho}$, reflecting the possibility of bankruptcy, and the risk premium, whose role we will explain further below. In simulations described below $\tilde{\bar{\rho}}$ is close to 1.23, and the long run average profitability is $\tilde{\bar{\rho}} - f \approx 0.23$. The firm value at this level of profitability is about 1.24. Thus the average return on assets is close to $0.23/1.24 = 18.5\%$. Also, setting $f = 1.0$ implies an operating leverage of $1.0/1.24 = 81\%$ which is consistent with recent estimates by Garca-Feijo and Jorgensen (2007). The value $\eta = 0.16$ makes the annualized standard deviation of earnings to be close to $\tilde{\bar{\rho}}^{\frac{1}{2}}\eta/0.23 = 77\%$ of the average earnings. This is in line with an analysis of COMPUSTAT data that we will describe more fully below. Finally, setting a long-term debt service at $q = 0.04$ implies an interest coverage ratio of approximately $(\tilde{\bar{\rho}} - f)/q = 5.75$. This number is broadly in line with an empirical summary, that we present in Table 2, and discuss further later.

The other parameters have direct economic interpretations. Notice that by setting $r = 6\%$ and $r_{in} = 5\%$ we are assuming that 1/6 of the market return on cash is dissipated by keeping the cash inside the firm and under the control of management. We view this as representing a significant problem of managerial moral hazard and a disincentive to holding cash. Similarly, setting $r_{bank} = 7\%$, we are assuming a significant relationship premium on short-term borrowing[11]. By setting $\theta = 0.8$ we assume 20% of the market value of newly issued equity is lost through transactions costs of one form or another. Given the direct costs plus underpricing of equity issues, we view these costs as substantial but not unreasonable in many settings[12]. Our assumption of bankruptcy costs of 30% is at the high end of estimates that can be found in the literature[13]. We explore the sensitivity of our solutions to these assumptions by examining alternative parameter values below.

Figure 1 gives the regions $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{I}$, defined above, for the model solved with our benchmark parameter values. In this figure,the x-axis represents expected revenues ρ, and the y-axis represents the cash holding, which corresponds to short term debt, at negative values. The 'save' region $\mathcal{S}$ is denoted by dots. As explained, within this region the firm does not pay dividends, and the solution will not stray beyond its upper and lower boundaries.

The upper barrier to the save region is the dividend region $\mathcal{D}$ indicated by the downward triangles in Figure 1. This can be regarded as the liquidity target. When (ρ_t, C_t) is below this target earnings are retained so as to increase C_t, and when (ρ_t, C_t) is above this target dividends are paid immediately, so as to reach this target.

The lower barrier to the save region is restricted to the lower boundary of cash $\{\underline{C}(\rho)\}$. For sufficiently high levels of expected revenue this coincides with the 'issue' region indicated by upward triangles in Figure 1. Here the firm chooses to issue equity to meet its cash flow needs. For lower values of ρ_t this coincides with the 'abandon' region indicated by small circles. Here it does not pay to issue equity, and the firm is allowed to go bankrupt. For the case depicted in Figure 1 we see that the firm is able to borrow short term when ρ greater than about 1.1. Also the lowest value of ρ at which the firm will issue shares if the cash reserve hits the lower boundary, is about 0.7.

Notice that for still lower values of ρ_t, the abandon region is met for strictly positive levels of cash. If revenues fall to these low levels while the stock of cash is sufficiently high, the firm will use the liquidity to pay operating losses, in the hope of surviving until business conditions improve. But at some critical level (indicated by the 'abandon' boundary) the prospects of surviving on the available cash are not sufficiently attractive, and the remaining cash reserve will be paid out, and the firm will be abandoned.

The solid line in Figure 1 shows the average cash holding displays a hump in the neighborhood of average earnings, $\bar{\rho}$. It is increasing in ρ for lower levels of profitability and decreasing for higher profitability. This line is calculated from the objectively realized average cash holding for each level of ρ, and taking 2000 simulations, as described in the next section. The average cash level is far below its target when ρ is below the long-term average, $\bar{\rho}$. In this range of ρ, the firm is attempting to increase its cash holdings through retentions, but the dynamics of cash are such that the firm settles down to an equilibrium cash holding short of its target level. The maximum average cash level of about 0.095 is attained slightly above the risk neutral average revenue rate of $\bar{\rho} = 1.1$. For revenue rates much above that level the firm is able to maintain cash at or just below target levels.[14]

Other properties of the model can be seen in Figure 2 where we present a portion of one history of the model when simulated under the optimal policy for the benchmark parameter values. This depicts the last ten years of the firm ending with its bankruptcy in its 120'th year. The period begins with good business conditions and the firm has built up a substantial cash cushion. Cash flows fall sharply in year 113 and the cash position of the firm deteriorates. Even though profitability returns to normal the firm is unable to fully restore its cash holdings, leaving it vulnerable to a renewed economic downturn in year 119 and leading to bankruptcy in the next year.

1.5 Benchmark Simulation of the Model

To explore our model further, we simulate it as in Figure 2 and summarize its implications for the liquid reserves, leverage ratios, credit spreads, default probabilities, and other relevant measures. Specifically, for the parameters above, we solve the model for the equity and debt values as functions of (ρ, C), and obtain the optimal regions $\mathcal{D}, \mathcal{I}$ and $\mathcal{S}$. Then we perform 2000 simulations of the ρ_t variable, the realized cash holding C_t, equity value J_t and debt value P_t. Each simulation starts from $\rho = 1.2, C = 0.0$ and then, after a burn-in period of 20 years, runs to the firm's bankruptcy, or to 1000 years, if no bankruptcy occurs before then.

Based on the realizations of the simulations we also calculate the average liquidity for each grid value of ρ, denoted $\overline{C}(\rho)$. Taking this average liquidity value as a function of ρ, we study the model conditional on five levels of profitability: $\rho = 0.90$, $\rho = 1.0$, $\rho = 1.1$, $\rho = 1.2$, and $\rho = 1.3$. At each level of ρ and average liquidity $\overline{C}(\rho)$, we present the equity value, net of the liquid reserve, $J^q(\rho, \overline{C}(\rho)) - \overline{C}(\rho)$, debt value $P^q(\rho, \overline{C}(\rho))$, net firm value $J^q(\rho, \overline{C}(\rho)) + P^q(\rho, \overline{C}(\rho)) - \overline{C}(\rho)$, and leverage (debt value divided by the value of the firm).

Taking net equity and firm values facilitates comparisons across different scenarios, since changing between scenarios would involve making up the difference in the liquidity reserve by cash. In addition we give the equity volatility calculated as $\sqrt{\left(\sqrt{\rho}\eta\frac{\partial}{\partial\rho}J\right)^2 + \left(\sigma\frac{\partial}{\partial C}J\right)^2}/J$ and the mean annual payout to shareholders ('dividend').

The simulations are done with respect to the objectively realized ("statistical") probabilities, and so we need to specify a risk premium. Referring to the risk associated with ρ, we can represent the risk premium by a parameter λ (assumed constant, for simplicity)[15], such that to transform from the risk neutral to the statistical measure, we should replace dW_t^ρ of Equation (2) by $dW_t^\rho + \lambda dt$. This λ can be thought of as a Sharpe Ratio: it is the extra return required, per unit of extra exposure to the risk represented by dW_t^ρ. To see how the risk premium affects the return of the firm's equity, note first that under the risk neutral measure, the expected return will be just the riskless return r. The equity value $J_t(\rho, C)$ is a smooth function of ρ, and using the Ito formula, we can write $dJ_t = (drift)dt + \frac{\partial}{\partial\rho}J_t d\rho \equiv rdt + \sqrt{\rho_t}\eta\frac{\partial}{\partial\rho}J_t dW_t^\rho$. The factor $\sqrt{\rho_t}\eta\frac{\partial}{\partial\rho}J_t$ here is the equity volatility, and we will calculate this in the tables below. Substituting dW_t^ρ by $dW_t^\rho + \lambda dt$, we can see that the risk premium increases the expected return by λ times this volatility. A reasonable value for the risk premium (Sharpe Ratio) of the market itself is $\lambda = 0.5$, corresponding to a market excess return of say 8%, and market volatility of say 16%. On the other hand a completely diversifiable risk would imply $\lambda = 0$. We take $\lambda = 0.3$, which is reasonable, if we assume that the risk of the firm has a systematic component, i.e. it is somewhat correlated with the market.

In addition we calculate the yield spread on zero-coupon bonds of 5 and 20 years until maturity. For this calculation, and following Duffie and Lando (2001), we assume that the perpetual debt is made up of a continuum of zero coupon bonds, and if the firm defaults, the these bonds are paid off in proportion to their value weight in the total debt. This calculation is done by adapting the perpetual debt valuation to accommodate this default rule, a payment of one dollar if there is no default before maturity, and the coupon being zero. We also calculate the probability of bankruptcy at 1, 5 and 20 year horizons. This calculation is again done by adapting the perpetual bond valuation, and we include the risk premium λ, since this probability is not risk neutral, but objectively realized.

Our results for liquidity, debt value, equity and leverage values are given in Table 1 Panel A, the credit relevant values are given in Table 1 Panel B. The calculations are done for a range of debt parameters, q. We take as the main reference for our calibration the case $q = 0.04$. In Table 2 we have summarized the results for this case in a way that can be compared to financial ratios of US non-financial firms as reported by Standard and Poors for the period 1997-1999. When evaluated at $\rho = 1.1$ these comparisons suggest that our benchmark firm has characteristics similar to a firm rated just below investment grade, at about BB+. When evaluated at $\rho = 1.2$, (i.e., close to the grand average realized value of ρ), the model corresponds to an investment grade firm rated about A- or BBB+.[16]

From Table 1A, for $q = 0.04$ we see that the average liquidity expressed as a function of net firm value is $0.095/1.02 = 9.3\%$ when calculated at $\rho = 1.1$. This is comparable to median level of liquidity holding for small to medium sized firms reported by during the 1980's and 1990's as depicted in Opler *et.al* (1999) Figure 2. Our model implies an equity volatility of about 35% under normal conditions ($\rho = 1.2$) which is in line with market experience.

Turning to the credit risk related indicators in Table 1B, we see that this firm in the reference case has a credit spread of 210 basis points on the 5-year pure discount bond when evaluated at $\rho = 1.1$ and 151 basis points at $\rho = 1.2$. By way of comparison, the annual average spread of the Moody's Baa non-financial corporate bonds over 10-year Treasuries ranged between 156 basis points and 229 basis points in the 1990's. Since 2000 this spread attained a high of about 321 basis points following the Dotcom Bust and 367 basis points in 2008 in the midst of the financial crisis.

In Table 1B we also report the probability of default, calculated under our assumption about the equity premium. Conditional on $\rho = 1.2$ this probability is 7.9% at the 5-year horizon. This is somewhat high for a firm that otherwise appears comparable to a firm in the investment grade range. For example, over the period 1983-2008, Moody's reports a 5-year cumulative default rate of 2.92% on bonds rated Baa3. However, in this same data set default rates rise sharply for non-investment grade bonds. Moody's reports a default rate of 6.28% for bonds rated Ba1 and 7.23% for those rated Ba2.

Finally, our model has implications of the value of defaulted bonds which can be compared to empirical recovery rates. Under the benchmark parameters the firm goes bankrupt when

$\rho = 0.7$ approximately. This is seen in Figure 1 as the boundary between the "abandon" region and the "issue" region. The collateral value paid to debt is $1 - \alpha$ times the value of the unlevered firm. Extrapolating to $\rho = 0.7$ in Table 1A and with $q = 0$, this unlevered firm value is about 0.3522, and so the debt value in bankruptcy is $(1 - 30\%) \times 0.3522 = 0.2465$. If we make the assumption that the firm's long-term debt was issued at par in average business conditions ($\rho = 1.2$) the face value of debt is about 0.500. Thus the implied recovery rate is $0.2465/0.500 = 49\%$. This number is consistent with empirical studies of BBB rated firms. Over the period 1988-2002 Standard and Poors found that the recovery rates on defaulted bonds were 30 per cent for Senior Subordinated Notes, 38 per cent for Senior Unsecured Notes, and 50 per cent for Senior Secured Notes. (See de Servigny and Renault (2004) Chapter 4.)

To summarize, by simulating our model under the optimal policy and for our chosen parameters we see that it produces plausible predictions in comparison to a very wide range of empirical benchmarks. These include cash holdings, leverage, equity volatility, yield spreads, default probabilities and recovery rates.

1.6 A comparison of some of our model features with empirical data

Another way to judge the plausibility of our model is to ask whether the empirical cross sectional evidence on cash holding and dividends corresponds to the policy reflected in Table 1 where we have depicted firms varying with respect to both degree of leverage and profitability.

In Table 3 we report average liquidity holdings and equity payout rates for US non-financial firms cross-tabulated by earnings quintile and degree of long-term leverage. This table is compiled from annual COMPUSTAT data from 1978 to 2008. For each firm we take the long term leverage for a given year, corresponding to maturity of more than 1 year, and the firm is allocated to the 'leverage categories' in the table according to the average long term leverage over all available years for the firm. We discard data for which the firm has been operating for less than 5 years. The numbers reported are averages weighted by total assets, with the maximum total assets capped at \$ 1000 million and firm years are discarded if total assets is less than \$ 10 million. In discussing our model we have called payouts to equity ‘dividends,’ but in fact, since we do not allow for negative share issues, these payouts are equivalent to a combination of dividends and share buy-backs in the data. Similarly, what we have referred to as cash really refers to a broad notion of liquid asset holding as represented by a firm’s net working capital.

By reading across rows of the upper panel of Table 3 we see that with the exception of firms in the two lowest leverage categories, liquidity tends to rise and then fall with profitability. We find a similar hump-shaped pattern in Table 1 comparing liquidity within columns, i.e., for a given value of q. Our model generates this pattern because under the optimal policy the target level of cash holdings is a decreasing function of profitability and because at low levels of profitability the firm faces a drain on cash which means their cash holdings will fall well short of the target (see Figure 1). The results in Table 3 suggest that this trade-off is a common feature in many firms. However, in the data this pattern does not hold for the firms with very low leverage. These firms tend to hold high levels of liquidity, independently of the level of current profitability. One possible explanation for this is that many low-debt firms maintain a high degree of liquidity in order to pursue growth opportunities as they arise. In Section 3 we will discuss how the predictions of our model would change if it were modified to explicitly incorporate fixed investment.

In the lower panel of Table 3 we present average payout to equity, again cross-tabulated by degree of leverage and by profitability. Reading across rows we see for firms of similar leverage, total payouts tend to be larger for more profitable firms. A similar pattern holds for our model as seen by comparing average dividend rates across different levels of ρ for firms with a given long-term debt, q. Again this pattern is a reflection of the optimal payout policy. Total payouts tend to be high for more profitable firms because for them cash targets are relatively low and are more easily attained.

While this evidence supports the empirical plausibility of the optimal payout policy we have derived, we would not suggest that our model is realistic in all respects. In particular, the received wisdom is that most firms smooth their dividends, and we have made no attempt to impose this in our model. Dividend smoothing would presumably lead to higher liquidity reserves, on average. On the other hand, since the increased popularity of share buy-backs starting in the 1980s, the degree to which firms actually smooth compensation to shareholders appears to have decreased. A recent survey by Brav, Graham, Harvey and Michaely (2005), concludes that "... maintaining the dividend level is on par with investment decisions while repurchases are made out of the residual cash flow after investment spending". Also, Allen and Michaely (2003) conclude that "Corporations smooth dividends relative to earnings. Repurchases are more volatile than dividends."

We have estimated the volatility of dividends and dividends plus share repurchases, using a panel regression on the of firms covered in Table 3. Specifically, we regress annual dividend (plus share repurchase) normalized by total assets, on its lagged value, and calculate volatility as the root mean squared residuals. We find this volatility is 3.3% for the dividends alone and 5.8% for the dividends plus share repurchases. When we carry out the same calculation for our model simulated under the optimal policy, we have a volatility of 8.3%.

2 Some Applications of the Model

2.1 The Dynamics of Leverage

Our model may be viewed as a further development in research on dynamic trade-off theory in which a firm's optimal dynamic policy balances off a variety of financial frictions. Important contributions to this literature include Fischer *et.al.* (1989), Hennessy and Whited (2005) and Strebulaev (2007). This literature was stimulated in part by the observed inability of traditional static trade-off theory to account for a variety of empirical regularities including mean reversion of firm leverage ratios and the negative correlation of the leverage ratio and lagged profitability. Our model has these regularities. In particular, using the data simulated as in Figure 2, and observing variables at yearly intervals, the correlation between leverage and lagged revenues (ρ) is -0.49.

In our model this behavior is the by-product of the firm's dividend and short-term borrowing policy. As seen in Figure 1, over a range of revenues close to their long-term mean an improvement in earnings will usually be saved, because the level of cash holdings is well below its target. Thus, in such moderate business conditions, higher earnings will result in higher levels of cash-holdings which reduces the book value of total debt, while firm value is increasing. In good business conditions, the cash holding is at its target, and so higher earnings will all be paid out, and in fact there will be a reduction in the cash holding. That is, there is a negative "cash flow sensitivity of cash", in the sense of Almeida, Campello and Weisbach (2004). Generally, the change in cash holding bears no simple relation to current cash flow, a point that has been emphasized by Riddick and Whited (2009). These results provide a way of understanding the ambiguous empirical findings concerning the effect on cash holdings, of the rate of cash flow.[17] Also, note that this behavior is independent of any consideration of investment incentives. Thus the dynamics induced by investment as in Riddick and Whited (2009) give an additional complicating factor in understanding changes in cash holdings. We will return to this issue below in Section 3 where we extend the model to allow for growth opportunities.

2.2 "Optimal" Capital Structure

Our model is distinguished from previous contributions (e.g., Hennessy and Whited (2005)) by the fact that we allow for both short-term and long-term debt. As such it is a natural framework to explore the relationship between capital structure, dividends and cash management policy. The key insights can be understood from Figure 3 which depicts the optimal policies for two levels of long-term debt. The "high debt" case depicted in the top panel corresponds to our benchmark where $q = 0.04$. In the lower panel we present the "low debt" case where we set $q = 0.0$ with all other parameters as in the benchmark case.

We see that the effect of a higher level of long-term debt is to raise the liquidity target for any given value of expected earnings. Similarly, at a given value of ρ, the equity issuance boundary is increased (i.e., short-term debt capacity decreased) and the average level of liquidity is increased. The economic intuition for this is that with higher long-term debt the firm requires a greater liquidity buffer to sustain itself through a period of low earnings. Note that in the low debt case the average level of cash holdings is negative for all ρ. That is, on average the firm engages in varying degrees of short-term borrowing as a function of business conditions. It is clear from this analysis that long-term and short-term debt are highly substitutable. With a reduction in long-term debt, a given firm will compensate by reducing its cash holdings and, possibly, borrowing short-term. In this way it will achieve a similar balance of debt tax shields and bankruptcy costs.[18]

What are the implications of this for the optimal "time zero" level of long-term debt?

For a firm with our benchmark parameters the answer is given in Table 1 Panel A where we present firm value for varying levels of q. It is notable that firm value is quite insensitive to q for a wide range of leverage values up to our benchmark $q = 0.04$. For example, at $\rho = 1.2$ by varying q from 0.01 to 0.04 the firm's leverage rises from 11% to 39% but firm value stays within a 1.2% range.

In our view this near-irrelevance of long-term capital structure is one of the important economic insights that emerges from our model and one of the benefits of incorporating long-term debt, short-term debt, cash management and equity issuance all into a single unified framework. For many firms the choice of its long-term debt structure is of second order importance because in choosing its dynamic cash/ short-term debt policy optimally it can achieve approximately the same value for many different values of long-term debt outstanding. However, the degree to which this near-irrelevance property applies across various firms may vary with their other characteristics. In particular, the degree of mean reversion matters. In Figure 4, we have plotted the firm value as a function of q for mean reversion parameter $\kappa = 0.2$, $\kappa = 0.4$ (our benchmark), and $\kappa = 0.6$. For $\kappa = 0.4$ the firm value is quite insensitive to q up to our benchmark value, as already mentioned. This property holds as well for lower speeds of mean reversion, as indicated by the case of $\kappa = 0.2$. For $\kappa = 0.6$, we see that the drop of firm value at high levels of leverage is more marked. A firm with fast mean reversion will find earnings highly concentrated around their long-term average. It will have a lesser need for active cash management and a bigger payoff to carefully choosing a long-term debt policy.

In Figure 4 we also see that for $\kappa = 0.2$, the firm value rises with q, up to the very high value $q = 0.06$. Thus for a slow mean reverting firm, firm value may be fairly insensitive to changes of long-term debt. Still the highest firm values are achieved at very high debt levels. This is similar to the result of of Leland (1994), in which the firm value follows geometric Brownian motion. More recently Strebulaev (2007) argues that the unreasonably high levels of leverage predicted by Leland disappear when the firm is allowed to rebalance its capital structure periodically. Our results show that mean-reverting firms, even with a moderate speed of mean reversion as for our benchmark case, do not need high leverage and do not need to engage in such rebalancing. These observations shed light on recent inconclusive work on whether firms empirically adjust their leverage to an optimum level, for example

Chang and Dasgupta (2009).

2.3 The Pecking Order Hypothesis

One of the first challenges to the static trade-off theory of capital structure came from Myers and Majluf (1983) who explicitly allowed for an informational asymmetry between incumbent shareholders and outside investors. This gave rise to a predicted hierarchy or "pecking order" among alternative modes of finance. While dynamic trade-off models such as ours make no explicit allowance for information asymmetries, the financial frictions such as bankruptcy costs and the cost wedge between inside and outside finance may be viewed as reduced form ways of representing such asymmetries.

Accordingly it is interesting to compare our model's implications to the recent study of pecking order theory by Leary and Roberts (2010). They attempt to overcome the problem of low power which characterizes earlier tests of the pecking order[19] by introducing an empirical model of the pecking order that allows a state dependent nature of financing policy. Depending upon the state, the firm will select financing according to three alternative hierarchies: (a. *pecking order*) internal first, then debt, then equity; (b. *debt capacity constrained*) cash first, then equity; and (c. *cash constrained*) debt first, then equity.

These three cases also emerge in our model where the relevant state variables are ρ and C as can be seen in Figure 1. A classic pecking order (case a) prevails in our model for relatively high values of expected revenue (ρ) where the equity issuance boundary corresponds to negative C. That is, as financing needs would increase (e.g., as a result of a larger negative cash flow shock) they would be met first by drawing down internal funds, then by borrowing short-term and then by issuing equity. Case b corresponds to lower values of ρ where the equity issuance boundary coincides with $C = 0$. And the cash constrained case (c) would correspond to high values of ρ where current cash is negative. Thus our dynamic trade-off model provides a rationale for their empirical model. It is worth noting that while in our model we have not introduced investment opportunities, the same state-dependent pecking order will emerge when the model is extended to allow for such opportunities. We will discuss this in section 3 below.

2.4 The Asset Substitution Effect

Since the analysis of Jensen and Meckling focussed attention on the incentive for shareholders of the levered firm to engage in *ex post* increases in asset volatility, the "asset substitution problem" has become one of the pillars of received wisdom in corporate finance. Based on this, there is a large literature on how this problem can be remedied either by security design (eg, Green (1984)) or by choice of capital structure (eg, Leland (1998)). In this section we re-examine the issue in our model and find that in a dynamic setting the asset substitution effect is reversed in many circumstances.[20]

In our model, there are two sources of earnings volatility: σ refers to a white noise type of volatility which represents a *transient* shock to profitability; η refers to the dynamic of the profitability ρ. Given the mean reverting nature of the profitability relation, η shocks are *persistent*.

In our model, increases in transient volatility decrease both the value of debt and the value of equity, net of cash in the firm. This can be seen from the first 2 diagrams in the top row of Figure 5. In this figure we calculate the value of equity and debt as a function of σ for different levels of profitability. From bottom to top these correspond to $\rho = 0.9$ (blue dots), $\rho = 1.0$ (green dot/dash), $\rho = 1.1$ (red line), $\rho = 1.2$ (turquoise dash), and $\rho = 1.3$ (black dot/dash), respectively. The reason for this is that an increase σ will not increase the firm's expected earnings under any future scenario, but will just increase their uncertainty. In response to this, the firm will increase its liquidity reserve, and equity values will be reduced as a result.

The matter is more complicated when it comes to increases in persistent volatility. There are two opposing forces at work here. First, increasing η increases the level of cash holdings, and this is harmful to both debt and equity. Second, increasing η increases the probability that the firm value will wander to relatively high or low earnings levels. Most of the benefit of the former accrues to equity, and most of the harm of the latter accrues to debt. This is the force reflected in the substitution effect of static theory. In this dynamic model the balance of the two forces depends on the strength of mean reversion. At our benchmark parameters, the equity values increase only slightly as η increases, but the debt and firm values decrease significantly. This can be seen from the third and fourth panels in the top row of Figure 5. For higher rates of mean reversion ($\kappa = 0.6$ first two figures in the bottom row of Figure 5) both equity and debt are hurt by increases in η. However, for much lower speeds of mean reversion, ($\kappa = 0.2$ in the third and fourth graphics of the second row of Figure 5) equity values increase significantly with η and debt values are reduced.

To summarize, the interests of debt and equity are aligned with respect to increases in transient earnings volatility. The interests of debt and equity are also aligned with respect to increases in persistent volatility if the speed of mean reversion is sufficiently high. Only when mean reversion is low do we find that increases in persistent volatility benefit equity at the expense of debt. These results help to understand the fact that empirical studies have been notably unsuccessful in finding any evidence in support of the asset substitution effect (see, Andrade and Kaplan 1998 and Rauh 2006).

2.5 Hedging

Our model also sheds light on corporate incentives to hedge. This has been addressed by Mello and Parsons (2000) and Rochet and Villeneuve (2004) in a continuous time framework. They characterize the benefits of hedging shocks to cash flow. Our results are similar to theirs, if we assume that σdW_t^σ represents a shock to earnings, that can be hedged costlessly, resulting in $\sigma = 0$. For example, if the source of transient shocks to earnings come from fluctuations in commodity prices it may be possible to eliminate these risks with positions in futures contracts. As in our discussion of the asset substitution effect, it is clear that such hedging is of benefit to shareholders, and the benefit comes from the fact that it allows the firm to reduce its use of costly cash balances.

What about hedging the noise represented by dW_t^ρ? The results of the previous section show that whether or not this would be of benefit to equity holders would depend on the firm's technological parameters, in particular the speed of mean reversion. A financial contract representing this risk, if such exists, will have its price, net of financing costs and under the risk neutral probability, given by a martingale. In fact hedging using such a contract can be represented in the model as including an extra term $\Delta\eta dW_t^\rho$ into equation (1), with Δ representing the hedge ratio. Including this term into the analysis results in replacing the term σ^2 in equations (6) and (9) by $\sigma^2 + \Delta^2\eta^2$. Thus, $\Delta\eta$ is playing the same role as σ in our analysis, and from the above, we can conclude that $\Delta = 0$ will give the best hedge.

Thus, the firm cannot benefit[21] from using a financial derivative to hedge the risk represented by the state variable ρ_t. The only way to hedge against this risk would be to alter the firms production technology. These results are suggestive of reasons why active use of derivatives to hedge long-term risks is not more widely observed.

2.6 Some Other Comparative Statics

We have also experimented with alternative values of our model parameters relating to other financial frictions faced by the firm. We briefly summarize the main effects here. The details are given in Appendix B.

In our benchmark parameters we have set bankruptcy costs at 30% of the value of assets in place at the time of bankruptcy. This may seem high compared to some studies in the literature, e.g., Warner (1977). However, those estimates pertain to direct bankruptcy costs typically for firms with large amounts of tangible assets and costs are expressed as a proportion of the book value of assets reflecting historical costs. Furthermore, more recent studies covering indirect costs and a wide range of industries (including those with substantial intangible assets) suggests that total bankruptcy costs may be very substantial. (See Franks and Torous (1989) and Weiss (1990)). By way of comparison, Leland (1994) assumes 50% proportional bankruptcy costs and Leland (1998) assumes 25%.

When we consider lower ($\alpha = 0.05$) and higher ($\alpha = 0.5$) values of this parameter, we find that the effect of higher bankruptcy costs is to depress the value of long-term debt. We find that higher bankruptcy costs also hurt equity values somewhat. This may seem surprising in our model where absolute priority is enforced in bankruptcy. The reason for this effect is that variations in bankruptcy costs affect the firm's capacity to borrow short-term and therefore has an impact on the optimal cash holding policy of the firm. In general, higher bankruptcy costs induce higher levels of cash holding which tend to depress equity values.

In our model the efficiency of external capital markets is captured by the parameter θ which in our benchmark case is set at $\theta = 0.8$. We also consider the extreme cases of $\theta = 0.01$, which may be taken as the case when capital markets very inefficient, and $\theta = 0.95$ which would seem to correspond to a highly efficient capital market and small agency costs.

While θ cannot be directly interpreted as the degree of underpricing, the estimates of the underpricing of IPO's, range from 5 per cent to often greater than 20 per cent. In the U.S. seasoned issues typically involve somewhat lower costs. From the estimates of Lee et.al. (1996) using the U.S. data, reasonable estimates might be 11% for IPO's and 7% for seasoned issues. This suggests that $\theta = 0.95$ would be appropriate only for very highly developed capital markets, and where there is no information asymmetry[22]. When capital market efficiency increases, the average level of liquid asset holding decreases. This result is consistent with Opler *et al* (1999) who document empirically that firms with greater access to capital markets carry smaller amounts of liquidity reserve.

In our framework the cost of holding liquidity inside the firm is represented by $r - r_{in}$. This is a proxy for the amount of rent extraction that is obtained through various forms of managerial moral hazard. Our benchmark value of r_{in} is 5%. In Appendix B we replace this by 4% and 6%.[23] As we would expect, the average level of liquid asset holding is increasing in r_{in}. The values of both equity and debt are increasing in r_{in}. The benefit to shareholders is direct since as residual claims on the firm's cash flows any increase in the return on a given level of liquid securities accrues to them. The benefit to creditors is indirect through the fact that the higher level of cash holding by the firm reduces the chances of bankruptcy.

Finally, we note that when the cash reserve is higher, either under the effect of increasing r_{in} or of decreasing θ, the equity volatility becomes lower. We thus have the counter-intuitive result that increasing market efficiency, represented by higher θ, is associated with higher volatility.

3 Extending Our Model to Include Growth Opportunities

Our basic framework is rich in modeling a wide range of capital market operations available to the firm. However, it is restrictive in that we have not allowed for growth opportunities, which potentially give an additional motive for holding cash. In the online Appendix C we discuss how to extend the model to account growth opportunities, and we will briefly summarize that work here.

The starting point of our extension is the observation that typically firms grow most when their profitability is highest. This statement is based on an analysis of the Compustat universe of non-financial firms that studies average annual capital expenditures as a function of profitability and leverage. Highest growth occurs in the highest profitability quintile. Furthermore, the amount of growth in this quintile is moderate, about 4% in excess of depreciation for all leverage categories combined. (See Appendix C for more details.)

With this observation in view, we first extend our model to allow for a one-time growth opportunity which takes the form of an increase of the scale of the firm's current asset. The size of the opportunity, and its cost, are fixed exogenously. In our extended model, it is optimal for the firm to pursue the growth opportunity only if current expected revenue (ρ) is above a given threshold. We calibrate the growth cost so that this threshold corresponds to the top quintile of ρ, agreeing with the empirical data.

The firm can finance the growth by drawing upon cash, engaging in short-term borrowing, if this will be riskless, or by share issues. Since internal finance is cheaper than external finance, there is an incentive to keep cash inside the firm as the investment opportunity approaches. This is the same behavior as depicted in Figure 1; though, the exact location of the "save", "issue" and "dividend" boundaries are modified taking into account the growth opportunity. However, since the investment opportunity is pursued only when profitability is high, the firm can accumulate the required funds in a short period before the investment opportunity arrives. If the firm faces a moderate size growth opportunity, for example of 4% in line of the average observed in the top earnings quintile of firms, and if its profitability is currently high, then it should target a 4% cash reserve at the time growth will occur. But since the firm can build up this reserve quickly, the growth opportunity has virtually no effect on the cash target, say 3 months before the opportunity is predicted to arrive. If the growth opportunity is exceptionally large, for example 24%, then the targeted cash holdings increase significantly, especially as the time of growth approaches. (See Appendix C for details.)

We also explore alternative forms of growth. For example, we consider a one-time, stochastic growth opportunity whose arrival probability is a constant, independently of the current profitability or size of cash holdings. Alternatively, we consider a non-stochastic growth opportunity that is implemented with a constant growth per month over a succession of months. For growth at 4% per year, either smooth ($\frac{1}{12} \times 4\%$ per month) or lumpy (4% with probability $\frac{1}{12}$ per month), the extra liquidity holding are insignificant. Again, more details are given in Appendix C.

Our conclusion is that the incentive to retain cash in order to pursue growth opportunities is quite insignificant at the growth rates that we see empirically, since when it is optimal to grow, the firm generally has enough current earnings (cash flow) to finance the growth. In fact the motivation to accumulate cash to facilitate growth is more a function of the lumpiness of the growth opportunity.

These extensions of our model to incorporate growth allow us to revisit the question of the firm's marginal propensity to out of cash flow discussed in section 2.1 above. When the firm faces a discrete investment opportunity, the firm accumulates cash ('saves') in anticipation of this, but at the investment date itself, high cash flow is associated with sharp cash *decumulation*. Thus, as emphasized by Riddick and Whited (2009), there is no simple relationship between saving and large, lumpy investment opportunities.

Having extended our model to incorporate growth we can better compare our model with that of Bolton, Chen and Wang (2011). Theirs is similar to ours in the they incorporate market frictions in terms of deadweight costs for retained cash, equity issuance, and bankruptcy. Their firm faces constant returns to scale, as in our Appendix C. They assume that the growth opportunity is available at all times but is subject to a quadratic adjustment cost. This contrasts with our basic case of a one-time, growth opportunity of given size which the firm can undertake at a fixed cost. However, as discussed in Appendix C, we can extend this framework to approximate a liquidity target of a firm that can grow permanently at a given rate every month in the future.

In our model growth is lumpy and will be undertaken when expected future profitability is sufficient attractive. In contrast in Bolton, Chen and Wang (2011)growth is smooth and is determined by the amount of available cash, W which is the only state variable in their model. Specifically the firm allocates its operating income to investment in its own growth, or to increasing the cash reserve W. Its growth rate increases with W, up to a point where operating income is paid as a dividend, rather than to further increasing W. When W is low, the firm chooses negative growth, and will liquidate its assets, if W becomes too low.

A crucial difference between their assumptions and ours, is that we have the profitability as a mean reverting state variable, whereas they assume that the expected revenue is simply proportional to the capital invested. This leads to a difference in the growth outcomes. Our benchmark firm is growing when the profitability is high, and at such times, the liquidity holding is low. In their model the firm must grow when the cash holding is high, because this is the only state variable governing the growth. More generally, in our model liquidity dynamics in a firm whose current earnings are below long-run mean are very different from those for a firm whose current earnings are above that mean. Bolton Chen and Wang (2011) cannot capture this at all due to their scale-invariance assumptions.

4 Conclusion

Using a structural dynamic model of the firm, we have studied the optimal policy for liquid asset holding, dividends and the issuance of both equity and short-term debt. Our model allows the firm profitability to be a mean reverting process, and this can capture the optimal liquidity policy over the business cycle. In this model it is in shareholders' interests to hold large amounts of cash inside the firm even if they have access to reasonably efficient capital markets and even if some of the return to cash is dissipated by insiders. The reason that forcing the firm to pay out "idle" cash in good times may be a bad idea is that it is myopic – it does not properly weight the future dilution costs to shareholders of raising funds, if the firm later were to approach financial distress.

The firm will set a target for the cash reserve, which is decreasing as the current profitability is increasing. Earnings will be paid into the cash reserve, until it has reached the target, after which earnings will be paid out as dividends. We show that on average the cash reserve reaches the target only for relatively high profitability, and the average profile of the cash holdings is a humped function in terms of the firm profitability. This humped shaped relation of cash and profitability is what is observed in data for firms with moderate leverage. Using one choice of parameters the model simultaneously agrees with a number of empirical bench marks including cash holdings, leverage, interest coverage, credit spreads, equity volatility, yield spreads, default probabilities and recovery rates of a typical levered firm.

Our model demonstrates the close link between capital structure and the firm's policy towards liquid asset holding. In particular, higher levels of long-term debt will result in higher levels of liquid asset holding and a reduction in the optimal use of short-term debt. The value of the firm is rather insensitive to the long-term debt level outstanding. The reason is that by adapting its liquidity policy appropriately, the firm is able to balance off its various contracting frictions in such a way as to achieve approximately the same value of the firm for a wide range of long-term debt levels. This result contributes to understanding the fact that empirical studies have not detected clear evidence of firms adapting long-term debt to a target level.

Our model has a state-dependent hierarchy among financial contracting alternatives, which is in line with results of recent tests of the pecking order hypothesis. In contrast, our model casts doubt on the generality of the intuition on the 'asset substitution problem' derived from static corporate finance models. We find that in many circumstances both debt and equity are harmed by increases in cash-flow volatility. A divergence of interests emerges only regarding persistent volatility and only in the case of low rates of mean reversion of earnings. Our model shows that in many circumstances firms have an incentive to hedge short-term shocks to earnings, but that they will not attempt to hedge long-term shocks both because this is difficult and because they have weak incentives to do so.

We extend the model to allow for increasing the scale of the fixed asset. In that extension, the effect of growth on cash holdings is usually fairly minor. The firm will increase its cash holding to prepare for a discrete growth opportunity but this effect is significant only when the growth opportunity large and is imminent. This is consistent with empirical evidence suggesting that firms tend to grow mostly when current profitability is high. Finally, in agreement with recent work we find the rate of corporate savings is highly variable and path dependent. Cash may be accumulated in anticipation of the arrival of a growth opportunity and then decumulated when the investment is realized. This implies that there is no stable relationship between cash flow and rate of cash accumulation.

Notes

[1]Sunday Times of London, April 23, 1995. This quote refers to the attempt by Kirk Kekorian to take over Chrysler Motors arguing that in doing so he could increase shareholder value by returning most of Chrysler's $7.5 billion cash reserve to shareholders.

[2]In section 3 we relax this assumption by allowing for growth opportunities.

[3]This equation also has the property that varying the volatility η does not alter the expectation of ρ_s for given ρ_t with $t < s$. See Duffie (2001). This will be useful when we discuss the asset substitution effect in our model.

[4]As in standard trade-off models, the tax deductibility of interest payments provides an incentive to issue debt. Recently, DeMarzo and Sannikov (2006) have studied the problem of security design in a continuous time model without taxes and find that the optimal capital structure involves a fixed amount of long-time debt and varying short-term debt as in our model.

[5]This is similar to the treatment of corporate taxes in Hennessey and Whited (2005). They model tax write-back and carry-forward provisions by allowing the tax rate to be progressive in terms of earnings. However, their estimated parameters imply that the marginal tax rate is essentially constant over the economically relevant range.

[6]Note that in our formulation bankruptcy occurs when our state variable evolving along a continuous sample path attains an absorbing barrier. Thus unlike reduced-form credit risk models there is no "jump-to-default risk", e.g., as in Duffie and Singleton (1999). They discount cash flows at the rate $r + \xi_t$, where ξ_t is the instantaneous hazard rate of default. In this terminology, our model assumes $\xi_t = 0$.

[7]Proof: The firm will issue shares if this increases the total equity value, after subtracting the liability represented by the newly issued shares, i.e. if $J(C + \delta C) - ns > J(C)$. Differentiating this, and using $\delta C = ns/\theta$, gives Equation (8) of the text.

[8]The solution must satisfy the value matching and smooth pasting condition across these boundaries. Note that we do not impose this condition directly in our numerical procedure described in Appendix A, but it is satisfied as a consequence of the maximum criterion, which we impose directly.

[9]To see that this boundary condition corresponds to reflection, i.e. to the fact that if C_t hits the boundary, then it is pushed back at infinite speed, note that if we had say $\frac{\partial}{\partial C} P_t^q > 0$, and it hit the barrier from below, then a long position in P would be sure to gain in value, which is an arbitrage opportunity.

[10]These estimates refer to mean reversion of the risk neutral cash flow process. We take this to apply to the physical default process in our model. In fact the two speeds of mean reversion will differ in general but in our model by an amount that is small for our purposes.

[11]A relatively high relationship premium may be appropriate for a smaller or younger firm. See Petersen and Rajan (1998).

[12]Smith(1977) estimates direct underwriting costs for seasoned equity issues to exceed 6% on average rising to over 13% on smaller issues. He also documents a significant price impact and other indirect costs.

[13]Like us Leland (1998) works with a model where absolute priority is respected in bankruptcy and assumes proportional bankruptcy costs of 25%. Anderson and Sundaresan (1996) show that observed yield spreads can be replicated assuming lower bankruptcy costs in a model allowing for strategic debt service.

[14]In Figure 1 the average cash holdings turns up slightly for very low levels of ρ. This reflects a survivorship bias in our sample in that firms that suffer a drain of cash at low profitability disappear from the sample through bankruptcy leaving only those firms that had amassed sufficient cash before a decline of profitability so that they were able to survive even as ρ declined to exceptionally low levels.

[15]Such λ has to exist, in the absence of arbitrage: see Duffie (2001).

[16]As seen in Table 2 different levels of ρ in our model yields ratios which would correspond to average ratios within different ratings classes. This is not to suggest that actual ratings fluctuate in step with business conditions in a similar manner, for it is well-known that ratings firms tend to rate "through the cycle".

[17]Kim *et al* (1998) find a negative influence of cash flow. Opler *et al* (1999) find a positive effect in most specifications. Dittmar *et al* (2003) and Anderson (2003)find the sign of the cash flow variable is sensitive to the data set used and the model specification.

[18]In the lower panel of Figure 3 average cash holdings coincide with the equity issue barrier for low levels of profitability. In this range of low profitability the firm has exhausted its ability to borrow (i.e., reduce cash holdings) and therefore survives by issuing equity. Therefore for this low long-term debt firm the average realized cash holdings displays more of an S-shape than a pronounced hump-shape seen in a firm with significant long-term debt issues.

[19]Most earlier tests of pecking order followed the approach introduced by Shyam-Sunder and Myers (1999).

[20]In their continuous time model with cash accumulation and short-term and long-term debt DeMarzo and Sannikov (2006) also find a convergence of interests of debt and equity. They characterize the result as "surprising" but do not pursue the issue further.

[21]As noted by Mello and Parsons (2000), a hedge that increases equity value, is different from one that reduces equity volatility. The latter could be achieved by shorting the equity index future, assuming that the firm has positive equity β, but this would not alter the equity value, unless it introduced some change in the firm's contracting frictions.

[22]Based on a sample of US, non-financial firms between 1993 and 2001 a period of very high stock market activity Hennessy and Whited (2005) estimate floatation costs of about 5.9%

[23]Note that the latter implies no moral hazard, but with this value, the liquidity holding is still bounded by its being tax-inefficient.

References

Allen, F. and R. Michaely, 2003, Payout Policy, *Handbook of the Economics of Finance Volume 1*, North Holland, pp 337-429.

Almeida,H. M. Campello, and M.S. Weisbach, 2004, The Cash Flow Sensitivity of Cash *The Journal of Finance* Vol 59, Issue 4, pp 17771804.

Anderson, R. W., 2003, Capital Structure, Firm Liquidity and Growth, in P.Butzen and C.Fuss (eds.), *Firms' Investment and Finance Decisions*, Edward Elgar.

Anderson, R.W. and S. Sundaresan, 1996, Design and Valuation of Debt Contracts, *Review of Financial Studies*, Vol 9, pp 37-68.

Andrade, G. and S.N. Kaplan, 1998, How Costly Is Financial (Non-Economic) Distress? Evidence from Highly Leveraged Transactions that Became Distressed, *Journal of Finance*, Vol 53, pp 1443-1493.

Bolton, P. H. Chen and N. Wang, 2011 A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management, Working paper, forthcoming *Journal of Finance*.

Bolton, P. and X. Freixas, 2000 Equity, Bonds and Bank Debt: Capital Structure and Financial Market Equilibrium under Asymmetric Information, *Journal of Political Economy*, Vol 108, pp 324-351.

Brav, A, Graham, J. R., Harvey, C. R. and Michaely, R, 2005 Payout Policy in the 21st Century, *Journal of Financial Economics*, Vol 77, pp 483 - 527.

Chang X. and Dasgupta, S, 2009, Target Behavior and Financing: How Conclusive is the Evidence?, *Journal of Finance*, Vol 64, pp 1767-1796.

De Marzo, P. and Y.Sannikov, 2006, Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model, *Journal of Finance*, Vol 61, pp 2681-2724.

de Servigny, A. and O. Renault, 2004, *Measuring and Managing Credit Risk.* McGraw-Hill.

Dittmar, A., J. Mahrt-Smith, and H. Servaes, 2003, International Corporate Governance and Corporate Cash Holdings, *Journal of Financial and Quantitative Analysis*, Vol. 38, 111-133.

Duffie, D., 2001, *Dynamic Asset Pricing Theory*, 3rd.ed. Princeton, Princeton University Press.

Duffie, D. and D. Lando, 2001, Term Structure of Credit Spreads with Incomplete Accounting Information, *Econometrica*, Vol 69, pp 633 - 664.

Duffie, D. and K.J. Singleton, 1999, Modeling Term Structures of Defaultable Bonds, *Review of Financial Studies*, Vol 12, pp 687-720.

Fama, E.F., and K. R. French, 2000, Forecasting Profitability and Earnings, *Journal of Business* 73, 161-175.

Franks, J. and W. Torous, 1989, An Empirical Investigation of U.S. Firms in Reorganization, *Journal of Finance*, Vol 44, 747-769.

Fischer, E.O., R. Heinkel, and J. Zechner, 1989, Dynamic Capital Structure Choice: Theory and Tests, *Journal of Finance*, Vol 44, pp 19-40.

Garca-Feijo, L, and R.D. Jorgensen, 2007, Can Operating Leverage be the Cause of the Value Premium?, Working Paper, CFA Institute.

Gomes, J., 2001 Financing Investment, *American Economic Review*, 91, 1263-1285.

Green, R, 1984, Investment Incentives, Debt and Warrants, *Journal of Finance*, Vol. 39, 115-136.

Hennessy, C. A. and T. M. Whited 2005 Debt Dynamics, *Journal of Finance*, 60, 1129-1164.

Kim C.-S., D. C. Mauer, and A. E. Sherman, 1998, The determinants of corporate liquidity: Theory and Evidence, *Journal of Financial and Quantitative Analysis* , Vol 33, pp 305-334.

Leary, M.T. and M.R Roberts, 2010, The Pecking Order, Debt Capacity and Information Asymmetry. *Journal of Financial Economics.* Vol 95, Issue 3, pp 332-355.

Lee, I., S. Lockhead, J. Ritter, and Q. Zhao, 1996, The Costs of Raising Capital, *Journal of Financial Research*, Vol 1.

Leland, H., 1994, Risky Debt, Bond Covenants and Optimal Capital Structure, *Journal of Finance*, Vol 49, pp 1213-1252.

Leland, H., 1998 Agency Costs, Risk Management and Capital Structure, *Journal of Finance.* 53, 1213-1243.

Mello, A. and J. Parsons, 2000, Hedging and Liquidity, *Review of Financial Studies*, Vol 13, pp 127-153.

Myers, S. C. and N.S. Majluf , 1984 Corporate Financing and Investment Decisions When Firms Have Information That Investors Do Not Have, *Journal of Financial Economics* Vol. 13, pp 187-221.

Myers, S. and R.G. Rajan, 1998, The Paradox of Liquidity, *Quarterly Journal of Economics*, Vol 5, pp 733-771.

Opler, T., L. Pinkowitz, R. Stulz, and R. Williamson, 1999, The determinants and implications of corporate cash holdings, *Journal of Financial Economics*, Vol 52, pp 3-46.

Pastor, L. and P. Veronesi, 2003 Stock Valuation and Learning about Profitability *Journal of Finance*, 58, 1749-1789.

Petersen, M.A. and R. Rajan 1994 The Benefits of Lending Relationships: Evidence from Small Business Data, *Journal of Finance.* Vol 49, 3-37.

Rauh, J. 2009 Risk Shifting versus Risk Management: Investment Policy in Corporate Pensions Plans, *Review of Financial Studies* Vol. 22, 2487-2534.

Riddick, L.A. and T.M Whited 2009 The Corporate Propensity to Save, *Journal of Finance* Vol. 64, 1729-1766.

Rochet, J.-C. and S. Villeneuve, 2004, Liquidity Risk and Corporate Demand for Hedging and Insurance, working paper, GREMAQ, Toulouse University.

Shyam-Sunder, L and S. Myers, 1999, Testing Static Trade-off Against Pecking Order Models of Capital Structure. *Journal of Financial Economics.* Vol 51, 219-244.

Smith, C. W., 1977, Alternative Ways of Raising Capital: Rights Versus Underwritten Offerings, *Journal of Financial Economics.* Vol 5, 273-307.

Strebulaev, I. 2007 Do Tests of Capital Structure Theory Mean What They Say? *Journal of Finance*, Vol 62, 1747-1787.

Warner, J., 1977, Bankruptcy Costs: Some Evidence, *Journal of Financial Economics*, Vol 5, pp 337-347.

Weiss, L., 1990, Bankruptcy Resolution: Direct Costs and the Violation of Priority of Claims, *Journal of Financial Economics*, Vol 27, pp 419-444.

TABLE 1

Panel A - Values of Equity, Debt and Liquidity

Payment to debt, q	0.00	0.01	0.02	0.03	0.04	0.05	0.06
Measures at $\rho = 0.90$							
Liquidity	-0.2015	-0.0864	-0.0002	0.0373	0.0399	0.0400	0.0388
Dividend	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Equity val	0.6979	0.5466	0.4063	0.2982	0.2153	0.1496	0.0987
Debt val	0.0000	0.1626	0.2976	0.3782	0.4294	0.4077	0.4054
Net firm val	0.6979	0.7092	0.7039	0.6765	0.6447	0.5573	0.5041
Leverage	0.0000	0.2611	0.4229	0.5299	0.6272	0.6826	0.7467
Equity volatility	0.5214	0.5704	0.6457	0.7521	0.9334	1.1478	1.3915
Measures at $\rho = 1.00$							
Liquidity	-0.1566	-0.0476	0.0279	0.0679	0.0718	0.0718	0.0679
Dividend	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Equity val	0.8670	0.7180	0.5779	0.4668	0.3784	0.3040	0.2415
Debt val	0.0000	0.1633	0.3035	0.3984	0.4676	0.4773	0.5003
Net firm val	0.8670	0.8813	0.8814	0.8652	0.8460	0.7813	0.7419
Leverage	0.0000	0.1959	0.3338	0.4270	0.5095	0.5595	0.6179
Equity volatility	0.3639	0.3893	0.4326	0.4834	0.5610	0.6480	0.7499
Measures at $\rho = 1.10$							
Liquidity	-0.1304	-0.0234	0.0486	0.0896	0.0958	0.0977	0.0939
Dividend	0.0042	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
Equity val	1.0268	0.8790	0.7394	0.6268	0.5358	0.4574	0.3898
Debt val	0.0000	0.1637	0.3066	0.4089	0.4875	0.5138	0.5490
Net firm val	1.0268	1.0427	1.0460	1.0357	1.0233	0.9712	0.9388
Leverage	0.0000	0.1606	0.2801	0.3634	0.4356	0.4807	0.5317
Equity volatility	0.2941	0.3096	0.3372	0.3680	0.4116	0.4587	0.5126
Measures at $\rho = 1.20$							
Liquidity	-0.1498	-0.0383	0.0282	0.0724	0.0843	0.0913	0.0907
Dividend	0.0322	0.0359	0.0561	0.0628	0.0423	0.0206	0.0335
Equity val	1.1830	1.0358	0.8967	0.7831	0.6903	0.6093	0.5384
Debt val	0.0000	0.1639	0.3085	0.4153	0.5000	0.5357	0.5801
Net firm val	1.1830	1.1997	1.2052	1.1984	1.1903	1.1450	1.1185
Leverage	0.0000	0.1411	0.2501	0.3268	0.3923	0.4333	0.4797
Equity volatility	0.2641	0.2743	0.2970	0.3195	0.3494	0.3808	0.4165
Measures at $\rho = 1.30$							
Liquidity	-0.2255	-0.0946	-0.0191	0.0178	0.0350	0.0449	0.0494
Dividend	0.1882	0.1352	0.0698	0.2192	0.2213	0.2094	0.2236
Equity val	1.3383	1.1915	1.0529	0.9388	0.8448	0.7619	0.6887
Debt val	0.0000	0.1640	0.3098	0.4196	0.5086	0.5506	0.6017
Net firm val	1.3383	1.3555	1.3628	1.3584	1.3533	1.3125	1.2903
Leverage	0.0000	0.1301	0.2306	0.3049	0.3663	0.4057	0.4491
Equity volatility	0.2547	0.2587	0.2751	0.2967	0.3203	0.3457	0.3729

TABLE 1 (continued)

Panel B - Credit Spreads and Bankruptcy Probabilities

Payment to debt, q	0.00	0.01	0.02	0.03	0.04	0.05	0.06
Measures at $\rho = 0.9$							
Credit spreads:							
5 year PDB	0.0001	0.0020	0.0105	0.0289	0.0516	0.0886	0.1212
20 year PDB	0.0001	0.0016	0.0075	0.0186	0.0271	0.0474	0.0463
Prob. bankrupt after:							
1 year	0.0000	0.0005	0.0066	0.0305	0.0807	0.1563	0.2733
5 years	0.0006	0.0098	0.0525	0.1396	0.2547	0.3768	0.5195
20 years	0.0020	0.0307	0.1450	0.3320	0.5231	0.6765	0.8080
Measures at $\rho = 1.0$							
Credit spreads:							
5 year PDB	0.0001	0.0011	0.0061	0.0172	0.0312	0.0531	0.0739
20 year PDB	0.0001	0.0013	0.0064	0.0158	0.0240	0.0409	0.0441
Prob. bankrupt after:							
1 year	0.0000	0.0001	0.0010	0.0059	0.0181	0.0400	0.0790
5 years	0.0003	0.0055	0.0309	0.0850	0.1600	0.2435	0.3463
20 years	0.0017	0.0259	0.1236	0.2872	0.4602	0.6056	0.7377
Measures at $\rho = 1.1$							
Credit spreads:							
5 year PDB	0.0000	0.0007	0.0040	0.0113	0.0211	0.0358	0.0514
20 year PDB	0.0001	0.0012	0.0057	0.0143	0.0223	0.0375	0.0432
Prob. bankrupt after:							
1 year	0.0000	0.0000	0.0001	0.0009	0.0033	0.0082	0.0182
5 years	0.0002	0.0034	0.0200	0.0564	0.1086	0.1693	0.2466
20 years	0.0015	0.0233	0.1118	0.2621	0.4243	0.5644	0.6960
Measures at $\rho = 1.2$							
Credit spreads:							
5 year PDB	0.0000	0.0005	0.0027	0.0079	0.0151	0.0260	0.0379
20 year PDB	0.0001	0.0011	0.0053	0.0134	0.0210	0.0352	0.0412
Prob. bankrupt after:							
1 year	0.0000	0.0000	0.0000	0.0001	0.0006	0.0016	0.0039
5 years	0.0001	0.0022	0.0136	0.0398	0.0788	0.1253	0.1866
20 years	0.0014	0.0216	0.1042	0.2462	0.4017	0.5384	0.6694
Measures at $\rho = 1.3$							
Credit spreads:							
5 year PDB	0.0000	0.0003	0.0019	0.0058	0.0113	0.0197	0.0290
20 year PDB	0.0001	0.0010	0.0051	0.0127	0.0201	0.0335	0.0394
Prob. bankrupt after:							
1 year	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0009
5 years	0.0001	0.0015	0.0096	0.0292	0.0593	0.0964	0.1464
20 years	0.0013	0.0204	0.0988	0.2348	0.3856	0.5198	0.6503

TABLE 2: Comparison of Model Solution to Empirical Benchmarks

The empirical benchmarks are 1997-1999 median values for US industrial firms as reported by Standard and Poors for firms rated A, BBB and BB (Source, Standard and Poors, "Adjusted Key Ratios," *Credit Week* September 20, 2000). The Model values are based on the simulation of the model under the benchmark parameters. The model equivalent of EBIT interest coverage is computed as $(\rho - f)/q$. Return on capital is $(\rho - f)/(P + J)$. The Long Term Debt to Capital ratio is $P/(P + J)$. The Total Debt to Capital ratio is $(P - C)/(P + J)$. Debt/Equity is $(P - C)/J$. Column 4 pertains to simulated model values evaluated at $\rho = 1.1$ and column 5 is evaluated at $\rho = 1.2$.

	A	BBB	BB	Model ρ=1.1	Model ρ=1.2
EBIT interest coverage	6.8	3.9	2.3	2.500	5.000
Return on capital	0.199	0.140	0.117	0.098	0.168
LT Debt/Capital	0.325	0.41	0.558	0.476	0.420
Total Debt/Capital	0.401	0.474	0.613	0.406	0.340
Debt/Equity	0.197	0.437	0.905	0.910	0.724

TABLE 3: Cross-sectional Evidence on Liquidity and Payout to Equity

This table reports firm liquidity as measured by new working capital and annual payout rates to equity holders measured as dividends plus share repurchases, cross tablulated by earnings quintile and long-term debt level. L is long-term debt divided by total assets. The firms included are the Compustat universe of non-financial firms with total assets of at least 100 million 2000 dollars. All variables are normalized by total assets at year end. The numbers reported are averages weighted by total assets (with total assets winsorized at 1000 million 2000 dollars). The time period is 1978-2008.

Net Working Capital						
	Profitability Quintile					
Leverage category	Lowest	2	3	4	Highest	Average
$0.0 < L \leq 0.1$	28.56%	25.15%	27.35%	29.10%	31.07%	28.25%
$0.1 < L \leq 0.2$	19.29%	18.35%	19.43%	19.59%	19.78%	19.29%
$0.2 < L \leq 0.3$	13.19%	14.97%	14.70%	12.92%	9.95%	13.14%
$0.3 < L \leq 0.4$	6.71%	9.58%	11.19%	10.75%	6.26%	8.90%
$0.4 < L \leq 0.5$	5.93%	6.47%	8.26%	4.86%	2.78%	5.66%
$0.5 < L$	9.22%	2.84%	2.28%	7.64%	4.13%	5.22%
Average	13.82%	12.89%	13.87%	14.14%	12.33%	13.41%
Dividend plus Share Repurchases						
	Profitability Quintile					
Leverage category	Lowest	2	3	4	Highest	Average
$0.0 < L \leq 0.1$	1.20%	2.06%	2.91%	5.26%	8.84%	4.05%
$0.1 < L \leq 0.2$	1.17%	1.70%	2.37%	4.00%	8.00%	3.45%
$0.2 < L \leq 0.3$	1.06%	1.71%	2.24%	3.69%	5.61%	2.86%
$0.3 < L \leq 0.4$	1.08%	1.14%	1.83%	2.77%	5.01%	2.37%
$0.4 < L \leq 0.5$	1.00%	1.23%	1.51%	2.03%	3.30%	1.81%
$0.5 < L$	2.29%	1.00%	2.09%	3.20%	3.40%	2.39%
Average	1.30%	1.47%	2.16%	3.49%	5.69%	2.82%

Figure 1

This figure depicts the regions of ‘save’ (i.e. do retain earnings, and do not pay dividends), ‘pay dividends’ and ’issue equity’, arising from the solution to the PDEs of Section 1.3. The ‘save’ region $\mathcal{S}$ is depicted by diamonds. The target liquid asset holding is the upper boundary of the $\mathcal{S}$ region and is depicted by downward triangles. In this region the firm pays dividends. The region in which the firm will issue new equity is indicated by upward triangles. These also indicate the lower limit of liquidity, or equivalently minus the limit of the short term borrowing facility. The points where shareholders would pay a liquidating dividend and abandon the firm to creditors are indicated with circles. The realized liquid asset holding, averaged over 2000 simulations of the firm history, as a function of ρ, is depicted by the solid line.

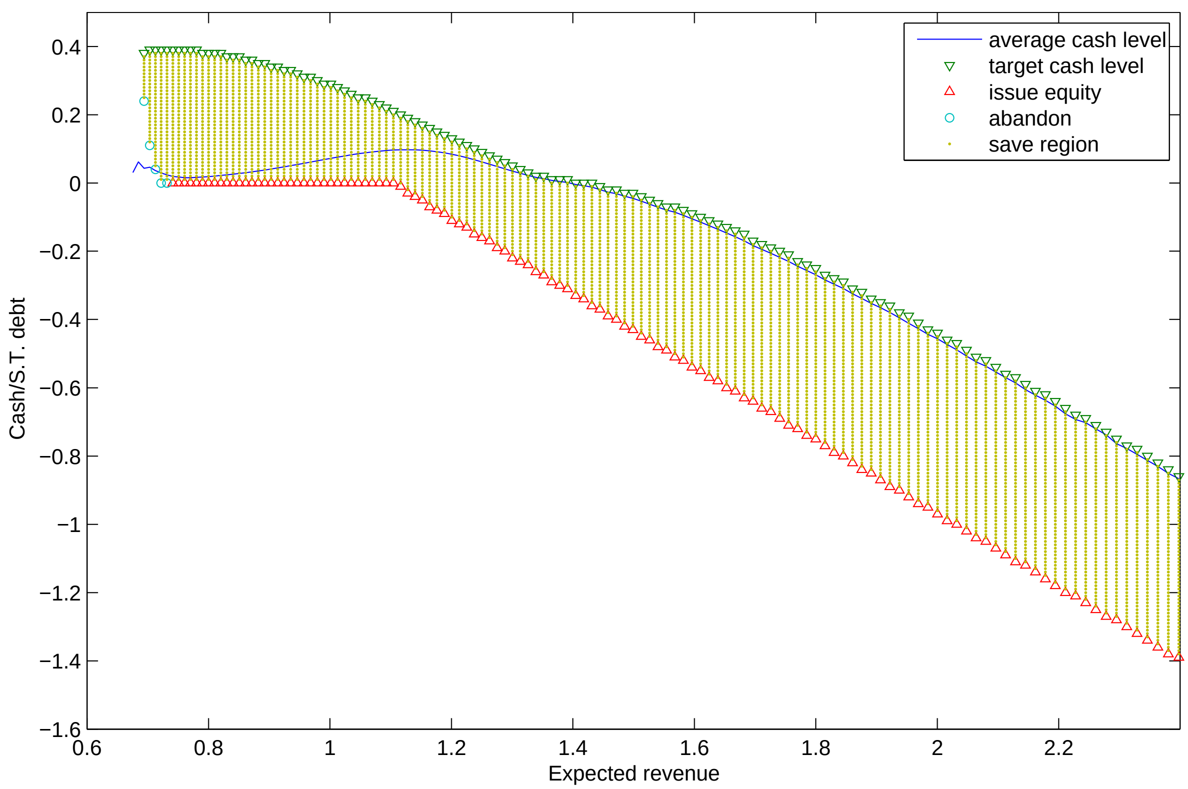
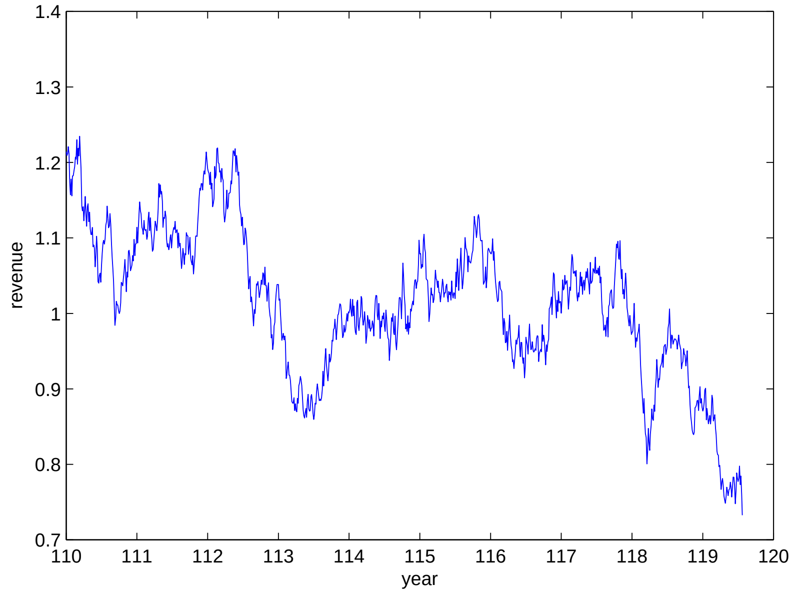


Figure 2

Simulated time series of the expected revenues ρ_t, (top graph), and the liquidity reserve C_t, equity value J_t, and total firm value $J_t + P_t$ (all on the bottom graph, in increasing order).



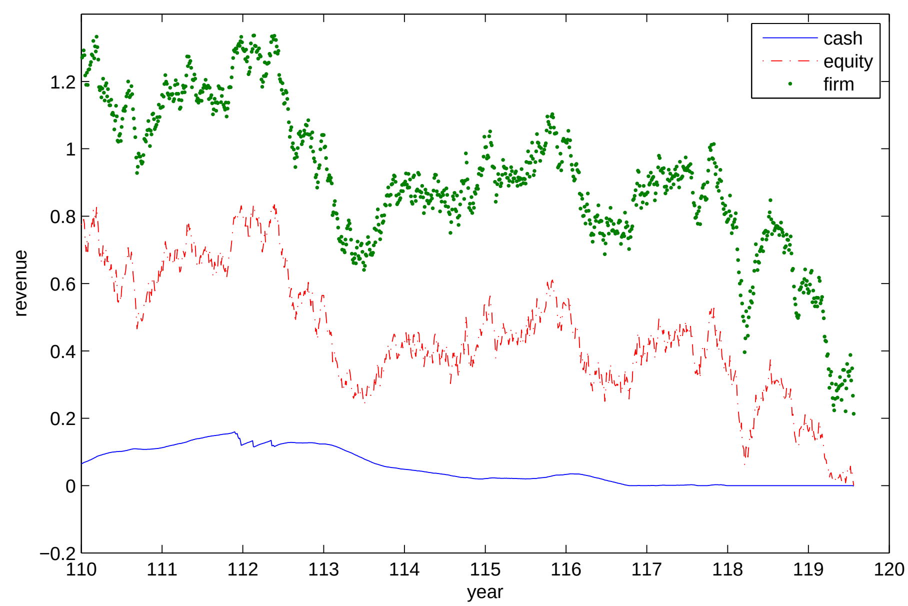
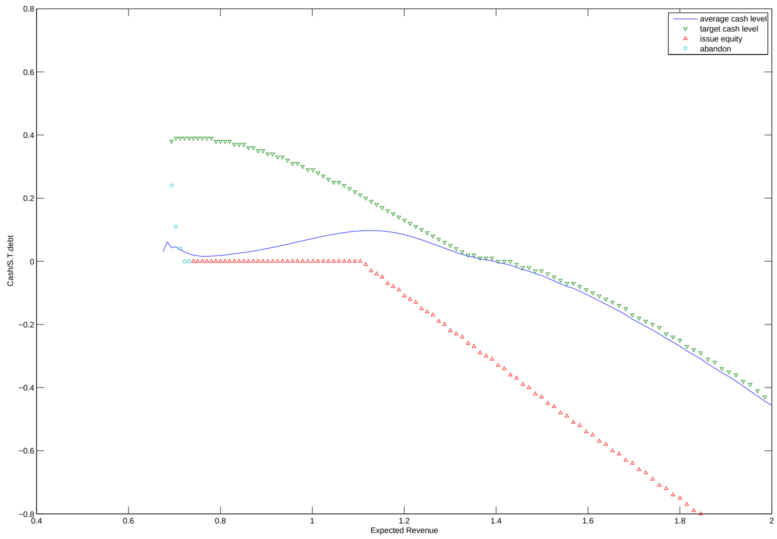


Figure 3

This figure depicts optimal policy as in Figure 1 for two different levels of long-term debt. The top panel is for the "high debt" case, $q = 0.04$. The bottom panel is for the "low debt" case, $q = 0.0$.



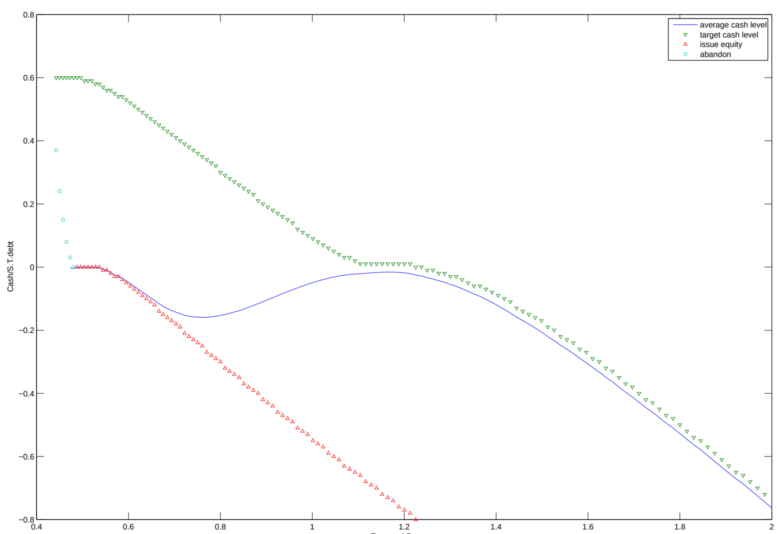
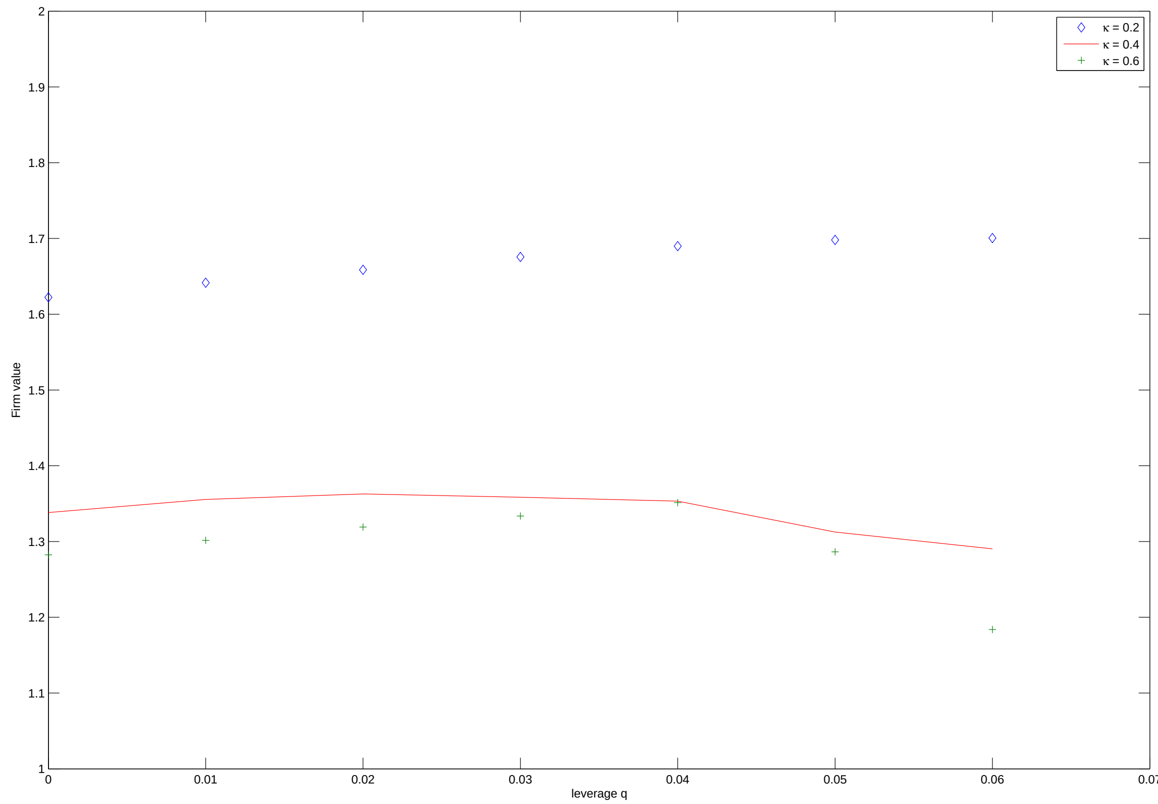


Figure 4:

This figure depicts total firm value as a function of long-term debt, q, with separate graphs for three values of mean reversion κ. The evaluations are all done for $\rho = 1.3$.



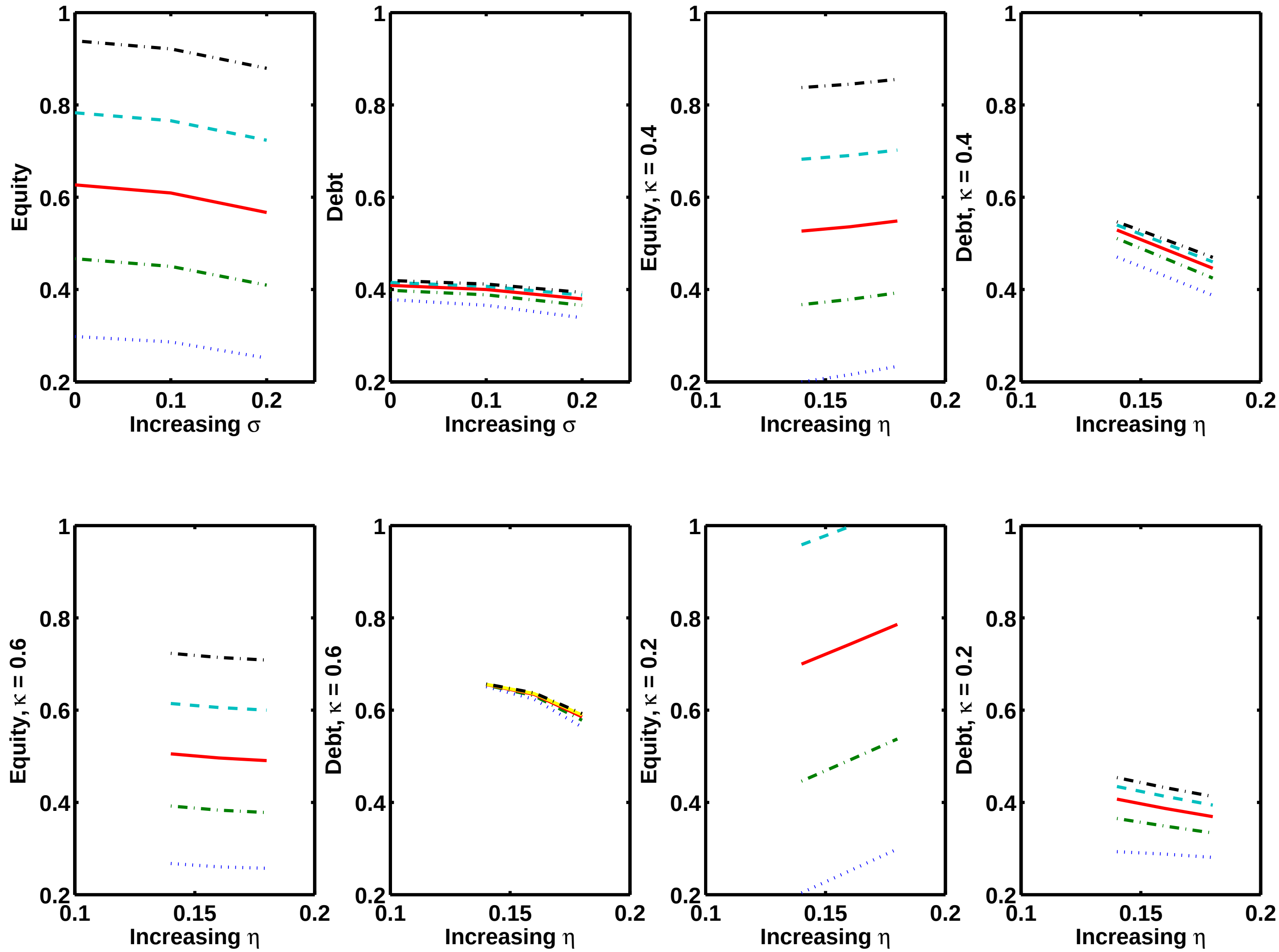


FIGURE 5: Asset Substitution

Liquidity and Capital Structure: Appendices

Ronald W. Anderson

Andrew Carverhill

A Appendix: Numerical Techniques for Solving the PDEs for Valuing Debt and Equity

Our strategy for valuing the equity is to solve Equation (6) numerically, by finite difference procedures, evolving backwards from a distant horizon time $t = T$, at which we assume the firm is liquidated. We have also changed the variable ρ to $\xi := \frac{1}{2}\rho^{\frac{1}{2}}$. This change helps the numerical stability of the scheme because after we have expressed Equation (2) in terms of η, the noise coefficient becomes constant. Also, under this transformation, the solution is more detailed for low values of ρ, which are more important.

We use the explicit finite difference scheme (see Ames (1992)), representing the ρ space by a grid with 401 points, ranging from 0.0 to $\rho_{max} = 5.0$, and we represent C by a grid with 401 points ranging from $C_{min} = -2.0$ to $C_{max} = 2.0$. With these parameters, and for all the parameters in the text, the finite difference scheme is numerically stable, if we take a time step of length 0.001. Also, we take $T = 50$ years. By experimentation, we have determined that the solutions are insensitive to first order variations of these parameters. To determine the regions $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{I}$, we test, at every time step and every grid point point representing (ρ, C), whether it is optimal to pay dividends, issue shares, or abandon the firm.

Our numerical scheme for valuing the debt is the same as for the equity. The debt boundary condition for bankruptcy is $(1-\alpha)J^0(\rho, C)$, and this condition is calculated using an implementation of the equity valuation for zero long term debt. Our calculations for the credit spread, the probability of default at horizon 20 years, etc, are not based on evolution to a steady state. These are obtained by taking $T = 20$ years, etc, in the above. Although it might be more usual to use the technique of Successive Over-Relaxation (SOR) to obtain our steady state solution (see Ames (1992)), our finite difference scheme is more useful for studying how quickly the steady state is achieved, and for dealing with these non-steady state calculations.

Our horizon boundary conditions for debt and equity at $t = T$ are as follows: To value unlevered equity for the valuations on bankruptcy, we take $J_T^{unlevered}(\rho, C) = \max\{C, 0\}$, reflecting abandonment at time T. For the debt horizon value we take $P_T^q(\rho, C) = \min\{q/r, (1-\alpha)J_0^{unlevered}(\rho, 0) + C\}$, and for the equity horizon value we take $J_T^q(\rho, C) = \max\{(1-\alpha)J_0^{unlevered}(\rho, 0) + C - q/r, 0\}$. These reflect the assumption that if bankruptcy has not occurred by time T, the productive asset is sold, incurring the bankruptcy costs. The face value of the debt, i.e.q/r, is paid out of the proceeds plus cash reserve to the extent possible, and the rest goes to the equity holders. Note that the choice of terminal time T condition does not matter, if T is sufficiently far away, but good choices of the horizon equity values, etc. allow T to be taken smaller, which is more efficient. Our choices prevent arbitrage at time T.

We must also choose boundary conditions at high and low values of $\xi \equiv \frac{1}{2}\rho^{\frac{1}{2}}$ and C. The lowest value of ρ is zero, and since $d\rho/d\xi = 0$, then any smooth boundary condition here for $\frac{\partial}{\partial \rho}J$ will translate to $\frac{\partial}{\partial \xi}J = 0$. We thus impose this condition. At the upper limit of ξ, we impose the boundary condition $\frac{\partial^2}{\partial \xi^2}J = 0$.

B Appendix: Sensitivity of the Model to Parameter Changes

Each column of the following table gives the firm characteristics as in Table 1, at the benchmark parameter values, except for the parameter value given in the first row.

Values of Equity, Debt and Liquidity

Parameter being varied	None	η		σ		κ		α		θ		r_{in}	
Parameter value	(Repeat	0.14	0.18	0.1	0.2	0.3	0.5	0.05	0.5	0.01	0.95	0.04	0.06
Benchmark value	Benchmark)	0.16	0.16	0.0	0.0	0.4	0.4	0.3	0.3	0.80	0.80	0.05	0.05
Measures at $\rho = 0.9$													
Liquidity	0.0399	0.0299	0.0485	0.1002	0.1932	0.0398	0.0394	0.0359	0.0408	0.2056	0.0023	0.0259	0.0663
Dividend	0.0000	0.0000	0.0000	0.0001	0.0103	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Net equity val	0.2153	0.1997	0.2335	0.2047	0.1753	0.2185	0.2314	0.2170	0.2148	0.1711	0.2735	0.2098	0.2238
Debt val	0.4294	0.4703	0.3874	0.3891	0.3602	0.3200	0.5224	0.4250	0.4184	0.2277	0.5154	0.3978	0.4204
Net firm val	0.6447	0.6700	0.6209	0.5938	0.5355	0.5385	0.7538	0.6420	0.6332	0.3989	0.7890	0.6076	0.6442
Leverage	0.6272	0.6719	0.5788	0.5607	0.4943	0.5534	0.6586	0.6269	0.6207	0.3768	0.6515	0.6279	0.5916
Equity volatility	0.9334	0.9344	0.9295	0.8432	0.8392	1.0626	0.7774	0.9451	0.9308	0.6115	0.8296	1.0115	0.8175
Measures at $\rho = 1.0$													
Liquidity	0.0722	0.0616	0.0806	0.1223	0.2049	0.0725	0.0728	0.0655	0.0736	0.2443	0.0124	0.0536	0.1047
Dividend	0.0000	0.0000	0.0000	0.0030	0.0234	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Net equity val	0.3784	0.3673	0.3930	0.3618	0.3250	0.4111	0.3727	0.3802	0.3777	0.3336	0.4263	0.3734	0.3858
Debt val	0.4676	0.5105	0.4247	0.4357	0.4112	0.3764	0.5469	0.4644	0.4586	0.3148	0.5402	0.4419	0.4613
Net firm val	0.8460	0.8778	0.8177	0.7975	0.7361	0.7874	0.9195	0.8446	0.8363	0.6484	0.9665	0.8153	0.8470
Leverage	0.5093	0.5435	0.4728	0.4737	0.4369	0.4377	0.5511	0.5103	0.5040	0.3527	0.5518	0.5085	0.4847
Equity volatility	0.5603	0.5250	0.5902	0.5606	0.6111	0.6322	0.4801	0.5680	0.5587	0.4306	0.5584	0.5943	0.5109
Measures at $\rho = 1.1$													
Liquidity	0.0956	0.0853	0.1039	0.1355	0.2106	0.0949	0.0965	0.0890	0.0974	0.2679	0.0227	0.0749	0.1310
Dividend	0.0000	0.0000	0.0000	0.0169	0.0608	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
Net equity val	0.5358	0.5267	0.5486	0.5177	0.4785	0.6039	0.5049	0.5380	0.5350	0.4894	0.5795	0.5315	0.5422
Debt val	0.4875	0.5291	0.4460	0.4599	0.4376	0.4087	0.5586	0.4850	0.4795	0.3582	0.5531	0.4649	0.4828
Net firm val	1.0233	1.0558	0.9946	0.9776	0.9161	1.0126	1.0635	1.0230	1.0145	0.8476	1.1325	0.9964	1.0250
Leverage	0.4357	0.4637	0.4060	0.4132	0.3884	0.3690	0.4815	0.4362	0.4313	0.3212	0.4788	0.4340	0.4177
Equity volatility	0.4118	0.3749	0.4445	0.4271	0.4800	0.4609	0.3606	0.4154	0.4107	0.3401	0.4259	0.4302	0.3845
Measures at $\rho = 1.2$													
Liquidity	0.0846	0.0673	0.0990	0.1196	0.1920	0.0837	0.0859	0.0784	0.0866	0.2539	0.0110	0.0629	0.1212
Dividend	0.0421	0.0698	0.0240	0.0551	0.1208	0.0341	0.0483	0.0382	0.0435	0.0469	0.0558	0.0466	0.0389
Net equity val	0.6903	0.6823	0.7020	0.6719	0.6321	0.7964	0.6334	0.6927	0.6893	0.6427	0.7324	0.6865	0.6961
Debt val	0.5000	0.5398	0.4598	0.4742	0.4535	0.4290	0.5653	0.4977	0.4924	0.3771	0.5610	0.4788	0.4955
Net firm val	1.1903	1.2221	1.1618	1.1461	1.0856	1.2254	1.1987	1.1903	1.1817	1.0197	1.2933	1.1653	1.1916
Leverage	0.3922	0.4186	0.3647	0.3747	0.3550	0.3277	0.4401	0.3923	0.3882	0.2961	0.4301	0.3899	0.3775
Equity volatility	0.3493	0.3179	0.3777	0.3646	0.4105	0.3849	0.3121	0.3513	0.3485	0.3002	0.3606	0.3622	0.3299
Measures at $\rho = 1.3$													
Liquidity	0.0351	0.0148	0.0547	0.0807	0.1536	0.0314	0.0360	0.0265	0.0384	0.1986	0.0194	0.0189	0.0671
Dividend	0.2201	0.2752	0.1865	0.2080	0.1864	0.2083	0.2361	0.1963	0.2314	0.1880	0.1730	0.3500	0.1705
Net equity val	0.8448	0.8376	0.8556	0.8261	0.7860	0.9903	0.7613	0.8472	0.8436	0.7964	0.8854	0.8413	0.8499
Debt val	0.5086	0.5468	0.4696	0.4839	0.4643	0.4431	0.5697	0.5063	0.5012	0.3876	0.5664	0.4883	0.5041
Net firm val	1.3533	1.3844	1.3252	1.3100	1.2503	1.4333	1.3310	1.3536	1.3448	1.1841	1.4518	1.3296	1.3540
Leverage	0.3663	0.3908	0.3403	0.3479	0.3307	0.3025	0.4168	0.3669	0.3624	0.2803	0.3954	0.3621	0.3547
Equity volatility	0.3203	0.2904	0.3467	0.3297	0.3690	0.3472	0.2924	0.3229	0.3191	0.2827	0.3225	0.3282	0.3064

C Appendix: Growth Opportunities

In this appendix we first extend the model of the paper to the case when the firm has a one-time growth opportunity. We derive the optimal cash holding associated with this, and then we extend to the case of time homogeneous growth opportunities. We confirm the robustness of the qualitative conclusions derived in the main paper.

C.1 Background and empirical context

It is useful first to examine some of the main empirical facts about growth and related issues of long-term financial contracts. Table C1 is an extension of Table 3 in the text, giving for the same firms, averages cross-tabulated by profitability quintile and leverage level. In the table we give the average annual capital expenditures, share issuance (net of repurchases), and long-term debt issuance, all as a fraction of preceding year's total assets. Also, we give cross-tabs of the profitability itself, represented as operating income before depreciation.

The first panel shows that capital expenditures tend to be highest for the firms with highest current profitability. This holds overall and for each leverage category. Overall the amount of growth of capital stock is moderate. Assuming a depreciation rate of 6% per year (see e.g., Nadiri and Prucha, 1996), firms in the highest quintile expand their fixed capital by about 4% on average. This is consistent with the notion that organic growth, increasing the scale of the firm's existing line of business, is the dominant mode of firm growth. Growth is not a monotone function of profitability. The lowest quintile by profitability tend to have greater capital expenditures relative to existing assets than firms in the second quintile of profitability. This might reflect the influence of younger firms just building up their productive capacity.

From the second panel we see that share issuance tends to be highest for the least profitable firms. This is consistent with the prediction of our baseline model whereby seasoned share issues occur optimally in times of relatively low profitability (see Figure 1). Interestingly, share issuance is negative among the highest quintile by profitability both overall and in all but one of the leverage categories. That is, share buy backs are widely used to payout earnings when economic conditions are exceptionally good. From the third panel we see it is also the case that long-term debt issuance is lowest for the highest earning firms. Taken together these results suggest that a dominant form of firm growth occurs among relatively profitable firms and that to a significant extent this growth is financed through internal resources rather than new issues of equity or long-term debt.

From the fourth panel, we see that on average, the operating income is sufficiently higher in the top quintile, to finance the higher capital expenditure.

C.2 Modelling growth

We analyze a firm with a one time opportunity at time T to increase productive capacity, up to a certain limit. The investment results in the rescaling of the firm's existing asset in place at the proportion $(\overline{V}+I)/\overline{V}$ where I is the expenditure on investment and $\overline{V}$ is scale prior to undertaking growth at time T. We assume $I \geq 0$. Note that since investment is measured in monetary terms, then so is $\overline{V}$, and in fact $\overline{V}$ can be thought of as the cost of setting up the original firm, or the cost of doubling its scale. Also, we will assume constant returns to scale, and in contrast most models of growth, we do not assume that there are dead-weight scale adjustment costs.

The new investment can be financed through some combination of equity issue, long-term debt issue, and drawing down cash (or short-term borrowing). For tractability, we assume that long-term debt issue takes the form of a rescaling of contractual coupon payments q in proportion to investment. The firm chooses the scale of investment, the amount of cash financing and the amount of equity financing to maximize the value of equity. As in section 1.2 we denote the value of equity without growth option as $J^q(\rho, C)$. As mentioned above, this firm has scale $\bar{V}$. Extending this notation, we will write the equity value after growth of physical assets and coupon to scale V when cash holding is C as $J^q(\rho, C, V)$.

Since we have assumed constant returns to scale, proportionate taxes, equity issuance costs, fixed costs and bankruptcy costs, the value of equity will be homogenous of degree 1 in the cash flow parameters, coupon and cash. That is, $J^q(\rho, kC, k\bar{V}) = kJ^q(\rho, C, \bar{V})$. By the properties of linear homogenous functions we can show,

$$\begin{aligned}
J^q(\rho, C, V) &= \tfrac{V}{\bar{V}} J^q(\rho, C\tfrac{\bar{V}}{V}, \bar{V}), && (A1)\\
\tfrac{\partial}{\partial C} J^q(\rho, C, V) &= \tfrac{\partial}{\partial C} J^q(\rho, C\tfrac{\bar{V}}{V}, \bar{V}), && (A2)\\
\tfrac{\partial}{\partial V} J^q(\rho, C, V) &= (1/V) J^q(\rho, C, V) - (C/V)\tfrac{\partial}{\partial C} J(\rho, C, V) && (A3)
\end{aligned}$$

These equations (and similar equations for P in place of J) enable us to calculate the values and derivatives of J^q at an arbitrary scale V, from their values at scale $\bar{V}$, where the latter can be calculated numerically, using the procedures of Section 1. This will prove useful in the analysis below. Note that growth by increasing the scale of fixed assets and coupon while keeping cash constant will exhibit decreasing returns to scale. The problem of irreversible investment with decreasing returns to scale was analysed in a certainty setting by Arrow (1968) using variational methods. Our formulation adapts this analysis to a stochastic growth setting using our computational framework.

We proceed recursively by first analyzing the growth decision at the instant T of the growth option. Then we will study optimal cash holding and dividend policy prior to growth when this opportunity is merely a prospect which might be stochastic. Specifically, at each grid point (ρ, C) we calculate the optimal action of the firm, in terms of how much to grow, how much cash to draw down, and how much equity to issue. This gives us the equity value $J^{q,GRW}(\rho, C)$ at the instant before growth. Second, we analyze the cash reserve policy that the firm will pursue before time T, trading off the inefficiency of holding cash against the inefficiency having to issue equity in order to exploit the growth opportunity. This second step simply involves developing the solution back from $J^{q,GRW}(\rho, C)$ at time T, using the same procedure as in Section 1.

We now deal with the first step above, and describe our procedure for finding the optimal growth decision at time T, from scale $\bar{V}$, and at any point (ρ, C), and the corresponding equity value $J^{q,GRW}(\rho, C)$. First, suppose the firm's current scale is $V \geq \bar{V}$, and it is considering to increase its scale by a small amount dV. For the time being, assume that the firm will not issue equity, but will finance the growth by drawing down cash in the amount dC, and raising the rest, denoted dP, from issuing long term debt. After growth to scale $V + dV$, the debt value will be $P^q(\rho, C - dC, V + dV)$, and the firm will have raised an amount $P^q(\rho, C - dC, V + dV)\frac{dV}{V+dV}$ from the issue. Thus

$$dV = dC + dP \equiv dC + P^q(\rho, C - dC, V + dV)\frac{dV}{V + dV}, \qquad (A4)$$

and letting $dV \to 0$ gives

$$\frac{dC}{dV} = 1 - \frac{1}{V}P^q(\rho, C, V). \qquad (A5)$$

This equation tells us how much cash must be drawn down, if the firm grows from scale V to scale $V + dV$, with no equity issue, but a proportionate increase in long term debt.

If the firm decides to grow by dV by drawing down dC from the cash reserve, then the gain to equity will be

$$J^q(\rho, C - dC, V + dV) - J^q(\rho, C, V) = \left[\frac{\partial}{\partial V}J^q(\rho, C, V) - \frac{dC}{dV}\frac{\partial}{\partial C}J^q(\rho, C, V)\right]dV + o(dV) \qquad (A6)$$

The firm will thus find it profitable to grow by issuing long-term debt in step with the growth of fixed assets and by drawing down cash as required to cover the costs of investment if

$$\frac{\partial}{\partial V}J^q(\rho, C, V) - \frac{dC}{dV}\frac{\partial}{\partial C}J^q(\rho, C, V) > 0 \qquad (A7)$$

where $\frac{dC}{dV}$ is as in Equation (A5).

Now instead suppose the firm decides to grow dV by issuing debt as above and equity by an amount dQ sufficient to cover the costs of investment while keeping cash constant. Then the amount raised by debt will be $P^q(\rho, C, V + dV)\frac{dV}{V+dV}$, and corresponding to Equation (A4), we have

$$dV = dQ + dP \equiv dQ + P^q(\rho, C, V + dV)\frac{dV}{V + dV}, \qquad (A8)$$

and letting $dV \to 0$, this gives

$$\frac{dQ}{dV} = 1 - \frac{1}{V}P^q(\rho, C, V), \qquad (A9)$$

whose RHS is the same as that of Eqn (A5). The firm will thus find it profitable to grow by issuing long-term debt in step with the growth of fixed assets and by issuing shares as required to cover the costs of investment if

$$\frac{\partial}{\partial V} J^q(\rho, C, V) - \frac{dQ}{dV}\frac{1}{\theta} > 0. \tag{A10}$$

In fact, the firm can choose to grow by issuing equity or by drawing down cash. It will choose the former if (A7) holds, and LHS(A7) > LHS(A10); and it will choose the latter if (A10) holds, and LHS(A10) > LHS(A7). Actually, our valuation of $J^q(\rho, C)$ in Section 1 includes the assumption that the firm will increase the cash reserve from C by issuing equity, if this is profitable, which corresponds to the criterion $\frac{\partial}{\partial C} J^q(\rho, C) > \frac{1}{\theta}$. This, and Equation (A2) give

$$\frac{\partial}{\partial C} J^q(\rho, C, V) \leq \frac{1}{\theta}, \tag{A11}$$

and if we have equality in Equation (A11), then the firm should issue equity to increase the cash reserve.

Our procedure for determining the growth decision at time T at each grid point (ρ, C) is as follows. Let V_{max} be the maximum possible scale. We take $J^q(\rho, C, \bar{V})$ as calculated in Section 1. Starting from the initial scale $\bar{V}$, we increase the scale in small steps of size dV, i.e. to $V_i := \bar{V} + idV$, for $i = 0, 2, 3, ...,$ and decreasing C by steps $\frac{dC_i}{dV} dV$, i.e so that $C_i = C_{i-1} - \frac{dC_{i-1}}{dV} dV$ for $i = 1, 2, 3, ...$ We continue until the highest i , say $i = n$, such that (16) holds or until $V_i = V_{max}$. The value $J^{q,GRW}(\rho, C, \bar{V})$ immediately before growth, is then the value $J^q(\rho, C_n, V_n)$

At this point it may be that cash has been depleted to the point that would make equity issuance preferable. Therefore, if at (C_n, V_n) calculated above, we have $\frac{\partial}{\partial C} J^q(\rho, C_i, V_i) = \frac{1}{\theta}$, then we take the firm to issue equity to replenish cash. This is done in successive increments dQ to $Q_i = i \times dQ$, $i = 1, 2, 3, ...$ resulting in $C_{n,i} = C_n + i \times dQ$, until we have $\frac{\partial}{\partial C} J^q(\rho, C_i, V_i) < \frac{1}{\theta}$.

We finally discuss the case when the growth opportunity is stochastic, in particular, when the firm is uncertain whether the opportunity with actually occur, but if it does, then the timing and size are known. In this case, the value the equity at each (ρ, C) and at the growth time T, is the probability weighted average of the value assuming that the growth does occur (calculated as above), and the value assuming that the growth does not occur (which is just the value in Section 1). This calculation assumes that the growth probability is taken as risk neutral. Also, we assume that the firm knows whether the growth opportunity has occurred, when it implements its cash draw down, equity and debt issuance, at time T.

This completes the analysis of the optimal growth and financing decision for each node (ρ, C) at time T. Given the resulting value of equity,$J^{q,GRW}(\rho, C, \bar{V})$, we then proceed to solve for the optimal dividend and share issuance policy in the pre-growth period, $t < T$. This is a non-stationary problem. However, given that the solution technique developed in section 1 is applicable to non-stationary problems as well as stationary, this is straight forward. We implement the dynamic program of Section 1, but working backwards in time from the final condition $J^{q,GRW}(\rho, C, \bar{V})$, at time T.

C.3 The Optimal Policy with Growth Opportunities

To implement the model with growth opportunities we assume that prior to growth the firm has the technology, capital structure, and cost parameters as in the benchmark solution described in Section 1. We have implemented the procedure above for a variety of choices of the pre-growth scale, $\bar{V}$ and the maximum post-growth scale, V_{max}. The scale and equity increments are set at $dV = 0.001$ and $dQ = 0.001$.

The solution to the growth problem is characterized by an investment threshold $(\rho^g(C), C)$ such that for a given level of cash holding C, there is a minimum rate of profitability given by $\rho^g(C)$, below which the firm will not invest. The *amount* of investment chosen, given C, is a weakly increasing function of ρ up to some point where V_{max} is attained. Virtually everywhere in our implementations, the firm grows by the maximum allowed.

The threshold of ρ above which the firm will invest depends crucially on the cost of investment, as reflected in the scale parameter $\bar{V}$. We present results for $\bar{V} = 1.3$. According to Table 1, this is close to the (no-growth) firm value for $\rho = 1.3$ (quite high), and so it will be profitable for the firm to grow for $\rho > 1.3$, approximately. This growth pattern is in line with that of the empirical Table C1.

Figure C1 gives the result of implementing the growth model for $\bar{V} = 1.3$. In panel A we have taken $V_{max} = 1.04 \times \bar{V}$, i.e. growth of 4%. We give the liquidity target 1, 3, 6, 9 and 12 months before the growth occurs, and also the no-growth liquidity target, as in Figure 1 of the text. This panel corresponds growth by a known 4%, and at a time known in advance, and we see that the firm builds up liquidity a few months before the event. However, it can do this quickly, and it does not have to start holding significant extra cash say 6 months before the event. The panel also includes the 'long-term' liquidity target, which represents the sum of the liquidity targets, in excess of the no-growth target, over all months before the growth event. This is close to the liquidity target that the firm will maintain permanently, if it can grow for certain by 4% every month. We consider that the firm is saving separately for growth at each future month. We take the sum over 12 months to calculate the aggregate liquidity target. This is sufficient to represent the sum over all time, because of the quickly decaying nature of the excess liquidity targets, as the time remaining to the growth event increases. Our aggregate target is only an approximation to the long term liquidity target, because we have ignored the (uncertain) scaling which should apply at each future month, if the growth occurs. Also, we have smoothed the aggregate liquidity target with respect to ρ. This is necessary, because the discretization error in the liquidity boundaries, coming from the discretization of the (ρ, C)-space, is amplified in the aggregate liquidity boundary. Such a growth option is a good approximation to the option to grow smoothly at any instant, at the very high rate of 48% per year.

If the firm is able to grow at 4%/12 per month for certain, or at 4% per month with probability $\frac{1}{12}$, then its cash holding is the same as the no-growth firm, and we do not present a figures for these results. These scenarios both agree with expected growth of 4%, as in Table A1. At such a modest rate, the firm can generate enough cash from its current earnings to finance the growth.

In Figure C1, Panel B we give the scenario for growth of 2% for certain each month, and in Panel C we give the scenario for growth of 24% with probability 1/12 each month. Both panels also include the corresponding 'long term' target. Both panels correspond to growth of 24% per annum, but in Panel C the growth is much more lumpy, and we see that in aggregate, the firm has a much higher cash target.

It is also notable that in all panels, the cash target is not much increased for very high ρ. At such ρ it is not necessary to save so much, since the current earnings can cover the growth cost. Looking more closely at the figure, at very high levels of ρ it is sometimes optimal to target a higher level of cash accumulation when the investment horizon is more distant than when it is close. This is because when current profitability is high, it is likely to decline by mean reversion as the investment horizon approaches. This also leads to the long term liquidity target being higher for very high ρ values.

We conclude that at the modest growth rates of Table C1, the firm will not significantly increase its cash holding to finance the growth, because it has a high current income, at times when it is profitable to grow. Investment for growth comes from reducing the dividend. The firm will increase the cash target if it anticipates high growth of say 24%, and then the cash target is strongly related to the lumpiness of the investment opportunity. The high cash holdings for the unlevered firms of Table 3 might the explained by such lumpiness of the investment opportunity.

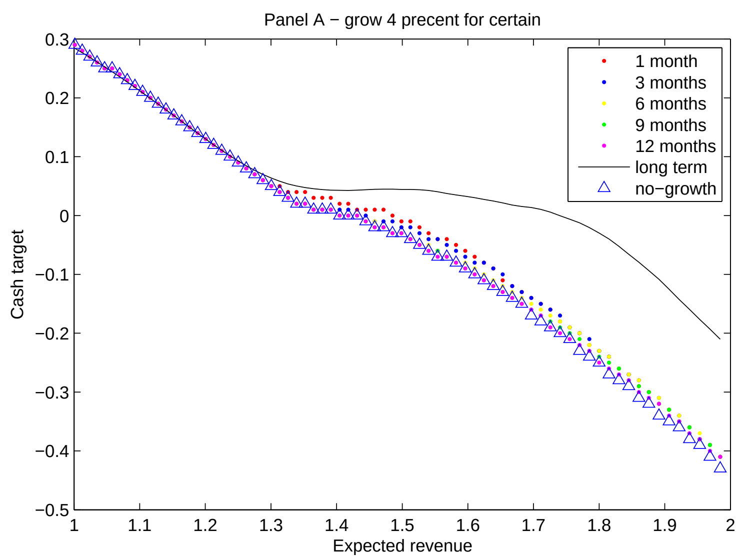
It is clear from these results that in the presence of large lumpy investment opportunities, the change in cash holding bears no simple relation to current cash flow. This point has been emphasized by Riddick and Whited (2009). In the run-up to the investment date high cash flow will be associated with high cash accumulation. At the investment date high cash flow is associated with sharp cash *decumulation*.

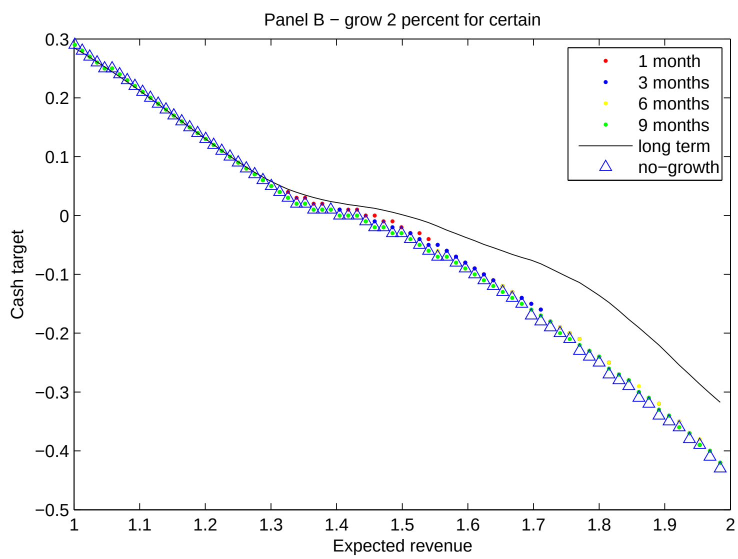
TABLE C1: Capital Expenditures, Share Issuance, LT Debt Issuance, and Profitability

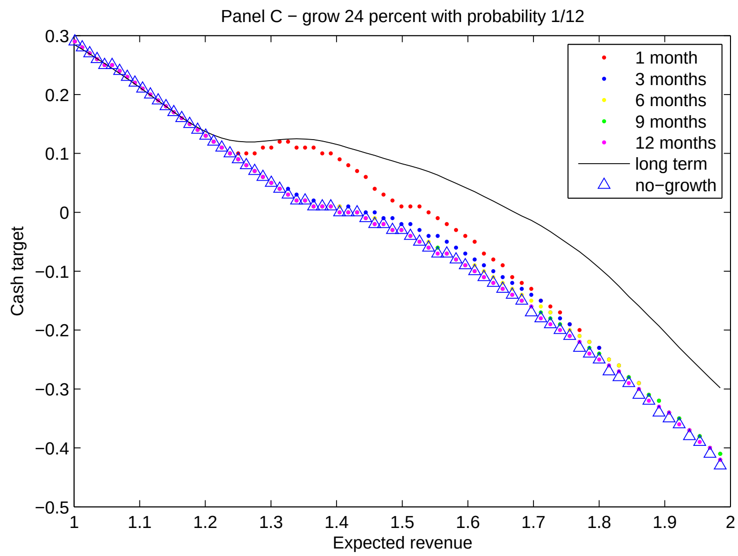
Average Capital Expenditures						
	Profitability Quintile					
Leverage category	Lowest	2	3	4	Highest	Average
$0.0 < L \leq 0.1$	4.20%	3.68%	4.76%	5.88%	7.82%	5.27%
$0.1 < L \leq 0.2$	4.41%	4.36%	5.47%	6.38%	8.15%	5.75%
$0.2 < L \leq 0.3$	4.92%	4.68%	5.85%	7.63%	10.31%	6.68%
$0.3 < L \leq 0.4$	6.44%	4.77%	6.50%	7.23%	11.21%	7.23%
$0.4 < L \leq 0.5$	5.45%	5.10%	5.93%	7.37%	10.97%	6.96%
$0.5 < L$	3.78%	5.94%	7.76%	6.62%	12.31%	7.28%
Average	4.87%	4.76%	6.04%	6.85%	10.13%	6.53%
Average Share Issuance						
	Profitability Quintile					
Leverage category	Lowest	2	3	4	Highest	Average
$0.0 < L \leq 0.1$	6.13%	0.41%	-0.17%	-1.26%	-3.58%	0.30%
$0.1 < L \leq 0.2$	2.55%	0.40%	0.03%	-1.06%	-3.39%	-0.29%
$0.2 < L \leq 0.3$	1.24%	0.42%	0.17%	-0.75%	-1.40%	-0.06%
$0.3 < L \leq 0.4$	1.83%	0.83%	0.33%	-0.31%	-1.91%	0.15%
$0.4 < L \leq 0.5$	0.43%	1.39%	0.84%	1.35%	-0.21%	0.76%
$0.5 < L$	2.66%	1.38%	0.69%	0.04%	0.09%	0.97%
Average	2.47%	0.80%	0.31%	-0.33%	-1.73%	0.31%
Average LT Debt Issuance						
	Profitability Quintile					
Leverage category	Lowest	2	3	4	Highest	Average
$0.0 < L \leq 0.1$	0.13%	0.78%	1.02%	0.66%	0.08%	0.53%
$0.1 < L \leq 0.2$	0.49%	1.30%	1.46%	0.86%	0.55%	0.93%
$0.2 < L \leq 0.3$	2.04%	1.93%	0.81%	1.00%	-0.30%	1.09%
$0.3 < L \leq 0.4$	0.70%	2.31%	1.89%	1.27%	0.49%	1.33%
$0.4 < L \leq 0.5$	2.91%	2.11%	1.47%	-0.16%	-1.91%	0.88%
$0.5 < L$	3.29%	4.64%	3.07%	0.22%	2.30%	2.70%
Average	1.59%	2.18%	1.62%	0.64%	0.20%	1.25%
Profitability						
	Profitability Quintile					
Leverage category	Lowest	2	3	4	Highest	Average
$0.0 < L \leq 0.1$	-13.79%	3.75%	11.06%	17.48%	26.41%	8.98%
$0.1 < L \leq 0.2$	-5.51%	7.00%	12.33%	16.95%	24.74%	11.10%
$0.2 < L \leq 0.3$	-0.93%	8.25%	12.46%	16.58%	24.29%	12.13%
$0.3 < L \leq 0.4$	-1.59%	7.92%	11.99%	15.71%	23.17%	11.44%
$0.4 < L \leq 0.5$	-1.12%	7.65%	11.56%	15.12%	23.04%	11.25%
$0.5 < L$	-3.87%	7.46%	11.41%	15.16%	22.57%	10.55%
Average	-4.47%	7.01%	11.80%	16.17%	24.03%	10.91%

Figure C1:

These figures each depict optimal cash targets at 1, 3, 6, 9 and 12 months prior to an uncertain growth opportunity for the firm, with size and probability stated in each panel. The ‘no-growth’ target is as in Figure 1 of the paper. The ‘long term’ target corresponds to the growth opportunity being present in every month in the future.







Panel C − grow 24 percent with probability 1/12
Cash target
Expected revenue
1 month
3 months
6 months
9 months
12 months
long term
no−growth
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

D Additional references

Ames, W. F., 1992, *Numerical Methods for Partial Differential Equations*, Academic Press.

Arrow, K., 1968 Optimal Capital Policy with Irreversible Investment, in J.N.Wolfe (ed.) *Value, Capital and Growth*, Aldine.

Nadiri, M.I. and I.R. Prucha, 1996 Estimation of the Depreciation Rate of Physical and R&D Capital in the U.S. Total Manufacturing Sector, *Economic Inquiry*, Vol. 34, pp 43-56.