Efficient Dynamic Coordination with Individual Learning*

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Abstract

We study efficiency of coordination in incomplete information games with private learning. Two players face the option to invest irreversibly in a project in one of many rounds. The project succeeds if some underlying state variable θ is positive and both players invest, possibly asynchronously. In each round they receive informative private signals about θ, and asymptotically learn the true value of θ. Players choose in each period whether to invest or to wait for more precise information about θ.

We show that with sufficiently many rounds, both players invest with arbitrarily high probability whenever investment is socially efficient, and delays in investment disappear when signals are precise. This result stands in sharp contrast to the usual static global game outcome in which players coordinate on the risk-dominant action. We provide a general foundation for these results in terms of higher order beliefs. We use this foundation to solve examples with varying degrees of asynchronicity, and show that little asynchronicity is required to achieve efficient coordination when learning is quick.

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1 Introduction

Coordination problems arise in a wide variety of economic situations. A typical example is of a setting where the successful implementation of some socially beneficial project depends on whether enough agents participate. Such settings may lead to coordination failure, which arises when a given group of agents fails to participate in the project despite the fact that it is in their collective interest to do so.

The traditional theoretical analysis of coordination problems, where payoffs are typically assumed to be commonly known, has been characterized by the existence of multiple equilibria. For a given payoff rule, there exists at least one equilibrium with coordination failure, and one without. Such analysis is unable, therefore, to identify the extent and relevance of coordination failure, since it is not possible to assign probabilities across equilibria.

The literature on global games (Carlsson and van Damme [3], Morris and Shin [20]) has made substantial progress in resolving the problem of multiplicity in the analysis of coordination problems. This literature has identified an important class of coordination games, in which underlying payoffs are observed with small amounts of idiosyncratic noise, where the multiplicity of equilibria is eliminated. The "refinement" thus achieved allows us to quantify the extent of coordination failure, and indeed coordination failures do occur in the unique equilibrium of the canonical global game. Whether coordination failure arises depends on the payoffs of the underlying complete information game. Roughly speaking, agents are only able to coordinate on some risky action in the unique equilibrium of a global game if that action is *risk dominant*, i.e., it is optimal for each agent to choose that action in the underlying complete information game even when there is only a "low" probability that his fellow players will choose that action.[1] This can only happen if the benefits that arise from the action conditional upon success are high relative to the cost of undertaking it. Thus, the global games literature has negative implications for the ability of agents to coordinate on socially beneficial actions: only projects that involve "little strategic risk" will be implemented in equilibrium. In all other cases, coordination failure will arise.

The canonical global game requires that all agents choose their actions simultaneously. Our analysis is motivated by the following question: To what extent would the incidence of coordination failure change if we allowed for some asynchronicity in the actions of potential participants in a coordination problem, while they privately learn about payoffs over time?

To be specific, consider a benchmark setting in which the success of a socially beneficial

[1] More precisely, equilibria of the underlying complete information game survive in the induced global game only if they are p-dominant (Morris, Rob, and Shin [19]) for "low" p. Exactly how low p must be depends on the structure of the game. In two player games, p-dominant equilibria for $p < \frac{1}{2}$ survive. See Kajii and Morris [17] for a generalization of this idea.

investment project depends on the total number of agents who invest over the course of T distinct periods. Two players choose at which period (if any) to invest irreversibly, while observing noisy private signals about the underlying state variable (θ). Players do not observe each other's actions; in Section 7 below, we show how to interpret this information structure as social learning with noisy observation of actions. At each period t, the information structure is that of a canonical global game. We assume that agents privately learn the fundamental θ asymptotically: if the number of periods gets arbitrarily large, each agent's cumulative individual information becomes arbitrarily precise. The project succeeds if the fundamental is good and each player invests in some period. Note that it is *not* necessary for success that both players invest in the same period. This is, therefore, an *asynchronous* investment game. The choice between early versus late investment is driven by a trade-off: early investment generates higher payoffs if the project succeeds, while late investors have more accurate private information about payoffs. As in a standard global game, we assume that there exist values of θ that make investment dominant ($\theta \geq 1$) or dominated ($\theta \leq 0$). To fix ideas, imagine that $0 < \theta < 1$ and the payoffs are such that investing is *not* risk-dominant. This means that if agents had to choose their actions simultaneously in some period, say T, and thus play a static global game, then, in the limit as noise vanishes, coordinated investment could not be supported as an equilibrium outcome, and coordination failure arises. To what extent will the possibility of choosing actions asynchronously affect the incidence of coordination failure?

We show that coordination failure almost never arises in a sufficiently long asynchronous investment game. For any $\varepsilon > 0$ there exists some $\underline{T}$ such that for any $T > \underline{T}$, investment succeeds with probability at least $1 - \varepsilon$ in the asynchronous game with T periods whenever $\theta > 0$. In addition, as noise in observation vanishes (i.e., $\sigma_t \to 0$ for all t) there is also no delay in investment: players successfully coordinate on implementing the project immediately, thus achieving the social optimum.

The forces driving our results can be cleanly characterized in terms of higher order beliefs in the asynchronous coordination game. Building on the standard belief operators of Monderer and Samet [18], we construct generalized belief operators that are suitable for characterizing behavior in asynchronous investment games. Using these operators, we show that by choosing sufficiently long asynchronous investment games, it is possible to generate adequate levels of generalized approximate common knowledge (i.e., generalized common p-belief for arbitrarily high p) in order to support asynchronous coordination. The generalization lies in allowing the required beliefs to be attained at different times.

If *synchronous* participation at the last round T was necessary for the success of the project, then players invest only if, at T, they individually believe $\theta \geq 1$ or *commonly* believe (in the sense of Monderer and Samet [18]) that $\theta > 0$, that is, that the fundamental allows success. It is now well understood (see, for example, Morris and Shin [20]) that, even when players receive arbitrarily precise private signals, the global games information structure does not generate common p-belief for $p > \frac{1}{2}$, and thus coordination fails whenever investment is not risk-dominant. In our setting, players do not have to participate synchronously at T, but both players must participate eventually by period T for investment to succeed. In such a situation, only a relaxed version of common beliefs is necessary for coordination. Fix a probability of success $p \in (0, 1)$ sufficient to induce players to invest in period t. Both players will invest by period T if each assigns probability at least p in *some* period to the fundamental being good, and in the same period, believes with probability p that the other assigns probability at least p in *some* period to the fundamental being good, etc. We refer to such an event as generalized common p-belief of the event $\theta > 0$. This variation of standard common belief turns out not to be very demanding in our setting.

The intuition for why generalized approximate common knowledge is attained would be relatively simple if we considered a game with infinitely many rounds in which each player asymptotically privately learns the fundamental. Then whenever the fundamental θ is positive, all players will eventually believe, at any arbitrarily high level of confidence, that θ is positive.

Analyzing only infinite-horizon games with asymptotically complete learning would effectively restrict attention to situations in which all fundamental uncertainty is completely resolved by the end of the game. In many applications, however, some uncertainty remains. For this reason, it is important to analyze long but finite games with incomplete learning. In such games, however, the argument outlined above becomes more subtle.

For example, consider the event that all players p-believe that fundamental is positive at least once during the game. One may reasonably be concerned that the length of game required for this event to have at least some fixed probability becomes infinite as θ approaches 0 from above. However, we show that this is *not* the case. Indeed, we show that, for any given confidence level, the probability that all players are confident that $\theta > 0$ at least once during the game converges to 1 *uniformly* across all $\theta > 0$ as $T \to \infty$. Such uniformity is essential for efficient coordination to occur.

Existing results on static global games show that there is an important discontinuity in the structure of standard common beliefs as information about the fundamental becomes infinitely precise. We find that such a discontinuity does *not* arise in asynchronous coordination problems. Generalized approximate common knowledge *is* attained even if a small amount of fundamental uncertainty remains.

In addition to explaining the efficiency result in the benchmark asynchronous investment game, our higher order belief framework may be applied to different games with varying degrees of asynchronicity. Section 4 introduces two such games. In the first of these games, when $\theta \in (0,1)$, each player prefers *not* to invest before the other player invests: there are small losses for being (exclusively) the first to invest. In the second, each player would prefer to invest *one period* before the other player invests: there is a small reward for being (exclusively) the first to invest. We show that coordination failure occurs in the first game but, when learning is sufficiently fast, efficient coordination arises in the second. Thus even a small change in tolerance to asynchronicity can have a dramatic effect on the ability to successfully coordinate. These further results reinforce the conclusion that the ability of agents to coordinate in dynamic settings is highly sensitive to the details of timing and payoffs. Thus, our analysis suggests that caution is warranted when drawing policy implications from the static analysis of coordination problems that may have dynamic elements. Simultaneously, the dynamic higher order beliefs framework of Section 5 provides some tools and methods which may be useful for analyzing a large class of dynamic coordination problems.

Our results suggest that allowing asynchronicity and private learning in coordination problems may substantially reduce the extent of coordination failure in global games. In addition to being of theoretical interest, our results are potentially widely applicable. For example, consider the problem of foreign direct investment (FDI) into a newly liberalizing emerging market. Payoffs from FDI depend on whether the emerging economy "takes off", which in turn depends on the amount of FDI. Thus, this is a coordination problem. In addition, it may not matter precisely that all FDI takes place at the same time, but simply that it occurs during the first several months to several years of the liberalization programme. It is not uncommon for liberalization to be accompanied by government subsidies to early investors. Yet, it is also likely that late investors will have better information about the state of the underlying emerging markets. Thus, the class of stylized games outlined above represents trade-offs that are not dissimilar to those outlined in this applied context. The FDI example is not unique. Indeed, it may be reasonable to argue that several of the applications studied to date using global games (e.g., currency crises, bank runs, financial contagion etc.[2]) may well have an element of asynchronicity to them.

The rest of the paper is organized as follows. In section 2 we outline the model. Section 3 states our main result for the benchmark game with pure asynchronicity. After introducting additional examples in section 4, section 5 examines the generalized common beliefs required to obtain efficient coordination. Section 6 applies the common belief framework of section 5 to the examples of section 4. Sections 7 and 8 discuss the role of our main assumptions,

[2]See Morris and Shin [21] for currency crises, Goldstein and Pauzner [12] for bank runs, and Dasgupta [7] and Golstein and Pauzner [13] for financial contagion.

consider potential extensions, and conclude. Before proceeding to the main model, we first outline the related literature.

1.1 Literature Review

Our analysis originated in the work of Dasgupta [6]. Dasgupta outlines conditions under which the provision of the option to delay combined with private learning improves efficiency in two-stage global games. We study a related framework with many periods and show that full efficiency can be achieved as the number of periods grows large. Our analysis highlights the role of asynchronicity in generating efficient coordination. Another example of a dynamic global game with private learning can be found in Heidhues and Melissas [15].

The current analysis bears a general connection to models of information dynamics in multi-stage global games (e.g., Chamley [4], Angeletos, Hellwig and Pavan [1]). In contrast to our work, papers in this strand of the literature focus on the robustness of equilibrium uniqueness in global games when learning is based on endogenously generated *public* signals and actions are reversible. We focus on characterizing the (lack of) incidence of coordination failure in pure private information settings with irreversibility in which there is a unique long-run outcome.

Our explanation of the efficiency result in terms of higher order beliefs builds on the work of Monderer and Samet [18] and Morris and Shin [22]. The analysis is also related to the work of Cripps, Ely, Mailath, and Samuelson [5] (henceforth CEMS). CEMS delineate general conditions under which agents asymptotically attain approximate common knowledge via private learning. The analysis of CEMS has important implications for long-run outcomes in situations which can be divided into two distinct phases: agents learn privately in the first phase, and attempt to coordinate *synchronously* in the second. We study situations in which those two phases are merged together: players attempt to coordinate *asynchronously* while they privately learn about payoffs. Both papers study whether private learning leads to approximate common knowledge. However, *different* concepts of approximate common knowledge are relevant for ensuring successful coordination in synchronous and asynchronous coordination games, because the payoff-relevant events differ in these two types of games. CEMS study standard common beliefs as defined in Monderer and Samet [18], while we study an asynchronous form of common beliefs. The two concepts turn out to have very different properties. In our model, private learning fails to deliver common knowledge in the standard sense as studied by CEMS, but succeeds in delivering asynchronous common beliefs. This explains why coordination failure arises in the synchronous coordination game but does not arise in our asynchronous game.[3]

[3]In an earlier paper Ely [8] informally discusses the notion of asynchronous common belief, but only to

Gale [10] provides an elegant analysis of the extent of inefficient delay in dynamic coordination games with complete information. He shows that inefficient delay can be eliminated when the period length becomes very small. Hörner [16] and Xue [24] similarly obtain efficient coordination in incomplete information games with observable irreversible actions. While our main result implies efficient delay-free coordination in the limit as private signals become accurate, and thus bears a resemblance to Gale's, the forces driving the results are very different. Gale's result builds on backward induction based on the observability of past actions in a game of perfect information, while we consider an asymmetric information setting in which players do not observe each others' actions. When the efficient action is irreversible, observability favors efficient coordination by eliminating strategic uncertainty once a player has chosen the irreversible action. Our results indicate that efficiency does not require observability *per se*; thus efficient coordination may arise even in games with large numbers of players where perfect observation of actions is not feasible. In Section 7 below, we describe how to reinterpret our model as one of noisy social learning.

2 Model

Two players $i \in \{1, 2\}$ play a joint investment game Γ_T, with $T \in \mathbb{N}$.[4] The game consists of T rounds, all of which may take place within a finite, possibly short time window. In each round $t \in \{1, \ldots, T\}$, each player chooses one of the two actions $a_t^i \in \{0, 1\}$; we interpret Action 1 as "invest", and Action 0 as "wait". Investment is irreversible.[5] To keep notation simple, we model irreversibility by making action choices payoff-irrelevant after a player has invested once. Whenever we refer to rationalizable actions below, we implicitly condition on the player not having invested in an earlier round.

The payoffs in the game depend on the action profiles and the value of a fundamental parameter $\theta \in \mathbb{R}$ describing the characteristics of the project. The fundamental θ is drawn before the first round according to an improper uniform distribution on $\mathbb{R}$, and remains fixed over all rounds.

The players do not observe the true value of the fundamental θ; instead, they receive private noisy signals of the value of θ in every round. Specifically, each player i receives a signal $\widetilde{x}^{(i,t)} = \theta + \tilde{\sigma}_t \varepsilon^{(i,t)}$ in round t, where the errors $\varepsilon^{(i,t)}$ are drawn from $N(0, 1)$ and are independent across players and rounds. The standard errors $\tilde{\sigma}_t$ are strictly positive for all t, and the sequence $(\tilde{\sigma})_{t=0}^{\infty}$ is fixed throughout independent of the value of T, which we will

contrast it to the standard common belief which is the relevant concept for the problems he considers.

[4]All of our results extend to games with any finite number of players.

[5]We discuss the role of irreversibility and other robustness issues in Section 7.

vary. Player i does not observe the choices of player $-i$ before the end of the game.[6]

Players form their beliefs in each period about the true value of the fundamental through Bayesian updating given their received signals. Conditional on a sequence of signals $(\widetilde{x}^{(i,t)})_{t'=1}^{t}$, player i believes θ is distributed as $N(x^{(i,t)}, \sigma_t^2)$, where

$$x^{(i,t)} = \frac{\sum_{t'=1}^{t} \widetilde{x}^{(i,t)} \frac{1}{\tilde{\sigma}_{t'}^2}}{\sum_{t'=1}^{t} \frac{1}{\tilde{\sigma}_{t'}^2}},$$

and

$$\frac{1}{\sigma_t^2} = \sum_{t'=1}^{t} \frac{1}{\tilde{\sigma}_{t'}^2}.$$

We will refer to $x^{(i,t)}$ as the *cumulative signal*, and to σ_t as the *cumulative standard error*.

We assume that players asymptotically *privately* learn the true fundamental, that is,

$$\lim_{t\to\infty} \sigma_t = 0.$$

Note that, since each standard error $\tilde{\sigma}_t$ is strictly positive, each cumulative standard error σ_t is also strictly positive. Thus even though players learn the true fundamental in the limit over all periods, some uncertainty remains in each round.

The success of the project is determined at the end of the game, based on the fundamental θ and the actions of the players:

- For $\theta \leq 0$, the project fails regardless of the players' actions.
- For $\theta \geq 1$, the project succeeds regardless of the players' actions.
- For $0 < \theta < 1$, the project succeeds if and only if both players invest by round T, *possibly asynchronously.*

Each player's payoff in the game depends on whether and in which round the player invested, and whether the project succeeded. The payoffs are

- 0 if the player never invests,
- $\delta^t b$ if the player invests in round t and the project succeeds, and
- $-\delta^t c$ if the player invests in round t and the project fails,

[6]However, as we demonstrate in Section 7, under certain circumstances, the signals $\widetilde{x}^{(i,t)}$ can be thought to arise from noisy social learning based on the observation of past actions.

where the parameters b and c are both strictly positive, and $\delta \in (0, 1)$.

We assume throughout that players' strategies depend only on their aggregate signal in each period, not on the full sequence of signals to date.[7]

Accordingly, a behavioral strategy for player i is a collection $s^i = \{s_t^i\}_{t=1}^T$ of measurable functions $s_t^i : X^{(i,t)} \longrightarrow \{0, 1\}$. Let $U^i(a, s^i, s^{-i}|x^{(i,t)})$ denote the expected payoff for type $x^{(i,t)}$ choosing action a conditional on player i having chosen action 0 in every period $1, \ldots, t-1$ and following strategy s^i in every period $t+1, \ldots, T$. Letting $\mathcal{S}$ denote the set of behavioral strategies for each player, define the (interim) best response correspondence $B(\cdot)$ over subsets $S \subseteq \mathcal{S}$ by

$$B(S) = \left\{ s^i \mid \forall x^{(i,t)} \; \exists s^{-i} \in S \text{ such that } s_t^i \left(x^{(i,t)} \right) \in \arg \max_{a \in \{0,1\}} U^i \left(a, s^i, s^{-i} | x^{(i,t)} \right) \right\}.$$

Note that, since we restrict our attention to symmetric games, we do not distinguish between the best response correspondences of the two players. The set of (interim) rationalizable strategies for each player is defined to be

$$\bigcap_{n=1}^{\infty} B^n(\mathcal{S}).$$

We say that action a is rationalizable for $x^{(i,t)}$ if there is exists a rationalizable strategy s^i such that $s^i(x^{(i,t)}) = a$.

Notions of rationalizability in dynamic games are typically more complicated than the one employed here. The main complication is how players revise conjectures about their opponents' strategies that are contradicted by their opponents' actions (see, e.g., Pearce [23]). This issue does not arise in our model because actions are unobservable.

We now proceed to analyze this game, and demonstrate how asynchronicity combined with asymptotic learning can eliminate coordination failure.

3 Analysis

The payoffs outlined above imply a simple property of the best response correspondence, which we describe in the following lemma.

Lemma 1 *There exists some $\overline{p} \in (0, 1)$ such that, in any round t, investing at t is the unique best response for any type that assigns probability greater than $\overline{p}$ to the success of the project.*

[7] This assumption simplifies notation but is not necessary for the results since the set of rationalizable actions depends only on the Δ-hierarchy of beliefs (see Ely and Peski [9]), which in turn depends only on the aggregate signal.

Proof. The payoff to investing at t is

$$\delta^t \left(pb + (1-p)(-c) \right).$$

The value to waiting at t is at most $\delta^{t+1} b$. Thus taking $\overline{p}$ to satisfy

$$\delta^t \left(\overline{p}b + (1-\overline{p})(-c) \right) = \delta^{t+1} b,$$

or equivalently,

$$\overline{p} = \frac{\delta b + c}{b + c},$$

gives the result. ■

We note that $0 < \overline{p} < 1$. Finally, we observe that the existence of $\overline{p} < 1$ implies that however great the amount of future information, any player will choose to invest whenever she is sufficiently optimistic.

We are now in a position to state our main results, which demonstrate the stark difference between synchronous and asynchronous coordination games. We begin with the benchmark synchronous case.

3.1 The failure of coordination in the synchronous game

As a benchmark to compare our results to the existing literature on static global games, consider the following static, synchronous version of the game Γ_T. In the synchronous version, which we label by Γ_T^S, for $0 < \theta < 1$, the project succeeds if and only if both players invest synchronously at round T. All other features remain unchanged. Let $\underline{p} = \frac{c}{b+c}$, and notice that player in the last round T prefers to invest only if she assigns probability at least $\underline{p}$ to the success of the project. We show that for any $\theta < 1$ coordinated investment fails with arbitrarily high probability as long as T is big enough, whenever $\underline{p} > \frac{1}{2}$.

Proposition 1 *Suppose $\underline{p} > \frac{1}{2}$. Suppose further that both players play interim rationalizable strategies. For any $\overline{\theta} < 1$ and $\varepsilon > 0$ there exists some $\underline{T}$ such that for any $T > \underline{T}$, the project fails with probability at least $1 - \varepsilon$ in Γ_T^S whenever $\theta \leq \overline{\theta}$.*

This result is a consequence of results from the extant literature on static global games (see Morris and Shin [20]), and so we only describe the intuition here.[8] In Section 6 we revisit the synchronous game and provide its formal analysis using our general higher order belief framework.

[8]Note that, when $\underline{p} > \frac{1}{2}$, not investing is the risk-dominant action whenever $\theta < 1$. Proposition 1 is therefore consistent with the usual selection of the risk-dominant equilibrium in 2×2 global games (Carlsson and Van Damme [3]).

Suppose that the two players use symmetric strategies characterized by a threshold aggregate signal x^* at time T above which each invests and below which each does not invest. A player who receives exactly the threshold signal must be indifferent between investing and not investing. The threshold type suffers from strategic uncertainty; by symmetry, she assigns probability exactly $1/2$ to her opponent receiving a signal above x^*, and hence probability $1/2$ to her opponent investing. Since $\underline{p} > 1/2$, not investing is a strict best response unless the player assigns high enough probability to θ being greater than 1. When T is large, this condition cannot be satisfied for signals bounded below 1 since the noise in the aggregate signals is small in period T. Therefore, the threshold x^* must be close to 1, and players are unlikely to invest if $\theta \leq \overline{\theta} < 1$.

We now show that coordination almost never fails in the asynchronous game for $\theta > 0$.

3.2 The success of coordination in the asynchronous game

The following proposition establishes that, in the game with many rounds, both players are likely to invest whenever the fundamental allows for success of the project ($\theta > 0$). The intuition behind the contrast between the outcomes of the synchronous and asynchronous games is as follows. When T is large, the noise in the aggregate signal is small in round T, so there is little payoff uncertainty (except for signals very close to 0 or 1). However, there is *strategic* uncertainty in equilibrium since player's actions are based on private information. This strategic uncertainty leads to coordination failure in the synchronous game. In the asynchronous game, strategic uncertainty is mitigated since a player who believes that the fundamental is good assigns high probability to the opponent sharing this belief in *some* period. When combined with irreversible investment, we show that these second-order beliefs allow each player to be confident that the other invests whenever the fundamental is good.

Proposition 2 *Suppose that both players play interim rationalizable strategies. For any $\varepsilon > 0$ there exists some $\underline{T}$ such that for any $T > \underline{T}$, the project succeeds with probability at least $1 - \varepsilon$ in Γ_T whenever $\theta > 0$.*

Proof. The result follows from two core lemmas (Lemmas 3 and 4). The proof consists of three steps: first, we briefly outline some notation and prove a technical lemma (Lemma 2). Second, we state the two core lemmas. Finally, we argue that the main result follows from the lemmas.

Let $\overline{p}$ be as in Lemma 1 and fix $q \in (\overline{p}, 1)$. For each (Lebesgue) measurable subset $E \subset \mathbb{R}$, let $B_q^{(i,t)}(E) = \left\{x^{(i,t)} \mid \Pr\left(\theta \in E | x^{(i,t)}\right) \geq q\right\}$, that is, $B_q^{(i,t)}(E)$ is the set of aggregate signals for player i at time t for which she assigns probability at least q to θ lying in the set E. Denote by $l_t^{\theta^*,q}(\theta)$ the probability that, when the fundamental is θ, the player q-believes that the fundamental exceeds θ^* in at least one round up to and including round t:

$$l_t^{\theta^*,q}(\theta) = \Pr\left(\bigcup_{t'=1,\ldots,t} B_q^{(i,t)}\left([\theta^*,\infty)\right) \middle| \theta \right).$$

Lemma 2 *The function $l_t^{\theta^*,q}(\theta)$ is continuous and nondecreasing in θ for each t, θ^*, and q.*

The proof of Lemma 2 is in the appendix.

The following lemma demonstrates one important consequence of our assumption that players asymptotically learn the true state. Such asymptotic learning guarantees that, when the true state is θ^*, each player will, at least once during the course of an arbitrarily long game, believe with arbitrarily high probability that the true state is not below θ^*.

Let $l^{\theta^*,q}(\theta)$ denote $\lim_{t\to\infty} l_t^{\theta^*,q}(\theta)$. This limit exists because $l_t^{\theta^*,q}(\theta)$ is non-decreasing in t and bounded above by 1. When $\theta^* = 0$, the following lemma implies that, as $t \to \infty$, the probability that each player p-believes the fundamental is positive converges to 1 *uniformly* across all $\theta \geq 0$.

Lemma 3 *For all $0 < q < 1$ and all $\theta^* \in \mathbb{R}$, $l^{\theta^*,q}(\theta)$ is continuous at θ^* and $l^{\theta^*,q}(\theta^*) = 1$.*

The proof of the lemma is provided in the appendix. The main idea of the proof of Lemma 3 is the following: conditional on θ^*, the probability that a player q-believes $\theta \geq \theta^*$ is $1-q$ in each round, but with the complication that the posterior probabilities $p^{(i,t)} = \Pr(\theta \geq \theta^* | x^{(i,t)})$ are correlated across rounds. We show, roughly, that beliefs across sufficiently distant rounds t and t' are approximately independent. The intuition is that if the amount of information that a player receives between t and t' is large relative to what she knew at t, then the information at t has only a negligible impact at t'. For long games, we can choose a long subsequence of rounds such that all rounds in the subsequence are sufficiently distant. Hence the probability of q-believing $\theta \geq \theta^*$ in at least one of these rounds approaches one as the number of rounds grows large.

Our next result shows that, in a sufficiently long (but finite) game, whenever a given player assigns probability greater than $\overline{p}$ to the fundamental being positive, the player invests immediately.

Lemma 4 *For each $q \in (\overline{p}, 1)$, there exists some $\underline{T}$ such that for any $T > \underline{T}$, investing is the unique rationalizable action in the game Γ_T for any type $x^{(i,t)} \in B_q^{(i,t)}([0,\infty))$.*

Proof. Fix $q \in (\overline{p}, 1)$ throughout the proof. Let $S(\underline{\theta})$ denote the following statement: there exists some $\underline{T}$ such that for every $T > \underline{T}$, investing is the unique rationalizable action in Γ_T for any type $x^{(i,t)}$ that q-believes $\theta \geq \underline{\theta}$.

We proceed by a contagion argument. The statement clearly holds for $\underline{\theta} \geq 1$. The proof consists of showing that if, for some $\theta^* \geq 0$, the statement $S(\underline{\theta})$ holds for all $\underline{\theta} > \theta^*$, then $S(\theta^*)$ holds, and, moreover, if $\theta^* > 0$, then there exists some $\varepsilon > 0$ such that the statement $S(\underline{\theta})$ holds for all $\underline{\theta} > \theta^* - \varepsilon$. Let θ^{**} denote the infimum of those θ^* for which the statement $S(\underline{\theta})$ holds for all $\underline{\theta} > \theta^*$. We must have $\theta^{**} = 0$, for otherwise taking $\theta^* = \theta^{**}$ contradicts the contagion step. It follows that $S(\underline{\theta})$ holds for all $\underline{\theta} > 0$ and hence $S(0)$ holds, as needed.

It remains to prove the contagion step. Suppose that, for some $\theta^* \geq 0$, $S(\underline{\theta})$ holds for all $\underline{\theta} > \theta^*$. Fix some $r \in \left(\frac{\overline{p}}{q}, 1\right)$. Lemma 3 implies that there exists $\underline{\theta} > \theta^*$ such that

$$l^{\underline{\theta},q}(\theta^*) > r.$$

Hence there exists some T' such that

$$l^{\underline{\theta},q}_{T'}(\theta^*) > r.$$

Since the function $l^{\underline{\theta},q}_{T'}(\cdot)$ is continuous, there exists some $\varepsilon > 0$ such that

$$l^{\underline{\theta},q}_{T'}(\theta^* - \varepsilon) \geq r.$$

Since $l^{\underline{\theta},q}_{T}(\theta)$ is non-decreasing in T and θ, we have

$$l^{\underline{\theta},q}_{T}(\theta) > r$$

for all $T > T'$ and $\theta > \theta^* - \varepsilon$.

Let T'' be such that the statement $S(\underline{\theta})$ holds with $\underline{T} = T''$. Consider a game Γ_T with $T > \max(T'', T')$. Suppose, for some t, player i q-believes at t that $\theta > \max\{0, \theta^* - \varepsilon\}$. Since $T > T'$, $\theta > \theta^* - \varepsilon$ implies that $l^{\underline{\theta},q}_{T}(\theta) \geq r$. Since $T > T''$, by hypothesis, player $-i$ invests in any period at which she $q-$believes that $\theta > \underline{\theta}$. Thus, conditional on $\theta > \theta^* - \varepsilon$, the probability that player $-i$ invests is no less than r. Therefore, at t, player i attaches probability at least rq to the event that the project succeeds. Since $rq > \overline{p}$, this implies that investing is the unique rationalizable action for player i at t. ■

Our main result follows immediately from Lemmas 3 and 4. Fix $\varepsilon > 0$. By Lemma 4, there exists some T' such that each player invests in the game Γ_T with $T > T'$ if she q-believes that $\theta \geq 0$. By Lemma 3, there exists some T'' such that for $T > T''$, when the fundamental is at least 0, the probability that both players q-believe that $\theta \geq 0$ in some round in Γ_T is greater than $1 - \varepsilon$. Taking $\underline{T} = \max\{T', T''\}$ gives the result. ■

Thus, in sharp contrast to the synchronous case, coordination failure arises with vanishing probability in the asynchronous case as the number of rounds grows large. In addition, if, as in the synchronous case, we let observation noise vanish, we get the even stronger implication that there is no delay in successful coordination. This is a corollary of Proposition 2. To make this idea precise, consider a family of sequences $(\sigma\sigma_t)_{t=1}^{\infty}$, where $\sigma > 0$ is a scaling factor, and $(\sigma_t)_{t=1}^{\infty}$ is some fixed sequence with strictly positive members converging to 0. We will denote by $\Gamma_T(\sigma)$ game with T rounds and noise parameters $(\sigma\sigma_t)_{t=1}^{T}$.

Corollary 1 *Suppose that both players play rationalizable strategies. For any $\underline{\theta} > 0$ and $\varepsilon > 0$ there exist some $\overline{\sigma} > 0$ and $\underline{T}$ such that for any $\sigma < \overline{\sigma}$ and any $T > \underline{T}$, both players invest in round 1 of game $\Gamma_T(\sigma)$ with probability at least $1 - \varepsilon$ whenever $\theta \geq \underline{\theta}$.*

While we have focused on simple payoffs to investment that can attain only values $-c$ or b, our main results hold much more generally. For each player i, let $a^i = 1$ if i invests in at least one round, and $a^i = 0$ otherwise. Suppose that each player i receives a payoff of 0 if she never invests, and $\delta^t u(\theta, a^{-i})$ if she invests in period t, where $u : \mathbb{R} \times \{0, 1\} \longrightarrow \mathbb{R}$ is bounded in absolute value by some $\overline{u}$. Suppose moreover that there exists $b > 0$ such that $u(\theta, a^{-i}) \geq b$ whenever $\theta > 0$ and $a^{-i} = 1$, or $\theta \geq 1$. Then Proposition 2 and Corollary 1 hold. The key step in extending the results to this broader class of payoffs is the following analogue to Lemma 1.

Claim 1 *There exists some $\overline{p} \in (0, 1)$ such that, in any round t, investing at t is the unique best response for any type of player i that assigns probability greater than $\overline{p}$ to*

$$\left\{ \left(\theta, a^{-i}\right) \mid \left(\theta > 0 \text{ and } a^{-i} = 1\right) \text{ or } \theta \geq 1 \right\}.$$

The proof of the claim is in the appendix. Given the claim, the proofs of Proposition 2 and Corollary 1 are the same as above since they rely only on the sufficient condition for investment from Lemma 1.

Note that the class of payoffs satisfying our conditions is large. In particular, payoffs need not be monotonic in actions or in θ.

What explains the stark difference in outcomes in the synchronous and asynchronous coordination games? One instructive way to interpret this difference arises out of characterizing the higher order beliefs of players in these games. We turn to such a characterization after describing several related examples in the next section.

4 Examples

The preceding section shows that the benchmark cases of purely synchronous and purely asynchronous coordination lead to starkly different predictions in terms of efficiency of coordination. We develop a framework below for analyzing higher order beliefs in these games that can also be applied to intermediate cases with partial asynchronicity. The intermediate cases we analyze involve games in which, even if both players do not have to invest at the same time, the relative timing of their investment affects their payoffs. In the first of these games, when $\theta \in (0, 1)$, each player prefers *not* to invest before the other player invests: there are small losses for being (exclusively) the first to invest. In the second, each player would prefer to invest *one period* before the other player invests: there is a small reward for being (exclusively) the first to invest.

For these intermediate cases, only the payoffs differ from the games of Section 2 above; the players, action sets, and information structures remain the same, and investment continues to be irreversible. Both cases consist of games in which players receive flow payoffs that depend on whether the other player has invested. The rules for success of the project are the same as in Section 2 but the success is determined during the coordination process instead of at the end.

Example 1 (game in which *neither* player wants to invest before the other) *Let s_t be a success indicator at round t, defined by*

$$s_t = \begin{cases} 1 & \text{if } (a_t^i = a_t^{-i} = 1 \text{ and } \theta > 0) \text{ or } \theta \geq 1; \\ 0 & \text{otherwise,} \end{cases}$$

recalling that, since investment is irreversible, if $a_t^i = 1$ for some t then $a_s^i = 1$ for all $s > t$. Each player i maximizes the discounted sum

$$\frac{1}{1-\delta} \sum_{t=1}^{T} \delta^t u^{(i,t)}$$

of flow payoffs $u^{(i,t)}$, where

$$u^{(i,t)} = \begin{cases} 0 & \text{if } a_t^i = 0, \\ b & \text{if } a_t^i = 1 \text{ and } s_t = 1, \\ -c & \text{if } a_t^i = 1 \text{ and } s_t = 0. \end{cases}$$

Unlike in the asynchronous game of Section 2, players in Example 1 care about the timing of their opponents' investment; neither wants to invest before the other does. One may readily verify that investing is a best response for player i at time t only if she $\underline{p}$-believes at t that the project will succeed at t if she invests, where $\underline{p} = \frac{c}{b+c}$.

The next example captures situations in which early investors receive a reward if the project succeeds.

Example 2 (game in which *each* player wants to invest before the other) *The game is almost identical to the previous one, except that players begin to receive the payoff b one period before the project succeeds. That is*

$$u^{(i,t)} = \begin{cases} 0 & \text{if } a_t^i = 0, \\ b & \text{if } a_t^i = 1, \text{ and } s_{t+1} = 1, \\ -c & \text{if } a_t^i = 1 \text{ and } s_{t+1} = 0, \end{cases}$$

where s_{T+1} is defined to be equal to s_T.[9]

As in Example 1, players in Example 2 care about the timing of their opponents' investment; neither wants to invest too early before the other does. One may readily verify that a result analogous to Lemma 1 holds: investing is a best response for player i at time t if she $\overline{p}$-believes at t that, if she invests, the project will succeed at or before time $t+1$.

In the next section, we develop tools that we use to characterize the outcomes in Examples 1 and 2. We find that the small difference in incentives can generate a large difference in behavior. Rationalizable outcomes in Example 1 turn out to be just as inefficient (for large T and small σ) as those in the fully synchronous game of Section 2. Thus private learning and irreversibility alone are not sufficient to increase efficiency relative to the synchronous benchmark. In contrast, when learning is sufficiently fast, the small incentive to invest just before the opponent leads to full efficiency in Example 2 (again for large T and small σ).

5 Asynchronous Higher Order Beliefs

It is well-known that the coordination failure arising in static global games can be explained by the lack of approximate common knowledge. The finding that coordination failure does not arise in our asynchronous global game indicates that some aspects of higher order beliefs differ between synchronous and asynchronous global games. The current section is devoted to examining this difference. In the process, we characterize general properties favoring efficiency or failure in coordination that can be applied to a large class of dynamic coordination games including those from Examples 1 and 2.

5.1 Preliminary notation and definitions

Throughout this section, we retain the same information and action structure as above, but allow for more general payoffs. Thus two players $i \in \{1, 2\}$ learn in periods $t = 1, 2, \ldots$ by

[9] For simplicity, we assume that the early investment reward corresponds to a flow payoff of b. The relevant feature for our analysis is only that belief that the project will succeed at $t+1$ induces investment at t. This feature also arises for many other reward specifications that differ in size and timing from that of the example.

receiving noisy private signals of a common payoff parameter $\theta \in \Theta = \mathbb{R}$. The parameter θ is drawn from an improper uniform distribution on $\mathbb{R}$, and the noise in players' signals is normally distributed and independent over time and across players. As above, we denote by $\sigma\sigma_t$ the standard deviation of the players' aggregate signals at time t where σ is a noise-scaling parameter. The sequence $(\sigma_t)_t$ is fixed throughout. The action space for player i at time t is $\{0, 1\}$, and action 1 is irreversible, which we capture by assuming that choices are payoff-irrelevant after choosing 1 once. We restrict attention to common value games in which payoffs depend only on θ and chosen actions, not on players' signals.

We focus on asymptotic results characterizing rationalizable behavior in games with many rounds and small noise. We denote by $\Gamma = (\Gamma_T(\sigma))_{T,\sigma}$ a class of games with varying scaling factor $\sigma > 0$ and number of rounds $T \in \mathbb{N}$. The payoff functions vary across T but not across σ.

We refer to each player-time pair (i, t) as an agent. The type of agent (i, t), denoted by $x^{(i,t)} \in X^{(i,t)} = \mathbb{R}$, consists of player i's aggregate signal at time t. The *ordinary state space* is $\Omega = \Theta \times_{(i,t)} X^{(i,t)}$. Let Σ_Ω denote the Borel σ-algebra on Ω (endowed with the usual Euclidean topology) and Σ_Θ the Borel σ-algebra on Θ. An element of Σ_Ω is an *ordinary event.* Note that the posterior beliefs of each agent (i, t) over ordinary events are well-defined and continuous in the uniform topology as the type $x^{(i,t)}$ varies.

In order to conveniently describe strategy profiles, we follow Morris and Shin [22] and introduce compound events. A *compound event* F is a vector $\left(F^{(i,t)}\right)_{(i,t)}$ of length $2T$ in which each component $F^{(i,t)}$ is a Borel-measurable subset of the type space $X^{(i,t)}$. Let $\mathcal{S}$ denote the class of all compound events. Each compound event $F = \left(F^{(i,t)}\right)_{(i,t)}$ can be interpreted as the strategy profile in which each agent (i, t) plays action 1 if and only if $x^{(i,t)} \in F^{(i,t)}$. We identify each compound event with the corresponding strategy profile. Note that we omit from the notation the dependence of compound events on T.

For two compound events $F, F' \in \mathcal{S}$, we write $F \subseteq F'$ if $F^{(i,t)} \subseteq F'^{(i,t)}$ for each (i, t). We define binary operations $\wedge$ and $\vee$ by

$$F \wedge F' = \left(F^{(i,t)} \cap F'^{(i,t)}\right)_{(i,t)},$$

$$F \vee F' = \left(F^{(i,t)} \cup F'^{(i,t)}\right)_{(i,t)}.$$

Negation of a compound event is defined as $\neg F = \left(\overline{F^{(i,t)}}\right)_{(i,t)}$, where $\overline{F^{(i,t)}}$ denotes the complement of the set $F^{(i,t)}$. Note that the class of compound events $\mathcal{S}$ is closed under the operations $\wedge$, $\vee$, and $\neg$. We abuse notation by writing $\emptyset$ for the empty compound event

$$(\underbrace{\emptyset, \ldots, \emptyset}_{2T \text{ times}}).$$

We denote the negation of $\emptyset$ by vector $\mathbf{X} = \left(X^{(i,t)}\right)_{(i,t)}$.

For each (i,t) and $p \in [0,1]$, define the belief operator $B_p^{(i,t)}$ mapping ordinary events to subsets of $X^{(i,t)}$ by

$$B_p^{(i,t)}(E) = \left\{x^{(i,t)} \in X^{(i,t)} : \Pr\left(E|x^{(i,t)}\right) \geq p\right\}.$$

Thus, $B_p^{(i,t)}(E)$ is the set of types of agent (i,t) that p-believe the event E. Note that, although the notation is the same, the domain of the belief operator $B_p^{(i,t)}$ differs from that in the proof of Proposition 2.

Define the *compound belief operator* $B_p : [\Sigma_\Omega]^{2T} \longrightarrow \mathcal{S}$ component-wise by

$$[B_p(E)]^{(i,t)} = B_p^{(i,t)}\left(E^{(i,t)}\right),$$

where $E = \left(E^{(i,t)}\right)_{(i,t)}$ is a vector of ordinary events $E^{(i,t)} \in \Sigma_\Omega$. Note that the notation omits dependence of the belief operators on σ and T.

5.2 Necessary and sufficient conditions for investment

In this section we introduce a formal method for describing necessary or sufficient conditions for investment in a large class of games which includes, but is not limited to, our benchmark games of Section 2 and the games in Examples 1 and 2.

In each class of games Γ, we assume: (i) for each $\theta \in U = [1,\infty)$, action 1 would be strictly dominant in each period if a player knew θ, and (ii) for each $\theta \in G = (0,\infty)$ and each agent (i,t), there exists an action profile for the opponent such that investing would be a best response for agent (i,t) if she knew θ. Below we abuse notation by identifying the sets $U, G \in \Sigma_\Theta$ with the ordinary events $U \times_{(i,t)} X^{(i,t)}$ and $G \times_{(i,t)} X^{(i,t)}$, respectively.

Unlike in the fully asynchronous game, we now consider conditions on best responses that are sensitive to the timing of investment. These conditions make use of operators $O^{T,(i,t)} : \mathcal{S} \longrightarrow \Sigma_\Omega$ that map compound events to ordinary events. We denote by $O^T = \left(O^{T,(i,t)}\right)_{(i,t)}$ the vector of such operators; thus $O^T : \mathcal{S} \longrightarrow [\Sigma_\Omega]^{2T}$. Finally, we denote by $O = \left(O^T\right)_T$ the sequence of operators O^T. When there is no risk of confusion, we sometimes omit dependence of O^T and $O^{T,(i,t)}$ on T. Let $p \in [0,1]$.

Definition 1 *We say that (O,p)* ***describes sufficient conditions for investment*** *in a class of games Γ if for any strategy profile F, in every game $\Gamma_T(\sigma)$, action 1 is the unique best response for $x^{(i,t)}$ whenever*

$$x^{(i,t)} \in B_p^{(i,t)}\left(\left[G \cap O^{T,(i,t)}(F)\right] \cup U\right). \tag{1}$$

Thus investing is a best response whenever agent (i,t) p-believes that the fundamental is very good ($\theta \in U$) or that the fundamental is good ($\theta \in G$) and that the timing of the opponent's investment satisfies certain requirements, which are summarized by O.

Definition 2 *We say that (O,p) **describes necessary conditions for investment** in a class of games Γ if for any strategy profile F, in every game $\Gamma_T(\sigma)$, action 1 is a best response for type $x^{(i,t)}$ only if*

$$x^{(i,t)} \in B_p^{(i,t)}\left(\left[G \cap O^{T,(i,t)}(F)\right] \cup U\right). \tag{2}$$

To illustrate the definitions, consider the classes of games in Sections 2 and 4. In each case, we abuse notation by identifying measurable subsets $F^{(i,t)} \subseteq X^{(i,t)}$ with the ordinary events in which every other component consists of the full space.

- In the fully asynchronous game of Section 2, by Lemma 1, $(O,\overline{p})$ describes sufficient conditions for investment where

$$O^{(i,t)}(F) = \bigcup_{t'=1}^{T} F^{(-i,t')}.$$

- In Example 1, $(O,\underline{p})$ describes necessary conditions for investment, where

$$O^{(i,t)}(F) = \bigcup_{t'=1}^{t} F^{(-i,t')}.$$

- In Example 2, $(O,\overline{p})$ describes sufficient conditions for investment, where

$$O^{(i,t)}(F) = \bigcup_{t'=1}^{\min\{t+1,T\}} F^{(-i,t')}.$$

We restrict attention to sufficient or necessary conditions for investment satisfying three natural properties that are assumed to hold throughout the remainder of the paper. This amounts to restricting the class of games that we consider.

Assumption 1 *For any F and F' such that $F^{(-i,t')} = F'^{(-i,t')}$ for all t', $O^{T,(i,t)}(F) = O^{T,(i,t)}(F')$ for all T and (i,t).*

The first assumption states that necessary and sufficient conditions for investment do not depend on the player's own strategy.

Assumption 2 *For each F and F' such that $F \subseteq F'$, $O^{T,(i,t)}(F) \subseteq O^{T,(i,t)}(F')$ for all T and (i,t).*

Assumption 2 states that if sufficient or necessary condition for investment is satisfied for a type $x^{(i,t)}$ under some strategy of the opponent $-i$, the condition continues to be satisfied if the opponent increases her strategy. This assumption is closely related to the action monotonicity assumption in static global games. Action monotonicity requires that best responses be non-decreasing in the opponent's strategy. Assumption 2 requires only that the lower or upper bound $B_p\left([G \wedge O(F)] \vee U\right)$ of the best response correspondence be non-decreasing in the opponent's strategy.

Before introducing the next assumption, we define a useful concept.

Definition 3 *Let $\mathcal{D}$ and $\mathcal{R}$ be classes of subsets of partially ordered sets. An operator $H : \mathcal{D} \longrightarrow \mathcal{R}$ is **point-monotone** if for every upper contour set $D \in \mathcal{D}$, $H(D)$ is also an upper contour set.*

Assumption 3 *The operator $O^{T,(i,t)}$ is point-monotone for each T and (i,t).*

The point-monotonicity assumption is related to the state monotonicity assumption in static global games that requires own-action payoff differences to be monotone in θ. State monotonicity implies that the best response correspondence is point-monotone. Assumption 3 requires only that the lower or upper bound $B_p\left([G \wedge O(F)] \vee U\right)$ of the best response correspondence be point-monotone.

5.3 Strategic optimism and pessimism

In Section 3 we found that, for high ratio c/b, investing is rationalizable for players in the synchronous game only if they believe that the fundamental is very good ($\theta \geq 1$); players turn out to be "strategically pessimistic" in the sense that they are only willing to invest when they believe that the project will succeed regardless of their opponent's actions. In the asynchronous game, players invest whenever they believe that the fundamental is good ($\theta > 0$). In this case, players turn out to be "strategically optimistic." The following definitions formalize notions of strategic pessimism and optimism.

Definition 4 *We say that there is **strategic pessimism** in a class of games Γ if there exist $\underline{T}$ and, for each T, $q(T) \in (0,1)$ and $\overline{\sigma}(T) > 0$ such that for all $\sigma < \overline{\sigma}(T)$ and $T > \underline{T}$, investing is rationalizable in $\Gamma_T(\sigma)$ for type $x^{(i,t)}$ only if*

$$x^{(i,t)} \in B^{(i,t)}_{q(T)}(U).$$

Definition 5 *We say that there is **strategic optimism** in a class of games Γ if there exist $\underline{T}$ and, for each T, $q(T) \in (0,1)$ and $\overline{\sigma}(T) > 0$ such that for all $\sigma < \overline{\sigma}(T)$ and $T > \underline{T}$ investing* *is the unique rationalizable action in $\Gamma_T(\sigma)$ for type $x^{(i,t)}$ whenever*

$$x^{(i,t)} \in B^{(i,t)}_{q(T)}(G).$$

Note that in both of the preceding definitions, q may depend on T but not on σ.

Remark 1 *Strategic optimism and pessimism are closely related to efficient and inefficient coordination respectively. If there is strategic pessimism, then players fail to coordinate on investing for fundamentals bounded away from the upper dominance region U in games with many rounds and precise signals. Similarly if there is strategic optimism, then players successfully coordinate on investing without delay for fundamentals bounded away from the lower dominance region $\Theta \setminus G$ in games with many rounds and precise signals. This connection is formalized as follows.*

1. *Suppose there is strategic pessimism in Γ. For each sufficiently large T, given any $\varepsilon > 0$ and $\overline{\theta} < 1$, there exists $\underline{\sigma} > 0$ such that for any rationalizable strategies in the game $\Gamma_T(\sigma)$ with $\sigma < \underline{\sigma}$,*
$$\Pr\left(s^{(i,t)}\left(x^{(i,t)}\right) = 0 \quad \forall i, t \mid \theta\right) > 1 - \varepsilon$$
whenever $\theta < \overline{\theta}$.

2. *Suppose there is strategic optimism in Γ. Given any $\varepsilon > 0$ and $\underline{\theta} > 0$ there exist some $\underline{\sigma}(T) > 0$ and $\underline{T}$ such that for any rationalizable strategies in any game $\Gamma_T(\sigma)$ with $\sigma < \underline{\sigma}(T)$ and $T > \underline{T}$,*
$$\Pr\left(s^{(1,1)}\left(x^{(1,1)}\right) = 1 = s^{(2,1)}\left(x^{(2,1)}\right) \mid \theta\right) > 1 - \varepsilon$$
whenever $\theta > \underline{\theta}$.

5.4 Generalized belief operators

We characterize rationalizable actions using a generalization of the common beliefs introduced by Monderer and Samet [18]. Our generalized belief operators and generalized common belief operators are closely related to those of Morris and Shin [22]. Morris and Shin identify the belief operator with the best response correspondence, which is convenient for analysis of rationalizability but complicates interpretation since the domain of beliefs does not include simple events describing θ. We define generalized belief operators for which the interpretation of common beliefs remains intuitive. In particular, we formulate necessary conditions for rationalizability of investment in terms of generalized common belief that the fundamental is good ($\theta > 0$).

Given a pair (O, p), define the *generalized belief operator* $\mathcal{B}_p : \Sigma_\Theta \cup \mathcal{S} \longrightarrow \mathcal{S}$ by

$$\mathcal{B}_p(F) = \begin{cases} B_p(F) & \text{if } F \in \Sigma_\Theta \\ B_p(O^T F) & \text{if } F \in \mathcal{S}. \end{cases}$$

Note that we suppress the dependence of $\mathcal{B}_p$ on O, T and σ from the notation. Also note that we abuse notation by identifying $F \in \Sigma_\Theta$ with the ordinary event $F \times_{(i,t)} X^{(i,t)}$.

Generalized belief $\mathcal{B}_p(F)$ is a vector with $2T$ components, one for each (i,t), where each is a set $\mathcal{B}_p^{(i,t)}(F)$ of types of agent (i,t). If $F \in \Sigma_\Theta$ is an event describing θ then $\mathcal{B}_p^{(i,t)}(F)$ is the set of types of (i,t) that p-believe F. If $F \in \mathcal{S}$ is a compound event — a strategy profile — then $\mathcal{B}_p^{(i,t)}(F)$ is the set of types of (i,t) that, given the profile F, p-believe that the timing of their opponent's investment satisfies the conditions described by O.

The following definition generalizes Monderer and Samet's common beliefs to our dynamic setting.

Definition 6 *For any operator $K : \mathcal{S}' \longrightarrow \mathcal{S}$ with $\mathcal{S} \subseteq \mathcal{S}'$, let $C_K^1(F) = K(F)$, and recursively define C_K^n for $n = 2, 3, \ldots$ by $C_K^n(F) = K(C_K^{n-1}(F)) \wedge C_K^{n-1}(F)$. The generalized common belief operator C_K is defined by*

$$C_K(F) = \bigwedge_{n=1}^{\infty} C_K^n(F).$$

We note without proof that the generalized common belief operator satisfies Monderer and Samet's belief operator axioms.

To intuitively understand the relationship between the generalized common belief operator and Monderer and Samet's common p-belief, recall that (standard) common p-belief of an event G refers to the event that both players p-believe G, both p-believe that both p-believe G, and so on. Since the standard notion focuses on fixed beliefs for each player, dynamic considerations do not arise.

Generalized common belief corresponds to the standard notion in static environments. Consider an information structure with only one round, $T = 1$, and let $O^{(i,1)}(F) = F^{(-i,1)}$ be the event that the opponent invests under strategy profile F. Then $C_{\mathcal{B}_p}$ may be identified with the common p-belief operator of Monderer and Samet.

The crucial difference between generalized and standard common beliefs is that the generalized formulation allows the timing of beliefs to vary across players and across orders of belief in dynamic setups. To fix ideas, we provide an informal discussion of the generalized belief operator in the context of a particular example with $O^{(i,t)}(F) = \bigcup_{t'=1}^{T} F^{(-i,t')}$, which corresponds to our fully asynchronous game. Generalized common belief $C_{\mathcal{B}_p}(G)$ of the event

G is a vector of length $2T$ with components $[C_{\mathcal{B}_p}(G)]^{(i,t)}$. If players follow the strategy profile $C_{\mathcal{B}_p}(G)$ then

$$\bigcap_i O^i(C_{\mathcal{B}_p}(G)), \tag{3}$$

is the event that each player invests during the game. The object (3) is, with a caveat that we explain below, the event that

1. each player p-believes G in some round,
2. each p-believes in some round that each p-believes G in some round,
3. etc.

It is easy to show that the component $[C_{\mathcal{B}_p}(G)]^{(i,t)}$ is the set of types of agent (i,t) that p-believe the above object.

The caveat is that the description does not specify the relationship among the timings of the various orders of beliefs held by different players. A more precise description is as follows: The object (3) is the event that

1. for each player i there exists $t_1^i \in \{1,\ldots,T\}$ such that player i p-believes G at t_1^i,
2. each player i p-believes at t_1^i that there exists $t_2^{-i} \in \{1,\ldots,T\}$ such that player $-i$ p-believes G at t_2^{-i} (and i p-believes at t_1^i that i herself p-believes G at t_1^i),
3. etc.

For this particular O, the times t_n^i are unrelated across players i and orders n. This corresponds to the feature of the fully asynchronous game that the specific timing of $-i$'s investment is irrelevant to player i. For more restrictive conditions O, corresponding to games in which the timing of $-i$'s investment is relevant to i, the timing of beliefs in the above list could be connected.

Next we define an operator, called the expansion operator, that is similar to the generalized common belief operator except with componentwise unions instead of componentwise intersections. Our use of the two operators is complementary; we use the generalized common belief operator to identify sufficient conditions for strategic pessimism and the expansion operator to identify sufficient conditions for strategic optimism.

Definition 7 *For any operator $K : \mathcal{S}' \longrightarrow \mathcal{S}$ with $\mathcal{S} \subseteq \mathcal{S}'$, let $E_K^1(F) = K(F)$, and recursively define E_K^n for $n = 2,3,\ldots$ by $E_K^n(F) = K(E_K^{n-1}(F)) \vee E_K^{n-1}(F)$. The **expansion operator** E_K is defined by*

$$E_K(F) = \bigvee_{n=1}^{\infty} E_K^n(F).$$

For $O^{(i,t)}(F) = O^i(F) = \bigcup_{t'=1}^{T} F^{(-i,t')}$, corresponding to the fully asynchronous game, the event $\bigcap_i O^i(E_{\mathcal{B}_p}(G))$ is described by the same list as above, but with the word "and" replaced everywhere with the word "or".

The generalized common belief and expansion operators are connected by a simple relation. We say that L is the dual operator to K if $L(F) = \neg K(\neg F)$ for all F. One may readily verify that if L is dual to K then

$$E_L(F) = \neg C_K(\neg F), \tag{4}$$

that is, E_L is dual to C_K.

The next proposition provides a characterization of rationalizability of investment using the generalized common belief operator $C_{\mathcal{B}_p}$ and the expansion operator $E_{\mathcal{B}_p}$. The first part of the proposition generalizes the well-known necessary condition for rationalizability of investment in static coordination games (see, for example, Morris and Shin 2003). The second part is derived from the first using the duality of the two operators.

Proposition 3 *1. Suppose that (O, p) describes* ***necessary*** *conditions for investment in a class of games Γ. If there exist $p' < p$, $\overline{\sigma}(T)$, and $\underline{T}$ such that $C_{\mathcal{B}_{p'}}(G) = \emptyset$ for all $\sigma < \overline{\sigma}(T)$ and $T > \underline{T}$, then there is strategic* ***pessimism*** *in Γ.*

2. Suppose that (O, p) describes ***sufficient*** *conditions for investment in a class of games Γ. If there exist $p' > p$, $\overline{\sigma}(T)$, and $\underline{T}$ such that $E_{\mathcal{B}_{p'}}(U) = \mathbf{X}$ for all $\sigma < \overline{\sigma}(T)$ and $T > \underline{T}$, then there is strategic* ***optimism*** *in Γ.*

The proof of the proposition is in Subsection A.1 of the appendix.

When generalized common belief is based on necessary conditions for investment, if in every state players fail to commonly believe that the fundamental is good, then the proposition says that players invest only if they individually believe that the fundamental is very good. Similarly, if the generalized belief operator (and hence also the expansion operator) is based on sufficient conditions for investment, then investment is the unique rationalizable action for any type that believes the fundamental is good if every type has, for some k, a kth order generalized belief that the fundamental is very good.

The following result describes a condition for strategic optimism that is easier to apply than that of part 2 of Proposition 3.

Proposition 4 *Suppose that (O, p) describes sufficient conditions for investment in a class of games Γ. If there exist $p' > p$, $q \in (0, 1)$, $\overline{\sigma}(T)$, and $\underline{T}$ such that for all $\sigma < \overline{\sigma}(T)$, $T > \underline{T}$, and all $\theta^* \in \mathbb{R}$*

$$B_q(\theta \geq \theta^*) \subseteq \mathcal{B}_{p'}\left(B_q(\theta \geq \theta^*)\right) \tag{5}$$

then there is strategic optimism in Γ.

The proof of the proposition is relegated to the appendix.

The discerning reader may notice that Proposition 4 generalizes the key step of the proof of Proposition 2. There we observed that event that both players q-believe $\theta \geq \theta^*$ is in some, asynchronous sense evident (in sufficiently long games) — whenever the event holds each player believes it at some time. Based on this observation we could use the contagion argument; the observation implied the contagion step: if investment is rationalizable whenever players q-believe $\theta \geq \theta^*$ then investment is rationalizable also whenever players q-believe $\theta \geq \theta^* - \varepsilon$. Proposition 4 states that if $B_q(\theta \geq \theta^*)$ is evident in a generalized sense that reflects sufficient conditions for investment (O, p) then strategic optimism arises. The proof in appendix is again based on the contagion argument. We could have also formulated a symmetric result based on a contagion of not investing from lower dominance region in games with necessary conditions for investment. We omit such result because we will not use in the next section where we apply the results of this section to particular games.

6 Applications

In this section we use the preceding results to characterize the rationalizable outcomes in several games. As above, the information structure in each game is identical to that in Section 2. For each game, we show that there is strategic optimism or strategic pessimism by analyzing a pair (O, p) that describes sufficient or necessary conditions for investment, respectively. By Remark 1, this establishes efficiency or inefficiency of coordination in each game.

6.1 The fully asynchronous game

First we return to the fully asynchronous game. The next proposition essentially replicates Lemma 4 to illustrate the relationship between the general method of Section 5 and the results of Section 3.

Proposition 5 *Suppose that (O, p) describes sufficient conditions for investment in a class of games Γ, where*

$$O^{T,(i,t)}(F) = \bigcup_{t'=1}^{T} F^{(-i,t')}. \tag{6}$$

Then there is strategic optimism in Γ.

Recall that if $p = \overline{p}$, then (O, p) describes sufficient conditions for investment in the fully asynchronous game if O is as in (6).

Proof. We will show that for each p' there exist q and $\underline{T}$ such that for each $T > \underline{T}$ and all $\sigma > 0$,

$$B_q(\theta \geq \theta^*) \subseteq \mathcal{B}_{p'}\left(B_q(\theta \geq \theta^*)\right).$$

Proposition 4 then implies strategic optimism in Γ.

By Lemma 3, we have

$$\lim_{T\to\infty} \Pr\left(O^{T,(i,t)}B_q(\theta \geq \theta^*) \mid \theta^*\right) = 1.$$

Choose $p' > p$, $q > p'$, and $\underline{T}$ such that

$$\Pr\left(O^{T,(i,t)}B_q(\theta \geq \theta^*) \mid \theta^*\right) \geq \frac{p'}{q}$$

for all $T > \underline{T}$ and $\sigma > 0$ (which is possible since the given probability is independent of σ). Every type $x^{(i,t)} \in B_q(\theta \geq \theta^*)$ assigns probability at least $q \times \frac{p'}{q} = p'$ to the event $O^{T,(i,t)}B_q(\theta \geq \theta^*)$, as needed. ■

6.2 Game with reward for early investment

Next we return to Example 2 of Section 4. We find that, if learning is sufficiently quick, strategic optimism arises in this class of games.

Proposition 6 *Suppose that (O,p) describes sufficient conditions for investment in a class of games Γ, where*

$$O^{T,(i,t)}(F) = \bigcup_{t'=1}^{\min\{t+1,T\}} F^{(-i,t')}. \tag{7}$$

If there exists $\tau \in (0,1)$ such that $\sigma_{t+1} \leq \tau\sigma_t$ for each t, then there is strategic optimism in Γ.

Recall that if $p = \overline{p}$ then (O,p) describes sufficient conditions for investment in Example 2 if O is as in (7).

Proof. We again show that for each p' there exist q and $\underline{T}$ such that for each $T > \underline{T}$ and $\sigma > 0$,

$$B_q(\theta \geq \theta^*) \subseteq \mathcal{B}_{p'}\left(B_q(\theta \geq \theta^*)\right). \tag{8}$$

Proposition 4 then implies strategic optimism in the class Γ.

We use the following lemma to prove (8).

Lemma 5 *Suppose there exists $\tau \in (0,1)$ such that $\sigma_{t+1} \leq \tau\sigma_t$ for every t. Then for each $p' \in (0,1)$, there exists $q \in (0,1)$ such that*

$$B_q^{(i,t)}(\theta \geq \theta^*) \subseteq B_{p'}^{(i,t)}\left(B_q^{(-i,t+1)}(\theta \geq \theta^*)\right)$$

for all $t = 1, \ldots, T-1$.

The proof of the lemma is in the Appendix. Roughly speaking, Lemma 5 states that if a player believes $\theta \geq \theta^*$, then she also believes that her opponent will believe $\theta \geq \theta^*$ in the next period. This property implies that the first-order belief $B_q(\theta \geq \theta^*)$ is evident in a generalized, asynchronous sense reflecting sufficient conditions for investment in this class of games.

Given p', let q be as in Lemma 5. The lemma implies that

$$B_q^{(i,t)}(\theta \geq \theta^*) \subseteq B_{p'}^{(i,t)}\left(O^{T,(i,t)}B_q(\theta \geq \theta^*)\right)$$

for each $t = 1, \ldots, T-1$ since $B_q^{(-i,t+1)}(\theta \geq \theta^*) \subseteq O^{T,(i,t)}B_q(\theta \geq \theta^*)$.

All that remains is to show that in the last round T

$$B_q^{(i,T)}(\theta \geq \theta^*) \subseteq B_{p'}^{(i,T)}\left(O^{T,(i,T)}B_q(\theta \geq \theta^*)\right)$$

when T is sufficiently large. This last containment follows from the proof of Proposition 5 since $O^{T,(i,T)} = \bigcup_{t'=1}^{T} F^{(-i,t')}$ is identical to the corresponding operator for the fully asynchronous game. ■

6.3 Fully synchronous game

Next we use the generalized common belief framework to prove that coordination failure arises in the synchronous game, as described in Section 3.1. The proof essentially replicates that for the failure of common knowledge in static global games (see Morris and Shin [20]).

Proposition 7 *Suppose that (O,p) describes necessary conditions for investment in class of games Γ, where*

$$\begin{aligned} O^{T,(i,t)}(F) &= \emptyset, \text{ for } t < T, \\ O^{T,(i,T)}(F) &= F^{(-i,T)}. \end{aligned}$$

and $p > \frac{1}{2}$. Then there is strategic pessimism in Γ.

Letting $p = \underline{p}$, the pair (O,p) of the proposition describe necessary conditions for investment in the synchronous game of Section 3.1, in which players invest only in round T.

Proof. Fix $p > \frac{1}{2}$. We will show that $C_{\mathcal{B}_p}(G) = \emptyset$ for any T and $\sigma > 0$. Strategic pessimism then follows from Proposition 3.

Clearly $\left[C_{\mathcal{B}_p}(G)\right]^{(i,t)} = \emptyset$ for any $t < T$ by the definition of O. Consider round T. Recall that $\mathcal{B}_p^{(i,T)}(G)$ is the set $\left[\sigma_T F^{-1}(p), \infty\right)$ of types of (i,T) that p-believe G. Turning to the second order of beliefs, $\left[\mathcal{B}_p^2\right]^{(i,T)}(G)$ is the set of types of (i,T) that p-believe that $(-i,T)$

p-believes G. Conditional on $x^{(i,T)}$, (i,T) believes that $x^{(-i,T)}$ is normally distributed with mean $x^{(i,T)}$ and variance $2\sigma_T^2$. Therefore, we have

$$\left[\mathcal{B}_p^2\right]^{(i,T)}(G) = \left[\sigma_T F^{-1}(p) + \sqrt{2}\sigma_T F^{-1}(p), \infty\right).$$

Proceeding inductively, we see that

$$\left[C_{\mathcal{B}_p}(G)\right]^{(i,T)} = \bigcap_n [\mathcal{B}_p^n]^{(i,T)}(G) = \bigcap_n \left[\sigma_T F^{-1}(p)\left(1 + (n-1)\sqrt{2}\right), \infty\right) = \emptyset,$$

as needed. ■

6.4 Game with loss for early investment

Next we return to Example 1 of Section 4. We find that strategic pessimism arises in this class of games.

Proposition 8 *Suppose that (O,p) describes necessary conditions for investment in class of games Γ, where*

$$O^{T,(i,t)}(F) = \bigcup_{t'=1}^{t} F^{(-i,t')} \tag{9}$$

and $p > \frac{1}{2}$. Then there is strategic pessimism in Γ.

Recall that if $p = \underline{p}$, (O,p) describes necessary conditions for investment in Example 1 if O is as in (9).

Proof. Fix any $p > \frac{1}{2}$. We will show that $C_{\mathcal{B}_p}(G) = \emptyset$ for any T and σ. Strategic pessimism then follows by Proposition 3.

First note that $\left[C_{\mathcal{B}_p}(G)\right]^{(i,1)}$ coincides with the usual static common p-belief of G, which is empty for $p > \frac{1}{2}$ (see the previous subsection). The result for periods $t \geq 2$ follows by induction. Suppose $\left[C_{\mathcal{B}_p}(G)\right]^{(i,t')} = \emptyset$ for $i = 1, 2$ and $t' = 1, \ldots, t-1$. Then $\left[C_{\mathcal{B}_p}(G)\right]^{(i,t)}$ coincides with the usual static common p-belief of G in period t, which is again empty. ■

7 Discussion

In this section, we discuss the roles of our major assumptions, and consider potential extensions.

First, we discuss the information structure. The uniform prior assumption simplifies the analysis but, as is well understood in the global games literature, is not important when signals are precise. Any prior with a continuous density leads to posteriors close to those arising from a uniform prior when the noise in signals is small.

A significant feature of the information structure is the (lack of) feedback the players receive during the course of the game. In particular, we assume that players do not observe each other's past choices. Instead, in each period, they observe a noisy private signal $\widetilde{x}^{(i,t)}$ of the state θ. The lack of learning from past actions may seem artificial. However, we can reinterpret our model as one with a continuum of players, as described in brief below. Under this reinterpretation, in the class of symmetric monotone equilibria, the per-period signals $\widetilde{x}^{(i,t)}$ can be shown to be equivalent to noisy observation of the past actions of other players.

To see this, we begin by describing how to reinterpret our fully asynchronous game as having a continuum of players of unit measure. Let l_t denote the measure of players who have invested by period t. Payoffs for investing at t are given by $\delta^t(bl_T - c(1 - l_T))$ if $\theta \in (0,1)$, by b if $\theta \geq 1$, and by $-c$ if $\theta \leq 0$. Each player has one investment option. At each time t, each player can either invest irreversibly or wait. At $t = 1$, let the information structure for each player be identical to that of the original two player game. If players follow symmetric monotone strategies at $t = 1$, then there exists an increasing function λ_1 such that $l_1 = \lambda_1(\theta)$. At $t = 2$, players observe a monotone statistic based on the measure of first period investors, with some private noise. In particular, assume that at $t = 2$, each player i observes $\widetilde{x}^{(i,2)} = \lambda_1^{-1}(l_1) + \widetilde{\sigma}_2\varepsilon^{(i,2)}$, where $\varepsilon^{(i,2)} \sim N(0,1)$ independently across players. Since $\lambda_1^{-1}(l_1) = \theta$, the observation of $\widetilde{x}^{(i,2)}$ is informationally equivalent to the observation of a noisy private signal directly about θ: $\widetilde{x}^{(i,2)} = \theta + \widetilde{\sigma}_2\varepsilon^{(i,2)}$. Thus, the exogenous private signal at $t = 2$ in the original model can be microfounded as a particular form of noisy social learning. But now, if players follow symmetric monotone strategies at $t = 2$, there will be an increasing function λ_2, such that $l_2 = \lambda_2(\theta)$. In turn, by allowing agents to observe $\lambda_2^{-1}(l_2)$ with private noise at $t = 3$, we can also reinterpret the private signal at $t = 3$ as arising out of the noisy observation of past play. Proceeding iteratively in this way, we can microfound the entire sequence of private signals in the original model. Thus, while our baseline analysis abstracts from the observability of past actions for tractability, our exogenous learning process can be viewed through the lens of endogenous noisy social learning. Although the precise translation between exogenous and social learning relies on particular parametric assumptions, this approach to modeling social learning has become common in the global games literature because of its tractability (see, for example, Angeletos, Hellwig, and Pavan [1] (online supplement), Angeletos and Werning [2], and Goldstein, Ozdenoren, and Yuan [11]).

Finally, we discuss the role of irreversibility in our arguments. Irreversibility itself is not necessary for the main results of Section 5. Even when actions are reversible, there may exist a pair (O, p) describing necessary or sufficient conditions for investment based on which we can derive conditions for strategic pessimism or optimism respectively. However, irreversibility plays an important role in the analysis of our specific games. The tendency towards efficiency in the fully asynchronous game is related to the fact that we chose the efficient rather than the inefficient action to be irreversible. This assumption forms a natural benchmark in the context of many applications, including the leading example of foreign direct investment which we used to motivate our stylized game. However, in other applications, alternative assumptions may be more appropriate. Had we chosen differently, that is, had we assumed that the project succeeds only if all players choose to invest in *all* rounds, the project would always fail outside the upper dominance region.

8 Conclusion

Static coordination games represent a useful abstraction for studying coordination problems in the real world. However, the associated requirement of synchronicity in participation may be a strong restriction: the outcomes generated in such models may not be good representations for real-world coordination problems where agents are able to participate at different points of time and can learn about payoffs while deciding when to participate. We illustrate the radical difference between synchronous and asynchronous coordination problems within the framework of global games. In canonical synchronous (one-shot) global games, the risk-dominant equilibrium of the underlying complete information game is selected. Thus, coordination failure is endemic in static global games: there exist a wide class of payoffs for which players fail to efficiently coordinate in the unique equilibrium of the canonical global game despite the fact that it is in their collective interest to do so.

We introduce a class of enriched asynchronous global games where agents have many opportunities to participate, while they asymptotically and privately learn the true payoffs. In our benchmark analysis, we consider an extreme version of such games, in which the specific timing of the other player's investment is irrelevant to one's own payoffs. We show that rationalizable play in such a game ensures Pareto dominant outcomes. Coordination failure is eliminated.

We also analyze intermediate cases in which players care to some extent about the timing of opponent's investment. We show that the coordination outcome is highly sensitive to the details of players' tolerance to asynchronicity of investment. A small reward for investing before one's opponent results in full efficiency, as in the benchmark fully asynchronous game. A small penalty for investing before one's opponent ensures complete coordination failure, as in the canonical static global game. A deeper understanding of timing incentives in dynamic coordination problems may pinpoint details in the design of coordination processes that could help to prevent coordination failures. Our results provide a starting point for such design exercises.

A Appendix

Proof of Lemma 2. First we prove continuity. Note that for every δ,

$$l_t^{\theta^*,q}(\theta+\delta) = l_t^{\theta^*-\delta,q}(\theta). \tag{10}$$

Hence $l_t^{\theta^*,q}(\theta)$ is continuous in θ if and only if it is continuous in θ^*. Since the density of posterior beliefs is uniformly bounded in each period t', given any $\eta > 0$, there exists some $\delta_{t'} > 0$ such that

$$\left|\Pr\left(B_q^{(i,t')}\left([\theta^*,\infty)\right)\middle|\,\theta\right) - \Pr\left(B_q^{(i,t')}\left([\theta',\infty)\right)\middle|\,\theta\right)\right| < \frac{\eta}{t}$$

whenever $|\theta^* - \theta'| < \delta_{t'}$. Letting $\delta = \min_{t'=1,\ldots,t} \delta_{t'}$, it follows that

$$\left|l_t^{\theta^*,q}(\theta) - l_t^{\theta',q}(\theta)\right| < \eta$$

whenever $|\theta^* - \theta'| < \delta$, as needed.

Monotonicity also follows from (10) since $l_t^{\theta^*,q}(\theta)$ is nonincreasing in θ^*. ■

Proof of Lemma 3. We begin by showing that $l^{\theta^*,q}(\theta^*) = 1$. For any infinite sequence $\tau = (t_1, t_2, \ldots)$, let

$$l^{\theta^*,q}(\theta^*;\tau) = \Pr\left(\bigcup_{t'=t_1,t_2,\ldots} B_q^{(i,t')}(\theta \geq \theta^*)\middle|\,\theta^*\right).$$

We have $l^{\theta^*,q}(\theta^*;\tau) \leq l^{\theta^*,q}(\theta^*)$, so it suffices to show that, for any $\varepsilon > 0$, there exists some sequence τ for which $l^{\theta^*,q}(\theta^*;\tau) \geq 1-\varepsilon$.

Given $\varepsilon > 0$, let $\underline{x} < \theta^*$ be such that $\Pr(x^{(i,1)} < \underline{x}|\theta^*) < \varepsilon(1-\varepsilon)$. Let $t_1 = 1$ and choose each subsequent t_k so that

$$\Pr(x^{(i,t_k)} < \underline{x}|\theta^*) < \varepsilon^k(1-\varepsilon).$$

For this sequence $t_1, t_2, \ldots$, we have

$$\Pr(x^{(i,t_k)} < \underline{x} \text{ for some } k|\theta^*) < \varepsilon.$$

Thus it suffices to show that, as long as $x^{(i,t_k)} \geq \underline{x}$ for all k, there almost surely exists some period t_k in this sequence at which the player q-believes that $\theta \geq \theta^*$. Hence the result follows if, for some $\delta > 0$ and some subsequence τ of this sequence, the player q-believes that $\theta \geq \theta^*$ with independent probability δ in each period in τ.

Player i q-believes in period t that $\theta \geq \theta^*$ as long as $\frac{x^{(i,t)}-\theta^*}{\sigma_t} > \Phi^{-1}(q)$, where $\Phi(\cdot)$ denotes the standard normal distribution function. Given that $x^{(i,t_k)} \geq \underline{x}$, for $t > t_k$, we have

$$\begin{aligned} x^{(i,t)} &= \frac{\sigma_t^2}{\sigma_{t_k}^2} x^{(i,t_k)} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} \\ &\geq \frac{\sigma_t^2}{\sigma_{t_k}^2} \underline{x} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}. \end{aligned}$$

Hence $\frac{x^{(i,t)}-\theta^*}{\sigma_t} > \Phi^{-1}(q)$ whenever

$$\frac{\frac{\sigma_t^2}{\sigma_{t_k}^2} \underline{x} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} - \theta^*}{\sigma_t} > \Phi^{-1}(q). \tag{11}$$

Since

$$\frac{1}{\sigma_t^2} = \frac{1}{\sigma_{t_k}^2} + \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}, \tag{12}$$

we have $\frac{1}{\sigma_t} = \frac{\sigma_t}{\sigma_{t_k}^2} + \sigma_t \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}$, and the left-hand side of Inequality (11) may be rewritten as

$$\begin{aligned} \frac{\frac{\sigma_t^2}{\sigma_{t_k}^2} \underline{x} + \sigma_t^2 \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} - \theta^*}{\sigma_t} &= \frac{\sigma_t}{\sigma_{t_k}^2} \underline{x} + \sigma_t \sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2} - \left(\frac{\sigma_t}{\sigma_{t_k}^2} + \sigma_t \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2} \right) \theta^* \\ &= \frac{\sigma_t}{\sigma_{t_k}^2} (\underline{x} - \theta^*) + \sigma_t \sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2} \left(\frac{\sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}}{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} - \theta^* \right) \\ &= \frac{\sigma_t}{\sigma_{t_k}^2} (\underline{x} - \theta^*) \\ &\quad + \left(\sigma_t \sqrt{\frac{1}{\sigma_t^2} - \frac{1}{\sigma_{t_k}^2}} \right) \left(\sqrt{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} \left(\frac{\sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}}{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} - \theta^* \right) \right), \end{aligned}$$

where the last equality follows again from Equation (12). Inequality (11) is therefore equivalent to

$$\frac{\sigma_t}{\sigma_{t_k}^2} (\underline{x} - \theta^*) + \left(\sigma_t \sqrt{\frac{1}{\sigma_t^2} - \frac{1}{\sigma_{t_k}^2}} \right) \left(\sqrt{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} \left(\frac{\sum_{s=t_k+1}^{t} \frac{\tilde{x}^{(i,s)}}{\tilde{\sigma}_s^2}}{\sum_{s=t_k+1}^{t} \frac{1}{\tilde{\sigma}_s^2}} - \theta^* \right) \right) > \Phi^{-1}(q).$$

The first term on the left-hand side of this inequality tends to zero as t grows large. The second term is a product of two factors, the first of which tends to one as t grows large, and the second of which is a standard normal random variable independent of the realizations of all signals up to period t_k. Therefore, for small enough $\delta > 0$, the inequality holds with probability at least δ when t is sufficiently large given δ, q, $\underline{x}$, and t_k. We may therefore construct the desired subsequence τ by selecting sufficiently distant elements of the sequence $t_1, t_2, \ldots$.

To prove that $l^{\theta^*,q}(\theta)$ is continuous at θ^*, note that for each t, $l_t^{\theta^*,q}(\theta)$ is continuous and bounded above by 1. Given $\varepsilon > 0$, let T be such that $l_t^{\theta^*,q}(\theta^*) > 1-\varepsilon/2$ for every $t \geq T$; such a T exists since $l^{\theta^*,q}(\theta^*) = 1$. Since $l_T^{\theta^*,q}(\theta)$ is continuous, there exists some $\delta > 0$ such that $l_T^{\theta^*,q}(\theta) > 1 - \varepsilon$ for every $\theta \in (\theta^* - \delta, \theta^* + \delta)$. Since $l_t^{\theta^*,q}(\theta)$ is nondecreasing in t, it follows that $l^{\theta^*,q}(\theta) > 1 - \varepsilon$ for every $\theta \in (\theta^* - \delta, \theta^* + \delta)$, and therefore $l^{\theta^*,q}(\theta)$ is continuous at θ^* since it is bounded above by 1. ■

Proof of Claim 1. Let $F^i = \{(\theta, a^{-i}) \mid \theta > 0 \text{ and } a^{-i} = 1 \text{ or } \theta \geq 1\}$. Consider a type $x^{(i,t)}$ that assigns probability at least p to F^i. The expected payoff to investing at t for type $x^{(i,t)}$ is at least

$$\delta^t \left(pE\left[u(\theta, a^{-i}) \mid F^i, x^{(i,t)}\right] + (1-p)(-\overline{u})\right).$$

The expected payoff to not investing is at most

$$\delta^{t+1} \left(pE\left[u(\theta, a^{-i}) \mid F^i, x^{(i,t)}\right] + (1-p)\overline{u}\right)$$

since, in round $t+1$, conditional on F^i, the payoff to investing always exceeds the payoff to not investing. Note that the last expression is also trivially an upper bound on the payoff to not investing if $t = T$. Therefore, investing is the unique best response for $x^{(i,t)}$ if

$$p\left((1-\delta)E\left[u(\theta, a^{-i}) \mid F^i, x^{(i,t)}\right] + (1+\delta)\overline{u}\right) > (1+\delta)\overline{u}.$$

Since $E\left[u(\theta, a^{-i}) \mid F^i, x^{(i,t)}\right] \geq b$, taking

$$\overline{p} = \frac{(1+\delta)\overline{u}}{(1-\delta)b + (1+\delta)\overline{u}}$$

gives the result. ■

A.1 Proof of Proposition 3

Before getting to the main proof we provide two lemmas.

Lemma 6 ***[Morris and Shin [22]]*** *Suppose that $K : \mathcal{S} \longrightarrow \mathcal{S}$ is an increasing operator such that for any strategy profile that prescribes action $a \in \{0,1\}$ on $F^{(i,t)}$ for each (i,t), action a is a best response for any agent (i,t) in $\Gamma_T(\sigma)$ only on a subset of $K^{(i,t)}(F)$. Then action a is rationalizable in $\Gamma_T(\sigma)$ for each agent (i,t) only on some subset of $[C_K(\mathbf{X})]^{(i,t)}$.*

There is a small difference between Lemma 6 and Proposition 10 of Morris and Shin [22]. Lemma 6 specifies necessary conditions for rationalizability based on a bound on the best response correspondence whereas Morris and Shin [22] specify necessary and sufficient conditions based directly on the best response correspondence.

Proof. Recall from the definition of generalized common beliefs that $C_K^1 = K(\mathbf{X})$ and $C_K^n = K\left(C_K^{n-1}\right) \wedge C_K^{n-1}$ for $n \geq 2$. Consider iterated deletion of never best responses. Action a is never a best response for types outside $K(\mathbf{X}) = C_K^1$ since K is increasing. After one round of deletion, action a is never a best response outside C_K^2. Proceeding in this fashion, action a is deleted for all types of each agent (i,t) outside $[C_K(\mathbf{X})]^{(i,t)} = \bigcap_n [C_K^n]^{(i,t)}$. ■

If (O,p) describes necessary conditions for investment, then Lemma 6 gives necessary conditions for rationalizability of investment. Define $\tilde{\mathcal{B}}_p : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$\tilde{\mathcal{B}}_p(F) = B_p\left([G \wedge O^T F] \vee U\right).^{10}$$

Note that the operator $\tilde{\mathcal{B}}_p(F)$ differs from the generalized belief operator $\mathcal{B}_p(F)$ defined in Section 5.4, which is independent of G and U.

One can express necessary conditions for rationalizability of investment in terms of generalized common belief $C_{\tilde{\mathcal{B}}_p}(\mathbf{X})$. However, such a characterization is complicated by the influence of the upper dominance region on the operator $\tilde{B}_p$. The following lemma provides a partial characterization of $C_{\tilde{\mathcal{B}}_p}(\mathbf{X})$ in terms of the simpler common belief operator $C_{\mathcal{B}_p}$, which is independent of the upper dominance region.

First we extend the definition of the operator $\tilde{\mathcal{B}}_p$ by replacing the particular sets G and U with arbitrary upper contour sets D and D' in Σ_Θ such that $D \subseteq D'$. For any O satisfying A1-3, define $\tilde{\mathcal{B}}_p : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$\tilde{\mathcal{B}}_p(F) = B_p\left([D' \wedge O^T F] \vee D\right).$$

Lemma 7 *For each $p' \in (0,p)$ and T, there exist $q \in (0,1)$ and $\overline{\sigma} > 0$ such that*

$$\neg C_{\mathcal{B}_{p'}}(D') \wedge C_{\tilde{\mathcal{B}}_p}(\mathbf{X}) \subseteq B_q(D),$$

for all $\sigma \leq \overline{\sigma}$.

To interpret the statement of the lemma, consider $D' = G$ and $D = U$. Roughly speaking, the lemma says that if the necessary conditions for rationalizability from Lemma 6 are satisfied in some state where (generalized) common belief of G fails, then players must assign high enough probability to U. The failure of common belief refers to the common belief operator based on $\mathcal{B}_p$, where $\mathcal{B}_p$ reflects sufficient conditions for investment (O,p).

[10] Note that we abuse notation by identifying G (resp. U) with vectors of ordinary events, each component of which is the event $G \times_{(i,t)} X^{(i,t)}$ (resp. $U \times_{(i,t)} X^{(i,t)}$).

Proof of Lemma 7. We prove the equivalent statement

$$\neg B_q(D) \wedge C_{\tilde{\mathcal{B}}_p}(\mathbf{X}) \subseteq C_{\mathcal{B}_{p'}}(D'). \tag{13}$$

To keep notation simple, we assume that O is symmetric across players; the proof for the asymmetric case is similar.

Let $\mathcal{T}(q,\sigma) \subseteq \{1,\ldots,T\}$ be the set of rounds for which the left hand side of (13) is empty, that is, $\mathcal{T}(q,\sigma)$ is the set of rounds t for which, given σ,

$$\left[C_{\tilde{\mathcal{B}}_p}(\mathbf{X})\right]^{(i,t)} \subseteq B_q^{(i,t)}(D). \tag{14}$$

Note that $\mathcal{T}(q,\sigma)$ is non-increasing in q and non-decreasing in σ. The first of these monotonicity properties is trivial. The second follows from the symmetry of the model with respect to the scale parameter σ. If $\sigma' > \sigma$, then $D' \setminus D$ is smaller relative to σ' than relative to σ, and hence if (14) holds for σ then it holds for σ'.

Let

$$\begin{aligned} \mathcal{T}(q) &= \bigcap_{\sigma>0} \mathcal{T}(q,\sigma) \\ \text{and} \qquad \mathcal{T} &= \bigcup_{q\in(0,1)} \mathcal{T}(q). \end{aligned}$$

Note that the lemma holds for all components (i,t) for which $t \in \mathcal{T}$: there exists q such that all rounds $t \in \mathcal{T}$ belong to the set $\mathcal{T}(q,\sigma)$ for sufficiently small σ, and (13) holds trivially for such t because the left-hand side is empty.

It remains to show that the lemma holds for all components (i,t) for which $t \in \mathcal{T}' = \{1,\ldots,T\} \setminus \mathcal{T}$. Take any $t \in \mathcal{T}'$. We have $t \notin \mathcal{T}(q)$ for each $q > 0$, and hence, by the monotonicity of $\mathcal{T}(q,\sigma)$ with respect to σ, for each $q > 0$ there exists $\overline{\sigma}$ such that $t \notin \mathcal{T}(q,\sigma)$ for each $\sigma \leq \overline{\sigma}$.

Let q' and $\overline{\sigma}'$ be such that $\mathcal{T} \subseteq \mathcal{T}(q',\sigma)$ for all $\sigma \leq \overline{\sigma}'$, which exist since $\mathcal{T}$ is finite. Let $q < q'$ and $\overline{\sigma}'' > 0$ be such that $\mathcal{T}(q,\sigma) \subseteq \mathcal{T}$ for all $\sigma \leq \overline{\sigma}''$. Letting $\overline{\sigma} = \min\{\overline{\sigma}', \overline{\sigma}''\}$, we have $\mathcal{T} = \mathcal{T}(q,\sigma)$ for all $\sigma \leq \overline{\sigma}$ since $\mathcal{T}(q,\sigma)$ is non-increasing in q. Henceforth we restrict σ to be at most $\overline{\sigma}$.

Let $\tilde{C}^n = C^n_{\tilde{\mathcal{B}}_p}(\mathbf{X})$ and $C^n = C^n_{\mathcal{B}_{p'}}(D')$, and similarly for $\tilde{C}$ and C. Let $x^{*(i,t)} = \inf \tilde{C}^{(i,t)}$. For each $t \in \mathcal{T}'$, we have $x^{*(i,t)} \notin B_q^{(i,t)}(D)$, for otherwise (14) would hold for t and t would belong to $\mathcal{T}(q,\sigma) = \mathcal{T}$, contradicting that $t \in \mathcal{T}'$.

We will prove by induction that $\tilde{C}^{n,(i,t)} \subseteq C^{n,(i,t)}$ for all n and all $t \in \mathcal{T}'$. This implies that $\tilde{C}^{(i,t)} \subseteq C^{(i,t)}$ for all $t \in \mathcal{T}'$, as needed.

The statement $\tilde{C}^{n,(i,t)} \subseteq C^{n,(i,t)}$ holds trivially for $n = 1$. Assume for induction that the statement holds for $n-1$ for all $t \in \mathcal{T}'$. We will show that $\tilde{C}^{n,(i,t)} \subseteq C^{n,(i,t)}$, that is, that

$$B_p^{(i,t)}\left(\left(D' \cap O^{(i,t)}\tilde{C}^{n-1}\right) \cup D\right) \cap \tilde{C}^{n-1,(i,t)} \subseteq B_{p'}^{(i,t)}\left(O^{(i,t)}C^{n-1}\right) \cap C^{n-1,(i,t)}.$$

Let $x_n^{(i,t)} = \inf \tilde{C}^{n,(i,t)}$. Note that, by induction, $\tilde{C}^n$ is a closed upper contour set for each n since $B_p(S)$ is closed for any upper contour set S, and hence $x_n^{(i,t)} \in \tilde{C}^{n,(i,t)}$. Since $x_n^{(i,t)} \leq x^{(i,t)}$, it follows that $x_n^{(i,t)} \notin B_q^{(i,t)}(D)$.

For q sufficiently small relative to q', the type $x_n^{(i,t)}$ assigns arbitrarily small probability to the event $D \cup_{t \in \mathcal{T}} B_{q'}^{(-i,t)}(D)$ independent of σ. Hence, for sufficiently small q, $x_n^{(i,t)} \in B_p^{(i,t)}\left(\left(D' \cap O^{(i,t)}\tilde{C}^{n-1}\right) \cup D\right)$ implies that

$$x_n^{(i,t)} \in B_{p'}^{(i,t)}\left(D' \cap \hat{O}^{(i,t)}\tilde{C}^{n-1}\right),$$

where the operator $\hat{O}^{(i,t)}$ is defined by

$$\hat{O}^{(i,t)}(F) = O^{(i,t)}(Q(F)),$$

where

$$Q(F)^{(i,t)} = \begin{cases} \emptyset & \text{if } t \in \mathcal{T}, \\ F^{(i,t)} & \text{otherwise.} \end{cases}$$

Next, we have

$$x_n^{(i,t)} \in B_{p'}^{(i,t)}\left(D' \cap \hat{O}^{(i,t)}\tilde{C}^{n-1}\right) \subseteq B_{p'}^{(i,t)}\left(\hat{O}^{(i,t)}\tilde{C}^{n-1}\right) \subseteq B_{p'}^{(i,t)}\left(\hat{O}^{(i,t)}C^{n-1}\right),$$

where the last containment follows from the induction hypothesis that $\tilde{C}^{n-1} \subseteq C^{n-1}$ for all components where $t \in \mathcal{T}'$, together with $\hat{O}$ being increasing. Finally, we have

$$x_n^{(i,t)} \in B_{p'}^{(i,t)}\left(\hat{O}^{(i,t)}C^{n-1}\right) \subseteq B_{p'}^{(i,t)}\left(O^{(i,t)}C^{n-1}\right)$$

because $\hat{O}^{(i,t)}F \subseteq O^{(i,t)}F$ for any F. Since $x_n^{(i,t)} \in \tilde{C}^{n-1,(i,t)} \subseteq C^{n-1,(i,t)}$ for all $t \in \mathcal{T}'$, we have $x_n^{(i,t)} \in B_{p'}^{(i,t)}\left(O^{(i,t)}C^{n-1}\right) \cap C^{n-1,(i,t)}$ for all $t \in \mathcal{T}'$.

All that remains is to show that $B_{p'}\left(OC^{n-1}\right) \wedge C^{n-1}$ is an upper contour set. This follows by induction on n since O and $B_{p'}$ are point-monotone and $A \wedge B$ is an upper contour set whenever A and B are. ■

Proof of Proposition 3. First we prove part 1. By Lemma 6, action 1 is rationalizable in $\Gamma_T(\sigma)$ only on a subset of $C_{\tilde{\mathcal{B}}_p}(\mathbf{X})$, where the operator $\tilde{\mathcal{B}}_p : \mathcal{S} \longrightarrow \mathcal{S}$ is defined by

$$\tilde{\mathcal{B}}_p(F) = B_p\left([G \wedge OF] \vee U\right).$$

Suppose that for some $q \in (0,p)$, $C_{\mathcal{B}_{p-q}}(G) = \emptyset$ whenever $\sigma \leq \overline{\sigma}(T)$ and T is sufficiently large. By Lemma 7 with $D = U$ and $D' = G$, we have $C_{\tilde{\mathcal{B}}_p}(\mathbf{X}) \subseteq B_{q(T)}(U)$. Therefore, action 1 is rationalizable in $\Gamma_T(\sigma)$ (for all $\sigma \leq \overline{\sigma}(T)$ and sufficiently large T) only if agent (i,t) $q(T)$-believes U, as claimed.

Next we prove part 2. Suppose (O,p) describes sufficient conditions for investment in a class of games Γ. Consider a strategy profile F in a game $\Gamma_T(\sigma)$. Action 0 is best response for type $x^{(i,t)}$ in $\Gamma_T(\sigma)$ only if

$$x^{(i,t)} \notin B_p^{(i,t)}\left(\left[G \cap O^{(i,t)}F\right] \cup U\right),$$

or equivalently, only if

$$\begin{aligned} x^{(i,t)} &\in B_{1-p}^{(i,t)}\left[\overline{\left(\left[G \cap O^{(i,t)}F\right] \cup U\right)}\right] \\ &= B_{1-p}^{(i,t)}\left(\left[\overline{G} \cup \overline{O^{(i,t)}F}\right] \cap \overline{U}\right) \\ &= B_{1-p}^{(i,t)}\left(\left[\overline{U} \cap \overline{O^{(i,t)}F}\right] \cup \overline{G}\right), \end{aligned}$$

where $\overline{X}$ denotes the complement of an event X.

By Lemma 6, action 0 is rationalizable in $\Gamma_T(\sigma)$ only on $C_{\hat{\mathcal{B}}_{1-p}}(\mathbf{X})$, where the operator $\hat{\mathcal{B}}_{1-p} : \mathcal{S} \longrightarrow \mathcal{S}$ is defined by

$$\hat{\mathcal{B}}_{1-p}(F) = B_{1-p}\left(\left[\overline{U} \wedge \neg OF\right] \vee \overline{G}\right),$$

where $\neg O$ is defined component-wise by $\neg O^{T,(i,t)}(F) = \overline{O^{T,(i,t)}(F)}$. Note that since $\neg O$ is decreasing, the operator $\hat{\mathcal{B}}_{1-p}$ satisfies the monotonicity requirement for K in Lemma 6: if F' prescribes action 0 to more types than does F, then $\hat{\mathcal{B}}_{1-p}(F')$ prescribes 0 to more types than does $\hat{\mathcal{B}}_{1-p}(F)$.

Applying Lemma 7 with $D = \overline{G}$ and $D' = \overline{U}$, we find that for each $q \in (0, 1-p)$,

$$\left(\neg C_{\mathcal{B}_{1-p-q}}(\overline{U}) \wedge C_{\hat{\mathcal{B}}_{1-p}}(\mathbf{X})\right) \subseteq B_q(\overline{G}). \tag{15}$$

Suppose that for some $q \in (0, 1-p)$,

$$\neg C_{\mathcal{B}_{1-p-q}}(\overline{U}) = \mathbf{X} \tag{16}$$

whenever $\sigma \leq \overline{\sigma}(T)$ and T is sufficiently large. Then (15) implies that

$$C_{\hat{\mathcal{B}}_{1-p}}(\mathbf{X}) \subseteq B_q(\overline{G}),$$

which in turn implies that $\neg B_q(\overline{G}) \subseteq \neg C_{\hat{\mathcal{B}}_{1-p}}(\mathbf{X})$, and therefore

$$B_{1-q}(G) \subseteq \neg C_{\hat{\mathcal{B}}_{1-p}}(\mathbf{X}).$$

Thus, given (16), if (i,t) $(1-q)$-believes G then not investing is not rationalizable and hence investing is the unique rationalizable action (for $\sigma \leq \overline{\sigma}(T)$ and sufficiently large T).

It remains to show that (16) is equivalent to

$$E_{\mathcal{B}_{p+q}}(U) = \mathbf{X}.$$

This equivalence follows from duality (as discussed on page 24). Noting that $\mathcal{B}_{1-p-q}(F) = \neg\mathcal{B}_{p+q}(\neg F)$, the equivalence follows from (4) since $\mathcal{B}_{1-p-q}$ is dual to $\mathcal{B}_{p+q}$. ■

A.2 Additional Proofs

Proof of Proposition 4. We prove that (5) implies that there exists $p'' > p$ such that $E_{\mathcal{B}_{p''}}(U) = \mathbf{X}$ whenever $\sigma \leq \overline{\sigma}(T)$ and T is sufficiently large. This result in turn implies strategic optimism by part 2 of Proposition 3.

Define the function $\pi^{(i,t)} : X^{(i,t)} \longrightarrow [0,1]$ by

$$\pi^{(i,t)}(x) = \Pr\left(O^{(i,t)}B_q(\theta \geq \theta^*) \mid x^{(i,t)} = x\right).$$

Note that the dependence of $\pi^{(i,t)}$ on σ and T is suppressed from the notation. The function $\pi^{(i,t)}(\cdot)$ is continuous since posterior beliefs are continuous in types.

Relation (5) implies that $\pi^{(i,t)}(x) \geq p'$ for all $x \geq \theta^* + \sigma_t F^{-1}(q)$. Choose $p'' \in (p,p')$. By the continuity of $\pi^{(i,t)}$, there exists $\varepsilon^{(i,t)} > 0$ such that $\pi^{(i,t)}(x) \geq p''$ for all $x \geq \theta^* + \sigma_t F^{-1}(q) - \varepsilon^{(i,t)}$. Thus we have

$$B_q^{(i,t)}\left(\theta \geq \theta^* - \varepsilon^{(i,t)}\right) \subseteq \mathcal{B}_{p''}^{(i,t)}\left(B_q(\theta \geq \theta^*)\right).$$

Letting $\varepsilon = \min_{(i,t)} \varepsilon^{(i,t)}$, it follows that

$$B_q(\theta \geq \theta^* - \varepsilon) \subseteq \mathcal{B}_{p''}\left(B_q(\theta \geq \theta^*)\right), \tag{17}$$

where ε depends on the length of game T but is strictly positive for any finite T.

Let $E_1 = B_{p''}(U)$, and recursively define $E_k = \mathcal{B}_{p''}(E_{k-1}) \vee E_{k-1}$ for $k = 2,3\ldots$.

We will prove that

$$B_q(\theta \geq 1 - k\varepsilon) \subseteq E_{k+1} \tag{18}$$

for each k. For $q > p''$, the statement holds trivially for $k = 0$. Suppose for induction that the statement holds for $k-1$. Then

$$B_q(\theta \geq 1-(k-1)\varepsilon - \varepsilon) \subseteq \mathcal{B}_{p''}\left(B_q(\theta \geq 1-(k-1)\varepsilon)\right) \subseteq \mathcal{B}_{p''}\left(E_k\right) \subseteq E_{k+1},$$

where the first containment follows from (17) with $\theta^* = 1-(k-1)\varepsilon$, and the second from the fact that $\mathcal{B}_{p''}$ is increasing.

By (18), we have

$$\bigvee_{k=0}^{\infty} B_q(\theta \geq 1 - k\varepsilon) \subseteq \bigvee_{k=0}^{\infty} E_{k+1} = E_{\mathcal{B}_{p''}}(U).$$

Since $\bigvee_{k=0}^{\infty} B_q(\theta \geq 1 - k\varepsilon) = \mathbf{X}$, it follows that $E_{\mathcal{B}_{p''}}(U) = \mathbf{X}$, as needed. ■

Proof of Lemma 5. We have

$$B_q^{(-i,t+1)}(\theta \geq \theta^*) = \left\{ x^{(-i,t+1)} \mid x^{(-i,t+1)} \geq \theta^* + \sigma_{t+1} F^{-1}(q) \right\}$$

and $x^{(-i,t+1)} \mid x^{(i,t)} \sim N\left(x^{(i,t)}, \sigma_t^2 + \sigma_{t+1}^2\right)$. Therefore

$$\Pr\left(B_q^{(-i,t+1)}(\theta \geq \theta^*) \mid x^{(i,t)}\right) = F\left(\frac{x^{(i,t)} - \theta^* - \sigma_{t+1} F^{-1}(q)}{\sqrt{\sigma_t^2 + \sigma_{t+1}^2}}\right). \tag{19}$$

The set $B_q^{(i,t)}(\theta \geq \theta^*)$ consists of all $x^{(i,t)}$ such that $x^{(i,t)} \geq \theta^* + \sigma_t F^{-1}(q)$. Thus for every $x^{(i,t)} \in B_q^{(i,t)}(\theta \geq \theta^*)$, the right-hand side of (19) is at least

$$F\left(\frac{\theta^* + \sigma_t F^{-1}(q) - \theta^* - \sigma_{t+1} F^{-1}(q)}{\sqrt{\sigma_t^2 + \sigma_{t+1}^2}}\right) = F\left(F^{-1}(q) \frac{1 - \tau_t}{\sqrt{1 + \tau_t^2}}\right),$$

where $\tau_t = \sigma_{t+1}/\sigma_t$. The last expression is decreasing in τ_t for $\tau_t \in (0,1)$. Since $0 < \tau_t \leq \tau < 1$ by assumption, each type $x^{(i,t)} \in B_q^{(i,t)}(\theta \geq \theta^*)$ assigns probability at least $F\left(F^{-1}(q)\frac{1-\tau}{\sqrt{1+\tau^2}}\right)$ to $B_q^{(-i,t+1)}(\theta \geq \theta^*)$. For q sufficiently large, $F\left(F^{-1}(q)\frac{1-\tau}{\sqrt{1+\tau^2}}\right) \geq p'$, as needed. ■

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