Why do risk premia vary over time?

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Abstract

This paper studies the dynamics of risk premia in a model with external habit formation. It shows that in order to generate countercyclically varying premia, as found in the data, one requires either hump shaped consumption dynamics or highly persistent shocks and slow-moving habits.

1 Introduction

Risk premia vary over time. Harvey (1989) showed that US equity risk premia are higher at business cycle troughs than they are at peaks. Subsequent results of Li (2001) confirm these findings. Cochrane and Piazzesi (2005) find that the term premium is countercyclical in the US while Lustig and Verdelhan (2007) document strong countercyclicality in the exchange rate risk premium. The two most popular asset pricing models attribute this variation either to countercyclical changes in risk aversion (Campbell and Cochrane, 1999) or to changes in the volatility of the consumption process (Bansal and Yaron, 2004). This paper uses a simple, standard, general equilibrium setup to demonstrate analytically that risk premia can be procyclical even though the volatility of consumption is constant and despite a countercyclically varying risk aversion coefficient. This is puzzling and it raises the question of what other factors can cause risk premia to vary over time? Identifying them and explaining why they matter are the focus of this paper.

We believe our study is important for several reasons. First, changes in premia substantially contribute to asset price volatility and so having a good understanding of factors driving them is crucial for modelling asset prices.\(^2\) Moreover, once the factors affecting

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2 Campbell and Cochrane (1999, p. 207), for instance, argue that a slowly time-varying, countercyclical risk premium is key for matching asset pricing data.

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risk premia are well understood, asset price data can be meaningfully used to decompose shocks affecting the economy. Finally, given the increasing frequency with which DSGE models are being used to address asset pricing puzzles, it is key to clarify how and why changes in standard modelling assumptions translate into different dynamics of premia.  

In our model we allow for consumption habits in the utility function. These have been helpful in accounting for asset price regularities - e.g. in Campbell and Cochrane (1999) and have also proved useful in many areas in macro. Unlike Campbell and Cochrane (1999), however, we use a linear, additive external habit specification which nests those used in Uhlig (2004) or Smets and Wouters (2007). Other than implying a time varying risk aversion coefficient, this setup has the added appeal that there is a single parameter directly controlling habit persistence.

So what determines the risk premium’s pattern of cyclical variation? In the remainder of the paper we demonstrate how premium cyclicity depends on agents’ assessment of future prospects. We show analytically that the standard external habit specification implies counterfactually procyclical premium variation unless shocks and habits are sufficiently persistent or unless consumption displays 'hump shaped' adjustment dynamics - i.e. the peak response is reached some periods after the shock hits, before returning to base.

Several existing contributions explore issues related to those we analyse. Li (2007) documents that premia in the framework of Campbell and Cochrane (1999) are not robustly countercyclical - a point similar to the one we make, though in a different setup. Den Haan (1995) shows that the slope of the yield curve changes with the endowment specification used - which is closely related to our findings on the role of shock persistence. Equally, there are many papers showing how habits in the utility help match empirical properties of anything from exchange rate risk premia (Verdelhan, 2006) to yield curve dynamics (Wachter, 2006), though we are unaware of any which explicitly investigate the impact of structural model characteristics on premium dynamics.

2 Model and Notation

Our analysis proceeds in the simplest possible setup - that of Lucas’ (1978) asset pricing model. Agents, indexed by \( i \in [0, 1] \) choose consumption \( C^i_t \), investment in riskless bonds \( B^i_t \) and investment in risky assets \( S^i_t \) to maximise the expected discounted value of lifetime utility:

\[
\max_{C^i_t, B^i_t, S^i_t} E \left( \sum_{t=0}^{+\infty} \beta^t \left( \frac{(C^i_t - hX_t)^{1-\rho}}{1-\rho} - 1 \right) \right) \tag{1}
\]

\[\text{For examples of such papers see Uhlig (2007) or Rudebusch and Swanson (2007).}\]

\[\text{Campbell and Cochrane (1999) demonstrate that up to a first order approximation the habit formation process they impose is equivalent to an autoregressive linear specification similar to the one we use.}\]
s.t. $X_t := (1 - \phi)C_{t-1} + \phi X_{t-1}$ \hspace{1cm} (2)

$C_t + V_f^t B_t^t + V_r^t S_t^t = B_t^{t-1} + S_t^{t-1}(V_r^t + D_t)$ \hspace{1cm} (3)

where $X_t$ denotes the external habit, $C_t$ corresponds to aggregate consumption, $V_f^t$ is the time $t$ price of a one-period bond paying a unit of the consumption good next period and $V_r^t$ is the price of a perfectly divisible risky asset entitling its owner to the stream of dividends $D_{t+1}, D_{t+2}, \ldots$. The parameter $\phi$ (where $0 \leq \phi < 1$) determines the persistence of habits. When $\phi = 0$ our specification simplifies to one in which habits are purely a function of last period’s aggregate consumption. We assume an external habit specification - in particular, $C_{-1}, C_{-2}, \ldots$ are assumed given.\(^5\)

The standard first order conditions with respect to asset holdings are

$$R_f^{t+1} \cdot E_t M_t^{t+1} = 1 \hspace{1cm} E_t M_t^{t+1} R_r^{t+1} = 1 \hspace{1cm} (4)$$

where the stochastic discount factor $M_t^i$, marginal utility of consumption $\Lambda_t^i$ and gross returns on bonds $R_f^t$ and risky assets $R_r^t$ are given by

$$M_{t+1}^i := \beta \cdot \frac{\Lambda_{t+1}^i}{\Lambda_t^i} \hspace{1cm} \Lambda_{t+1}^i := (C_{t+1}^i - hX_{t+1})^{-\rho}$$

$$R_f^{t+1} := \frac{1}{V_f^{t+1}} \hspace{1cm} R_r^{t+1} := \frac{V_r^{t+1} + D_{t+1}}{V_r^{t+1}}.$$  

We let consumption $C_e^t$ and the surplus ratio $S_t$ be defined as

$$C_e^t := C_t - hX_t \hspace{1cm} S_t := \frac{C_t - hX_t}{C_t} = \frac{C_e^t}{C_t}. \hspace{1cm} (5)$$

The textbook definition of the equity risk premium $r_p^t$ is

$$r_p^t := E_t \left( \log \left( R_r^{t+1} \right) \right) - \log \left( R_f^{t+1} \right) = E_t r_r^{t+1} - r_f^{t+1} \hspace{1cm} (6)$$

with lower-case letters denoting logs - i.e. $r_i^{t+1}, i \in \{f, r\}$ corresponds to log-returns.

We define the coefficient of relative risk aversion as\(^6\)

$$\eta(C_t, X_t) := -C_t \cdot \frac{U_{cc}(C_t, X_t)}{U_c(C_t, X_t)}$$

where $U_y(\cdot , \cdot)$ denotes the partial derivative of utility function $U(\cdot, \cdot)$ with respect to $y$. Since this coefficient measures agents’ willingness to enter pure consumption gambles,

\(^5\) We have also investigated the implications of internal habit formation for risk premium cyclical-ity. While for some parameter combinations the conclusions we reach are similar to the ones detailed here, this need not generally be the case. Discussing this case is beyond the scope of our paper.

\(^6\) As noted in Campbell, Lo, and MacKinlay (1997) ”risk aversion may also be measured by the normalised curvature of the value function [...] or by the volatility of the stochastic discount factor [...]. While these measures of risk aversion are different from each other in this model, they all move inversely with $S_t$.”
given habits fixed at reference level $X_t$, this can be referred to as consumption risk aversion. It is easy to show, that if the utility function and external habits are as in equations (1) - (2) then the coefficient of consumption risk aversion is countercyclical.

To analyse the determinants of risk premia in the model, we can derive a second order approximation to the first order conditions. This approximation implies

$$rp_t + \frac{1}{2} \text{var}_t r_{t+1}^r \approx \rho \text{cov}_t(c_{t+1}^c, r_{t+1}^r).$$

(7)

Jensen’s inequality term aside, the risk premium is proportional to the excess consumption relative risk aversion coefficient $\rho$ and the conditional covariance of returns $r_{t+1}^r$ with excess consumption $c_{t+1}^c$. Following Li (2001), and under the assumptions discussed therein (see Appendix A for a brief discussion), we can apply Stein’s lemma and express $\text{cov}_t(c_{t+1}^c, r_{t+1}^r)$ as

$$\text{cov}_t(c_{t+1}^c, r_{t+1}^r) = \text{cov}_t(c_t^c, r_{t+1}^r) \mathbb{E}_t \frac{S_t}{S_{t+1}}.$$  

(8)

Equation (8) demonstrates that agents’ expectations about the surplus ratio matter because they affect the covariance of excess consumption and returns. Combined with equation (7) this shows that if $\text{cov}_t(c_{t+1}^c, r_{t+1}^r)$ is time invariant, then only changes in these expectations are going to affect risk premium cyclicity.\(^7\) Equation (8) can be further rewritten as

$$rp_t + \frac{1}{2} \text{var}_t r_{t+1}^r \approx \eta_t \text{cov}_t(c_{t+1}^c, r_{t+1}^r) \mathbb{E}_t \frac{S_t}{S_{t+1}}.$$  

(9)

which demonstrates that the risk premium is determined by the coefficient of risk aversion $\eta_t$, the covariance of consumption and returns as well as expectations about the growth of the surplus ratio. Importantly, if agents expectations of the future improve following a bad shock then the risk premium can be procyclical even though the risk aversion coefficient is countercyclical.

3 Results

3.1 Case of Auto-Regressive Consumption Dynamics

In this section we investigate factors driving risk premium cyclicity under the assumption that consumption following a shock converges back to steady state exponentially or follows a unit root process. The following proposition formalises our results.

\(^7\) Many partial-equilibrium finance papers assume that the covariance of consumption and returns is constant. Numerical simulations conducted on our model suggest that fluctuations in these covariances are small and for this reason we impose the assumption of time-invariance in subsequent propositions.
**Proposition 1** If the conditional variance of returns \( \text{var}(r_{t+1}) \) and their conditional covariance with consumption \( \text{cov}(r_{t+1}, c_{t+1}) \) are constant and log-consumption follows\(^8\)

\[
c_t = \gamma c_{t-1} + \varepsilon_t \quad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2), \quad \gamma \in [0, 1] \tag{10}
\]

then the derivative of the risk premium \( r_{pt} \) with respect to the current shock realisation can be expressed as\(^9\)

\[
\frac{\partial r_{pt}}{\partial \varepsilon_t} \approx E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1 - \phi) \left[ C_t - \gamma \sum_{s=0}^{t} \phi^s C_{t-s} \right]. \quad \square \tag{11}
\]

The proof of this and all subsequent propositions can be found in Appendix A. Since, in general, the sign of the risk premium is ambiguous, in order to build some intuition we now focus on two popular specifications - one in which habits fully adjust in a single period and the other in which consumption is a random walk.

**Corollary 2** If habits only depend on past periods’ consumption \( (\phi = 0) \) then the risk premium is procyclical

\[
\frac{\partial r_{pt}}{\partial \varepsilon_t} = E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1 - \gamma) \geq 0. \quad \square \tag{12}
\]

Under the assumptions of Corollary 2, following an adverse shock, agents expect consumption to improve (given the AR(1) nature of the consumption process) while habits will unambiguously fall. Since habits adjust fully in a single period, excess consumption, which is all agents care about, is expected to increase following the negative shock. Thus, even risk averse agents will require lower compensation for bearing risk - i.e. premia will fall. The fact that the risk premium is procyclical if \( \phi = 0 \) is important as that assumption is frequently used in macro-models.

**Corollary 3** Under the assumptions of Proposition 1 if log-consumption follows a random walk \( (\gamma = 1) \) and habits are persistent \( (\phi > 0) \) then the risk premium is countercyclical as

\[
\frac{\partial r_{pt}}{\partial \varepsilon_t} = -E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1 - \phi) \sum_{s=1}^{t} \phi^s C_{t-s} < 0. \quad \square \tag{13}
\]

Under the assumptions of Corollary 3, shocks to consumption are permanent and habits adjust gradually. In this setting, after adverse shocks, expected excess consumption falls leading to an increase in the risk premium. This shows that a combination of permanent shocks and persistent habits generates countercyclically varying risk premia. Equation (11) generalises this point and shows that a sufficiently persistent shock yields countercyclical

\(^8\) While the finance literature typically sets \( \gamma = 1 \), macro models often assume a persistent but trend stationary process for consumption. For this reason we allow \( \gamma \) to be a free parameter.

\(^9\) Equality (11) holds exactly under the additional assumption that excess consumption and risky returns are jointly conditionally log-normal and that consumption is also conditionally log-normal. See Appendix A for details.
premium variation. While the effect of increasing habit persistence $\phi$ might seem less clear cut, evaluating expression (13) for plausible parameter values suggests that raising $\phi$ has a similar effect.

3.2 Case of Hump Shaped Consumption Dynamics

In the data and in most large-scale dynamic general equilibrium models, consumption does not necessarily behave as posited in Equation (10). In particular consumption has been shown to exhibit ‘hump shaped’ dynamics - see for instance Fuhrer (2000). To address this issue, we now investigate how the cyclicity of the risk premium is determined when the peak response of consumption is not reached immediately after the shock hits. To capture this idea in the simplest possible way, we now model log-consumption as an ARMA(1,1) process, i.e.\(^{10}\)

$$c_t = \gamma c_{t-1} + \varepsilon_t + \theta \varepsilon_{t-i}, \quad \varepsilon_t \sim N.i.d.(0, \sigma^2), \quad \gamma \in [0, 1].$$

**Proposition 4** If the conditional variance of returns $\text{var}_t(r_{t+1}^r)$ and their conditional co-variance with consumption $\text{cov}_t(r_{t+1}^r, c_{t+1})$ are constant and log-consumption follows an ARMA process as in Equation (14) then

$$\frac{\partial r_p}{\partial \varepsilon_t} \approx E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1 - \phi) \left[ C_t - (\gamma + \theta) \sum_{s=0}^{t} \phi^s C_{t-s} \right].$$

In particular, if habits only depend on past periods’ consumption ($\phi = 0$) and $\gamma + \theta \geq 1$ then

$$\frac{\partial r_p}{\partial \varepsilon_t} \approx E_t S_{t+1}^{-2} C_{t+1}^{-1} h C_t \left[ 1 - (\gamma + \theta) \right] \leq 0. \Box$$

Equation (15) demonstrates that $\theta$ can play a similar role to $\gamma$. It suggests that models with hump-shaped consumption responses may be able to generate countercyclical premia without persistent habits or shocks. To understand the intuition we focus on the simpler case described in Equation (16). This expression shows that if $\gamma + \theta > 1$ the risk premium is unambiguously countercyclical. Notably, this condition implies that log-consumption increases (decreases) further in the period after a positive (negative) shock, following which it converges back to its steady state (i.e. it reaches its peak in period two). In this case, after a bad shock, agents expect the future to get worse (i.e. excess consumption to decrease) and therefore require a higher compensation for bearing risk.

4 Summary and Conclusions

We have used a simple model to analyse the determinants of risk premium dynamics. We demonstrated that risk premia can be procyclical even though the volatility of consumption is constant and despite a countercyclically varying risk aversion coefficient. We\(^{10}\) The results discussed in this section naturally generalise to arbitrary ARMA(1,K) processes.
have documented, however, that persistent habits, shocks or a hump shaped consumption process are all likely to make the premium countercyclical. In other words, the countercyclicality of the premium rests on agents’ belief that changes in economic conditions are persistent. Our results thus suggest that factors which help match activity data - i.e. persistent shocks or consumption habits - are also likely to help along the asset pricing dimension.

References


A Appendix

Proof of Proposition 1

Li (2001) demonstrates that under certain distributional assumptions on \(e_t, r^r_t\) and \(C_t\)

\[
rp_t = -\frac{1}{2} \text{var}_t \left( r^r_{t+1} \right) + \rho \text{cov}_t \left( c_{t+1}, r^r_{t+1} \right) + \lambda_t \rho \text{cov}_t \left( c_{t+1}, r^r_{t+1} \right) \tag{A.1}
\]

where \(\lambda_t = E_t(\partial s_{t+1}/\partial c_{t+1}) = E_t(1/S_{t+1}) - 1\). Notably, under the assumptions of Propositions 1 and 4, equation (A.1) also holds up to a second order approximation. Repeated use of the definition of habits - Equation (2) - and the endowment specification - Equation (10) - makes it possible to express \(S_t\) as

\[
S_t = \left[ 1 - \left( C_t \gamma e^{s_{t+1}} \right)^{-1} \frac{h(1 - \phi)}{\phi} \sum_{s=0}^{+\infty} \phi^s C_{t-s} \right]. \tag{A.2}
\]

Thus, computing the derivative of the above expression we have

\[
\frac{\partial \lambda_t}{\partial C_t} = E_t - S^{-2}_{t+1} \frac{\partial S_{t+1}}{\partial C_t} = E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1 - \phi) \left[ 1 - \gamma C^{-1}_t \sum_{s=0}^{+\infty} \phi^s C_{t-s} \right]
\]

which, can be plugged into the chain rule \(\partial \lambda_t/\partial \varepsilon_t = \partial \lambda_t/\partial C_t \cdot \partial C_t/\partial c_t \cdot \partial c_t/\partial \varepsilon_t\) to yield

\[
\frac{\partial \lambda_t}{\partial \varepsilon_t} \approx E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1 - \phi) \left[ C_t - \gamma \sum_{s=0}^{+\infty} \phi^s C_{t-s} \right]. \tag{A.3}
\]

Proof of Proposition 4

Given that

\[
\lambda_t = E_t \left[ 1 - C^{-1}_{t+1} h(1 - \phi) \sum_{s=0}^{+\infty} \phi^s C_{t-s} \right]^{-1} - 1 \tag{A.4}
\]

and since

\[
\frac{\partial \lambda_t}{\partial C_t} = E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1 - \phi), \quad \frac{\partial \lambda_t}{\partial C_{t+1}} = -E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1 - \phi) \sum_{s=0}^{+\infty} \phi^s C_{t-s}
\]

we can apply the chain rule

\[
\frac{\partial \lambda_t}{\partial \varepsilon_t} = \frac{\partial \lambda_t}{\partial C_t} \cdot \frac{\partial C_t}{\partial c_t} \cdot \frac{\partial c_t}{\partial \varepsilon_t} + \frac{\partial \lambda_t}{\partial C_{t+1}} \cdot \frac{\partial C_{t+1}}{\partial \varepsilon_t} \left( \frac{\partial c_{t+1}}{\partial \varepsilon_t} + \frac{\partial c_{t+1}}{\partial c_t} \frac{\partial c_t}{\partial \varepsilon_t} \right)
\]

to obtain that

\[
\frac{\partial \lambda_t}{\partial \varepsilon_t} \approx E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1 - \phi) \left[ C_t - \left( \theta + \gamma \right) \sum_{s=0}^{+\infty} \phi^s C_{t-s} \right]. \tag{A.5}
\]