Slides 2: Extending the RBC Model Endogenous Labour Supply and Real Rigidities

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1 The social planner's maximisation problem

In previous lectures presented the following problem:

$$\max E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} \right) \right\}$$

s.t.

$$\begin{aligned}
Y_{t+i} &= C_{t+i} + K_{t+i} - (1-\delta) K_{t+i-1}, \\
Y_{t+i} &= Z_{t+i} K_{t+i-1}^{\alpha}, \\
Z_{t+i} &= \overline{Z}^{(1-\rho)} Z_{t+i-1}^{\rho} \exp(\varepsilon_{t+i}).
\end{aligned}$$

Labour supply is fixed at 1 for all t.

We solved for the decision rules (**for all** t):

$$C_t^{-\sigma} = \beta E_t \left\{ C_{t+1}^{-\sigma} \left(1 + \alpha Z_{t+1} K_t^{\alpha-1} - \delta \right) \right\},$$

$$Z_t K_{t-1}^{\alpha} = C_t + K_t - (1-\delta) K_{t-1},$$

and derived the deterministic steady state

$$\overline{R} = \frac{1}{\beta};$$

$$\overline{K} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}; \overline{Y} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{\alpha}{\alpha-1}};$$

$$\overline{I} = \delta \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}; \overline{C} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{\alpha}{\alpha-1}} - \delta \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}}.$$

Endogenising labour supply -why?

- A model with capital accumulation alone can only have an important effect on the dynamics of the economy when the underlying technology shock is persistent;
- Technology shocks do not have strong effects on realised or expected returns on capital (relating to obsereved low capital share in output);
- Capital accumulation does not generate a short- or long-run multiplier in the sense that the output response to an underlying shock is never larger (in percentage terms) than the shock itself.

So we need to introduce an endogenous labour choice. This model is a simple extension of the neoclassical model studied in the previous lectures.

The social planner's maximisation problem

The maximisation problem now becomes

$$\max E_t \left\{ \sum_{i=0}^{\infty} \beta^i U\left(C_{t+i}, N_{t+i}\right) \right\}$$

s.t.

$$\begin{aligned}
Y_{t+i} &= C_{t+i} + K_{t+i} - (1-\delta) K_{t+i-1}, \\
Y_{t+i} &= Z_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{(1-\alpha)}, \\
Z_{t+i} &= \overline{Z}^{(1-\rho)} Z_{t+i-1}^{\rho} \exp\left(\varepsilon_{t+i}\right).
\end{aligned}$$

We set up utility as follows

$$U\left(C_{t+i}, N_{t+i}\right) = \frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \theta \frac{N_{t+i}^{1+\varphi}}{1+\varphi}.$$

- In this particular set up, the agent derives disutility from supplying labour because doing so leaves less time to derive utility from leisure.

- But this is not a universal formulation - e.g. other specifications may be nonadditively separable or may incorporate agents that derive utility from leisure, rather than disutility from working.

- Two new parameters: θ and φ represent the preference weight of leisure in utility (used maily to calibrate the steady state value for N) and the inverse of the Frisch elasticity of labour supply, respectively.

Solving for decision rules

The Lagrangean for this problem is

$$\max E_t \left[\sum_{i=0}^{\infty} \beta^i \left\{ \begin{array}{c} \frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \theta \frac{N_{t+i}^{1+\varphi}}{1+\varphi} \\ -\Lambda_{t+i} \left(K_{t+i} - Z_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{(1-\alpha)} - (1-\delta) K_{t+i-1} + C_{t+i} \right) \end{array} \right. \right.$$

- N_t, C_t and K_t are now the planner's *choice* variables.
- Z_t and K_{t-1} are again the *state* variables.
- Λ_t is the Lagrange multiplier.

The first order conditions to this problem **for all** t are as follows:

$$\begin{aligned}
C_t &: \qquad C_t^{-\sigma} = \Lambda_t. \\
K_t &: \qquad \Lambda_t = \beta E_t \left\{ \Lambda_{t+1} \left[1 + \alpha Z_{t+1} K_t^{(\alpha-1)} N_{t+1}^{(1-\alpha)} - \delta \right] \right\}. \\
N_t &: \qquad \theta N_t^{\varphi} = \Lambda_t (1-\alpha) Z_t K_{t-1}^{\alpha} N^{-a}. \\
\Lambda_t &: \qquad Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} = C_t + K_t - (1-\delta) K_{t-1}.
\end{aligned}$$

The last condition is the resource constraint. Combining the first two conditions, we obtain the familiar Euler equation (EE)

$$C_t^{-\sigma} = \beta E_t \left\{ C_{t+1}^{-\sigma} \left[1 + \alpha Z_{t+1} K_t^{\alpha-1} N_{t+1}^{(1-\alpha)} - \delta \right] \right\}.$$

Combining the first and third conditions we get

$$\theta N_t^{\varphi} = C_t^{-\sigma} (1-\alpha) Z_t K_{t-1}^{\alpha} N^{-\alpha}.$$

We can express the previous condtion in terms of log deviations from mean

$$\hat{n}_t = \frac{1}{\varphi}\left[\hat{z}_t + \alpha\hat{k}_{t-1} - \alpha\hat{n}_t - \sigma\hat{c}_t\right]$$

which shows that:

- $\uparrow$ Marginal product of labour ($\equiv$ real wage) $\longrightarrow$ $\uparrow$ Labour effort;
- $\uparrow$ C_t (marginal utility$\downarrow$) $\longrightarrow$ $\downarrow$ Labour effort;
- We can think of these in terms of income and substitution effects.

The System

Rearranging conditions above gives

$$C_t^{-\sigma} = E_t \left\{ \beta C_{t+1}^{-\sigma} \left[1 + \alpha Z_{t+1} K_t^{\alpha-1} N_{t+1}^{(1-\alpha)} - \delta \right] \right\},$$

$$\frac{\theta N_t^{\varphi}}{C_t^{-\sigma}} = (1-\alpha) Z_t K_{t-1}^{\alpha} N^{-a},$$

- These first two equations of the system impose equality between marginal rates of substitution and transformation.
- The first means that the marginal rate of intertemporal substitution in consumption equals the marginal product of capital net of depreciation.
- The second equates the marginal rate of substitution between lesiure and consumption with the marginal product of labour.

$$Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} = C_t + K_t - (1-\delta) K_{t-1},$$

$$Z_{t+i} = \overline{Z}^{(1-\rho)} Z_{t+i-1}^{\rho} \exp(\varepsilon_{t+i}).$$

- The third equation is the resource constraint while the fourth is the law of motion for productivity.
- Other variables are defined recursively:

$$\begin{aligned}
\text{Output:} \quad & Y_t = Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} \\
\text{Investment:} \quad & I_t = K_t - (1-\delta) K_{t-1} \\
\text{Return on capital:} \quad & R_t^k = 1 + \alpha Z_t K_t^{\alpha-1} N_t^{(1-\alpha)} - \delta \\
\text{Real wages:} \quad & W_t = (1-\alpha) Z_t K_{t-1}^{\alpha} N_t^{-\alpha}
\end{aligned}$$

Solving the deterministic steady state (DSS)

The DSS for this model differs slightly from that derived in the previous lecture. We now set the steady state level of labour supply at 0.25 (as suggested in Cochrane (2001) - but Prescott (1996) suggest a higher number as households alocate 1/3 of their time to market activities), so

$$\overline{N} = 0.25.$$

(but Prescott (1996) suggest $\overline{N} = 0.33$ as households allocate 1/3 of their time to market activities)

The steady state gross return on capital and level of productivity are

$$\overline{R} = \frac{1}{\beta}; \quad \overline{Z} = 1.$$

Using this in R_t we obtain

$$\frac{\overline{K}}{\overline{\overline{N}}} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha - 1}},$$

We can express the other steady state values in terms of $\frac{\overline{K}}{\overline{\overline{N}}}$ or, indeed, the deep parameters.

$$\begin{aligned}
\overline{Y} &= \overline{N} \left(\frac{\overline{K}}{\overline{\overline{N}}} \right)^{\alpha} = \overline{N} \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}} \\
\overline{I} &= \delta \overline{N} \left(\frac{\overline{K}}{\overline{\overline{N}}} \right) = \delta \overline{N} \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha - 1}}
\end{aligned}$$

$$\overline{C} = \overline{N}\left[\left(\frac{\overline{K}}{\overline{N}}\right)^{\alpha} - \delta\left(\frac{\overline{K}}{\overline{N}}\right)\right] = \overline{N}\left[\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}\right].$$

However we still have a free paramater, θ. In steady state,

$$\theta = \overline{N}^{-\varphi}\,\overline{C}^{-\sigma}\,(1-\alpha)\left(\frac{\overline{K}}{\overline{N}}\right)^{\alpha}.$$

This expression can be expressed entirely in terms of deep paramaters, for a given $\overline{N}$, so it can be calibrated.

The responses to a productivity shock:

$$C_t^{-\sigma} = \beta E_t \left\{ C_{t+1}^{-\sigma} \left(R_{t+1} \right) \right\},$$

$$R_t = 1 + \alpha Z_t K_t^{\alpha - 1} N_t^{(1-\alpha)} - \delta,$$

$$\theta N_t^{\varphi} = C_t^{-\sigma} W_t,$$

$$W_t = (1 - \alpha) Z_t K_{t-1}^{\alpha} N^{-a}$$

$$Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} = C_t + K_t - (1 - \delta) K_{t-1},$$

$$Z_{t+i} = \overline{Z}^{(1-\rho)} Z_{t+i-1}^{\rho} \exp \left(\varepsilon_{t+i} \right).$$

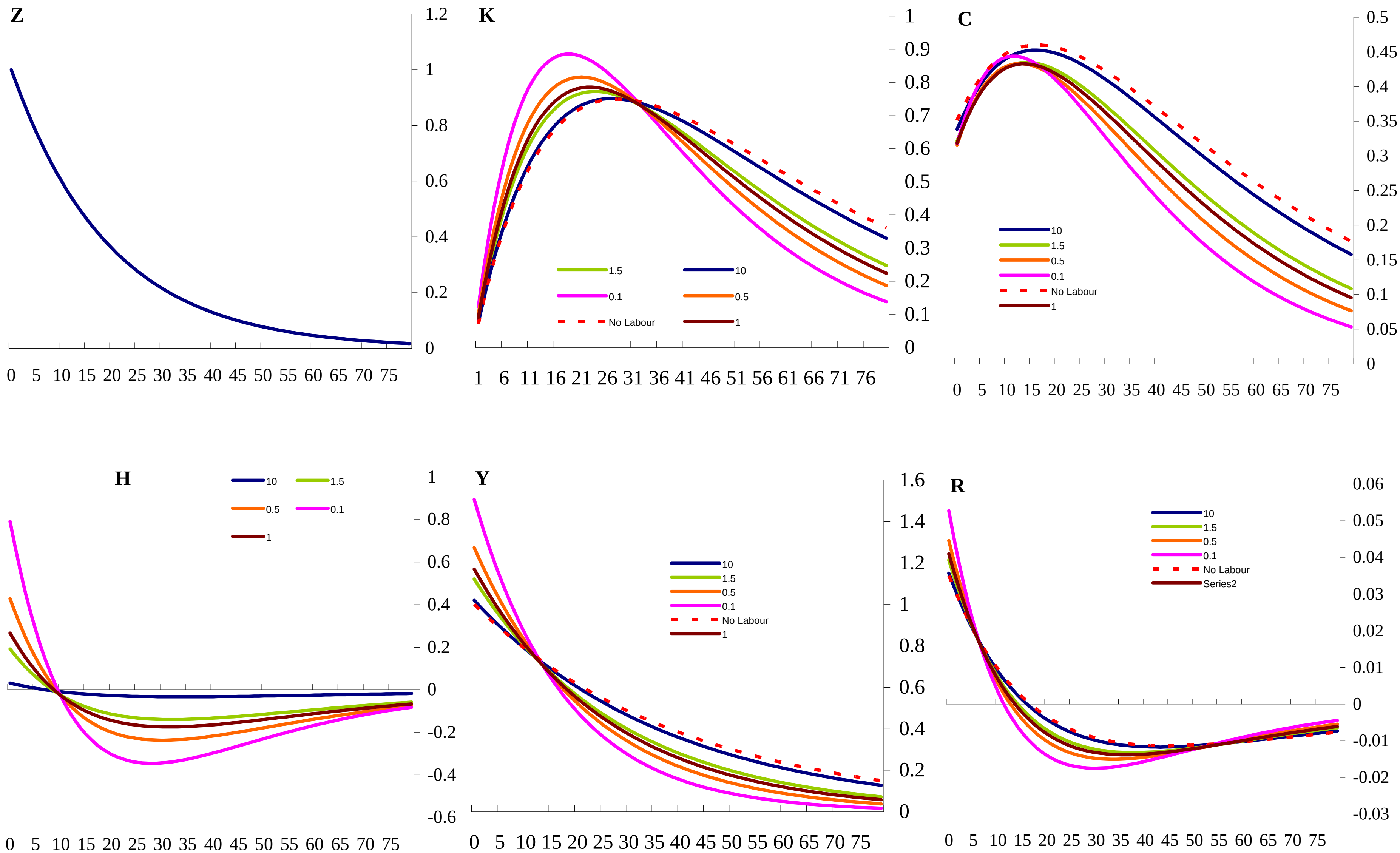
Varying the Frisch elasticity:

Calibration:

$$\alpha = 0.3; \delta = 0.025; \sigma = 0.5.$$

- We vary φ to see what happens to the responses.

$$\varphi = [0.1, 0.5, 1.0, 1.5, 10]$$


Z
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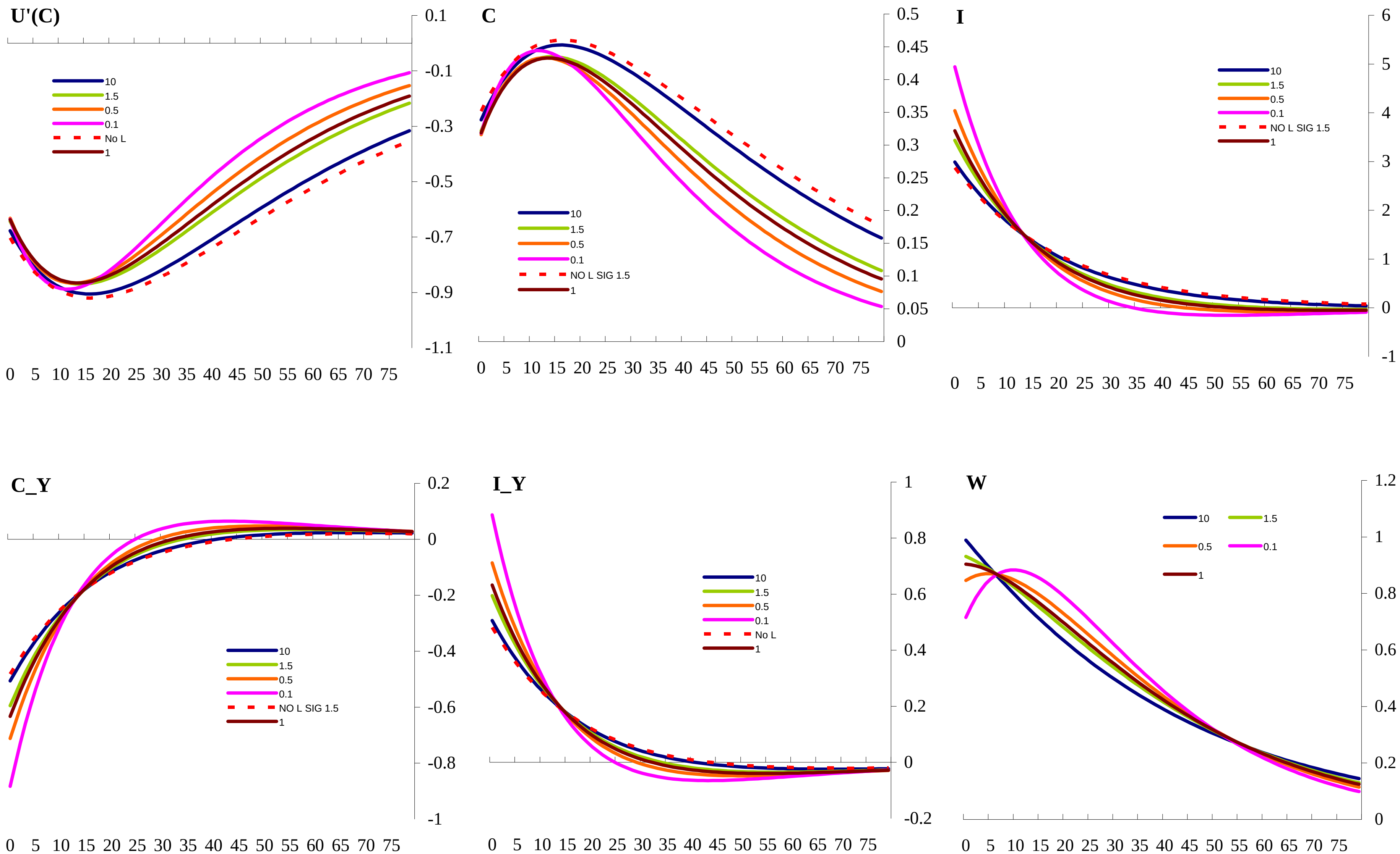
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Discussion

- The red dotted line in each chart shows the response from the model we discussed in the last session - i.e a model without labour and $\sigma = 0.5$.
- When $\varphi = 10$, labour supply is inelastic. The respsonse of the economy is similar to that obtained from a model with an exogenous labour supply.
- We see that at high values of the Frisch elasticity (i.e when $\frac{1}{\varphi}$ is high, so φ is low), labour supply initially increases sharply to take advantage of higher productivity.
- With labour supply higher, the capital stock is increased with sharp increases in in investment. With higher labour and capital, output also initially rises and we observe an output multiplier effect.
- To balance desired investment with savings the real interest rate rises.

Alternative discussion:

- The initial increase in productivity raises both the marginal products of capital and labour;
- Therefore capital is gradually increased (as it cannot jump), while labour supply (which can jump) increases sharply.
- This results in higher output.
- Both consumption and investment are higher, but investment more so, as agents realise that the higher productivity environment is only temporary.
- Again, the interest rate adjusts to attract savings.
- Income vs Substitution effect.

Homework

Consider the case in which the utility function is given by:

$$U\left(C_{t+i}, N_{t+i}\right) = \frac{(C_{t+i}^{\theta}(1 - N_{t+i})^{1-\theta})^{1-\sigma}}{1-\sigma}$$

-Calculate the equilibrium conditions and the steady state (which value of θ is consistent with $\overline{N} = 0.25$?)

- Simulate the impulse response to a productivity shock for different values of σ and ρ

- How does the responses compare with the case of separable labour

Real Rigidities:

Consumption Habits

Habits make agents very sensitive to small changes in consumption. They increase agents' motivation to smooth (near) changes not levels. Technically, habits make the utility function temporally nonseparable.

- Habits can be specified as differences or as a ratio. We use the difference form

$$U(C_{j,t+i}) = \frac{\left(C_{j,t+i} - \phi X_{j,t+i}\right)^{1-\sigma}}{1-\sigma},$$

  in which $C_{j,t+i}$ and $X_{j,t+i}$ represent the consumption and habit levels of an individual, or cohort, j at time $t+i$, respectively.

- The parameter ϕ measures preference weight for reference level of habit.

- Habits can be Internal or External ("keeping up with the Joneses"), and this is reflected by the way in which we specify $X_{j,t}$:

- Habits are *Internal* if e.g. $X_{j,t} = C_{j,t-1} \rightarrow U(C_{j,t+i}) = \frac{(C_{j,t} - \phi C_{j,t-1})^{1-\sigma}}{1-\sigma} \rightarrow$ when optimising w.r.t.$C_{j,t}$, we would need to worry about 2nd term in brackets, as in this case, $C_{j,t+i}$ appears in consecutive time periods.

- Habits are *External* if $X_{j,t} = C_{t-1} \ \forall j,t \rightarrow U(C_{j,t+i}) = \frac{(C_{j,t} - \phi C_{t-1})^{1-\sigma}}{1-\sigma} \rightarrow$ when optimising w.r.t.$C_{j,t}$, we would ignore 2nd term in brackets.

- We use the difference form and assume habits are external.

- NB: we will have FOCs for individuals - need to derive aggregate decision rules.

Capital adjustment costs

- Adding capital adjustment costs to a model with habits ensures that not only do agents care a lot about changes in consumption, they are also prevented from easily smoothing through fluctuations. Consumers cannot costlessly adjust their production to see out bad states of the world.

- Many different specifications for adjustment costs - could be in terms of capital or investment.

- Provide a formal framework for q theory of investment.

- Often they are quadratic - implying it is: (i) increasingly costly to change investment with size of change; and (ii) as costly to decumulate as accumulate capital.

- We use the following quadratic form:

$$Y_{t+i} = Z_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{(1-\alpha)} - \frac{\chi}{2} \left(K_{t+i} - K_{t+i-1} \right)^2 .$$

- χ dictates how costly it is to change capital.

The social planner's maximisation problem (for an individual, j)

The maximisation problem incorporating both habits and adjustment costs becomes

$$\max E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left(\frac{\left(C_{j,t+i} - \phi X_{j,t+i}\right)^{1-\sigma}}{1-\sigma} - \theta \frac{N_{j,t+i}^{1+\varphi}}{1+\varphi} \right) \right\}$$

s.t.

$$\begin{aligned} & Z_{t+i} K_{j,t+i-1}^{\alpha} N_{j,t+i}^{(1-\alpha)} - \frac{\chi}{2} \left(K_{j,t+i} - K_{j,t+i-1}\right)^2 \\ = \; & C_{j,t+i} + K_{j,t+i} - (1-\delta) K_{j,t+i-1}, \end{aligned}$$

$$Y_{j,t+i} = Z_{t+i} K_{j,t+i-1}^{\alpha} N_{j,t+i}^{(1-\alpha)} - \frac{\chi}{2} \left(K_{j,t+i} - K_{j,t+i-1}\right)^2,$$

$$Z_{t+i} = \overline{Z}^{(1-\rho)} Z_{t+i-1}^{\rho} \exp\left(\varepsilon_{t+i}\right).$$

Solving for decision rules

The Lagrangean for this problem is

$$\max E_t \sum_{i=0}^{\infty} \beta^i \{ \frac{\left(C_{j,t+i} - \phi X_{j,t+i}\right)^{1-\sigma}}{1-\sigma} - \theta \frac{N_{j,t+i}^{1+\varphi}}{1+\varphi}$$
$$-\Lambda_{j,t+i}[K_{j,t+i} - Z_{t+i} K_{j,t+i-1}^{\alpha} N_{j,t+i}^{(1-\alpha)}$$
$$-(1-\delta) K_{j,t+i-1} + C_{j,t+i} + \frac{\chi}{2}\left(K_{j,t+i} - K_{j,t+i-1}\right)^2]\}.$$

- As stated, we treat the habit as *external* $\rightarrow$ so no need to optimise w.r.t $X_{j,t+i}$
- $N_{j,t}$, $C_{j,t}$ and $K_{j,t}$ are the planner's *choice* variables for the jth individual.
- $\Lambda_{j,t}$ is the Lagrange multiplier for the jth individual.

The first order conditions, for the jth individual, and treating the habit as *external*, are as follows (**for all** *t*) :

$$
\begin{aligned}
C_{j,t} \quad &: \qquad \left(C_{j,t} - \phi X_{j,t}\right)^{-\sigma} = \Lambda_{j,t}. \\
K_{j,t} \quad &: \qquad \Lambda_{j,t}\left[1 + \chi\left(K_{j,t} - K_{j,t-1}\right)\right] \\
&= \beta E_t\left\{\Lambda_{j,t+1}\left[1 + \alpha Z_{t+1} K_{j,t}^{\alpha-1} N_{j,t+1}^{1-\alpha} - \delta + \chi\left(K_{j,t+1} - K_{j,t}\right)\right]\right\}. \\
N_{j,t} \quad &: \qquad \theta N_{j,t}^{\varphi} = \Lambda_{j,t}\left(1-\alpha\right) Z_t K_{j,t-1}^{\alpha} N_{j,t}^{-\alpha}. \\
\Lambda_{j,t} \quad &: \qquad Z_t K_{j,t-1}^{\alpha} N_{j,t}^{(1-\alpha)} - \frac{\chi}{2}\left(K_{j,t+i} - K_{j,t+i-1}\right)^2 = C_{j,t} + K_{j,t} - (1-\delta)
\end{aligned}
$$

- Marginal utility now depends on how far the agent is from his or her habit level.

Aggregate Decision Rules

To return to the representative agent framework we have to aggregate each decision rule and express it in *per capita terms*. Defining

$$C_t = \sum_{j=1}^{n} \frac{1}{n} C_{j,t} \ , \ \Lambda_t = \sum_{j=1}^{n} \frac{1}{n} \Lambda_{j,t} \ , \ K_t = \ , \ Y_t = ...etc$$

and writing the habit level as

$$X_{j,t} = C_{t-1} \ \forall j, t$$

we can aggregate the first FOC to give

$$(C_t - \phi C_{t-1})^{-\sigma} = \Lambda_t.$$

We can aggregate the other conditions in similar fashion:

$$\left(C_t - \phi C_{t-1}\right)^{-\sigma} = \Lambda_t.$$

$$\begin{aligned} & \Lambda_t \left[1 + \chi\left(K_t - K_{t-1}\right)\right] \\ = \ & \beta E_t \left\{ \Lambda_{t+1} \left[1 + \alpha Z_{t+1} K_t^{\alpha - 1} N_{t+1}^{1-\alpha} - \delta + \chi\left(K_{t+1} - K_t\right) \right] \right\} \end{aligned}$$

$$\theta N_t^{\varphi} = \Lambda_t \left(1 - \alpha\right) Z_t K_{t-1}^{\alpha} N_t^{-\alpha}.$$

$$Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} - \frac{\chi}{2}\left(K_{t+i} - K_{t+i-1}\right)^2 = C_t + K_t - \left(1 - \delta\right) K_{t-1}.$$

- Z_t, K_{t-1} and additionally C_{t-1} are now the *state* variables.

- Combining the first two conditions, we obtain an Euler equation (EE) which is still equating $MRS = MRT$

$$
\begin{aligned}
& \left(C_t - \phi C_{t-1}\right)^{-\sigma} \\
= \;\; & E_t \left\{ \frac{\beta \left(C_{t+1} - \phi C_t\right)^{-\sigma} \left[1 + \alpha Z_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} - \delta + \chi \left(K_{t+1} - K_t\right)\right]}{\left[1 + \chi \left(K_t - K_{t-1}\right)\right]} \right\}
\end{aligned}
$$

- Labour supply condition from 1st and 3rd condition (very similar to last week's condition)

$$
N_t = \left[\frac{(1-\alpha)}{\theta} \left[\left(C_t - \phi C_{t-1}\right)^{-\sigma} Z_t K_{t-1}^{\alpha} N_t^{-\alpha} \right] \right]^{\frac{1}{\varphi}}.
$$

- Resource constraint (written as net output) and law of motion for productivity are written as

$$Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} - \frac{\chi}{2}\left(K_{t+i} - K_{t+i-1}\right)^2 = C_t + K_t - (1-\delta) K_{t-1},$$

$$Z_{t+i} = \overline{Z}^{(1-\rho)} Z_{t+i-1}^{\rho} \exp\left(\varepsilon_{t+i}\right).$$

- Capital adjustment costs leave less resources available for consumption/investment.

- The recursive definitions for other variables are now

$$\begin{aligned}
\text{(Net) Output} &: \quad Y_t = Z_t K_{t-1}^{\alpha} N_t^{(1-\alpha)} - \frac{\chi}{2}\left(K_{t+i} - K_{t+i-1}\right)^2 \\
\text{Investment} &: \quad I_t = K_t - (1-\delta) K_{t-1} \\
\text{Return on capital} &: \quad R_{t+1}^{k} = \frac{\left[1 + \alpha Z_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} - \delta + \chi\left(K_{t+1} - K_t\right)\right]}{\left[1 + \chi\left(K_t - K_{t-1}\right)\right]} \\
\text{Real wages} &: \quad W_t = (1-\alpha) Z_t K_{t-1}^{\alpha} N_t^{-\alpha}
\end{aligned}$$

Solving the deterministic steady state (DSS)

In steady state we know that $K_t = K_{t-1} \forall t$ & $C_t = C_{t-1} \forall t$, therefore the adjustment cost terms disappear from the expressions above and the DSS in this model is the same as in a model without adjustment costs or habits.

$$\overline{N} = 0.25.$$

The steady state gross return on capital and level of productivity are

$$\overline{R} = \frac{1}{\beta}; \quad \overline{Z} = 1.$$

Using this in R_t we obtain

$$\frac{\overline{K}}{\overline{N}} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha} \right)^{\frac{1}{\alpha - 1}},$$

We can express the other steady state values in terms of $\frac{\overline{K}}{\overline{N}}$ or, indeed, the deep parameters.

$$
\begin{aligned}
\overline{N} &= 0.25;\ \overline{R} = \frac{1}{\beta};\quad \overline{Z} = 1;\ \frac{\overline{K}}{\overline{N}} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}};\\
\overline{I} &= \delta\overline{N}\left(\frac{\overline{K}}{\overline{N}}\right) = \delta\overline{N}\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}};\ \overline{Y} = \overline{N}\left(\frac{\overline{K}}{\overline{N}}\right)^{\alpha} = \overline{N}\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}};\\
\overline{C} &= \overline{N}\left[\left(\frac{\overline{K}}{\overline{N}}\right)^{\alpha} - \delta\left(\frac{\overline{K}}{\overline{N}}\right)\right] = \overline{N}\left[\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}\right].
\end{aligned}
$$

We still have a free paramater, θ. In steady state,

$$\theta = \overline{N}^{-\varphi} \left((1-\phi)\,\overline{C} \right)^{-\sigma} (1-\alpha) \left(\frac{\overline{K}}{\overline{N}} \right)^{\alpha}.$$

This expression can be expressed entirely in terms of deep paramaters, for a given $\overline{N}$, so it can be calibrated.

The responses to a productivity shock:

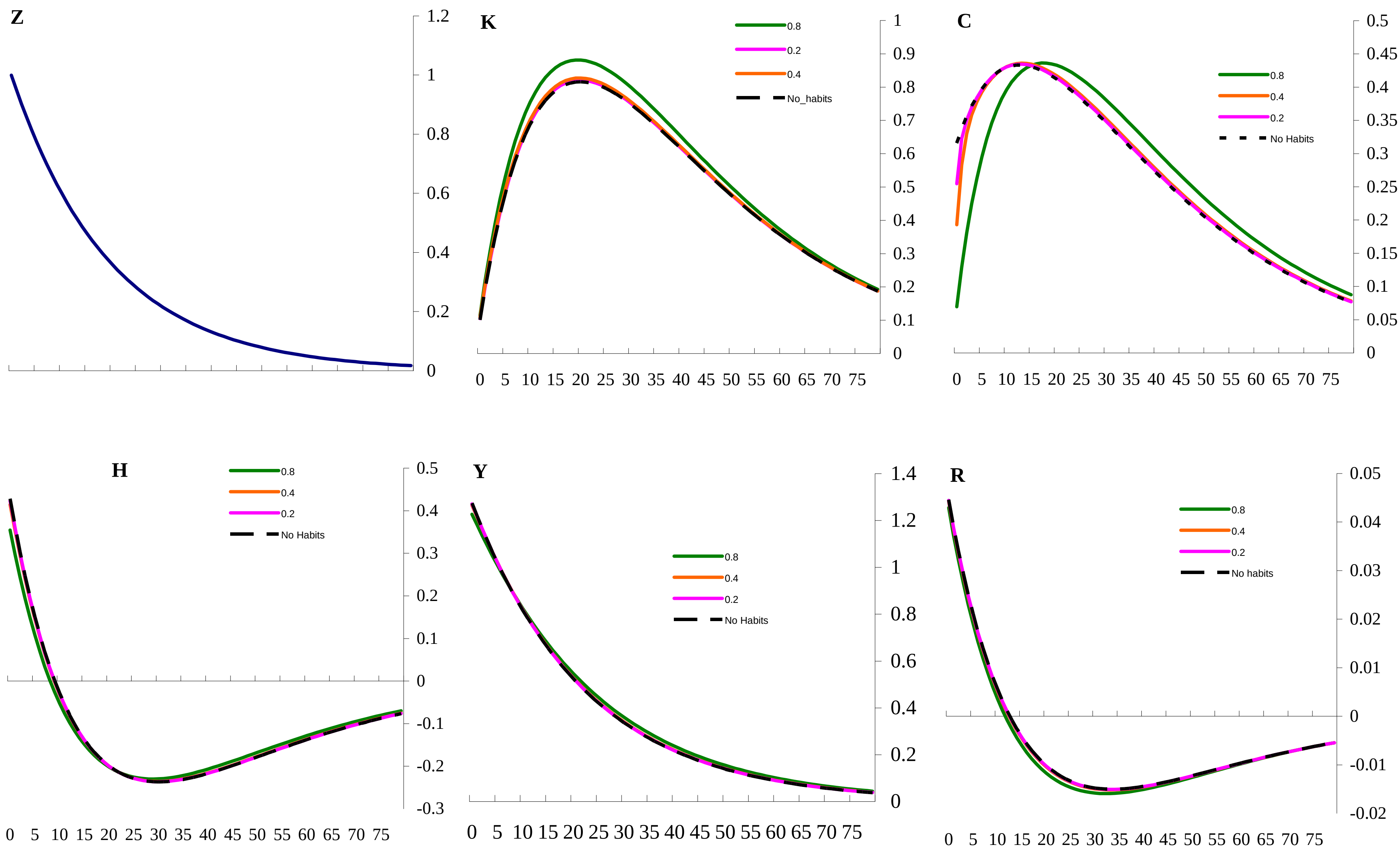
Varying the habit weight (in a model without capital adjustment costs)

Calibration:

$$\alpha = 0.3; \delta = 0.025; \sigma = 0.5;\ \varphi = 0.5$$

- We vary ϕ to see what happens to the responses.

$$\phi = [0.2, 0.4, 0.8]$$


Z
0
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C
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No Habits
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0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75
Y
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No Habits
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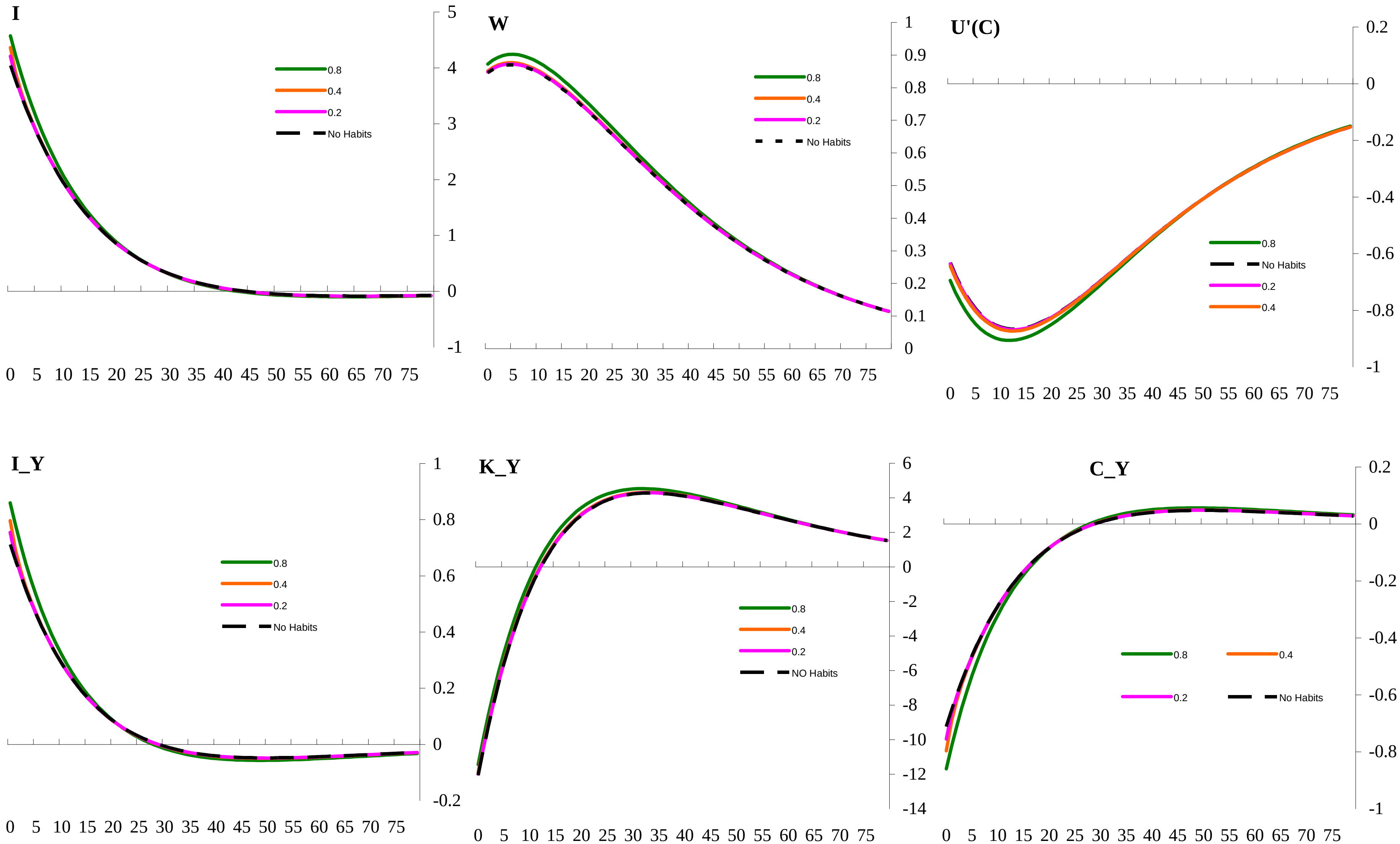
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K_Y
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0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75
C_Y
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No Habits
0.2 0 -0.2 -0.4 -0.6 -0.8 -1
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75

Varying the cost of adjusting capital (in a model with habits):

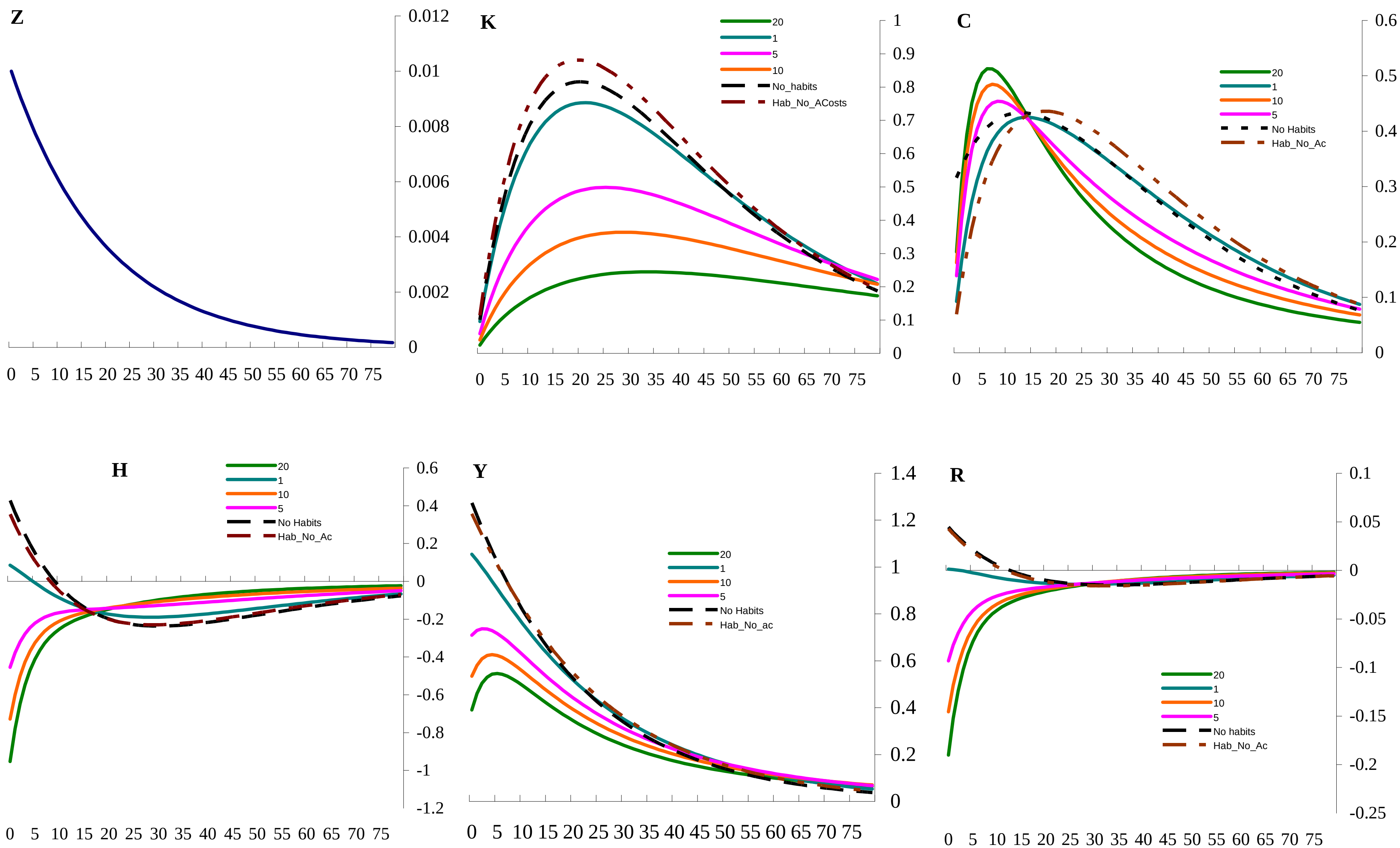
Calibration:

$$\alpha = 0.3; \delta = 0.025; \sigma = 0.5; \ \varphi = 0.5; \phi = 0.8$$

- Now we vary χ to see what happens to the responses.

$$\chi = [1, 5, 10, 20]$$

- We also show responses from (1) a model without habits or adjustment costs; and (2) a model with habits, but no adjustment costs.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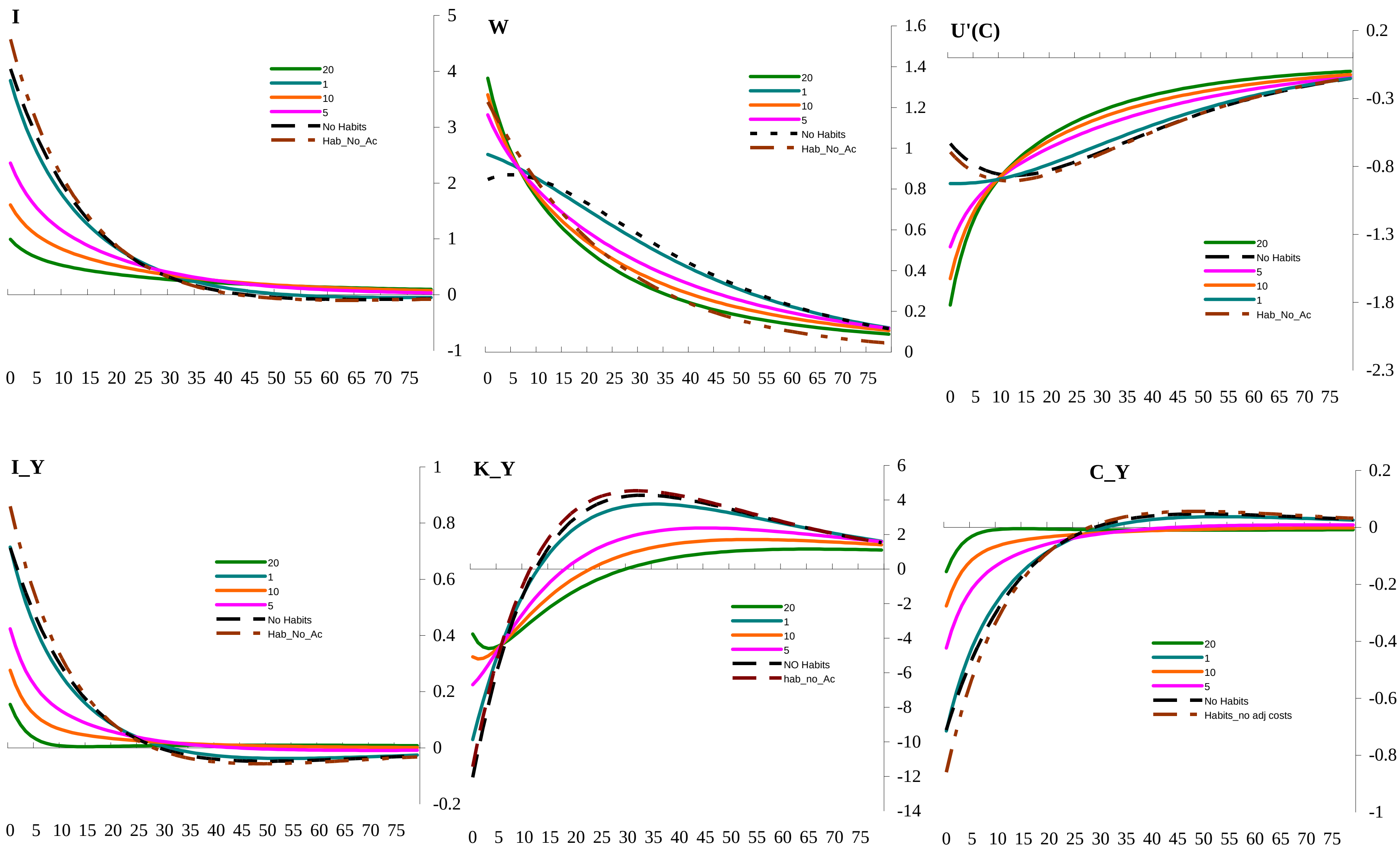

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Hab_No_Ac
I_Y
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5
No Habits
Hab_No_Ac
K_Y
20
1
10
5
NO Habits
hab_no_Ac
C_Y
20
1
10
5
No Habits
Habits_no adj costs

Discussion - a model with habits

- The black dotted line in each chart shows the response obtained from a model without habits or adjustment costs, while the brown dotted line shows the responses from a model with habits, but still no adjustment costs.
- Habits, in isolation, do not seem to have a great effect on the responses.
- Interestingly, even though the output response is little changed from a model without habits, there is a compositional difference - look at the investment- and consumption to output ratios.

Discussion - adding capital adjustment costs

- Immediately we see that the investment and capital responses are muted. The greater the value of χ, the less pronounced are the responses of these two variables.

- The positive output response is also lower.

- We find that even though there is a substitution towards investment from current consumption, it is not as dramatic as in a model without adjustment costs. And the more costly it is to change capital, the lower the substituion towards investment.

- The productivity shock enables agents to consume more without having to supply extra labour. In fact, as the capital stock rises and remains above steady state for a prolonged period, agents can achieve higher than steady state level of consumption by working less - i.e. the income effect dominates.

- Importantly, we see that marginal utility is far more volatile when adjustment costs are added to a habits model. This will lead to increased volatility in the stochastic discount factor, which is one of the prerequisites for producing a plausible risk premium in these kinds of models.

2 Centralised and decentralised representations

Three equivalent representations:

- Social planner problem: benevolent social planner allocates capital and labour to ensure maximum utility

- Rental model: households own factors of production and pay the costs associated with them. They rent these to firms in rental markets and get capital and labour income in return to use to finance consumption.

- "Capital ownership" model: firms own their own capital and pay capital adjustment costs. They issue shares which entitle holders to whatever dividends are paid. Households get wage income and firms' profits.

These are all exactly equivalent!

Some important assumptions:

- Perfectly competitive goods markets and rental markets for factors of production.
- No distortions from, eg, taxes.
- Closed economy.

The rental model

- Social planner: total production output directed to consumption. One agent consumes, sleeps, works, invests.
- Rental version: total production output divided into rental streams (c.f. National Accounts). "Firms" demand factors and "households" rent them to firms.

Firms' maximisation problem

Assume that a large number of firms seeks to maximise period-by-period profits, subject to technology constraints and rental rates for labour and capital:

$$\max Y_{a,t+i} - W_{t+i}N_{a,t+i} - r^{k}_{t+i}K_{a,t+i-1}$$

s.t.

$$Y_{a,t+i} = Z_{t+i}\left(K_{a,t+i-1}\right)^{\alpha}\left(N_{a,t+i}\right)^{(1-\alpha)}$$

and

$$Z_{t+i} = \overline{Z}^{(1-\rho)}Z^{\rho}_{t+i-1}\exp\left(\varepsilon_{t+i}\right).$$

- $K_{a,t+i-1}$ and $N_{a,t+i}$ represent the demand by firm for capital and labour, respectively, at the *real* capital rental rate r^{k}_{t+i} and *real* wage rate W_{t+i} .

- The firm chooses capital and labour - the FOC's (for firm a) are

$$r_t^k = \alpha \frac{Y_{a,t}}{K_{a,t-1}}$$

$$W_t = (1-\alpha) \frac{Y_{a,t}}{N_{a,t}}$$

Households' maximisation problem

The maximand for this problem is the same as previously:

$$\max E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left(\frac{\left(C_{j,t+i} - \phi X_{j,t+i}\right)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \theta \frac{N_{j,t+i}^{1+\omega}}{1+\omega} \right) \right\}.$$

$$\begin{aligned} B_{j,t+i} + K_{j,t+i} + C_{j,t+i} &= \left(1 + r_{t+i}\right) B_{j,t+i-1} + W_{t+i} N_{j,t+i} + r^k_{t+i} K_{j,t+i-1} \\ &\quad + \left(1-\delta\right) K_{j,t+i-1} - \frac{\chi}{2} \left(K_{j,t+i} - K_{j,t+i-1}\right)^2. \end{aligned}$$

- $B_{j,t}$ represent consumer $j's$ holdings of real (private) consumption bonds at time t. These are not government bonds – in aggregate they are in *zero net supply.*

- Note that it is the household that finds it costly to adjust the capital stock – it maintains the capital stock.

Euler's Theorem on Homogenous functions tells us that

$$Y = F(K, N) = F_K(K, N) K + F_L(K, N) N$$

if the function $F(\cdot)$ is homogenous of degree one - i.e.

$$F(aK, aN) = aF(K, N) \;\; \forall a$$

This is necessary in order to have the competitive equilibrium solution identical to the social planner's version of the economy. This holds in our set up.

Household budget constrain in centralized model

$$\begin{aligned} Y_{j,t+i} &= Z_{t+i} K_{j,t+i-1}^{\alpha} N_{j,t+i}^{(1-\alpha)} = C_{j,t+i} + B_{j,t+i} + K_{j,t+i} \\ & - (1-\delta) K_{j,t+i-1} - (1 + r_{t+i}) B_{j,t+i-1} + \frac{\chi}{2} \left(K_{j,t+i} - K_{j,t+i-1} \right)^2 \end{aligned}$$

Household budget constrain in decentralized model

$$\begin{aligned} W_{t+i} N_{j,t+i} + r_{t+i}^{k} K_{j,t+i-1} &= B_{j,t+i} + K_{j,t+i} + C_{j,t+i} - (1-\delta) K_{j,t+i-1} \\ & - (1 + r_{t+i}) B_{j,t+i-1} + \frac{\chi}{2} \left(K_{j,t+i} - K_{j,t+i-1} \right)^2 \end{aligned}$$

Equivalents, given that

$$W_{t+i} N_{j,t+i} + r_{t+i}^{k} K_{j,t+i-1} = (1-\alpha) \frac{Y_{j,t+i}}{N_{j,t+i}} N_{j,t+i} + \alpha \frac{Y_{j,t+i}}{K_{j,t+i-1}} K_{j,t+i-1} = Y_{j,t+i}$$

The Lagrangean for *households* is

$$\max E_t \left[\sum_{i=0}^{\infty} \beta^i \left\{ \begin{array}{c} \frac{\left(C_{j,t+i} - \phi X_{j,t+i}\right)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \theta \frac{N_{j,t+i}^{1+\omega}}{1+\omega} \\ -\Lambda_{j,t+i} \left(\begin{array}{c} B_{j,t+i} + K_{j,t+i} + C_{j,t+i} \\ +\frac{\chi}{2}\left(K_{j,t+i} - K_{j,t+i-1}\right)^2 - \left(1 + r_{t+i}\right) B_{j,t+i-1} \\ -W_{t+i} N_{j,t+i} - r_{t+i}^{k} K_{j,t+i-1} - \left(1-\delta\right) K_{j,t+i-1} \end{array} \right) \end{array} \right\} \right.$$

- $C_{j,t}$, $K_{j,t}$, $N_{j,t}$ and also $B_{j,t}$ are the *choice* variables for the j^{th} individual.

- The first order conditions, for the jth individual, treating the habit as external, are as follows (**for all** *t)* :

$$
\begin{aligned}
C_{j,t} \quad &: \qquad \left(C_{j,t} - \phi X_{j,t}\right)^{-\frac{1}{\sigma}} = \Lambda_{j,t}. \\
K_{j,t} \quad &: \qquad \Lambda_{j,t} = \beta E_t \left\{ \Lambda_{j,t+1} \frac{\left[1 + r^k_{t+1} - \delta + \chi\left(K_{j,t+1} - K_{j,t}\right)\right]}{\left[1 + \chi\left(K_{j,t} - K_{j,t-1}\right)\right]} \right\}. \\
N_{j,t} \quad &: \qquad \theta N^{\omega}_{j,t} = \Lambda_{j,t} W_t. \\
B_{j,t} \quad &: \qquad \Lambda_{j,t} = E_t \left\{ \Lambda_{j,t+1} \beta \left(1 + r_{t+1}\right) \right\}. \\
\Lambda_{j,t} \quad &: \qquad \left[\begin{array}{c} B_{j,t} + K_{j,t} + C_{j,t} = \left(1 + r_{t-1}\right) B_{j,t-1} + W_t N_{j,t} \\ + r^k_t K_{j,t-1} + \left(1 - \delta\right) K_{j,t-1} - \frac{\chi}{2}\left(K_{j,t} - K_{j,t-1}\right)^2. \end{array} \right]
\end{aligned}
$$

- The FOC w.r.t $C_{j,t}$ and $N_{j,t}$ are identical to the conditions obtained from a social planner's version.
- The FOC w.r.t $K_{j,t}$ looks different, but we will show later that it implies the same consumption Euler equation as before.
- The FOC w.r.t $B_{j,t}$ implies that at an optimum the consumer equates the marginal cost of consuming a bit less today with the discounted marginal benefit of saving today and consuming a bit more tommorow.

- In a deterministic model (which eliminates the expectation operator from the above system), return on bonds equal's return on capital. Dividing the 2nd condition by the 4th gives

$$R_{t+1} = (1 + r_{t+1}) = \frac{\left[1 + r_{t+1}^{k} - \delta + \chi\left(K_{j,t+1} - K_{j,t}\right)\right]}{\left[1 + \chi\left(K_{j,t} - K_{j,t-1}\right)\right]}.$$

and in steady state

$$R - 1 = r = r^{k} - \delta$$

Aggregate Decision Rules

Again, we have to aggregate each decision rule and express it in per capita terms. Defining

$$C_t = \sum_{j=1}^{n} \frac{1}{n} C_{j,t} \text{ , } \Lambda_t = \sum_{j=1}^{n} \frac{1}{n} \Lambda_{j,t} \text{ , } K_t = \text{ , } Y_t = ...etc.$$

Also in aggregate,

$$B_t = 0 \forall t.$$

- **NB The conditions and recursive definitions are exactly the same as those obtained from a social planner's model.**

- In steady state we know that $K_t = K_{t-1} \forall t$ & $C_t = C_{t-1} \forall t$, therefore the adjustment cost terms disappear from the expressions above and the DSS in this model is the same as in the social planner's version

The capital ownership model

- Rental version: total production output divided into rental streams (c.f. National Accounts). Firms demand factors and households rent them to firms.
- Capital ownership: firms own capital - they invest and face adjustment costs. Shares, representing claims on production, are issued to households, who still supply labour at the real wage rate.
- We can now explicitly solve for the value of equity issued by firms.
- We will show that in terms of aggregate dynamics, this version of the model is also identical to the social planner's version.

Firms' maximisation problem

- Large number of firms seek to maximise the sum of discounted lifetime dividends, subject to a period-by-period resource constraint.
- The 'value of the firm' is normally assumed to be the sum of discounted lifetime dividends - so we can think that each firm wishes to maximise its own value.

$$\max E_t \sum_{i=0}^{\infty} \beta^i \Psi_{a,t+i} \left(D_{a,t+i} \right)$$

$$D_{a,t+i} = Y_{a,t+i} - W_{t+i} N_{a,t+i} - I_{a,t+i}$$

$$Y_{a,t+i} = Z_{t+i} K^{\alpha}_{a,t+i-1} N^{(1-\alpha)}_{a,t+i} - \frac{\chi}{2} \left(K_{a,t+i} - K_{a,t+i-1} \right)^2$$

$$I_{a,t+i} = K_{a,t+i} - (1-\delta) K_{a,t+i-1}$$

- $N_{a,t+i}$ represents labour used by firm a at real wage rate W_{t+i} .
- $D_{a,t+i}$ represents dividends paid by firm a - they are the difference between output ($=$ revenue) and outlays on total wages and investment.
- $\boldsymbol{\Psi}_{a,t+i}$ can be thought of as the shadow value of capital.
- The firm chooses capital and labour - the FOC's (for firm a,and for all t) are

$$
\begin{aligned}
N_{a,t} &: \quad W_t = (1-\alpha)\frac{Y_{a,t}}{N_{a,t}} \\
K_{a,t} &: \quad 1 = \beta E_t \left\{ \frac{\boldsymbol{\Psi}_{a,t+1}}{\boldsymbol{\Psi}_{a,t}} \left[\frac{\alpha \frac{Y_{a,t+1}}{K_{a,t}} + (1-\delta) + \chi\left(K_{a,t+1} - K_{a,t}\right)}{1 + \chi\left(K_{a,t} - K_{a,t-1}\right)} \right] \right\} \\
&= \quad \beta E_t \left\{ \frac{\boldsymbol{\Psi}_{t+1}}{\boldsymbol{\Psi}_t} R^k_{t+1} \right\}
\end{aligned}
$$

Households' maximisation problem

The maximand for this problem is the same as previously:

$$\max E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left(\frac{\left(C_{j,t+i} - \phi X_{j,t+i}\right)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \theta \frac{N_{j,t+i}^{1+\omega}}{1+\omega} \right) \right\}.$$

The budget constraint now takes account of equity share holdings and lack of investment decisions,

$$\begin{aligned} B_{j,t+i} + C_{j,t+i} + V_{t+i} S_{j,t+i} &= (1 + r_{t+i}) B_{j,t+i-1} + W_{t+i} N_{j,t+i} \\ &\quad + (V_{t+i} + D_{t+i}) S_{j,t+i-1} \end{aligned}$$

- V_{t+i} and $S_{j,t+i}$ represent the value and quantity of shares. D_{t+i} are the dividends earned from holding shares.

- We define the total return from holding a share as $R^y_{t+1} = \left(\frac{V_{t+1}+D_{t+1}}{V_t}\right)$ i.e it is the sum of the capital gain from holding the share and its dividend yield.

- Private bonds are again in zero net supply.

- The first order conditions are as follows (**for all** t) :

$$
\begin{aligned}
C_{j,t} &: \quad \left(C_{j,t} - \phi X_{j,t}\right)^{-\frac{1}{\sigma}} = \Lambda_{j,t}. \\
S_{j,t} &: \quad \Lambda_{j,t} = E_t \left\{ \Lambda_{j,t+1} \beta \frac{(V_{t+1} + D_{t+1})}{V_t} \right\} = E_t \left\{ \Lambda_{j,t+1} \beta R^y_{t+1} \right\}. \\
N_{j,t} &: \quad \theta N^{\omega}_{j,t} = \Lambda_{j,t} W_t. \\
B_{j,t} &: \quad \Lambda_{j,t} = E_t \left\{ \Lambda_{j,t+1} \beta \left(1 + r_{t+1}\right) \right\}. \\
\Lambda_{j,t} &: \quad \Big[B_{j,t+i} + C_{j,t+i} + V_{t+i} S_{t+i} = \left(1 + r_{t+i}\right) B_{j,t+i-1} + W_{t+i} N_{j,t+i}
\end{aligned}
$$

- The FOCs w.r.t $C_{j,t}$,$N_{j,t}$ and $B_{j,t}$ are identical to the conditions obtained from the rental model - see last lecture for intuition.

- The additional FOC w.r.t $S_{j,t}$ implies that at an optimum the consumer equates the marginal cost of consuming a bit less today with the discounted marginal benefit of holding an equity claim today, and selling it and consuming its proceeds tommorow.

Aggregate Decision Rules

We aggregate as before (please see previous notes) but with the additional condition that share holdings of all j households must sum to 1.

$$\sum_{j=1}^{\infty} S_{j,t} = 1 \forall t$$

- Additionally because the households collectively own the firms

$$\frac{\Psi_{t+1}}{\Psi_t} = \frac{\Lambda_{t+1}}{\Lambda_t}$$

- the FOCs of the firm with respect to capita, and households FOCs for $B_{j,t}$ and $S_{j,t}$ imply that

$$E_t \left\{ \Lambda_{j,t+1} \beta R^k_{t+1} \right\} = E_t \left\{ \Lambda_{j,t+1} \beta R_{t+1} \right\} = E_t \left\{ \Lambda_{j,t+1} \beta R^y_{t+1} \right\}.$$

- Again, in a deterministic setting,

$$R_{t+1} = R^y_{t+1} = R^k_{t+1}$$

The Value of the Firm

So, in a deterministic setting

$$V_t = \frac{V_{t+1} + D_{t+1}}{R^y_{t+1}} = \frac{V_{t+1} + D_{t+1}}{(1 + r_{t+1})}$$

Iterating this forward -e.g.

$$V_t = \frac{1}{(1 + r_{t+1})}\frac{1}{(1 + r_{t+2})}V_{t+2} + \frac{1}{(1 + r_{t+1})}D_{t+1} + \frac{1}{(1 + r_{t+1})}\frac{1}{(1 + r_{t+2})}D_{t+2}$$

We can therefore write

$$V_t = \sum_{s=t+1}^{\infty}\left(\prod_{j=t+1}^{s}\left(\frac{1}{1 + r_j}\right)\right)D_s$$

Now in steady state

$$D = Y - \delta K - WN.$$

From last week's session: Euler's Theorem tells us that

$$Y = r^k K + WN;$$

and in steady state that

$$r^k = r + \delta.$$

So,

$$D = (r + \delta) K + WN - \delta K - WN = rK.$$

Therefore

$$V = rK \sum_{s=1}^{\infty} \left(\frac{1}{1+r} \right)^s$$

Expanding this gives

$$V = K$$

Asset pricing:

We have equities and bonds in the model above, so have a candidate asset pricing model

$$1 = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} \right\}$$

$$1 = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R^y_{t+1} \right\}$$

If we assume that these bonds are risk-free - i.e. they deliver a unity of consumption in each period, the first equation can be written as:

$$1 = \beta R_{t+1} E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} \right\}$$

- $R_{t+1}$is known in period t
- So, can we compute the equity risk premium?
- Let's use our numerical solution (Note: have to advise the program that $R_{t+1}$is known in period t- how?)
- *What will be the outcome? Homework*