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CORRIGENDUM: LONG-TERM CONTRACTING WITH TIME-INCONSISTENT AGENTS

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THE MAIN RESULT in Gottlieb and Zhang (2021, Theorem 1) showed that in any equilibrium of the game between firms and a time-inconsistent agent, the inefficiency arising from naive present-bias vanishes as the number of periods grows. While this result is correct, the paper failed to note that an equilibrium may not exist. This document corrects this issue and provides general conditions for existence.

Equilibrium may not exist in the model because each firm's strategy space (history-dependent consumption vectors) is not compact. This means that the equilibrium program, defined on page 800, may not have a solution. While the auxiliary program always has a solution, the equivalence between this program and the equilibrium program (Lemma 2) only holds when the equilibrium program has a solution.[1]

Recall that the agent's utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, strictly concave, and twice continuously differentiable in the interior of its domain. Without loss of generality, we normalize $u(0) = 0$. We assume that the agent's net present value of future income I is finite:

$$I := E\left[\sum_{t=1}^{+\infty} \frac{w(s_t)}{R^{t-1}}\right] < +\infty, \tag{1}$$

where $w(s_t) \geq 0$ for all s_t. Let $I_T := E[\sum_{t=1}^{T} \frac{w(s_t)}{R^{t-1}}]$ denote the T-period truncation of I.

We are interested in comparing the equilibrium welfare of time-inconsistent agents with the solution of the standard consumption-savings problem of a time-consistent agent. We

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[1]More specifically, in the proof of Lemma 2, one needs to ensure that the payoffs off the equilibrium path lie in the image of the utility function. While this is always the case if utility is unbounded, it may fail when the utility function is bounded above. When it fails, the equilibrium program has no solution and the game has no equilibrium. Since other results in the paper use the auxiliary program, they hold whenever the equivalence result holds (i.e., whenever an equilibrium exists). For the version of the model with one-sided commitment (Section 3.1), assumptions (1) and (2) below must be made conditional on each state.

therefore assume that the consumption-savings problem of a time-consistent agent is *well posed*:[2]

$$\sup_{\{c_t\}_{t=1}^{+\infty}} \left\{ E\left[\sum_{t=1}^{\infty} \delta^{t-1} u(c_t)\right] : E\left[\sum_{t=1}^{\infty} \frac{c_t}{R^{t-1}}\right] \leq I \right\} < +\infty. \tag{2}$$

As in the paper, let W_T^I denote the equilibrium welfare of the time-inconsistent consumer and let W_T^C denote the welfare of the time-consistent consumer (see pages 797 and 803).

THEOREM 1: *Suppose an equilibrium exists. Then*, $\lim_{T \nearrow +\infty}(W_T^I - W_T^C) = 0$.

While the theorem establishes that the inefficiency vanishes in any equilibrium of the game, it does not guarantee that an equilibrium exists. The proposition below obtains necessary and sufficient conditions for existence. Recall that the auxiliary program is a straightforward maximization problem that admits a unique solution $\{c_t^*\}_{t=1}^T$ (we omit the dependence of c_t^* on the parameters of the model to simplify notation). Let $S_1 := u(c_{T-1}^*) + \beta\delta u(c_T^*)$ and let $S_k := u(c_{T-k}^*) + \delta S_{k-1}$ for $k = 2, \dots, T-2$.

PROPOSITION 1: *An equilibrium exists if and only if* $\frac{1-\delta}{\beta\delta(1-\delta^k)} S_k < \sup_{c\in\mathbb{R}_+}\{u(c)\}$ *for* $k \in \{1, \dots, T-2\}$.

It follows immediately from Proposition 1 that an equilibrium exists when the utility function is unbounded (that is, $\sup_{c\in\mathbb{R}_+}\{u(c)\} = +\infty$).

COROLLARY 1: *Suppose* $u(\cdot)$ *is unbounded. Then, an equilibrium exists.*

When the utility function is bounded, an equilibrium may not exist. Intuitively, lack of existence is due to the firm's inability to exploit enough variation in utils to relax the incentive constraints. Recall that firms exploit a time-inconsistent consumer by offering a contract with a low baseline utility in the immediate future in exchange for a high baseline utility in subsequent periods. This is always possible if the utility space is unbounded. However, if the utility function is bounded and the agent has a high enough initial income, such a high utility may not be feasible. Then, for any contract, a firm can always obtain positive profits by shifting additional baseline consumption into the future, and an equilibrium fails to exist. This is not an issue if the initial income is low enough and the equilibrium consumption is non-increasing:

COROLLARY 2: *Suppose* $u(\cdot)$ *is bounded. If* $\delta R \leq 1$, *there exists* $\bar{I} > 0$ *such that, for any* T, *an equilibrium exists whenever* $I_T < \bar{I}$.

We now present an algorithm to verify existence of equilibrium. Recall that an equilibrium always exists when the utility function is unbounded from above (Corollary 1). Suppose $u(\cdot)$ is bounded and, without loss of generality, normalize $\sup_{c\in\mathbb{R}_+}\{u(c)\} = 1$.

[2]This assumption is common in macroeconomic models, as otherwise a household's intertemporal consumption problem would have no solution. It always holds if $u(\cdot)$ is bounded or if the utility is logarithmic. For the power function $u(c) = c^{\alpha}$, where $0 < \alpha < 1$, the assumption holds whenever $\delta R^{\alpha} \leq 1$. A general sufficient condition is $\delta R \leq 1$.

COROLLARY 3: *Suppose $u(\cdot)$ is bounded. An equilibrium exists if and only if*

- $S_1 < \beta\delta$ *if* $\delta R \geq 1$;
- $S_1 < \beta\delta$ *and* $S_{T-2} < \beta\delta\frac{1-\delta^{T-2}}{1-\delta}$ *if* $\delta R < 1$.

The auxiliary program is a standard convex optimization program that can be easily computed. The algorithm from Corollary 3 shows how one can verify in at most two steps whether an equilibrium exists after calculating the solution to the auxiliary program. If the conditions are met, the solution of the auxiliary program determines the consumption on the equilibrium path. Otherwise, no equilibrium exists.

PROOFS

PROOF OF THEOREM 1: Consider the following program:[3]

$$V_T(\beta, I_T) \equiv \max_{\{c_t\}_{t=1}^T} \sum_{t=1}^{T-1} \delta^{t-1} u(c_t) + \beta\delta^{T-1} u(c_T)$$

$$\text{subject to } \sum_{t=1}^{T} \frac{c_t}{R^{t-1}} \leq I_T.$$

Since an equilibrium exists, the proof of Lemma 2 in Gottlieb and Zhang (2021) is valid.[4] Thus, the equivalence to the auxiliary program applies, and the time-inconsistent agent's consumption path is given by the solution to the program associated with $V_T(\beta, I_T)$, denoted by $c^*(\beta, I_T) \equiv (c_1^*(\beta, I_T), \dots, c_T^*(\beta, I_T))$. Since the time-consistent infinite horizon program is well-posed, $V_T(1, I_T)$ is bounded from above. Since $\{V_T(1, I_T)\}_{T=2}^{\infty}$ is an nondecreasing sequence bounded from above, the sequence converges to a finite limit. As a result, the sequence is Cauchy, and $\lim_{T\to\infty} V_T(1, I_T) - V_{T-1}(1, I_{T-1}) = 0$.

For a fixed I_T, the inefficiency of type β is

$$L_T(\beta) \equiv V_T(1, I_T) - \sum_{t=1}^{T} \delta^{t-1} u\big(c_t^*(\beta, I_T)\big) \geq 0.$$

By revealed preferences, by definition of V and since $I_T \geq I_{T-1}$ and $\beta < 1$, while $u(c) \geq 0$,

$$\sum_{t=1}^{T} \delta^{t-1} u\big(c_t^*(\beta, I_T)\big) \geq V_T(\beta, I_T) \geq V_{T-1}(1, I_{T-1}).$$

Therefore,

$$V_T(1, I_T) - V_{T-1}(1, I_{T-1}) \geq L_T(\beta) \geq 0,$$

which implies the claim. *Q.E.D.*

[3] To simplify notation, we consider the model without uncertainty. With uncertainty, the proof is essentially the same except that the budget constraint features the expectation of discounted future income.

[4] The proof of Lemma 2 uses a perturbation argument to show the equivalence between the equilibrium program and the auxiliary program. For any contract that does not solve the auxiliary program, we can find another contract that increases the objective function of the equilibrium program. Therefore, when an equilibrium exists, consumption on the equilibrium path must solve the auxiliary program. If no off-path consumption supports the solution of the auxiliary program, the same perturbation argument implies that any candidate consumption vector can be improved upon. Therefore, the equilibrium program does not have a solution in that case.

PROOF OF PROPOSITION 1: We first establish sufficiency. Let $\{c_t^*\}_{t=1}^T$ be the solution to the auxiliary program. To show that under the stated assumption an equilibrium exists, we proceed in two steps. First, we construct a consumption vector from the consumption on the equilibrium path $\{c_t^*\}_{t=1}^T$ by constructing a perceived consumption path so that all constraints in the equilibrium program (P) are satisfied. Second, we show that there cannot be any consumption vector in the constraint set of P dominating the extension of $\{c_t^*\}_{t=1}^T$.

Given $\{c_t^*\}_{t=1}^T$, we construct the perceived consumption in the following manner. The key constraints are the ICs. At time $t < T$, the baseline option at $(t+1)$ offers zero consumption. We pick the baseline options from period $t+2$ to period T to offer an identical level of consumption x_t, which is left to be determined. The IC at time t requires that, when binding,

$$\sum_{\tau=t}^{T-1} \delta^{\tau-1} u(c_\tau^*) + \beta\delta^{T-1} u(c_T^*) = \beta \sum_{\tau=t+1}^{T} \delta^{\tau-1} u(x_t), \tag{3}$$

or equivalently,

$$S_{T-t} = \beta\delta \frac{1-\delta^{T-t}}{1-\delta} u(x_t).$$

A finite x_t can be found solving this equation if the stated condition holds.

We now show that this consumption vector solves the equilibrium program P. Suppose by contradiction that there exists a dominating consumption vector in the constraint set of program P. We obtain the following inequalities:

$$\begin{aligned}
& u(c_1^*) + \sum_{t=2}^{T-1} \delta^{t-1} u\big(c_t^*(\overbrace{A\dots A}^{t-1})\big) + \beta\delta^{T-1} u\big(c_T^*(\overbrace{A\dots A}^{T-1})\big) \\
&= u(c_1^*) + \beta \sum_{t=2}^{T} \delta^{t-1} u\big(c_t^*(\overbrace{B\dots B}^{t-1})\big) \\
&< u(c_1) + \beta \sum_{t=2}^{T} \delta^{t-1} u\big(c_t(\overbrace{B\dots B}^{t-1})\big) \\
&\le u(c_1) + \sum_{t=2}^{T-1} \delta^{t-1} u\big(c_t(\overbrace{A\dots A}^{t-1})\big) + \beta\delta^{T-1} u\big(c_T(\overbrace{A\dots A}^{T-1})\big) \\
&\le u(c_1^*) + \sum_{t=2}^{T-1} \delta^{t-1} u\big(c_t^*(\overbrace{A\dots A}^{t-1})\big) + \beta\delta^{T-1} u\big(c_T^*(\overbrace{A\dots A}^{T-1})\big),
\end{aligned}$$

where the first equality (second line) substitutes the binding ICs, the first inequality (third line) follows from the assumption of domination, the second inequality follows from the perturbation argument in Lemma 2 in the paper (see page 802), and the last inequality uses the optimality of $\{c_t^*\}_{t=1}^T$ for the auxiliary program. This is a contradiction, concluding the sufficiency part of the claim.

We now turn to necessity. If an equilibrium exists, then by the perturbation argument in Lemma 2, the perceived (i.e., baseline) consumption immediately following any consumption on the equilibrium path (i.e., alternative) must be zero,

$$c_2(B) = c_3(AB) = c_4(AAB) = c_5(AAAB) = \cdots = c_{T-1}\big(\overbrace{A\ldots A}^{T-3}B\big) = 0,$$

and must solve the binding ICs. Thus, there exists x_t solving equation (3), and this can occur only if the stated condition holds. *Q.E.D.*

PROOF OF COROLLARY 2: Without loss of generality, we normalize $\lim_{c\nearrow\infty} u(c) = 1$. To emphasize that consumption depends on the present discounted value (PDV) of wages, we write $c^A(I) = (c_1^A(I), \ldots, c_T^A(I))$ to denote the solution to the auxiliary program when the PDV is given by I. Since $\delta R \leq 1$, it implies that the solution to the auxiliary program features a weakly decreasing consumption stream: $c_1^A(I) \geq c_2^A(I) \geq \cdots \geq c_T^A(I)$. Moreover, from the zero-profit condition, $c_1^A(I) \leq I$. Thus, each consumption on the actual consumption path is weakly lower than I. To show the corollary, we need to show that there exists $\bar{I}$ such that if $I < \bar{I}$, we can find a solution to all the perceived consumption while maintaining all constraints.

Starting from the IC constraint at $T-1$, it requires $u(c_{T-1}^A(I)) + \beta\delta u(c_T^A(I)) = u(0) + \beta\delta u(c_T(A\ldots AB))$. For this equation to have a solution of $c_T(A\ldots AB)$, it suffices to have $u(c_{T-1}^A(I)) + \beta\delta u(c_T^A(I)) < \beta\delta$. Since the left-hand side is smaller than $u(I) + \beta\delta u(I)$, a sufficient condition is $u(I) < \frac{\beta\delta}{1+\beta\delta}$. Moving to the IC constraint at $T-2$, it requires that $u(c_{T-2}^A(I)) + \delta u(c_{T-1}^A(I)) + \beta\delta^2 u(c_T^A(I)) = u(0) + \beta\delta u(c_{T-1}(A\ldots ABB)) + \beta\delta^2 u(c_T(A\ldots ABB))$. A sufficient condition for this equation to have a solution of perceived consumption is that $u(I) < \frac{\beta\delta+\beta\delta^2}{1+\delta+\beta\delta^2}$. Iterating backward, a sufficient condition for the existence of perceived consumption that satisfy all constraints is

$$u(I) < \frac{\beta\delta + \cdots + \beta\delta^t}{1 + \delta + \cdots + \delta^{t-1} + \beta\delta^t}, \quad \forall t \geq 1.$$

As t goes to infinity, the right-hand side converges to β. So there exists a uniform lower bound for the right-hand side for any t. Thus, there exists $\bar{I}$ such that if $I < \bar{I}$, then we can find perceived consumption that satisfies all constraints, in which case, an equilibrium exists. *Q.E.D.*

PROOF OF COROLLARY 3: By Proposition 1, an equilibrium exists if and only if

$$S_k < \beta\delta\left(\sum_{j=0}^{k-1}\delta^j\right), \quad k = 1, \ldots, T-2.$$

Setting $k = 1$ and $k = T-2$ establishes necessity.

To establish sufficiency, we first consider the case of $\delta R \geq 1$. We argue by contradiction. Suppose that $S_1 < \beta\delta$, while $S_k \geq \beta\delta(1 + \cdots + \delta^{k-1})$ for some $k > 1$. Let $k^* > 1$ be the smallest such index. Then, $S_{k^*-1} < \beta\delta(\sum_{j=0}^{k^*-2}\delta^j)$ and $S_{k^*} \geq \beta\delta(\sum_{j=0}^{k^*-1}\delta^j)$. Since $u(c_{T-k^*}^*) + \delta S_{k^*-1} = S_{k^*}$, it follows that

$$u\big(c_{T-k^*}^*\big) > \beta\delta.$$

Since $\delta R \geq 1$, it follows from a standard Euler equation that $u(c^*_{T-k})$ is non-increasing in k. Thus, $u(c^*_{T-k}) > \beta\delta$ for all $0 < k < k^*$. Then, using $u(c) \geq 0$, we find that $S_1 = u(c^*_{T-1}) + \beta\delta u(c^*_T) > \beta\delta$, a contradiction.

Next, consider the case of $\delta R < 1$. We argue by contradiction again. Suppose k^* is the smallest index such that $S_k \geq \beta\delta(1 + \cdots + \delta^{k-1})$. It follows that $1 < k^* < T - 2$. When $\delta R < 1$, the Euler equation implies that $u(c^*_{T-k})$ is nondecreasing in $k = 2, \dots, T-1$. Therefore, we find that $u(c^*_{T-k}) > \beta\delta$ for all $k \geq k^*$, which implies $S_{T-2} \geq \beta\delta(1 + \cdots + \delta^{T-3})$, a contradiction. *Q.E.D.*

REFERENCES

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