ELSEVIER

Mathematical Social Sciences 35 (1998) 69–73

mathematical social sciences

A brief note on a further refinement of the Condorcet Jury Theorem for heterogeneous groups

Satoshi Kanazawa*

Department of Sociology, Uris Hall, Cornell University, Ithaca, NY 14853-7601, USA

Received 29 April 1997; received in revised form 1 August 1997; accepted 16 September 1997

Abstract

I extend Boland's (1989) work on the Condorcet's Jury Theorem (CJT) for heterogeneous groups. I demonstrate that, as long as CJT holds (in that the mean individual competence $\geq (1/2) + (1/2n)$), heterogeneous groups are better at making the correct decision than homogeneous groups for any given level of mean competence. I also extend CJT to collective decision rules other than simple majority, and show that CJT holds for groups with supermajority decision rules if the mean individual competence is at least $(\pi(n+1)/n)$ (where $\pi =$ required majority).
© 1998 Elsevier Science B.V.

Keywords: The Condorcet Jury Theorem; Heterogeneity of competence; Democracy; Majority rule

1. Introduction

The Condorcet Jury Theorem (CJT) demonstrates that, under specified conditions, a majority of a group is always better at choosing the superior of two alternatives than any single individual (Black, 1958; Condorcet, 1785, pp. 164–165; McLean and Hewitt, 1994, pp. 34–40). In its original formulation, CJT assumes the following conditions:

(1) There are exactly two alternatives.

(2) All individuals share a common preference such that one of the alternatives is superior to all in light of full information.

(3) The individual decisions are independent of one another.

(4) Each individual makes the right decision with the probability $p > 0.5$; individuals are homogenous in p.

(5) The collective decision rule is simple majority.

*Tel: 607-255-0706; fax: 607-255-8473; e-mail: Satoshi.Kanazawa@cornell.edu

PII S0165-4896(97)00028-0

If all of these conditions hold, then the probability that the majority of a group of size n makes the right decision, P_N, is always greater than p, and very quickly approaches 1 as n or p increases (see Ladha, 1992, Appendix for proof of CJT in its original formulation; see Miller, 1986, p. 176, Table 1, for values of P_N for selected values of p and n). CJT has often been used to defend democracy, diversity of opinions, free speech and majoritarian rule (Grofman and Feld, 1988; Ladha, 1992; Miller, 1986; however, for the meritocratic, not democratic, implications of CJT, see Karotkin and Nitzan, forthcoming; Nitzan and Paroush, 1982; Paroush, 1997; Shapley and Grofman, 1984).

There have been many extensions of the original formulation of CJT. Miller (1986) relaxes Condition (2) and extends CJT to groups that contain two opposing subgroups where there are no ''right'' or ''wrong'' decisions. He shows that P'_N, the probability that the majority opinion will prevail, is always greater than p for large n. Thus Miller extends CJT from ''juries'' (where the right decision is the same for all jurors) to democratic ''electorates'' (where there are conflicting opinions about which is the right decision).

Ladha (1992) relaxes Condition (3) and allows individual decisions to be correlated (through interpersonal influence or concurrence of opinions). He shows that CJT still holds and $P_N > p$ as long as the decisions are not too correlated. (See Ladha, 1992 for proof and the exact upper limit for decision correlation.) Estlund (1994) argues that interpersonal influence (where some individuals defer to others' opinions) does not necessarily eliminate the independence of individual decisions necessary for CJT.

Owen et al. (1989) and Grofman et al. (1983) relax Condition (4) and allow individuals to be heterogeneous in their competence (p). First, Grofman et al. (1983) (Theorem V) show that if the distribution of p_i in the group is symmetric, then CJT holds for heterogeneous groups as long as $\bar{p} > 0.5$ (where $\bar{p} =$ the mean of individual p's in a heterogeneous group). Then, Owen et al. (1989) (Theorem II) show that, for any distribution of p_i, $\lim_{n\to\infty} P_N \to 1$ if $\bar{p} > 0.5$. Finally, Boland (1989) (Theorem 3) uses Hoeffding (1956) theorem to generalize CJT to heterogeneous groups of finite size with unknown distribution of p_i's. He demonstrates that CJT holds (in that $P_N > \bar{p}$) as long as $\bar{p} > (1/2) + (1/2n)$. Similarly, Paroush (1997) proves that $p_i > (1/2)\ \forall i$ is *not* a sufficient condition for CJT to hold. The minimum average competence for heterogeneous groups must therefore be higher than that for homogeneous groups.

In this brief note, I will build on Boland's (1989) work and derive another theorem from Hoeffding (1956) to demonstrate that, for any given level of average competence ($\bar{p}$), heterogeneous groups are more capable of arriving at the right decision than homogeneous groups (Theorem 1 below). I will also relax Condition (5), which has hitherto been unattempted to my knowledge, and show the implications for CJT of various collective decision rules other than simple majority (Theorem 2 below).

2. Superiority of heterogeneous to homogeneous groups

Let $X_1,...,X_n$ denote independent Bernoulli random variables with probability of success $p_1,...,p_n$. Independent Bernoulli random variables are analogous to a heterogeneous group where individuals make the right decision with p_i.

Let $X \equiv X_1 + ... + X_n$ and $\bar{p} \equiv (p_1 + ... + p_n / n)$.

Let Y denote a binomial $(n, \bar{p})$ random variable. A binomial random variable is analogous to a homogeneous group where individuals make the right decision with $p = \bar{p}$.

Hoeffding (1956) (Theorem 5) shows that

$$\Pr(a \leq Y \leq b) \leq \Pr(a \leq X \leq b), \tag{1}$$

$$\text{if } 0 \leq a \leq n\bar{p} \leq b \leq n. \tag{2}$$

In other words, $\Pr(a \leq X \leq b)$ attains its minimal value when $p_1 = p_2 = ... = p_n$ (when the independent Bernoulli random variables reduces to a binomial).

If we let $a = (n + 1/2)$ and $b = n$ in (1), then

$$Pr\left(\frac{n+1}{2} \leq Y \leq n\right) \leq Pr\left(\frac{n+1}{2} \leq X \leq n\right), \tag{3}$$

$$\text{if } \frac{n+1}{2} \leq Y \leq n\bar{p} \leq n, \tag{4}$$

$$\text{or } \frac{n+1}{2n} \leq \bar{p} \leq 1, \tag{5}$$

$$\text{or } \frac{1}{2} + \frac{1}{2n} \leq \bar{p} \leq 1. \tag{6}$$

Thus for any given $\bar{p}$ where $(1/2) + (1/2n) \leq \bar{p} \leq 1$, heterogeneous groups are more likely to make the right decision than homogeneous groups.

Theorem 1. (*The superiority of heterogeneous to homogeneous groups*). *If* $p^* \geq (1/2) + (1/2n)$, *then* $P_{N_{HET}} > P_{N_{HOM}}$ *where*
$P_{N_{HET}} = P_N$ *for a heterogeneous group with* $\bar{p} = p^*$, *and*
$P_{N_{HOM}} = P_N$ *for a homogeneous group with* $p = p^*$.

3. The effect of different decision rules

I will now relax Condition (5) of the original formulation.

Let $\pi =$ the proportion required for collective decision, where $(1/2) \leq \pi \leq 1.0$.[1] ($\pi = (1/2)$ for simple majority.)

Let $a = \pi(n + 1)$ and $b = n$ in (1) above.

Then, similarly,

$$\Pr(\pi(n+1) \leq Y \leq n) \leq \Pr(\pi(n+1) \leq X \leq n), \tag{7}$$

[1] When $\pi > (1/2)$, there are possibilities where a group fails to reach a collective decision because neither alternative gains the support of pn individuals, whereas a simple majority rule *always* leads to a unique collective decision (in the absence of abstention) as long as ties are broken with a coin toss. I will not explore the implications of this observation for CJT here.

$$\text{if } \pi(n+1) \leq n\bar{p} \leq n, \tag{10}$$

$$\text{or } \frac{\pi(n+1)}{n} \leq \bar{p} \leq 1. \tag{11}$$

$$\text{Since } \Pr(\pi(n+1) \leq Y \leq n) \geq \bar{p} \tag{12}$$

$$\text{and } \frac{\pi(n+1)}{n} \geq \frac{1}{2}\ \forall \pi \geq \frac{1}{2}, \tag{13}$$

$$\Pr(p(n+1) \leq X \leq n) \geq \bar{p}\ \forall \bar{p} \geq \frac{\pi(n+1)}{n}. \tag{14}$$

Theorem 2. (*CJT for supermajority decision rules*).
If $\bar{p} \leq (\pi(n+1/n)$, *then* $P_N > \bar{p}$.

Theorem 2 demonstrates that CJT holds for heterogeneous groups (in that $P_N > \bar{p}$) *regardless of the collective decision rule*, as long as $\bar{p}$ is at least $(\pi(n+1)/n)$. It is obvious that Boland's (1989) Theorem 3 (CJT for heterogeneous groups) is a special case of my Theorem 2 where $\pi = (1/2)$.

For a heterogeneous group of 100 (e.g. the U.S. Senate), the mean individual competence can be as low as 0.505 if the collective decision rule is simple majority. However, it must be at least 0.606 if the collective decision rule requires three-fifth majority, and it must be at least 0.673 for a two-thirds majority decision rule.

4. Conclusion

I have extended the work of Boland (1989) to demonstrate that, where CJT holds (i.e. $\bar{p} > (1/2) + (1/2n)$), heterogeneous groups, where individuals have different competencies, are better at arriving at the correct binary decision than homogeneous groups, where individuals have the same level of competence, for any given $\bar{p}$. This result lends further support for diversity of opinions and dissent as a means of arriving at the correct collective decision under democratic majoritarian rules (Grofman and Feld, 1988; Ladha, 1992; Miller, 1986). I have also extended CJT from simple majority to other majority decision rules. Relative to the original formulation (which assumes simple majority rule), the sufficient condition for CJT to hold (in terms of the mean individual competence) is more restrictive for supermajority decision rules. Thus, if a group's goal is to arrive at the correct collective decision, it is better off adhering to the simple majority rule ($\pi = (1/2)$) than increasing the proportion necessary for such decisions.

Acknowledgements

I thank Scott L. Feld, Martin T. Wells and anonymous reviewers for their comments on earlier drafts.

References

Black, D., 1958. The Theory of Committees and Elections. Cambridge University Press, Cambridge.

Boland, P.J., 1989. Majority Systems and the Condorcet Jury Theorem. The Statistician 38, 181–189.

Condorcet, Marquis de, 1785. Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des vox. Imprimerie Royale, Paris.

Estlund, D.M., 1994. Opinion Leaders, Independence, and Condorcet Jury Theorem. Theory and Decision 36, 131–162.

Grofman, B., Feld, S.L., 1988. Rousseau's General Will: A Condorcetian Perspective. American Political Science Review 82, 567–576.

Grofman, B., Guillermo, O., Feld, S.L., 1983. Thirteen Theorems in Search of the Truth. Theory and Decision 15, 261–278.

Hoeffding, W., 1956. On the Distribution of the Number of Successes in Independent Trials. Annals of Mathematical Statistics 27, 713–721.

Karotkin, D., Nitzan, S., 1997. On Two Properties of the Marginal Contribution of Individual Decisional Skills. Mathematical Social Sciences 34, 29–36.

Ladha, K.K., 1992. The Condorcet Jury Theorem, Free Speech, and Correlated Votes. American Journal of Political Science 36, 617–634.

McLean, I., Hewitt, F., 1994. Condorcet: Foundations of Social Choice and Political Theory. Edward Elgar, Brookfield.

Miller, N.R., 1986. Information, electorates, and democracy: Some extensions and interpretations of the Condorcet Jury Theorem. In: Grofman, B., Owen, G. (Eds.), Information Pooling and Group Decision Making: Proceedings of the Second University of California, Irvine. Conference on Political Economy. JAI Press, Greenwich, pp. 173–192.

Nitzan, S., Paroush, J., 1982. Optimal Decision Rules in Uncertain Dichotomous Choice Situations. International Economic Review 23, 289–297.

Owen, G., Grofman, B., Feld, S.L., 1989. Proving a Distribution-Free Generalization of the Condorcet Jury Theorem. Mathematical Social Sciences 17, 1–16.

Paroush, J., 1998. Stay Away from Fair Coins: A Condorcet Jury Theorem. Social Choice and Welfare 15, 15–20.

Shapley, L., Grofman, B., 1984. Optimizing Group Judgmental Accuracy in the Presence of Interdependence. Public Choice 43, 329–343.