Complementary Proofs to "Careerist Judges"

Proof of Lemma 6: (i) $q_n(n,t^*)$ increases with t^* :

$$\begin{aligned}
&\text{sign}\frac{\partial q_n(n,t^*)}{\partial t^*}\\
=&\,\text{sign}(1-q)qf(t^*)[-t^*\int_{t^*}^1(1-v)f(v)dv+(1-t^*)\int_{t^*}^1 vf(v)dv]\\
=&\,\text{sign}-t^*\int_{t^*}^1 f(v)dv+t^*\int_{t^*}^1 vf(v)dv+\int_{t^*}^1 vf(v)dv-t^*\int_{t^*}^1 vf(v)dv\\
=&\,\text{sign}(1-q)qf(t^*)[-t^*\int_{t^*}^1 f(v)dv+\int_{t^*}^1 vf(v)dv]>0
\end{aligned}$$

Similarly, I will show that $q_n(y,t^*)$ decreases with t^* :

$$\begin{aligned}
&\text{sign}\frac{\partial q_n(s^*=y,t^*)}{\partial t^*}\\
=&\,\text{sign}(1-t^*)(\int_{.5}^1(1-v)f(v)dv+\int_{.5}^{t^*}vf(v)dv)-t^*(\int_{.5}^1 vf(v)dv+\int_{.5}^{t^*}(1-v)f(v)dv)\\
=&\,\text{sign}(1-q)qf(t^*)(\int_{.5}^{t^*}(1-2t^*)f(v)dv+\int_{t^*}^1(1-t^*-v)f(v)dv)<0.\Box
\end{aligned}$$

(ii) The proof of this is analogous to part (i).□

(iii) Follows from (i) and (ii).□

(iv) By the above claims, it is enough to show that $q_y(n,t^*)>q$, i.e., that:

$$\frac{q(\int_{.5}^1 vf(v)dv+\int_{.5}^{t^*}(1-v)f(v)dv)}{q(\int_{.5}^1 vf(v)dv+\int_{.5}^{t^*}(1-v)f(v)dv)+(1-q)(\int_{.5}^1(1-v)f(v)dv+\int_{.5}^{t^*}vf(v)dv)}>q$$

which holds if

$$\begin{aligned}
\int_{.5}^1 vf(v)dv+\int_{.5}^{t^*}(1-v)f(v)dv>\int_{.5}^1(1-v)f(v)dv+\int_{.5}^{t^*}vf(v)dv\iff\\
\int_{t^*}^1 vf(v)dv>\int_{t^*}^1(1-v)f(v)dv\iff\int_{t^*}^1(2v-1)f(v)dv>0.
\end{aligned}$$

The second part, i.e., that $q_n(s^*,t^*)>1-q$, has an analogous proof.□

(v) To show that $q_y(n,t^*)>\Pr(w=y|q,n,t^*)$, we need to show that:

$$q_y(n,t^*)>\frac{q(1-t^*)}{q(1-t^*)+t^*(1-q)}$$

which, after re-arranging, is analogous to:

$$t^*(\int_{.5}^1 vf(v)dv+\int_{.5}^{t^*}(1-v)f(v)dv)>(1-t^*)(\int_{.5}^1(1-v)f(v)dv+\int_{.5}^{t^*}vf(v)dv)$$

where the last inequality is satisfied using the proof in part (iv). To show that $q_y(y,t^*) > \Pr(w=y|q,y,t^*)$, we need to show that:

$$q_y(y,t^*) > \frac{qt^*}{qt^*+(1-t^*)(1-q)} \Longleftrightarrow \frac{\int_{t^*}^1 vf(v)dv}{\int_{t^*}^1 (1-v)f(v)dv} > \frac{t^*}{1-t^*},$$

but

$$\int_{t^*}^1 vf(v)dv > t^* \; and \; \int_{t^*}^1 (1-v)f(v)dv < 1-t^*,$$

hence the above is satisfied. We can use the analogous proof to show that $q_n(s^*,t^*) > \Pr(w=n|q,s^*,t^*)$.□

(vi) Note that $q_y(y,\frac{1}{2}) > q_n(y,\frac{1}{2})$, and by the above claims this holds for all $t^* > \frac{1}{2}$ and $s^*=y$. On the other hand, $q_y(n,t^*\to 1) \to q$ and $q_n(n,t^*\to 1)\to 1$. Since $q_n(n,t^*)$ increases with t^* and $q_y(n,t^*)$ decreases with t^*, there exists a unique $\tilde{t}(q)$ such that $q_y(n,\tilde{t}(q)) = q_n(n,\tilde{t}(q))$ and for all $t^* < (>)\tilde{t}(q)$, $q_y(n,t^*) > (<) q_n(n,t^*)$. With the uniform distribution, i.e., $f(v)=2$, then $q_y(n,q) = \frac{2-q}{3-2q} \geq \frac{1+q}{1+2q} = q_n(n,q)$ for all $q \geq .5$ which implies that $\tilde{t}(q) > q$.□

(vii) To see this, recall that:

$$\tilde{p}(y) = q_y^2 + (1-q_y)\Pr(w=y|q,s,t),$$

hence:

$$\frac{\partial \tilde{p}(y)}{\partial t}|_{s^c=n} = (2q_y - \Pr(w=y|q,n,t))\frac{\partial q_y}{\partial t} + (1-q_y)\frac{\partial \Pr(w=y|q,n,t)}{\partial t}$$

but when $s^c=n$, $\frac{\partial q_y}{\partial t} < 0$. Also, $\frac{\partial \Pr(w=y|q,n,t)}{\partial t}$ and $2q_y - \Pr(w=y|q,s,t) > 0$ by the above. Similarly,

$$\frac{\partial \tilde{p}(n)}{\partial t} = (2q_n - \Pr(w=n|q,n,t))\frac{\partial q_n}{\partial t} + (1-q_n)\frac{\partial \Pr(w=n|q,n,t)}{\partial t} > 0.$$

An analogous analysis holds for the derivatives w.r.t. q.

This completes the proof of Lemma 6.■

Proof of Proposition 1, Step 2 (Uniqueness). I show a sufficient condition for uniqueness, i.e., that at the equilibrium value of t^e, whenever $\frac{\partial}{\partial t}\frac{q_y(n,t)-\theta\beta(n,t)}{q_n(n,t)+\theta\beta(n,t)} < 0$, then:

$$|\frac{\partial}{\partial t}\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)}| > |\frac{\partial}{\partial t}\frac{q_y(n,t)-\theta\beta(n,t)}{q_n(n,t)+\theta\beta(n,t)}|.$$ [1]

Consider first $\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)} = \frac{q(1-t)}{t(1-q)}$. Then:

$$|\frac{\partial}{\partial t}\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)}| = \frac{q}{(1-q)t^2}$$

Now consider

$$|\frac{\partial}{\partial t}\frac{q_y(n,t)-\theta(\beta(n,t)}{q_n(n,t)+\theta\beta(n,t)}|$$
$$=\frac{1}{q_n+\theta\beta}[\frac{\partial q_y}{\partial t}(-1+\theta\frac{\partial\beta}{\partial q_y}+\theta\frac{\partial\beta}{\partial q_y}\frac{q_y-\theta\beta}{q_n+\theta\beta})+\frac{\partial q_n}{\partial t}((1+\theta\frac{\partial\beta}{\partial q_n})\frac{q_y-\theta\beta}{q_n+\theta\beta}+\theta\frac{\partial\beta}{\partial q_n})]$$

but since

$$\frac{\partial\beta}{\partial q_y}>0,\ \frac{\partial\beta}{\partial q_n}<0\ ,\ \frac{\partial q_n}{\partial t}>0 \text{ and } \frac{\partial q_y}{\partial t}<0,$$

it is enough to show that

$$\frac{q}{(1-q)t^2}>\frac{1}{q_n}[-\frac{\partial q_y}{\partial t}+\frac{\partial q_n}{\partial t}\frac{q_y-\theta\beta}{q_n+\theta\beta}]$$

Plugging in the equilibrium condition and the expressions for the derivatives, the right-hand-side becomes:

$$\frac{1}{q_n}[\frac{2q_y(1-q_y)}{(2-t)t}+\frac{q}{(1+t)}\frac{2q_n(1-q_n)}{t(1-q)}]$$

Let $q_x\in\{q_y,q_n\}$ such that $q_x(1-q_x)=\max\{q_y(1-q_y),q_n(1-q_n)\}$. It is therefore sufficient to prove that:

$$\frac{q}{t}>\frac{2q_x(1-q_x)}{q_n}[\frac{(1+t)(1-q)+q(2-t)}{(2-t)(1+t)}]$$

But the above equation holds both when $q_x=q_n$ and when $q_x=q_y$.■

Proof of Proposition 1, Step 4 ($\frac{\partial t^e(q,\theta)}{\partial q}>0$).

By total differentiation of the equilibrium condition:

$$\frac{\partial t}{\partial q}|_{t=t^e}=\frac{\frac{\partial}{\partial q}\frac{q_y(n,t)-\theta\beta(n,t)}{q_n(n,t)+\theta\beta(n,t)}-\frac{\partial}{\partial q}\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)}}{\frac{\partial}{\partial t}\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)}-\frac{\partial}{\partial t}\frac{q_y(n,t)-\theta\beta(n,t)}{q_n(n,t)+\theta\beta(n,t)}}|_{t=t^e}$$

I show that when $\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)}=\frac{q_y(n,t)-\theta\beta(n,t)}{q_n(n,t)+\theta\beta(n,t)}$,

$$\frac{\partial}{\partial q}\frac{\Pr(w=y|q,n,t)}{\Pr(w=n|q,n,t)}>\frac{\partial}{\partial q}\frac{q_y(n,t)-\theta(n,t)}{q_n(n,t)+\theta(n,t)}$$

which, along with step 2 of the Proposition, implies that $\frac{\partial t}{\partial q}|_{t=t^e}>0$. As in step 2, it is enough to show the inequality for $\theta=0$, i.e., we have to show that:

$$\frac{(1-t)}{t(1-q)^2}>\frac{1}{q_n}[\frac{\partial q_y}{\partial q}-\frac{\partial q_n}{\partial q}\frac{q(1-t)}{t(1-q)}]\iff$$
$$\frac{(1-t)}{t(1-q)^2}>\frac{1}{q_n}[\frac{q_y(1-q_y)}{q(1-q)}+\frac{q_n(1-q_n)(1-t)}{t(1-q)^2}]$$

but since $q_y > q_n$ in equilibrium, and for all q and t, $q_y > 1 - q_n$, it is sufficient to show that:

$$\frac{(1-t)}{t(1-q)^2} > \frac{q_n(1-q_n)}{q_n}\left[\frac{1}{q(1-q)} + \frac{(1-t)}{t(1-q)^2}\right] \iff$$
$$\frac{q(1-t)}{t(1-q)+q(1-t)} > \frac{q(1-t)}{(1-q)(1+t)+q(1-t)}$$

which holds for all $t, q \in [.5, 1]$. This implies that $\frac{\partial t}{\partial q}|_{t=t^e} > 0$.■

Proof of Proposition 2, Step 2 ($t^c(q) < \hat{t}$ for all q) :

Let $s^c = n$ and $t = t^c$. I will show that there is a unique $\hat{t} < 1$ satisfying $\tau_{\hat{t}}(y, y, \sigma^c) = \tau_{\hat{t}}(n, y, \sigma^c)$, and that for all $t > \hat{t}$, $\tau(y, y, \sigma^c) < \tau(n, y, \sigma^c)$. This implies that an equilibrium with $t > \hat{t}$ cannot exist, since then the expected utility from ruling n, an average over $\tau(n, y, \sigma^c)$ and $\tau(n, n, \sigma^c)$ where $\tau(n, n, \sigma^c) > \tau(n, y, \sigma^c)$, is higher than the expected utility from ruling y, an average over $\tau(y, n, \sigma^c)$ and $\tau(y, y, \sigma^c)$ where $\tau(y, y, \sigma^c) > \tau(y, n, \sigma^c)$.

The expression for $\tau(y, y, \sigma^c)|_{s^c=n}$ is

$$\tau(y, y, \sigma^c)|_{s^c=n} = \frac{\int_{.5}^{1} t^2 f(t)dt + \int_{.5}^{t^c} t(1-t)f(t)dt}{\int_{.5}^{1} tf(t)dt + \int_{.5}^{t^c} (1-t)f(t)dt}$$

Taking the derivative of $\tau(y, y, \sigma^c)|_{s^c=n}$ w.r.t t^c, it is

$$\frac{d\tau(y, y, \sigma^c)}{dt^c}|_{s^c=n} = \frac{(1-t^c)f(t^c)(t^c - \tau(y, y, \sigma^c))}{(\int_{.5}^{1} tf(t)dt + \int_{.5}^{t^c} (1-t)f(t)dt)^2}$$

Therefore, $\tau(y, y, \sigma^c)$ is a monotonously decreasing function as long as $t^c < \tau(y, y, \sigma^c)$ and a monotonously increasing function when $t^c > \tau(y, y, \sigma^c)$. When $t^c \to .5$, $t^c < \tau(y, y, \sigma^c)$ and when $t^c \to 1$, $\tau(y, y, \sigma^c) < t^c$. Therefore, there exists t' such that $t' = \tau_{t'}(y, y, \sigma^c)$. Moreover, t' is unique, since when $t^c > \tau(y, y, \sigma^c)$, $\frac{d\tau(y,y,\sigma^c)}{dt^c} < 1$. Thus, for all $t^c < (>)t'$, $t^c < (>)\tau(y, y, \sigma^c)$ and $\frac{d\tau(y,y,\sigma^c)}{dt^c} < (>)0$.

On the other hand, $\tau(n, y, \sigma^c)$ is an average over t for $t > t^c$ and thus increases with t^c for all $t^c > .5$. Also, since only values of $t > t^c$ are included in the computation of $\tau(n, y, \sigma^c)$, then $\tau(n, y, \sigma^c) > t^c$ for all t^c. By the above, when $t^c \to 1$, $\tau(n, y, \sigma^c) > t^c > \tau(y, y, \sigma^c)$. When $t^c = .5$, by Lemma 3, $\tau(y, y, \sigma^c) = \tau(n, n, \sigma^c) > \tau(n, y, \sigma^c)$. Then, there must exist some $\hat{t} \in (.5, 1)$ satisfying $\tau_{\hat{t}}(y, y, \sigma^c) = \tau_{\hat{t}}(n, y, \sigma^c)$. Moreover, it must be that $\hat{t} < t'$ because for all $t^c > t'$, $\tau(n, y, \sigma^c) > t^c > \tau(y, y, \sigma^c)$. Because $\tau(y, y, \sigma^c)$ decreases monotonously for $t^c < \hat{t}$ and $\tau(n, y, \sigma^c)$ increases monotonously in t^c, $\hat{t}$ is unique.■

Proof of Proposition 4, Part (ii). We have to show that $t^f(q) < t^c(q)$. $t^f(q)$ solves:

$$\Pr(w=y|q,n,t^f(q))\Gamma_y = \Pr(w=n|q,n,t^f(q))\Gamma_n + \Gamma \tag{1}$$

where $\Gamma_y = \tau(y,y,\sigma^f) - \tau(y,n,\sigma^f)$, $\Gamma_n = \tau(n,n,\sigma^f) - \tau(n,y,\sigma^f)$ and $\Gamma = \tau(n,y,\sigma^f) - \tau(y,n,\sigma^f)$. I will show that at $t^f(q)$,

$$\tilde{p}(y)\Gamma_y > \tilde{p}(n)\Gamma_n + \Gamma \tag{2}$$

for $\tilde{p}(d) = (1-q_d)\Pr(w=d|q,n,t^f(q)) + q_d^2$, which implies that at $t^f(q)$, the utility from y is higher than the utility from n if appeals are allowed, meaning that the equilibrium solution $t^c(q)$ must admit $t^c(q) > t^f(q)$.

Plugging (1) into (2), we have to show that:

$$q_y(q_y - \Pr(w=y|q,n,t^f(q)))\Gamma_y > q_n(q_n - \Pr(w=n|q,n,t^f(q)))\Gamma_n \Leftrightarrow$$
$$\frac{\Gamma_y}{\Gamma_n} > \frac{q_n(q_n - \Pr(w=n|q,n,t^f(q)))}{q_y(q_y - \Pr(w=y|q,n,t^f(q)))}$$

However, I now show that for all values of t, $\frac{\Gamma_y}{\Gamma_n} > 1$, whereas for all values of t, $\frac{q_n(q_n - \Pr(w=n|q,n,t^f(q)))}{q_y(q_y - \Pr(w=y|q,n,t^f(q)))} < 1$, which implies the desired result.

To see that $\frac{\Gamma_y}{\Gamma_n} > 1$, I calculate the reputations for a cutoff point t from each action and state of the world:

$$\begin{aligned}\Gamma_y &= \tau(y,y,\sigma) - \tau(y,n,\sigma)\\ &= \frac{\int_{.5}^1 2v^2dv + \int_{.5}^t 2v(1-v)dv}{\int_{.5}^1 2vdv + \int_{.5}^t 2(1-v)dv} - \frac{\int_{.5}^1 2v(1-v)dv + \int_{.5}^t 2v^2dv}{\int_{.5}^1 2(1-v)dv + \int_{.5}^t 2vdv}\\ &= \frac{\frac{1}{2}t - \frac{1}{3}t^3 - \frac{1}{6}}{t^2(2-t)}\end{aligned}$$

and similarly

$$\Gamma_n = \tau(n,n,\sigma) - \tau(n,y,\sigma) = \frac{-t + t^2 + \frac{1}{3} - \frac{1}{3}t^3}{(1+t)(1-t)^2}$$

and therefore

$$\Gamma_y > \Gamma_n \Longleftrightarrow \frac{1}{6}\frac{-4t^4 + 4t^3 - t^2 + 2t - 1}{(t+1)\,t^2\,(2-t)} > 0$$

which holds for all $t > \frac{1}{2}$.

To see that $\frac{q_n(q_n - \Pr(w=n|q,n,t))}{q_y(q_y - \Pr(w=y|q,n,t))} < 1$ for all t, I simplify the expression and find that this holds iff:

$$\frac{(1-q)\,(1+t)\,(1-t)}{t^2(1-2q)^2 + 1 + 2t(1-2q)} < \frac{q\,(2-t)\,t}{t^2(1-2q)^2 + 4q^2 + 4qt(1-2q)}$$

which holds for all $t \geq \frac{1}{2}$ and $q \geq \frac{1}{2}$. This completes the proof.■