Adding and Subtracting Black-Scholes: A New Approach to Approximating Derivative Prices in Continuous-Time Models*

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Abstract

We develop a new approach to approximating asset prices in the context of continuous-time models. For any pricing model that lacks a closed-form solution, we provide a closed-form approximate solution, which relies on the expansion of the intractable model around an "auxiliary" one. We derive an expression for the difference between the true (but unknown) price and the auxiliary one, which we approximate in closed-form, and use to create increasingly improved refinements to the initial mispricing induced by the auxiliary model. The approach is intuitive, simple to implement, and leads to fast and extremely accurate approximations. We illustrate this method in a variety of contexts including option pricing with stochastic volatility, computation of Greeks, and the term-structure of interest rates.

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1. Introduction

The last decade has witnessed an ever-increasing demand for new models addressing a number of empirical puzzles in financial economics which relate to pricing, hedging, and spanning derivatives contracts (e.g., Bakshi and Madan, 2000; Duffie, Pan, and Singleton, 2000), the term structure of interest rates (e.g., Ahn, Dittmar, and Gallant, 2002; Dai and Singleton, 2002), or the aggregate stock market (e.g., Menzly, Santos, and Veronesi, 2004; Gabaix, 2008). The vast majority of these models rely on a continuous-time framework which is by now one of the most celebrated analytical tools in financial economics. Market practitioners have also increasingly relied on continuous-time models (e.g., Brigo and Mercurio, 2006). The reason for this consensus about the benefits of continuous-time modeling is that within this framework, we are able to provide elegant representations for the price of a variety of contingent claims. At the same time, continuous-time models call for an old and well-known practical issue: how do we implement models that cannot be solved in closed-form?

To cope with prices not available in closed-form, one typically relies on either of the following two alternative approaches. The first approach hinges upon the numerical solution to a partial differential equation obtained through, say, finite-difference, Fourier-inversion, or tree methods (Schwartz, 1977; Hull and White, 1990; Scott, 1997; Figlewski and Gao, 1999). The second approach, initiated by Boyle (1977), relies on Monte Carlo simulations in which a large number of trajectories need to be generated for the state variables underlying the asset pricing model. Both methods can be cumbersome to implement and, computationally, quite time-consuming.

This paper develops a new conceptual framework to compute asset prices in nonlinear, multifactor diffusion settings. We develop closed-form approximations to any given contingent claim model, which are easy to implement and require very little computer power. Our main idea is to choose an "auxiliary" pricing model for which a solution is available in closed-form. For example, we can choose affine models (e.g., Heston, 1993; Duffie, Pan, and Singleton, 2000) to be the auxiliary models, as we shall illustrate throughout the whole paper. Additional examples of candidate auxiliary models are the quadratic models studied by Ahn, Dittmar, and Gallant (2002) or the linearity-generating processes introduced by Gabaix (2008, 2009). Given any auxiliary model, we derive an expression for the difference between the unknown price of the model of interest and the auxiliary one. This expression takes the form of a conditional expectation taken under the risk-neutral probability, which, under regularity conditions, can be cast in terms of a Taylor series expansion. We approximate the unknown price by retaining a finite number of terms from this series. Our method is highly general and therefore applicable in a wide range of settings including the pricing of options, computation of the associated Greeks, and the pricing of bonds. We develop several examples to illustrate how to use our general insights and provide numerical results that show that our methods are quite precise and easily implemented.

Our closed-form approximations to asset prices rely, as explained, on Taylor series expansions of conditional expectations. Similar expansions are widely used in financial econometrics and empirical finance (see, e.g., Aït-Sahalia, 2002; Schaumburg, 2004; Aït-Sahalia and Yu, 2006; Bakshi, Ju, and Ou-Yang, 2006; Aït-Sahalia and Kimmel, 2007; Xiu, 2010). A key feature in this literature is the expansion of a conditional expectation of a continuous-time variable, say, some conditional moment related to the short-term interest rate expected to prevail over a small time-span—e.g., one day or one week at most. Such "small time expansions" are relatively less useful, when the objective is to approximate option pricing models, either because (i) the presence of optionality leads to payoff functions that are not differentiable, as for example, in the simple European option pricing case, or because (ii) the maturity of the derivative contracts might occur at long maturity dates, as for example, in the term structure of interest rates. For these reasons, small time expansions have not been applied to asset pricing models previously,[1] although they are reconsidered in recent work by Kimmel (2008), which we shall discuss in a moment.

Our approach still relies on series expansions of conditional expectations, but works differently. Rather than being applied directly to payoff functions, our expansions apply to pricing errors that summarize the mispricing between the true pricing function and the auxiliary pricing function we choose to approximate the true model by. These pricing errors are typically differentiable even if the payoffs are not. In fact, after completing this paper, we came across the work of Kimmel (2008), who develops a clever method to deal with expansions of payoff functions, which can be used to address the issues related to long maturity dates. Although Kimmel's method cannot be applied to deal with payoffs that are not differentiable, it can be used in efficient conjunction with ours, to implement closed-form approximations to our pricing errors, which, as noted, are typically differentiable.

Our trick to expand asset prices around prices computed in closed-form, shares similarities with Yang's (2006) expansion around "base-models." However, the approximation arising through Yang's method is quite different from ours. Heuristically, Yang's expansion relies on terms improving upon the base-models, which are conditional expectations of the base-model mispricing, taken under the base-model probability. Instead, our corrective terms are conditional expectations of the auxiliary model's mispricing, taken under the true probability. Both methods, which we numerically compare, carry obvious advantages over perturbation methods (e.g., Fouque, Papanicolaou, and Sircar, 2000; Lewis, 2000), which need to rely on expansions of pricing functions around "small" values of some of the model's parameters.

[1] One early and isolated exception appears in Chapman, Long, and Pearson (1999), which is indeed a special case of our method, as we shall explain in Section 5.2. However, this special case does not allow one to deal with derivatives written on non-differentiable payoffs.

Finally, the method introduced in this article can be also interpreted as an expansion of the risk-neutral probability implied by the model of interest, around that of some auxiliary model chosen by the user. As such, our approach shares similarities with the strand of literature where option prices are computed through an approximation of the risk-neutral density underlying the true pricing model, as in Abadir and Rockinger (2003), or in the "saddlepoint approximations" considered by Rogers and Zane (1999), Xiong, Wong, and Salopek (2005), or Aït-Sahalia and Yu (2006). In fact, approximating the risk-neutral probability is a special case of our approach as we shall explain.

Our method carries some advantages over approximations of conditional densities, when applied to asset pricing. First, because it relies on a direct expansion of asset prices, our method avoids the numerical computation of multidimensional Riemann integrals against an approximate conditional density. This feature is attractive in multifactor models such as those that involve stochastic interest rates, stochastic volatility, or macro-finance determinants of the yield curve. Second, the expansion we provide carries new and interesting economic content, as we shall illustrate. For example, we shall see that approximating stochastic volatility models through our approach leads to errors, which we can interpret as hedging costs arising through the use of misspecified Black-Scholes deltas. Finally, we provide an explicit expression for the difference between the pricing function of the true and the auxiliary model, which leads to a more direct analysis of the pricing error and simpler approximations.

The paper is organized as follows. In the next section, we illustrate our methods through an example relating to the pricing of options with stochastic volatility. In Section 3, we develop a general framework to approximate asset prices and provide extensions that allow for the computation of sensitivities of derivative prices. Section 4 relates our approach to the existing literature on expansion methods for both asset prices and risk-neutral probabilities. In Section 5, we assess the numerical performance of our methods in concrete applications including option pricing with stochastic volatility and the yield curve. Section 6 explores further extensions relating to models with jumps and barriers. Section 7 concludes. The appendix provides details omitted from the main text.

2. The gist of the approximation method

We illustrate the basic ideas underlying our method with an empirically relevant example, arising in the context of the pricing of European options. It is well-known that the volatility of stock returns is stochastic, and that this feature can account for many of the empirical puzzles stemming from the Black and Scholes (1973) (Black-Scholes, henceforth) model, such as, for example, the tendency of out-of-the-money put options to be more expensive than at-the-money options—the volatility skew (see, e.g., Lewis, 2000). In a stochastic volatility model, the price of a stock, $S(t)$ say, is the solution to:

$$\frac{dS(t)}{S(t)} = rdt + \sqrt{v(t)}dW(t), \tag{1}$$

where $W(t)$ is a standard Brownian motion under the risk-neutral probability, r is the short-term rate, taken to be a constant, and $v(t)$ is the instantaneous return variance. For example, $v(t)$ can be a mean-reverting process, and have constant elasticity of variance (CEV, henceforth), as in the following model:

$$dv(t) = \kappa(\alpha - v(t))\,dt + \omega\,|v(t)|^{\xi}\,dW_v(t), \tag{2}$$

where $W_v(t)$ is a Brownian motion correlated with $W(t)$, with instantaneous correlation ρ, $\xi > 0$ is the CEV parameter, and, finally, (κ, α, ω) are three additional constants. The properties of this model are summarized by Lewis (2000, Chapter 9), and further discussed by Aït-Sahalia and Kimmel (2007).

Consider a European call option written on this asset. The option payoff is $b(S(T)) \equiv \max\{S(T) - K, 0\}$ at maturity time $T > 0$, where $K > 0$ is the strike price. Let $w(S, v, t)$ be the option price as of time $t \in [0, T]$, when the stock price is S and the instantaneous variance is v. Come time T, $w(S, v, T) = b(S)$ for all v. Subject to this boundary condition, the pricing function satisfies,

$$0 = Lw(x, v, t) - rw(x, v, t), \tag{3}$$

where L is the infinitesimal generator associated with Eqs. (1) and (2),

$$Lw = \frac{\partial w}{\partial t} + rx\frac{\partial w}{\partial x} + \frac{1}{2}vx^2\frac{\partial^2 w}{\partial x^2} + \kappa(\alpha - v)\frac{\partial w}{\partial v} + \frac{1}{2}\omega^2 v^{2\xi}\frac{\partial^2 w}{\partial v^2} + \rho\omega v^{\xi+\frac{1}{2}}x\frac{\partial^2 w}{\partial x \partial v}. \tag{4}$$

Apart from Heston's (1993) affine model, which sets $\xi = 1/2$, the solution to Eq. (3), provided it exists, is not known in closed-form. Yet a CEV model with $\xi > 1/2$ might be quite important, empirically. As noted by Jones (2003) and Yang (2006), the instantaneous volatility of volatility, $\sqrt{v(t)}$, equals $\frac{1}{2}\omega v(t)^{\xi-\frac{1}{2}}$, which is constant for $\xi = 1/2$, thereby ruling out situations where the volatility of volatility is time-varying.

Our objective is to obtain a closed-form approximation of the unknown price, $w(x, v, t)$, relying upon an "auxiliary" model that can be solved in closed-form. The Black-Scholes model is the simplest auxiliary model in this context. According to this model, the asset price is a geometric Brownian motion, and the Black-Scholes option price, $w^{\mathrm{bs}}(x, t; \sigma_0)$, say, is the solution to

$$0 = L_0 w^{\mathrm{bs}}(x, t; \sigma_0) - rw^{\mathrm{bs}}(x, t; \sigma_0), \tag{5}$$

where again, $w^{\mathrm{bs}}(x, T; \sigma_0) = b(x)$, and the associated infinitesimal operator, L_0, is the same as in Eq.

(4), but with the constant σ_0^2 replacing the instantaneous stochastic variance v, and all the partial derivatives with respect to v set equal to zero, $\frac{\partial w}{\partial v} \equiv \frac{\partial^2 w}{\partial v^2} \equiv \frac{\partial^2 w}{\partial x \partial v} \equiv 0$.

Our key idea, now, is to subtract Eq. (5) from Eq. (3). The result is that the price difference, $\Delta w\,(x, v, t; \sigma_0) \equiv w\,(x, v, t) - w^{\text{bs}}\,(x, t; \sigma_0)$, satisfies,

$$0 = L\Delta w\,(x, v, t; \sigma_0) - r\Delta w\,(x, v, t; \sigma_0) + \delta\,(x, v, t; \sigma_0)\,, \tag{6}$$

with boundary condition $\Delta w\,(x, v, T; \sigma_0) = 0$ for all x and v, and "mispricing function" δ given by:

$$\delta\,(x, v, t; \sigma_0) \equiv \frac{1}{2}\left(v - \sigma_0^2\right) x^2 \frac{\partial^2}{\partial x^2} w^{\text{bs}}\,(x, t; \sigma_0)\,. \tag{7}$$

Since $w^{\text{bs}}\,(x, t; \sigma_0)$ is known, we can compute $\delta\,(x, t; \sigma_0)$. By relying on the Feynman-Kac representation of the solution to Eq. (6) (see, e.g., Karatzas and Shreve, 1991), and recalling the definition of the price difference, $\Delta w = w - w^{\text{bs}}$, the unknown pricing function, w, can be expressed as the sum of the Black-Scholes price plus a conditional moment, which we shall interpret in a moment:

$$w\,(x, v, t) = w^{\text{bs}}\,(x, t; \sigma_0) + \mathbb{E}_{x,v,t}\left[\int_t^T e^{-r(u-t)} \delta\,(S(u), v\,(u)\,, u; \sigma_0)\, du\right]. \tag{8}$$

The interpretation of the mispricing function δ in Eq. (7) relates to the hedging cost arising while evaluating and hedging the option through the Black-Scholes formula. Precisely, suppose a trader sells the option and wishes to hedge against it through a self-financing strategy, in which he trades the underlying stock using the Black-Scholes delta, $\partial w^{\text{bs}}\,(x, t; \sigma_0)\,/\partial x$. Then, as shown by El Karoui, Jeanblanc-Picqué, and Shreve (1998), and further elaborated by Corielli (2006), δ can be interpreted as the instantaneous increment in the total hedging cost arising from the use of a wrong model (the Black-Scholes model) to hedge against the true model in Eqs. (1) and (2).

The conditional moment in Eq. (8) is taken under the stock price dynamics given by Eqs. (1)-(2). Therefore, it is in general impossible to obtain a closed-form expression for the second term in Eq. (8). However, under regularity conditions, the very same conditional moment can be explicitly written as a series expansion in terms of the infinitesimal generator associated with the model of interest, L in Eq. (4). As shown in Appendix A (see Proposition A.3.), Eq. (8) is indeed equivalent to:

$$w\,(x, v, t) = w^{\text{bs}}(x, t; \sigma_0) + \sum_{n=0}^{\infty} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n\,(x, v, t; \sigma_0)\,, \tag{9}$$

where δ_n satisfies the recursive equation: $\delta_{n+1}\,(x, v, t; \sigma_0) = L\delta_n\,(x, v, t; \sigma_0) - r\delta_n\,(x, v, t; \sigma_0)$, with

$\delta_0 \equiv \delta$. In practice, this formula is truncated to a finite number of terms, yielding:

$$w_N\left(x, v, t; \sigma_0\right) \equiv w^{\text{bs}}(x, t; \sigma_0) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x, v, t; \sigma_0), \tag{10}$$

for some $N \geq 0$. For example, a first-order approximation ($N = 0$) is given by $w_0\left(x, v, t; \sigma_0\right) \equiv w^{\text{bs}}(x, t; \sigma_0) + (T-t)\, \delta(x, v, t; \sigma_0)$. Naturally, the unknown option price w in Eq. (9) does not depend on σ_0, although its "truncation" w_N does. In Section 5.1, we discuss choices of the nuisance parameter, σ_0, and find that the numerical accuracy of $w_N\left(x, v, t; \sigma_0\right)$ does not crucially depend on the choice of σ_0.

Finally, note that even though the Black-Scholes model has constant volatility, our methods still allow to feed information about stochastic volatility. The reason is that our power series expansions hinge upon the initial mispricing arising from the use of the Black-Scholes model, and this mispricing is a function of the initial state, price, and volatility. The expansions, then, deliver refinements that are increasingly more informative about stochastic volatility, as we shall illustrate analytically in Section 5.1.2. Needless to mention, one could rely on models with stochastic volatility as auxiliary devices, and this choice might only improve the numerical accuracy of our methods.

3. A general approximating pricing formula

In this section, we derive a general approximation formula for asset prices in models not solved in closed-form, following the same lead as that of the example of Section 2. In Section 3.1, we introduce notation for the model we approximate and its auxiliary counterpart, and provide our approximating formula. In Section 3.2, we discuss approximations of the sensitivities of pricing functions with respect to the state variables underlying the evaluation framework, which are useful for the purpose of derivative hedging.

3.1. The model and its approximation

We consider a multifactor model in which a d-dimensional vector of state variables $x\left(t\right)$ affects all asset prices in the economy. We assume that under the risk-neutral probability, $x\left(t\right)$ satisfies:

$$dx\left(t\right) = \mu\left(x\left(t\right), t\right) dt + \sigma\left(x\left(t\right), t\right) dW\left(t\right), \tag{11}$$

where $W\left(t\right)$ is a d-dimensional standard Brownian motion under the risk-neutral probability, and $\mu\left(x, t\right)$ and $\sigma\left(x, t\right)$ are some drift and diffusion functions. This general framework covers most popular asset price specifications in the literature. For example, in the stochastic volatility model of Section 2, Eqs. (1) and (2), the state vector is $x(t) = [S(t) \;\; v(t)]$, where $S(t)$ is the stock price and $v(t)$ is the instantaneous stock return variance. Similarly, most interest rate and bond price models take as given a set of factors driving the entire yield curve, such as $x(t)$ in Eq. (11) (see Section 5.2 for more details and examples).

Let $w(x,t)$ be the price of a derivative written on the realization of $x(T)$, for some $T > t$, when the current state is $x(t) = x$. The price of this derivative is determined by three exogenous components: (i) its payoff at T as given by $b(x(T))$ for some function $b(x)$; (ii) the instantaneous coupon rate paid off by the asset at time t denoted by $c(x(t),t)$; and (iii) the instantaneous short-term interest rate at time t, $R(x(t),t)$ say, with which the final expected payoff and expected coupon payments are discounted back, under the risk-neutral probability.

Define the infinitesimal generator L associated with Eq. (11),

$$Lw(x,t) = \frac{\partial w(x,t)}{\partial t} + \sum_{i=1}^{d} \mu_i(x,t) \frac{\partial w(x,t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_{ij}^2(x,t) \frac{\partial^2 w(x,t)}{\partial x_i \partial x_j}, \tag{12}$$

where $\sigma^2(x,t) = \sigma(x,t)\,\sigma(x,t)^\top \in \mathbb{R}^{d\times d}$. The derivative price, $w(x,t)$, can then be expressed as the solution to the following partial differential equation:

$$Lw(x,t) + c(x,t) = R(x,t)\,w(x,t), \tag{13}$$

with boundary condition $w(x,T) = b(x)$ for all x. In words, an investment into this asset must be such that the expected instantaneous capital gain under the risk-neutral probability, $Lw(x,t)$, plus the instantaneous coupon rate, $c(x,t)$, equal the instantaneous yield on a safe asset.

To approximate the unknown price $w(x,t)$, we introduce an auxiliary model,

$$dx_0(t) = \mu_0(x_0(t),t)\,dt + \sigma_0(x_0(t),t)\,dW(t), \tag{14}$$

for some drift and diffusion functions $\mu_0(x,t)$ and $\sigma_0(x,t)$. Our objective is a suitable expansion of the model of interest around this auxiliary model. We assume that the dimension of the auxiliary model is the same as that of the actual model, i.e., x_0 is a d-dimensional vector. This assumption does not entail any loss of generality, since we can always add constant components, as we now explain. Suppose, for example, that we wish to consider an auxiliary model with a lower dimension, where the state vector $y(t) \in \mathbb{R}^m$, with $m < d$, solves, for some drift and diffusion functions μ_Y and σ_Y:

$$dy(t) = \mu_Y(y(t),t)\,dt + \sigma_Y(y(t),t)\,dW_1(t),$$

and $W_1(t)$ is an m-dimensional standard Brownian motion. The vector process $\left[y^\top \; x_{m+1} \; \cdots \; x_d\right]^\top$, where the last $d-m$ components remain constant over time, is then a solution to Eq. (14) with:

$$\mu_{0,i}(x,t) = \begin{cases} \mu_{Y,i}(y,t), & 1 \le i \le m \\ 0, & \text{otherwise} \end{cases} \qquad \sigma_{0,ij}(x,t) = \begin{cases} \sigma_{Y,ij}(y,t), & 1 \le i,j \le m \\ 0, & \text{otherwise} \end{cases}$$

For example, in Section 2, we rely upon this modeling trick and use a low dimensional model, Black-Scholes, as an auxiliary device to approximate the price of options in markets with higher dimension, i.e., with stochastic volatility.

As for the derivative associated with the auxiliary market, we assume that the derivative is worth $b_0(x_0(T))$ at time T, for some function $b_0(\cdot)$. This complication helps illustrate a few properties of our approximation methods arising within the pricing of bonds, as we shall explain in Section 5.2. However, in most cases, one will choose $b_0(x) = b(x)$ such that the auxiliary pricing function, $w_0(x,t)$, mimics $w(x,t)$ at expiration T. Finally, and crucially, we assume that we have a closed-form solution $w_0(x,t)$ for the pricing function in the markets where the state vector satisfies Eq. (14). To save on notation, we do not make explicit that the pricing function $w_0(x,t)$ depends on nuisance parameters, as we did in the introductory example of the previous section.

The price difference, $\Delta w(x,t) \equiv w(x,t) - w_0(x,t)$, satisfies,

$$L\Delta w(x,t) + \delta(x,t) = R(x,t)\,\Delta w(x,t), \tag{15}$$

with boundary condition $\Delta w(x,T) = d(x)$. The two adjustment terms are given by the two mispricing functions:

$$d(x) = b(x) - b_0(x), \tag{16}$$

and $\delta(x,t) = (L - L_0)\,w_0(x,t)$, that is,

$$\delta(x,t) = \sum_{i=1}^{d} \Delta\mu_i(x,t)\,\frac{\partial w_0(x,t)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} \Delta\sigma_{ij}^2(x,t)\,\frac{\partial^2 w_0(x,t)}{\partial x_i \partial x_j}, \tag{17}$$

where

$$\Delta\mu_i(x,t) = \mu_i(x,t) - \mu_{0,i}(x,t), \quad \Delta\sigma_{ij}^2(x,t) = \sigma_{ij}^2(x,t) - \sigma_{0,ij}^2(x,t).$$

The first adjustment term, $d(x)$, arises due to the use of an incorrect payoff function, and obviously drops out once the payoff function in the auxiliary market is matched to that in the market of interest. The second term captures discrepancies between the auxiliary and the true model, relating to the underlying factors driving the market.[2]

Given some $N \geq 1$, we assume $d(x)$ to be $2N$ times differentiable with respect to x, and $\delta(x,t)$ to be $2N$ times differentiable with respect to x and N times differentiable with respect to t. The number N is, basically, the order of the approximation of our asset price expansion in Definition 1 below, and these assumptions on d and δ are needed for this expansion to be well-defined. We note that the smoothness conditions imposed on $d(x) = b(x) - b_0(x)$ in general rule out choosing $b_0(x) \neq b(x)$, if the payoff function $b(x)$ is non-differentiable (as, for example, in the case of plain vanilla options).

Under standard regularity conditions reviewed in Appendix A, we can apply the Feynman-Kac representation of the solution to the derivative mispricing in Eq. (15), $\Delta w(x,t)$, to obtain the following representation of the asset price, $w(x,t)$, in terms of that arising through the auxiliary model, $w_0(x,t)$:

Theorem 1 (Asset Price Representation) *Assume that the two solutions, $w(x,t)$ and $w_0(x,t)$ to the asset pricing Eqs. (13) and (15) exist. Then the following identity holds:*

$$\begin{aligned} w(x,t) = w_0(x,t) + \mathbb{E}_{x,t}\left[\exp\left(-\int_t^T R(x(s),s)\,ds\right) d(x(T))\right] \\ + \int_t^T \mathbb{E}_{x,t}\left[\exp\left(-\int_t^s R(x(u),u)\,du\right)\delta(x(s),s)\right] ds, \end{aligned} \tag{18}$$

where $x(t)$ satisfies Eq. (11), and d, δ are given in Eqs. (16)-(17).

The above representation formula holds under standard regularity conditions.[3] The right-hand side delivers an exact expression for the error due to the use of the auxiliary model to price the claim, instead of the true model. This representation is useful in its own right, as it precisely shows how the pricing error is related to the auxiliary model.

Yet our main goal is to look for an approximation of the error term in order to adjust the price $w_0(x,t)$ for the error involved. Accordingly, our next step is to approximate the two expectations on the right-hand side of Eq. (18) using series expansions. Consider the following definition:

[2] Asset price representations such as those in Eq. (15) were noticed by both Mele (2003, Appendix B) in the scalar case, and Kristensen (2008, Appendix A) in a multifactor setting, although used for quite different purposes.

[3] These regularity conditions are summarized by conditions (A3)-(A5) in Appendix A.2. Among these, the condition that could not possibly hold in standard asset pricing models is the linear growth condition imposed on the drift and diffusion terms. However, this condition is only needed to ensure that the solutions $x(t)$ and $x_0(t)$ to Eqs. (11) and (14) exist. Other conditions than the linear growth can be used to ensure these solutions do actually exist.

Definition 1 (Asset Price Approximation) *The N-th order approximation $w_N(x,t)$ to the unknown price $w(x,t)$ in Eq. (18), at time t and state x, is given by:*

$$w_N(x,t) = w_0(x,t) + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n(x,t) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t), \tag{19}$$

where $d_0(x,t) = d(x)$, $\delta_0(x,t) = \delta(x,t)$, and

$$d_n(x,t) = Ld_{n-1}(x,t) - R(x,t)\, d_{n-1}(x,t), \quad \delta_n(x,t) = L\delta_{n-1}(x,t) - R(x,t)\, \delta_{n-1}(x,t).$$

In Appendix A, we provide additional regularity conditions under which our asset price approximation formula is valid, asymptotically, in that $w_N(x,t) \to w(x,t)$ as $N \to \infty$. Appendix A also provides error bounds applying to any fixed approximation order, $N \geq 1$.

Note, finally, that the approximation in Definition 1 is only a means to estimate the right-hand side of Eq. (18) in Theorem 1. Other methods might be available. For example, one could approximate the two conditional expectations appearing in the right-hand side of Eq. (18) through simulations. Note that one might then just use simulations to directly compute the conditional expectation appearing in the Feynman-Kac representation of $w(x,t)$. However, a potential advantage of simulating Eq. (18), rather than the Feynman-Kac representation of $w(x,t)$, is that the auxiliary pricing function, $w_0(x,t)$, might play a role similar to that of a control variate, thereby increasing the precision of the price estimate. The attractive feature of the power expansion in Eq. (19), over and above simulations, is, naturally, that once implemented, it requires virtually no computation time.

3.2. Approximating Greeks

We outline how our expansions can be used to obtain closed-form approximations to the partial derivatives of asset prices, which can be useful to estimate Greeks. The approximations to these partial derivatives are readily obtained indeed, by differentiating the approximating formula in Eq. (19) of Definition 1 with respect to the variables of interest.

The approximation of the k-th order derivative of $w(x,t)$ is given by,

$$\frac{\partial^k w_N(x,t)}{\partial x^k} = \frac{\partial^k w_0(x,t)}{\partial x^k} + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n^{(k)}(x,t) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n^{(k)}(x,t), \tag{20}$$

where

$$d_n^{(k)}(x,t) = \frac{\partial^k d_n(x,t)}{\partial x^k}, \quad \delta_n^{(k)}(x,t) = \frac{\partial_n^k \delta_n(x,t)}{\partial x^k}.$$

The two sequences, $d_n^{(k)}(x,t)$ and $\delta_n^{(k)}(x,t)$, can be evaluated either numerically (using, say, finite-difference methods) or analytically. For example, to compute the approximation to the first-order derivatives, $k=1$, we use the following recursion: $d_0^{(1)}(x,t)=\partial d(x)/\partial x$, $\delta_0^{(1)}(x,t)=\partial\delta(x,t)/\partial x$ and,

$$d_n^{(1)}(x,t)=Ld_{n-1}^{(1)}(x,t)-R(x,t)d_{n-1}^{(1)}(x,t)+L^{(1)}d_{n-1}(x,t)-\frac{\partial R(x,t)}{\partial x}d_{n-1}(x,t),$$

$$\delta_n^{(1)}(x,t)=L\delta_{n-1}^{(1)}(x,t)-R(x,t)\delta_{n-1}^{(1)}(x,t)+L^{(1)}\delta_{n-1}(x,t)-\frac{\partial R(x,t)}{\partial x}\delta_{n-1}(x,t),$$

where

$$L^{(1)}\phi(x,t)=\sum_{i=1}^{d}\frac{\partial\mu_i(x,t)}{\partial x}\frac{\partial\phi(x,t)}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial\sigma_{ij}^2(x,t)}{\partial x}\frac{\partial^2\phi(x,t)}{\partial x_i\partial x_j}.$$

For $k=2$, the recursive scheme needed to compute the second-order partial derivatives of $d_n(x,t)$ and $\delta_n(x,t)$ with respect to x, is: $d_0^{(2)}(x,t)=\partial^2 d(x)/\partial x^2$, $\delta_0^{(2)}(x,t)=\partial^2\delta(x,t)/\partial x^2$ and,

$$\begin{aligned}d_n^{(2)}(x,t) &= Ld_{n-1}^{(2)}(x,t)-R(x,t)d_{n-1}^{(2)}(x,t)+2L^{(1)}d_{n-1}^{(1)}(x,t)-2\frac{\partial R(x,t)}{\partial x}d_{n-1}^{(1)}(x,t)\\ &\quad+L^{(2)}d_{n-1}(x,t)-\frac{\partial^2 R(x,t)}{\partial x^2}d_{n-1}(x,t),\end{aligned}$$

$$\begin{aligned}\delta_n^{(2)}(x,t) &= L\delta_{n-1}^{(2)}(x,t)-R(x,t)\delta_{n-1}^{(2)}(x,t)+2L^{(1)}\delta_{n-1}^{(1)}(x,t)-2\frac{\partial R(x,t)}{\partial x}\delta_{n-1}^{(1)}(x,t)\\ &\quad+L^{(2)}\delta_{n-1}(x,t)-\frac{\partial^2 R(x,t)}{\partial x^2}\delta_{n-1}(x,t),\end{aligned}$$

where

$$L^{(2)}\phi(x,t)=\sum_{i=1}^{d}\frac{\partial^2\mu_i(x,t)}{\partial x^2}\frac{\partial\phi(x,t)}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\sigma_{ij}^2(x,t)}{\partial x^2}\frac{\partial^2\phi(x,t)}{\partial x_i\partial x_j}.$$

4. Discussion

This section discusses how our work relates to existing approximation methods. Section 4.1 compares our asset price expansions with that in Yang (2006), and succinctly discusses alternative expansions, such as the small volatility of volatility expansions in Lewis (2000), or mean-reverting approximations, such as those in Fouque, Papanicolaou, and Sircar (2000). In Section 4.2, we explain how our approach relates to methods hinging upon the expansion of risk-neutral densities.

4.1. Asset price expansions

4.1.1. Yang's expansion

Asset price expansions have been considered in the literature, typically as a means to approximate the solution to models of option pricing with stochastic volatility. The asset price approximation developed by Yang (2006) relies on the idea that the unknown solution to the model of interest can be expressed in terms of the corresponding solution to a "base-model" and some suitably behaved "residual." While Yang's "base-model" is similar to our "auxiliary model," our series expansions are substantially different from those in Yang. We illustrate the differences by relying on the general setup in Section 3. However, for the purpose of simplifying the presentation, we set the instantaneous short-term rate R and the coupon c in Eq. (13) to be identically zero, $R(x,t) = c(x,t) \equiv 0$, and take the auxiliary market to be one for the same payoff function as the true market, i.e., $d(x) = 0$ in Eq. (16).

The starting point of Yang's method is, similarly to ours, to specify a base-model characterized by an infinitesimal generator such that $L_0 w^{(0)} = 0$, for some pricing function $w^{(0)}$ satisfying the same boundary condition as the price of interest, w. Note, then, that by Eq. (13), the unknown price function can always be written as the solution to:

$$0 = Lw(x,t) \equiv L_0 w(x,t) + (L - L_0) w(x,t). \tag{21}$$

Consider, for example, the stochastic volatility model of Section 2, where the state variables are the stock price S, and the instantaneous return variance v, $x = [S\ v]$. With the base-model chosen to be the Black-Scholes model, we obtain

$$L_0 w = \frac{\partial w}{\partial t} + rS\frac{\partial w}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 w}{\partial S^2} \quad \text{and} \quad (L - L_0) w = \kappa(\alpha - v)\frac{\partial w}{\partial v} + \frac{1}{2}\omega^2 v^{2\xi}\frac{\partial^2 w}{\partial v^2} + \rho\omega v^{\xi+\frac{1}{2}} S\frac{\partial^2 w}{\partial S \partial v}. \tag{22}$$

Decomposing Eq. (3) in this way is convenient because the price for the base-model, w_0, is then simply Black-Scholes, with the interesting feature that the variance variable, v, is plugged into the pricing formula, in lieu of the constant variance, σ_0^2—i.e., $w_0(x,t) = w^{\text{bs}}(S, t; \sqrt{v})$, in terms of the notation in Section 2.

The operator $L - L_0$ in Eq. (21), once applied to the unknown price w, leads to a term bearing the interpretation of a mispricing function, $(L - L_0) w$, similarly as our mispricing function δ does in Eq. (15). Formally, Eq. (21) together with the fact that $L_0 w^{(0)} = 0$ imply that the difference between the price in the true market and that in the base market, $\Delta w \equiv w - w^{(0)}$, satisfies $L_0 \Delta w(x,t) +$

$(L - L_0)\, w\,(x, t) = 0$, or:

$$w\,(x, t) = w^{(0)}\,(x, t) + \int_t^T \mathbb{E}^0_{x,t}\left[(L - L_0)\, w\,(x\,(u)\,, u)\right] du, \tag{23}$$

where $\mathbb{E}^0_{x,t}$ denotes the expectation taken under the probability underlying the base-model, as defined by the infinitesimal generator L_0.

While this expression is akin to the identity stated in Theorem 1, our representation of the unknown price in terms of the mispricing function $\delta = (L - L_0)\, w_0$ is different. Under the simplifying assumptions of this section, our asset price representation is:

$$w\,(x, t) = w_0\,(x, t) + \int_t^T \mathbb{E}_{x,t}\left[(L - L_0)\, w_0\,(x\,(u)\,, u)\right] du. \tag{24}$$

While both Eqs. (23) and (24) rely on auxiliary models solved in closed-form, the corrective terms are different. In Yang's (2006) representation, Eq. (23), the corrective term is the expectation of the *unknown* mispricing term, $(L - L_0)\, w$, taken under the *auxiliary model's* probability. In our representation, Eq. (24), the corrective term is the expectation of the *known function* $(L - L_0)\, w_0$ taken under the *true model* probability.

These differences have implications, when it comes to approximating either of these corrective terms. In our case, the adjustment term is based on $\delta = (L - L_0)\, w_0$, which is known in closed-form, thereby allowing a direct closed-form approximation of the integrated expectation. Yang's adjustment term is based on $(L - L_0)\, w$, where w is unknown, and the resulting approximation relies on an expansion where:

$$w\,(x, t) = w_0\,(x, t) + \sum_{m=1}^{\infty} w^{(m)}\,(x, t)\,, \tag{25}$$

and the terms $w^{(m)}\,(x, t)$ can be solved for recursively, as solutions to: $L_0 w^{(m)} + (L - L_0)\, w^{(m-1)} = 0$, with boundary conditions $w^{(m)}\,(x, T) = 0$, $m = 1, 2, \cdots$. Yang proposes to approximate w by only including the first, say M, leading terms in the infinite sum of Eq. (25). This approximation requires the computation of $w^{(m)}\,(x, t)$, for $m = 1, \cdots, M$, which can be done either numerically or analytically. Yang suggests to utilize their Feynman-Kac representations,

$$w^{(m)}\,(x, t) = \int_t^T \mathbb{E}^0_{x,t}\left[(L - L_0)\, w^{(m-1)}\,(x\,(u)\,, u)\right] du, \quad m = 1, \cdots, M,$$

which can be calculated through standard symbolic software packages, at least for the first several order terms. Section 5.1.2 compares the numerical performance of our series expansion with that of

Yang (2006) in the context of option pricing with stochastic volatility.

4.1.2. Perturbations

Perturbation methods provide an alternative means to approximate asset prices, yielding expansions of the unknown pricing function around particular values of some of the model's parameters. For example, in the context of the stochastic volatility model in Eqs. (1) and (2), Lewis (2000, Chapter 3) considers an expansion of the option price around the volatility of volatility parameter ω, as follows: $w(S,v,t) = \sum_{m=0}^{\infty} \omega^m w_{(m)}(S,v,t)$, for some functions $w_{(m)}$. When ω is small, a truncation of this expansion to a few terms is quite accurate.

Fouque, Papanicolaou, and Sircar (2000, Chapter 5) consider perturbations in a model with stochastic volatility, where, for $r \equiv 0$, Eq. (1) is replaced by $dS(t) = \nu(y(t))S(t)dW(t)$, for some instantaneous return volatility function $\nu(y)$, and $y(t)$ is a mean-reverting Ornstein-Uhlenbeck process,

$$dy(t) = \alpha(m - y(t))dt + \sqrt{\alpha}\sigma_{\infty} dW_y(t),$$

where $W_y(t)$ is a Brownian motion under the risk-neutral probability, possibly correlated with $W(t)$, α is the persistence parameter, and m and $\frac{1}{2}\sigma_{\infty}^2$ are the ergodic mean and variance of $y(t)$, i.e., $m = \lim_{\tau\to\infty} \mathbb{E}_{y,t}[y(t+\tau)]$ and $\sigma_{\infty}^2 = 2\lim_{\tau\to\infty} \mathbb{V}_{y,t}[y(t+\tau)]$, with $\mathbb{V}_{y,t}$ denoting the variance operator. A small value of α^{-1} means $y(t)$ is a "fast" mean-reverting process. It might be a convenient assumption while pricing options with relatively large maturities, in which case we might consider mean-reversion to act relatively fast. The authors show that given this setup, option prices can be expanded as: $w(S,v,t) = \sum_{m=0}^{\infty} \alpha^{-m/2} w_m(S,v,t)$ for some functions w_m. Naturally, one may also consider "slow" mean-reversion expansions (as in Fouque, Papanicolaou, Sircar and Solna, 2003), which are more suited to deal with relatively short-term options, as in this case, volatility is less likely to experience large swings before maturity.

While perturbation methods are attractive, base-model expansions such as those in this paper and in the previous section have the relative merit that they do not rely on specific assumptions about "small" values of any of the model's parameters, and are flexible enough to deal with models possibly more complex than those in this section.

4.2. Risk-neutral probabilities

Asset prices are conditional expectations taken under the risk-neutral probability. Approximating asset prices, then, does necessarily entail approximating risk-neutral probabilities. How do our approximation methods precisely relate to those approximating risk-neutral probabilities? In this section, we link the expansion in Theorem 1 of the price $w(x,t)$ about the auxiliary price $w_0(x,t)$, to the expansion of the risk-neutral probability of the asset pricing model around that of the auxiliary pricing model.[4]

To simplify the discussion, we keep on assuming that in Eq. (13), the short-term rate and the coupon c are both zero, $R(x,t) = c(x,t) \equiv 0$, and take the payoff functions in the auxiliary and the true markets to be the same, i.e., $d(x) = 0$ in Eq. (16), just as in Section 4.1. Then, the two prices, $w(x,t)$ and $w_0(x,t)$, are simply:

$$w(x,t) = \int_{\mathbb{R}^d} b(y)\, p(y,T|x,t)\, dy, \quad w_0(x,t) = \int_{\mathbb{R}^d} b(y)\, p_0(y,T|x,t)\, dy,$$

where p and p_0 are the risk-neutral conditional densities underlying the two models: the true, p, and the auxiliary, p_0. Clearly, we have:

$$w(x,t) = w_0(x,t) + \int_{\mathbb{R}^d} b(y)\, \Delta p(y,T|x,t)\, dy, \tag{26}$$

where $\Delta p \equiv p - p_0$ is the difference between the two conditional densities, the risk-neutral "transition discrepancy," using a terminology due to Aït-Sahalia (1996). It is easy to see that the asset price representation in Theorem 1 implies that the following identity holds true:

$$\int_{\mathbb{R}^d} b(y)\, \Delta p(y,T|x,t)\, dy = \int_t^T \mathbb{E}_{x,t}\left[\delta(x(s), s)\right] ds, \tag{27}$$

where δ is as in Eq. (17) (see Appendix B). Therefore, our expansion of $\int_t^T \mathbb{E}_{x,t}\left[\delta(x(s), s)\right] ds$ in Definition 1 is related to a corresponding expansion of the risk-neutral transition discrepancy, Δp. In fact, in Appendix B, we derive an explicit representation of $\Delta p(y,T|x,t)$ in terms of a conditional expectation (see Eq. (B1)), which highlights the fact that the representation and approximations of w in Theorem 1 and Definition 1 rely on equivalent representations and approximations of the risk-neutral conditional density.

In spite of this equivalence, our methods are, in general, more easily implemented as they lead to closed-form approximations for pricing errors that are easy to compute. To illustrate, the right-hand side of Eq. (27), which is the pricing error arising from the use of an *auxiliary asset price*, can be easily computed through a power series expansion as that in Definition 1. In contrast, the left-hand side of Eq. (27), which is the pricing error arising from the use of *an auxiliary risk-neutral density*, requires the computation of a Riemann integral. This computation can be cumbersome, especially

[4] A few more technical details, available upon request, show that the evaluation of risk-neutral densities based on "saddlepoint approximations" are also special cases of our approximation methods.

when the dimension of the model, d, is large. Finally, the previous equivalence was derived assuming that the short-term interest rate and the coupon are both zero. In general, it is unclear as to how to use approximations of risk-neutral probabilities to deal with conditional expectations such as,

$$\mathbb{E}_{x,t}\left[\exp\left(-\int_t^T R\left(x\left(s\right),s\right)ds\right)b\left(x\left(T\right)\right)\right].$$

These cases need to be dealt with in many instances, especially those including the pricing of fixed income products, or derivatives in the presence of stochastic interest rates. Our methods, which rely on approximations directly obtained through auxiliary *pricing* functions (instead of auxiliary *risk-neutral probabilities*), do handle these cases in a quite natural manner.

5. Numerical accuracy of approximation

We assess the performance of our asset price approximations in two natural contexts: (i) option pricing in models with CEV and stochastic volatility, in Section 5.1, and (ii) the term-structure of interest rates, in Section 5.2.

5.1. Option pricing with CEV and random volatility

We consider increasingly challenging experiments, aiming to assess the resilience of our methods to the approximation of option prices in increasingly complex models. Section 5.1.1 explores a first example, where we use the Black-Scholes model to approximate the price of an option as predicted by a CEV model. Section 5.1.2 contains results for the case where the option price is determined within a stochastic volatility model. We show how the use of the simple Black-Scholes model is capable to deliver accurate approximations, and apply our methods to (i) the stochastic volatility model of Heston (1993), which, for the purpose of the experiment, we assume to be unknown; and, to the general stochastic volatility model with CEV in Eqs. (1) and (2) of Section 2, for which a closed-form solution is unavailable.

5.1.1. The generalized Black-Scholes option pricing model

We investigate the performance of our approximation methods when applied to a simple model, where the price of a stock, $S\left(t\right)$, is the solution to:

$$\frac{dS\left(t\right)}{S\left(t\right)}=rdt+\sigma\left(S\left(t\right),t\right)dW\left(t\right), \tag{28}$$

for some volatility function $\sigma(x,t)$. We wish to approximate the European call option price predicted by this model using the Black-Scholes model as an auxiliary pricing device, where volatility is constant and equal to σ_0. By the expansion set forth in Section 3 (Theorem 1 and Definition 1), the approximation to the supposedly unknown option price relating to the model in Eq. (28) is, for some $N \geq 0$:

$$w_N(x,t;\sigma_0) \equiv w^{\mathrm{bs}}(x,t;\sigma_0) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t;\sigma_0), \tag{29}$$

where $\delta_n(x,t;\sigma_0) = L\delta_{n-1}(x,t;\sigma_0) - r\delta_{n-1}(x,t;\sigma_0)$, $\delta_0(x,t;\sigma_0) \equiv \delta(x,t;\sigma_0)$, the mispricing function, and L is the same infinitesimal generator as in Eq. (4), but with the function $\sigma^2(x,t)$ replacing v, and all the partial derivatives with respect to v set equal to zero, $\partial w/(\partial v) \equiv \partial^2 w/\left(\partial v^2\right) \equiv \partial^2 w/(\partial x \partial v) \equiv 0$. Finally, the mispricing function is:

$$\delta(x,t;\sigma_0) = \frac{1}{2}\left(\sigma^2(x,t) - \sigma_0^2\right) x^2 \frac{\partial^2}{\partial x^2} w^{\mathrm{bs}}(x,t;\sigma_0), \tag{30}$$

and still bears the interpretation of an instantaneous hedging cost arising from the use of a wrong model, the Black-Scholes model, as explained in Section 2 in the case of the stochastic volatility model of Eqs. (1) and (2).

We consider approximating the CEV model, for which $\sigma(x,t) = \sigma_{\mathrm{cev}} x^{\gamma-1}$, where σ_{cev} is constant and $\gamma > 0$. For this model, the option price is known in closed-form (Schroder, 1989), which allows us to achieve a precise quantitative assessment of our approximations. As anticipated in Section 2, the use of an auxiliary model inevitably leads to a nuisance parameter—a parameter that does not affect the unknown price, but does enter the pricing formula for the auxiliary model. In the Black-Scholes case, the nuisance parameter is the instantaneous volatility σ_0 in Eq. (5). There are several alternatives to deal with this parameter. For example, let $\hat{\sigma}_0$ be some estimate of σ_0. Then, we may approximate $w(x,t)$ with $w_N(x,t;\hat{\sigma}_0)$. Alternatively, we may consider,

$$\hat{\sigma}_N(x,t) = \arg\min_{\sigma} \left(w_N(x,t;\sigma) - w_0(x,t;\sigma)\right)^2, \tag{31}$$

where $w_0(x,t;\sigma) \equiv w^{\mathrm{bs}}(x,t;\sigma)$, in terms of the notation in Section 2.2. As a simple example, we have that for $N=0$, $\hat{\sigma}_0(x,t) = \sigma(x,t)$. Clearly, $\lim_{N\to\infty} \hat{\sigma}_N(x,t) = \mathrm{IV}(x,t)$, where $\mathrm{IV}(x,t)$ denotes the Black-Scholes implied volatility, defined by $w(x,t) = w^{\mathrm{bs}}(x,t;\mathrm{IV}(x,t))$. For fixed N, then, the unknown option price can be approximated by $w_N(x,t;\sigma_N(x,t))$, or more generally, $w_N(x,t;\sigma_M(x,t))$, where $M \leq N$, as we do in the numerical experiments reported below.

Insert Fig.1 and Fig. 2 near here

In Fig. 1, we depict the approximation errors resulting from our method, arising for different levels of the asset price, when the parameter values are those displayed in the figure legend, and for a time-to-maturity equal to three months. The approximating price is obtained as $w_N(x,t;\hat{\sigma}_0(x))$, where $\hat{\sigma}_0(x) = \sigma_{\text{cev}} x^{\gamma-1}$. The errors are several orders of magnitude lower than 1% with only a very small number of correction terms. Fig. 2 depicts the errors arising whilst pricing the option with a larger maturity, one year: our approximation is still quite accurate in this case, even for the more extreme far-in and far-out of the money options.

5.1.2. Option pricing with stochastic volatility

Next, we study the numerical performance of our method when approximating the solution to a European option price predicted by a model where volatility is stochastic. As in the previous example, we employ the Black-Scholes model, where stock volatility is constant, as auxiliary model in our expansion. Naturally, to approximate the unknown price in the market of interest, we might have relied upon an auxiliary market where stock volatility is random rather than constant. Our experiment to approximate a market with a given state space (that with stochastic volatility) through a market with a lower state space (that with constant volatility) serves the purpose to make a strong case for our methods. All in all, the numerical issue we wish to investigate links to how many terms in the expansion are needed, in practice, to feed information about stochastic volatility so as to make our approximation reasonable. In Section 5.1.2.1, we provide intuition about the ability of the Black-Scholes model to feed information about stochastic volatility, by providing the first two terms of the expansion in Eq. (10). Section 5.1.2.2 provides numerical results for both the Heston model and the non-affine models stemming out of Eqs. (1) and (2).

5.1.2.1 Approximating markets with stochastic volatility through Black & Scholes The approximation in Eq. (10) is seemingly identical to that in the Generalized Black-Scholes model of Section 5.1.1 (see Eq. (29)). In particular, the mispricing function $\delta(x,v,t;\sigma_0)$ in Eq. (7) has a functional form similar to that of $\delta(x,t;\sigma_0)$ in Eq. (30). However, the infinitesimal operator L in Eq. (4), which the mispricing $\delta(x,v,t;\sigma_0)$ iterates upon, provides increasingly precise information about random volatility, as the iterations develop. To illustrate, consider the first-order approximation to the unknown price, $w(x,v,t)$,

$$w_1(x,v,t;\sigma_0) = w^{\text{bs}}(x,t;\sigma_0) + \delta(x,v,t;\sigma_0)(T-t) + \frac{1}{2}\delta_1(x,v,t;\sigma_0)(T-t)^2 . \qquad (32)$$

The first term on the right-hand side is the Black-Scholes price. The second, is the first adjustment, which is proportional to time-to-maturity, $T-t$, with proportionality factor equal to the mispricing function, δ. Intuitively, consider Eq. (18) in Theorem 1. The payoffs in both Heston's and Black-Scholes markets are obviously the same and, hence, $d = 0$. Therefore, in the context of this section, it is only the third term on the right-hand side of Eq. (18) that matters. By approximating the integrand of this third term with its value taken at t, we obtain the second term in Eq. (32). This approximation is quite rough: for example, the coefficients of the stochastic volatility process, κ, α, ω, and ρ, do not enter δ. These coefficients enter the third, and final, term on the right-hand side of Eq. (32). This term is the product of a quadratic adjustment for time-to-maturity and the function δ_1, obtained through one iteration upon the mispricing function $\delta(x, v, t; \sigma_0)$ in Eq. (7):

$$\delta_1(x, v, t; \sigma_0) = \left(v - \sigma_0^2\right) x^2 \varphi(x, v, t; \sigma_0) + \frac{1}{2}\kappa(\alpha - v) x^2 \frac{\partial^2 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^2}$$
$$+ \omega\rho v^{\xi+\frac{1}{2}} x \left(x \frac{\partial^2 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^2} + \frac{1}{2} x^2 \frac{\partial^3 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^3} \right) - r\delta(x, v, t; \sigma_0),$$

where the function φ is defined as:

$$\varphi(x, v, t; \sigma_0) \equiv \frac{1}{2} r x \frac{\partial^3 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^3} + \frac{1}{2} \frac{\partial^3 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^2 \partial t} + r \frac{\partial^2 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^2}$$
$$+ v \left[\frac{1}{2} \frac{\partial^2 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^2} + x \left(\frac{\partial^3 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^3} + \frac{1}{4} x \frac{\partial^4 w^{\mathrm{bs}}(x, t; \sigma_0)}{\partial x^4} \right) \right].$$

The first-order approximation in Eq. (32) is not expected to be accurate. We make use of it to illustrate, analytically, how the approximating price becomes more informative as we add new terms. For example, the volatility of volatility parameter, ω, enters δ_1 only through a correlation channel: if $\rho = 0$, ω does not enter $w_1(x, v, t; \sigma_0)$ anymore. Yet one iteration is sufficient to feed the approximating price with information about the parameters of the drift function of volatility, κ and α. In the next section, we show indeed that we can obtain quite accurate approximations to the unknown price, with only a very few additional corrective terms to Eq. (32).

5.1.2.2 Experiment results We begin with analyzing the Heston (1993) model, where the stock price, $S(t)$, is solution to Eqs. (1) and (2), with $\xi = 1/2$. The N-order approximation to the true price $w(S, v, t)$ is given by Eq. (10) in Section 2, where the infinitesimal generator in Eq. (4) has, now, obviously, $\xi = 1/2$. Fig. 3 depicts the percentage approximation error, as a function of the current stock price S, when: time-to-maturity is one year, $T - t = 1$, the current value of volatility is such that $v = 0.05$, and the parameter values are as those displayed in the figure legend—roughly the same as those in Heston (1993). Finally, we set the nuisance parameter, the Black-Scholes σ_0, equal to $\sqrt{v}$. Note, then, that this particular choice for σ_0 implies the first term of the right-hand side of

Eq. (8) is the same as the first corrective term proposed by Yang (2006), although our expansion hinges upon a quite different series, as explained in Section 4.1.1. Furthermore, we might choose a different value for σ_0, and this would lead even the first term of the right-hand side of Eq. (8) to differ from the first term in Yang's expansion. For example, we could set σ_0 to be the minimizer of a criterion generalizing that in Eq. (31) in Section 5.1.1:

$$\hat{\sigma}_N(x, v, t) = \arg\min_{\sigma} \left(w_N(x, v, t; \sigma) - w_0(x, t; \sigma)\right)^2,$$

where the notation for the approximating pricing functions w_N and w_0 is the usual one, that in Section 2. We choose, however, $\sigma_0 = \sqrt{v}$, i.e., $\hat{\sigma}_N(x, v, t) = \hat{\sigma}_0(x, v, t) = \sqrt{v}$ for all N, to produce numerical results comparable with those in Yang (2006). Moreover, in what follows, we compute the price from Heston's (1993) model through the Fourier transform methods of Madan and Carr (1999).

Insert Fig. 3 near here

Fig. 3 confirms that the first-order approximation in Eq. (32), while improving over that obtained for $N = 0$ (i.e., that stemming from the use of the first two terms in the right-hand side of Eq. (32)), still produces significant pricing errors. At the same time, the approximation in Eq. (10) considerably improves, and quite quickly, as we add new terms. With $N = 3$, for example, Eq. (32) provides a reasonable approximation to the Heston model's price, with pricing errors amounting to less than 1% from the truth, over a realistic range of variation for the underlying stock price. With $N = 4$, our approximation produces percentage pricing errors as small as 0.2%, even for far-out-of-the-money options.

Table 1 compares the performance of our approximations with that of Yang (2006). We use the parameter values estimated by Bollerslev and Zhou (2002) in an empirical study of foreign exchange rate volatility. These parameter values, reported in the table legend, differ from those leading to the results in Fig. 3. This further experiment helps stress-test the robustness of our methods to alternative parameter specifications, within a market—currency options—which is, by far, orders of magnitude more important than equity, in terms of transactions. We price options with strikes equal to $K = 1,000$ and expiring in one month, using four leading terms. Both Yang's and our method produce quite sensible results for a variety of values of the underlying, and its volatility, with very small percentage pricing errors. Our method produces better approximations than Yang's, with percentage pricing errors never exceeding 0.0065% but in three cases, when $S = 1,000$ and $v = 0.1, 0.2, 0.3$. The largest error occurs when $S = 1,000$ and $v = 0.1$, where our error is 0.10%, and Yang's is 1.16%.

Insert Table 1 and Table 2 near here

Are these results robust to nonlinear models, going beyond the affine class? We address this question by setting the CEV parameter ξ in Eq. (2) equal to 0.6. We price options with strikes equal to $K = 1,000$ and time-to-maturity equal to one month. Since no closed-form solutions are available, the benchmark we use now is the option price computed through Monte Carlo integration. We design the experiment in such a way to keep less than 0.5% discrepancy between the prices computed through Monte Carlo integration and the previous prices obtained through Fourier transforms, for $\xi = 1/2$. Table 2 compares our approximations with those of Yang (2006). The performance of both methods deteriorates, compared to that for the affine model shown in Table 1, although the percentage price errors are still quite small. Yang's method performs better than ours in this case, with percentage errors never exceeding 0.70%. Our method, instead, produces percentage pricing errors mostly around 0.80% – 1.00%, with the highest error being 1.23% (occurring when $S = 1,000$ and $v = 0.1$, in which case Yang's error is 0.21%).

Next, we gauge the ability of our methods to approximate Greeks, as developed in Section 3.2, through Eq. (20). While our approximate Greeks are straightforward to implement, the computation of benchmark values that we can compare our approximation to proves quite challenging when the solution for the option price is unknown. In order to compute benchmark values for the unknown Greeks, we need to first numerically estimate the unknown prices and then, numerically evaluate the sensitivities of these price estimates. In particular, as regards our benchmarks, we need to rely on numerical derivatives, the accuracy of which tends to deteriorate as higher-order sensitivities (such as gamma) are considered.

Insert Table 3 and Table 4 near here

We implement two experiments. In a first experiment, we consider Heston's (1993) model, and compute the first two partials of the option price with respect to both the stock price, $\partial C/\partial S$, the delta, and $\partial^2 C/\partial S^2$, the gamma, as well as the first partial of the option price with respect to the return variance, $\partial C/\partial v$. These quantities can be computed in closed-form, up to a Riemann integration, thereby mitigating our previous concerns about the precision of an established benchmark. Table 3 shows that our method produces quite precise estimates, with percentage errors for delta and $\partial C/\partial v$ not exceeding 0.006% and 0.6%, respectively, and with percentage errors for gamma mostly around 2.5%, with the highest errors being 4.55% and 5.91% (occurring when $S = 1000$ and, $v = 0.1$ and $v = 0.2$, respectively).

To estimate benchmark values for the Greeks within the non-affine model, arising when the CEV parameter ξ in Eq. (2) equals 0.6, we need to rely on numerical derivatives. Our approach comprises three steps: first, we compute Monte Carlo prices over a suitable grid of (S, v) values; second, we approximate the pricing surface by local polynomials; third, we proceed to take symbolic derivatives of the fitted surface with respect to both S and v. We refrain from presenting additional results for gamma, as these might be seriously contaminated by estimation errors induced by our fitting procedure, as well as the increased number of approximations needed to estimate theoretical quantities. Table 4 shows that for this non-affine model, the distance between the benchmark and our approximation methods is higher than for Heston's, albeit still quite small, with percentage errors for delta not exceeding 1.51%, and errors for $\partial C/\partial v$ mostly around 0.80% - 1.20%, with the highest error being 1.87% (occurring when $S = 1,000$ and $v = 0.1$).

5.2. *The term structure of interest rates*

This section investigates how our approximation methods perform when applied to approximate bond prices in both one-factor (in Section 5.2.1) and multifactor models (in Section 5.2.2).

5.2.1. *One Factor Models*

This section illustrates our approximation methods and their numerical performance whilst dealing with one-factor models. By choosing $x(t) = r(t)$ as the short-term interest rate, the bond price solves Eq. (13) with $c(x,t) = 0$, $R(x,t) = x$, and $b(x) = 1$.

We take as starting point the (supposedly unknown) solution to the Cox, Ingersoll, and Ross (1985) (CIR, henceforth) model of the yield curve, where the short-term rate is solution to:

$$dr(t) = \beta(\alpha - r(t))\,dt + \sigma\sqrt{r(t)}dW(t)\,, \tag{33}$$

where $\alpha > 0$, $\beta > 0$, and $\sigma > 0$ are constants.

The first auxiliary model we analyze is simply one where: (i) the asset payoff is just zero, rather than one; and (ii) the short-term rate is the same as in the true data generating mechanism, that in Eq. (33). We show that in this case, our approximating formulae collapse to those provided by Chapman, Long, and Pearson (1999). We first analyze this case, in Section 5.2.1.1. In Section 5.2.1.2, we investigate the performance of our methods when the auxiliary model is such that: (i) the payoff of the bond equals the true payoff, one; and (ii) the auxiliary model is the Vasicek (1977) model, where the short-term rate is solution to:

$$dr(t) = \beta_0(\alpha_0 - r(t))\,dt + \sigma_0 dW(t)\,, \tag{34}$$

for three constants α_0, β_0, and σ_0.

5.2.1.1 A simple power expansion We consider a quite straightforward auxiliary market, one where drift and diffusion terms coincide with the drift and diffusion of the CIR (1985) short-term rate in Eq. (33), that is, $\mu = \mu_0$, $\sigma = \sigma_0$, in terms of the general framework of Section 3. We assume, however, that in this auxiliary market, the final payoff is identically zero,

$$b_0(x) \equiv 0.$$

The price of the contract in the auxiliary market is, naturally, zero, $w_0(x,t) = 0$, and we also have $d(x) = 1$, $\delta(x,t) = 0$. By simple computations, then, we obtain that the approximation in Eq. (19) of Definition 1 collapses to:

$$w_N(x,t) = \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n(x,t), \tag{35}$$

where:

$$\begin{aligned} d_0(x,t) &= 1, \quad d_1(x,t) = -x, \quad d_2(x,t) = -\left(\mu(x,t) - x^2\right), \\ d_3(x,t) &= -\frac{\partial \mu(x,t)}{\partial t} + \mu(x,t)\left(2x - \frac{\partial \mu(x,t)}{\partial x}\right) + \frac{1}{2}\sigma^2(x,t)\left(2 - \frac{\partial^2 \mu(x,t)}{\partial x^2}\right) + x\left(\mu(x,t) - x^2\right). \end{aligned}$$

Eq. (35) is a slight generalization to the power series expansion appearing in Chapman, Long, and Pearson (1999, Proposition 3), and Wilmott (2003, p. 572). We assess the accuracy of this expansion to approximate the bond prices predicted by the CIR (1985) model, by fixing $\alpha = \alpha_0$ and $\beta = \beta_0$ and choosing them to match the average, standard deviation and first-order autocorrelation of the US overnight rate, using post-war data.

Insert Fig. 4 near here

Fig. 4 plots the percentage pricing error arising for $N = 2, 4, 6, 8$, and 10, with parameter values fixed to those displayed in the figure legend, and the initial level of the interest rate equal to 10%. A truncation of Eq. (35) based on a few terms provides a quite accurate approximation to *short maturity* bond prices, which was indeed the main purpose in Chapman, Long and Pearson (1999). Many more terms are needed for the resulting approximation to be accurate at longer maturities, as also shown by Kimmel (2008). As an example, the quality of the approximation based on only the first three terms deteriorates for $T - t \geq 3$. We now turn to an expansion based on a richer auxiliary market, i.e. one where the final payoff is not zero.

5.2.1.2 A better expansion: the Vasicek model as auxiliary pricing device

The results pertaining to the previous example can be improved, once we use a more informative auxiliary market, where the payoff of the bond is one, i.e., $b_0(x) = 1$, such that $d(x) = 0$. Consider, then, an auxiliary market where the short-term rate is as in Vasicek (1977), and is solution to Eq. (34). The solution for the bond price, denoted by w_0, is well-known as this is the simplest example of an exponential affine model. The mispricing function, δ, is now,

$$\delta(x,t;\theta_0) = (\mu(x,t) - \beta_0(\alpha_0 - x)) \frac{\partial w_0(x,t;\theta_0)}{\partial x} + \frac{1}{2}\left(\sigma^2(x,t) - \sigma_0^2\right) \frac{\partial^2 w_0(x,t;\theta_0)}{\partial x^2}, \tag{36}$$

where now $\theta_0 = [\alpha_0 \; \beta_0 \; \sigma_0]^\top$ is the nuisance parameter vector arising from the use of the misspecified Vasicek (1977) model. Note, the function summarizing the mispricing arising from the use of the auxiliary model, $\delta(x,t;\theta_0)$, has now a more complex structure than that we find for the option pricing case (see Sections 2 and 5.1). Its second component, the convexity adjustment, is now familiar, by the results in Sections 2 and 5.1. Its first term, which is new, arises because the short-term rate is obviously not a traded risk, which makes the two drifts under the risk-neutral probability, μ and μ_0, differ. In the option example dealt with in Sections 2 and 5.1, instead, the asset underlying the contract is tradable, and is expected to appreciate at an instantaneous rate of rdt, under the risk-neutral probability, independently of the evaluation model. A choice that simplifies the function δ in Eq. (36) is $\mu(x,t) = \beta_0(\alpha_0 - x)$, which we use in our numerical experiments discussed below.

Using the approximating formula in Eq. (19) of Definition 1, delivers the following approximating formula to the CIR (1985) bond price function $w(x,t)$:

$$w_N(x,t;\theta_0) = w_0(x,t;\theta_0) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t;\theta_0), \tag{37}$$

where now, $\delta_n(x,t;\theta_0) = L\delta_{n-1}(x,t;\theta_0) - x\delta_{n-1}(x,t;\theta_0)$, and $\delta_0(x,t;\theta_0) \equiv \delta(x,t;\theta_0)$, with mispricing function δ given in Eq. (36).

As in the option pricing problem of the previous section, we have a nuisance parameter vector to choose, arising through the auxiliary model. Because the two models, Vasicek (1977) and CIR (1985), both have a linear drift, we perfectly match these drifts, as anticipated, by setting α_0 and β_0 equal to the numerical values we use for the CIR (1985) α and β, i.e., $\alpha = \alpha_0$ and $\beta = \beta_0$. Accordingly, $\theta_0 \equiv \sigma_0$ is the only remaining nuisance parameter and the mispricing function δ simplifies to:

$$\delta(x,t;\sigma_0) \equiv \frac{1}{2}\left(\sigma^2 x - \sigma_0^2\right) \frac{\partial^2 w_0(x,t;\theta_0)}{\partial x^2}.$$

For a given value of the short-term rate x, we set σ_0^2 so as to equal $\sigma_0^2 = \sigma^2 x$, which corresponds to choosing $\sigma_0 = \arg\min_\sigma \delta^2(x, t; \sigma)$.

Insert Fig. 5 near here

To illustrate the performance of the resulting approximation, we set the current short-term rate to $x = 10\%$, as in the previous section. Fig. 5 plots the approximation error against time-to-maturity for different values of N, and the same parameter values of the CIR (1985) model used for the numerical analysis summarized in Fig. 4. Compared to the approximation error of the simple expansion, the approximation based on the Vasicek (1977) model works considerably better, as it only needs a few terms to achieve a quite high level of precision. Numerical results not reported here confirm that the approximation works equally well for other initial values of the short-term rate, x.

5.2.2. A nonlinear two-factor model

Similarly as for the analysis in Section 5.1.2.2, we wish to examine the robustness of the previous results to models going beyond the affine class. We consider the non-affine two-factor model proposed by Fornari and Mele (2006). In their specification, the short-term rate $r(t)$ is solution to:

$$\begin{cases} dr(t) = \beta(\alpha - r(t))\,dt + \sigma(t)\sqrt{r(t)}dW(t) \\ d\sigma(t) = \kappa(\mu - \sigma(t))\,dt + \omega\sigma(t)\,dW_\sigma(t) \end{cases} \tag{38}$$

where $W(t)$ and $W_\sigma(t)$ are Brownian motions with correlation ρ. This model is an extended version of the CIR (1985) model in Eq. (33), where the diffusion coefficient of the short-term rate is stochastic and depends on the level of the short-term rate, but also features mean-reverting stochastic volatility, through $\sigma(t)$. In this setting, the bond price is driven by two factors, $x(t) = (r(t), \sigma(t))$, and solves $Lw(x,t) - rw(x,t) = 0$, with boundary condition $w(x,T) = 1$, where L is the infinitesimal generator associated with Eqs. (38).

This model is non-affine for two reasons. First, the diffusion coefficient for the short-term rate is $\sigma\sqrt{r}$, and given σ is stochastic, the model cannot generate exponential affine bond prices. Second, even if the diffusion coefficient for the short-term was σ, instead of $\sigma\sqrt{r}$, the model would not be affine anyway, because the second equation in (38) is about the dynamics of σ, rather than σ^2.

To approximate the solution to this model, we use, again, the one-factor Vasicek (1977) model as the auxiliary one. The resulting approximation is then formally the same as that for the CIR (1985) model in Eq. (37), although the infinitesimal operator L associated to Eq. (38) is more complex, comprising the stochastic volatility terms in (38), and the mispricing function is given by:

$$\delta(r, \sigma, t; \theta_0) = (\beta(\alpha - r) - \beta_0(\alpha_0 - r)) \frac{\partial w_0(r, t; \theta_0)}{\partial r} + \frac{1}{2}\left(\sigma^2 r - \sigma_0^2\right) \frac{\partial^2 w_0(r, t; \theta_0)}{\partial r^2},$$

where θ_0 denotes the same parameter vector used in Eq. (36). As for the single factor model in Section 5.2.1.2, we set the parameter values of the Vasicek (1977) model to $\beta_0 = \beta$, $\alpha_0 = \alpha$, thereby removing the first term of $\delta(x, t; \theta_0)$. We choose the remaining nuisance parameter, σ_0, so as to match the diffusion coefficients for the Vasicek (1977) model to that for the short-term rate in Eqs. (38), $\sigma_0^2 = \sigma^2 r$.

Insert Table 5 near here

Insert Fig. 6 near here

Table 5 reports the percentage approximation errors for a variety of levels of the short-term rate, r, and volatility σ, obtained using four leading terms, as for all the numerical experiments of this section, and for time-to-maturity equal to six months, and one, two, three and five years. Our theoretical benchmarks are bond prices calculated through Monte Carlo integration. Our percentage errors have the tendency to increase, as time-to-maturity increases, although they are quite small. Even for five year bonds, the percentage errors are around 0.80% – 1.10%, with the highest error being -1.37% (occurring when $r = 6\%$ and $\sigma^2 = 0.60\%$). Finally, we explore the accuracy of our methods in correspondence of varying degrees of approximations and expiration dates. We compute pricing errors arising when the number of corrective terms equals $N = 1$, 2, 3, 4, and 5, and for all the values of r and σ^2 considered in Table 5. We find that increasing the number of corrective terms reduces the approximation errors for maturities less than three years, but not for larger maturities, where we would presumably need significantly more terms to obtain results of the same quality as that relating to lower maturities. Fig. 6 reports results for the case $r = 6\%$ and $\sigma^2 = 0.50\%$, and maturities up to three years, which are quite representative of the results relating to all the remaining combinations of r and σ^2. The figure shows that the percentage pricing errors monotonically decrease as N increases, and that the approximation is highly precise when $N = 5$.

6. Extensions

This section outlines two possible extensions where our approach could be applied, which relate to: (i) option evaluation when the underlying asset returns follow a jump-diffusion model with stochastic volatility (in Section 6.1); and (ii) pricing of barrier options in the presence of stochastic volatility (in Section 6.2).

6.1. Jumps

We consider a model where stock returns have both stochastic volatility and jumps: under the risk-neutral probability, the asset price evolves according to,

$$\frac{dS(t)}{S(t)} = \left(r - \lambda\left(S(t), v(t)\right)\bar{j}\right)dt + \sqrt{v(t)}dW(t) + jdN(t), \tag{39}$$

where the stochastic variance, $v(t)$, is still solution to Eq. (2). The jump component consists of: (i) $N(t)$, a Cox process with a bounded intensity function given by $\lambda(S, v)$, and (ii) j, a random variable with probability measure on $[-1, \infty)$, density p, and expectation $\bar{j}$ [see, e.g., Jacod and Shiryaev (1987, p. 142-146), for a succinct discussion of diffusion processes with jumps]. For example, we may take $j + 1$ to be log-normal distributed, an assumption we maintain in the remainder of the section. Special cases of this model include Merton (1976), where both volatility and jumps intensity are constant and equal to σ_0 and, say, $\underline{\lambda}$, respectively; and Broadie, Chernov and Johannes (2009), where $\xi = 1/2$ in Eq. (2). Yang (2006, Section 6) considers a number of models with jumps, including the one in this section, and illustrates how to use his expansion to deal with jumps.

The price of a European call is unknown in the general setting of this section. Our method can still be used to approximate this unknown price, $w(x, v, t)$, say. We implement our expansion through the infinitesimal generator for the jump-diffusion model of Eqs. (39) and (2):

$$L^J w(x, v, t) = Lw(x, v, t) + \lambda(x, v)\int_{-1}^{\infty}\left[w(x(1+j), v, t) - w(x, v, t)\right]p(dj), \tag{40}$$

where L is as in Eq. (4). We may choose Merton's model as an auxiliary device, and then show, formally, that $\Delta w(x, v, t; \sigma_0)$, defined as the difference between the unknown price, and Merton's, $w^{\mathrm{m}}(x, t; \sigma_0)$, satisfies:

$$0 = L^J \Delta w(x, v, t; \sigma_0) - r\Delta w(x, v, t; \sigma_0) + \delta_{J1}(x, v, t; \sigma_0) + \delta_{J2}(x, v, t; \sigma_0), \tag{41}$$

where $\Delta w(x, v, T) = 0$, and δ_{J1} is the same mispricing function as δ in Eq. (7), but with w^{m} replacing w^{bs}; and, finally, δ_{J2} is an additional mispricing term, arising due to the presence of jumps in Eq. (39):

$$\delta_{J2}(x, v, t; \sigma_0) \equiv \left(\lambda(x, v) - \underline{\lambda}\right)\int_{-1}^{\infty}\left[w^{\mathrm{m}}(x(1+j), t; \sigma_0) - w^{\mathrm{m}}(x, t; \sigma_0)\right]p(dj).$$

Note that this term is, obviously, zero, once we assume the intensity for the true market is constant and equal to that for the auxiliary, Merton's market.

By the usual arguments, we can represent the solution $\Delta w(x,v,t)$ to Eq. (41) in terms of a conditional moment, which can be expanded, leaving:

$$w(x,v,t) = w^{\mathrm{m}}(x,v,t;\sigma_0) + \sum_{n=0}^{\infty} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n^J(x,v,t;\sigma_0),$$

where $\delta_{n+1}^J(x,v,t;\sigma_0) = L^J \delta_n^J(x,v,t;\sigma_0) - r\delta_n^J(x,v,t;\sigma_0)$, and $\delta_0^J \equiv \delta_{J1} + \delta_{J2}$. The theoretical validity of this expansion could be proved by extending the arguments given in Appendix A, so as to include generators having the same form as L^j in Eq. (40).

6.2. Barriers

This section explores how to approximate the price of derivatives with path-dependent payoffs through our method. We consider barrier options. An *out* type barrier option promises a payoff $b(S(T))$, provided the stock price $S(t)$ does not hit a given value before maturity T: for up-and-out contracts, this value is an upper barrier, $\bar{B}$, say, and for down-and-out contracts, it is a lower barrier. Instead, barrier options of the *in* type promise a payoff $b(S(T))$, only if the stock price hits a given lower barrier (for down-and-in options) or a given upper barrier (for up-and-in options).

In general, the price of barrier options is unknown, and we discuss how our method can be extended to deal with this type of option when the underlying asset price follows the stochastic volatility model of Section 2, Eqs. (1) and (2) with $\xi = 1/2$—the Heston's (1993) model. Consider the basic up-and-out contract, and let $w_{\mathrm{O}}(x,v,t)$ be its price when the stock price is x and the variance is v. The price of an in-contract follows by the in-out parity and equals: $w(x,v,t) - w_{\mathrm{O}}(x,v,t)$, where $w(x,v,t)$ is the price of a plain vanilla option paying off $b(x) = \max\{x-K, 0\}$ at T. By arguments similar to those in Shreve (2004, Chapter 7), w_{O} satisfies $Lw_{\mathrm{O}}(x,v,\tau) - rw_{\mathrm{O}}(x,v,\tau) = 0$ for $(x,v,\tau) \in (0,\bar{B}) \times \mathbb{R}_{++} \times [t,T)$, where L is the infinitesimal generator in Eq. (4) with $\xi = 1/2$, subject to the following boundary conditions:

$$\begin{cases} w_{\mathrm{O}}(x,v,T) = b(x) & \text{for } (x,v) \in \mathcal{D} \equiv (0,\bar{B}) \times \mathbb{R}_{++} \\ w_{\mathrm{O}}(\bar{B},v,\tau) = 0 & \text{for } (v,\tau) \in \mathbb{R}_{++} \times [t,T) \end{cases} \tag{42}$$

Under regularity conditions, such as those in Shreve (2004, Chapter 7), we have that:

$$w_{\mathrm{O}}(x,v,t) = e^{-r(T-t)} \mathbb{E}_{x,v,t}\left[b(S(T))\, \mathbb{I}_{T \le \tau_u}\right],$$

where $\tau_u \equiv \inf \left\{ \tau : S(\tau) = \bar{B} \right\}$. While no analytical solutions are known for $w_{\text{O}}(x, v, t)$, solutions are instead available under the assumption the stock price is as in the Black-Scholes market (see, e.g., Eq. (7.3.20), p. 307, in Shreve, 2004). We can use the barrier option price from the Black-Scholes market as an auxiliary device, $w_{\text{O}}^{\text{bs}}(x, t; \sigma_0)$ say, similarly to what we have done in the previous sections, and show the unknown price is given by:

$$w_{\text{O}}(x, v, t) = w_{\text{O}}^{\text{bs}}(x, t; \sigma_0) + \mathbb{E}_{x,v,t} \left[\int_t^{\min\{\tau_u, T\}} e^{-r(s-t)} \delta_{\text{O}}(S(s), v(s), s; \sigma_0) \, ds \right], \tag{43}$$

where the mispricing function δ_{O} is as δ in Eq. (7), but with w_{O}^{bs} replacing w^{bs}. Unfortunately, expanding the expectation in Eq. (43) similarly as we did with Eqs. (8) and (9), is challenging, as the stopping time τ_u in the upper integration limit is obviously random. We mitigate this issue as follows. Define the Arrow-Debreu state price density $G(x', v', s; x, v, t)$, i.e. the value as of time t, in state (x, v), of a unit of numéraire at time $s > t$, should the future state lie in a neighborhood of (x', v'). Then, Eq. (43) can be written as:

$$w_{\text{O}}(x, v, t) = w_{\text{O}}^{\text{bs}}(x, t; \sigma_0) + \int_t^T \int_{\mathcal{D}} G\left(x', v', s; x, v, t\right) \delta_{\text{O}}\left(x', v', s; \sigma_0\right) dx' dv' ds, \tag{44}$$

where $\mathcal{D}$ is as in the first of Eqs. (42). The function G, also known as Green's function, satisfies the same partial differential equation as $w_{\text{O}}(x, v, t)$ in the backward variables (x, v, t), with the boundary condition in the first of Eqs. (42) replaced by $G(x', v', T; x, v, T) = \text{Dir}(x' - x)$, for all v, where $\text{Dir}(\cdot)$ is the Dirac's function. Solutions for Green's functions relating to affine models with barriers are known in closed-form (see Chapter 3 in Chen, 1996), and can be used to compute w_{O} from Eq. (44).

7. Conclusion

We have developed a novel method to approximate the price of derivative assets in the context of multifactor continuous-time models. The idea underlying our approach is quite simple: given a model with no closed-form solution, we select an "auxiliary" model, which has a closed-form solution, and expand the unknown price around the auxiliary one. We apply this method to asset pricing problems spanning multifactor models of the yield curve and models of stochastic volatility option pricing, and show that a truncation of our expansions up to a few terms is quite accurate. Naturally, our approach does not require any simulation, and once implemented, requires a small amount of computational time.

Our method can be used in a variety of related contexts such as those pertaining to pricing exotic contracts through local volatility models *à la* Dupire (Dupire, 1994; Derman and Kani, 1997). Prices of these exotic derivatives predicted by local volatility models are unknown in closed-form. They are typically computed through either simulations of the asset price underlying the derivative, or closed-form approximations based on quite specific parametric assumptions (e.g., Hagan, Kumar, Lesniewski, and Woodward, 2002). But local volatility models are, simply, those where the volatility of the asset return is a function of the underlying asset price and calendar time, calibrated through liquid options data—a more general version of the CEV model analyzed in Section 5.1.1. Our approach, therefore, is a viable alternative to approximate the solution to these models, exactly as to any other continuous-time model without closed-form solution.

A second example where our approach has a potential is the estimation and calibration of asset pricing models. Estimation of continuous-time models given, say, option or bond prices, typically centers around inverting pricing formulae for the model's parameters. This numerical task can be performed via our approximating pricing formulae. More generally, our methods can be used to generate closed-form approximations to conditional moments of the state variables in asset pricing models, which can then be used as inputs in the implementation of Generalized Method of Moments type of estimators.

Appendix

A. The expansion

We develop theoretical properties of the asset price approximation formula of Section 3, as given in Definition 1. Given the asset price $w(x,t)$ that solves Eq. (13), we provide conditions under which we can state error bounds for a fixed approximation, and also establish that $w_N(x,t) \to w(x,t)$ as $N \to \infty$, where $w_N(x,t)$ is our approximation to the asset price in Definition 1. The approach of this appendix relies heavily on previous work that Schaumburg (2004) developed in a different context.

A.1. Properties

The next proposition establishes an error bound for the approximation, which holds for any fixed approximation order $N \geq 1$. We have:

Proposition A.1. *Assume that μ, σ^2 are time-homogeneous and that $\mu, \sigma^2 \in C^{2N}(\mathbb{R}^d)$ and $d, \delta \in C^{2(N+1)}(\mathbb{R}^d)$. Then w_N given in Definition 1 satisfies:*

$$|w(x,t) - w_N(x,t)| \leq E_N(x) \frac{(T-t)^{N+1}}{(N+1)!}, \text{ for all } (x,t) \in \mathbb{R}^d \times [0,T],$$

where

$$\begin{aligned} E_N(x) \quad \equiv \quad & \sup_{0\leq s\leq T} \mathbb{E}_{x,t}\left[\left\|L^{N+1} d(x(s))\right\|\right] + \sup_{0\leq s\leq T} \mathbb{E}_{x,t}\left[\left\|L^{N+1}\delta(x(s),s)\right\|\right] \\ & + \sup_{0\leq s\leq T} \mathbb{E}_{x,t}\left[\left\|\partial^{N+1}\delta(x(s),s)/\partial s^{N+1}\right\|\right], \end{aligned}$$

and L is the infinitesimal generator of $x(t)$. In particular, if μ, σ^2, d, and δ are polynomially bounded, then:

$$E_N(x) \leq (1 + \|x\|^{q_N}) e^{c_N T},$$

for some constants c_N and q_N.

This result tells us that in great generality, the error decreases at a geometric rate uniformly over (x,t) in any compact interval as N increases. Florens-Zmirou (1989, Lemma 1) and Aït-Sahalia (2002) develop similar error bounds for approximations of conditional moments of diffusion processes in different contexts.

Proposition A.1 is not informative about the asymptotic behavior of the error terms. In particular, we have not been able to establish bounds on q_N and c_N as N increases. To deal with the error terms for large N, we rely, instead, on results from the literature on operator theory. First, we introduce some additional notation and definitions. First, for a given operator A, we define its spectrum and resolvent as

$$\sigma(A) = \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not a bijection}\} \quad \text{and} \quad R_\lambda(A) = (\lambda - A)^{-1}, \ \lambda \in \sigma(A).$$

Second, we introduce a function space $\mathcal{H}$, which is equipped with some function norm $\|\cdot\|_{\mathcal{H}}$. We impose the following conditions on the spectrum and resolvent of the infinitesimal operator L of $\{x(t)\}$ in order to show that our power series expansion converges:

(A.1) Given three constant $m, \omega > 0$ and $M \in (e^{-1}, \infty)$, the infinitesimal operator L given in Eq. (12) satisfies

$$\sigma(L) \subset \bar{\sigma} \equiv \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| > \pi/2 + m\},$$

and its resolvent satisfies $\|R_\lambda(L)\| \leq M/|\lambda|$ for $\lambda \in \mathbb{C} \backslash \bar{\sigma}(L)$.

(A.2) There exists $\bar{\tau} > 0$ and $\phi_\delta, \phi_d \in \mathcal{H}$ such that $d : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ and $\delta : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ defined in Eqs. (16)-(17) satisfy:

$$\mathbb{E}\left[\phi_\delta\left(x\left(\bar{\tau}\right)\right)|x\left(0\right)=x\right] = \delta\left(x,\bar{\tau}\right) \text{ and } \mathbb{E}\left[\phi_d\left(x\left(\bar{\tau}\right)\right)|x\left(0\right)=x\right] = d\left(x\right).$$

Moreover, the function $t \mapsto \delta(x,t)$ is analytic uniformly in $\|\cdot\|_{\mathcal{H}}$ and, in Eq. (13), $R(x,t) = R(x)$, with $\sup_x |R(x)| < \infty$.

Condition (A.1) relates to the infinitesimal operator and requires that its spectrum is within $\bar{\sigma}$. The second condition, (A.2), imposes conditions on the two functions d and δ, the conditional moments of which, we wish to expand about. It basically requires that each of the two functions can be matched through conditional moments. Both assumptions are abstract, and not easily verified for specific models. The following proposition develops more primitive conditions for (A.1) to hold that are typically met in many diffusion models.

Proposition A.2. *The generator L satisfies (A.1), under the following conditions:*

(i) *L has a transition density $p_t(y|x)$ with respect to Lebesgue measure.*

(ii) *L has an invariant measure π satisfying: $\pi(x)\,p_t(y|x) = \pi(y)\,p_t(x|y)$.*

Conditions (i)-(ii) in Proposition A.2 are satisfied by many standard processes used in finance. Most diffusion models have a transition density, while the second condition is a generalization of time-reversibility. In particular, if the process is univariate and stationary, it is necessarily time-reversible and therefore satisfies the second condition. In conclusion, condition (A.1) holds under fairly weak conditions.

We are unaware of more primitive conditions for (A.2) to hold. For one example where (A.2) is not satisfied, we refer to Schaumburg (2004, Example 1), who also provides additional discussion about this condition.

The theoretical foundations to the approximation in Definition 1 are in the following proposition:

Proposition A.3. *Let $\{x(t)\}_{t\geq 0}$ be a homogeneous diffusion process with infinitesimal operator L. Assume that L satisfies (A.1), the function $R : R^d \mapsto R_+$ is analytic, and d and δ satisfy (A.2). Then for any $|t-T| < \bar{\tau}/(Me)$, where M and $\bar{\tau}$ are given in (A.1)-(A.2), we have:*

(i) *The following equality holds:*

$$\mathbb{E}\left[e^{-\int_t^u R(x(s),s)ds}\delta(x(u),u)\Big| x(t)=x\right] = \sum_{n=0}^{\infty}\frac{(u-t)^n}{n!}\delta_n(x,t),$$

and

$$\int_0^T \mathbb{E}\left[e^{-\int_t^u R(x(s),s)ds}\delta(x(u),u)\Big| x(t)=x\right]du = \sum_{n=0}^{\infty}\frac{(T-t)^{n+1}}{(n+1)!}\delta_n(x,t),$$

where $\delta_0 \equiv \delta$ and

$$\delta_{n+1}(x,t) = L\delta_n(x,t) - R(x,t)\,\delta_n(x,t), \quad n \geq 0,$$

and similarly for the function d.

(ii) *The approximation w_N given in Definition 1 satisfies, for all $|t-T| < \bar{\tau}/(Me)$,*

$$\|w_N(\cdot,t) \to w(\cdot,t)\|_{\mathcal{H}} \to 0, \quad N \to \infty.$$

To establish Propositions A.1.-A.3., we introduce the following Cauchy problem:

$$-\frac{\partial w\left(x,t\right)}{\partial t}=Aw\left(x,t\right)+b\left(x,t\right), \tag{A1}$$

for $(x,t)\in\mathbb{R}^{d}\times[0,T]$, where A is a general linear operator, and w satisfies the boundary condition:

$$w\left(x,T\right)=c\left(x\right).$$

We define the semigroup associated with A (see, e.g., Pazy, 1983) as:

$$U\left(t\right)=e^{tA},$$

and let $\mathcal{D}\left(A\right)$ denote the domain of A defined as the set of functions for which

$$A\phi\left(x,0\right)=\lim_{t\to 0}\frac{U\left(t\right)\phi\left(x,t\right)-\phi\left(x,0\right)}{t}$$

is well-defined. We note that Eq. (15) with time-homogeneous coefficients and $R\left(x,t\right)=R\left(x\right)$ can be cast in the same format as Eq. (A1), with

$$A\phi\left(x,t\right)=L\phi\left(x,t\right)-R\left(x\right)\phi\left(x,t\right), \tag{A2}$$

$$b\left(x,t\right)=\delta\left(x,t\right),\quad c\left(x\right)=d\left(x\right). \tag{A3}$$

With this specification of A, we have:

$$U\left(t\right)\phi\left(x,t\right)=\mathbb{E}\left[\exp\left(-\int_{0}^{t}R\left(x\left(s\right)\right)ds\right)\phi\left(x\left(t\right),t\right)\bigg|\,x\left(0\right)=x\right].$$

It is easily seen that the solution to the inhomogenous problem of Eq. (A1) can be represented as:

$$w\left(x,t\right)=U\left(T-t\right)c\left(x\right)+\int_{0}^{T-t}U\left(s\right)b\left(x,s\right)ds. \tag{A4}$$

Next, we obtain an approximate solution, w_{N}, through a series expansion of $U\left(t\right)$. In particular, we wish to give conditions under which $U\left(t\right)$ satisfies:

$$U\left(t\right)\phi(x)=e^{tA}\phi(x)=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}A^{n}\phi(x), \tag{A5}$$

in which case we define the approximation:

$$U_{N}\left(t\right)\phi(x)=\sum_{n=0}^{N}\frac{t^{n}}{n!}A^{n}\phi(x). \tag{A6}$$

Suppose that the function $t\mapsto\phi(x,t)$ is analytic for all x, such that

$$\phi(x,t)=\sum_{k=0}^{\infty}\frac{t^{k}}{k!}B^{k}\phi(x,0),\quad B\phi\left(x,t\right)\equiv\frac{\partial\phi(x,t)}{\partial t}.$$

Then,

$$U(t)\,\phi(x,t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\phi(x,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^{n+k}}{n!k!}A^nB^k\phi(x,0)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(A+B\right)^n\phi(x,0).$$

Thus, we shall use the following approximation:

$$U_N(t)\,\phi(x,t)=\sum_{n=0}^{N}\frac{t^n}{n!}\left(A+B\right)^n\phi(x,0). \tag{A7}$$

By plugging the two approximations in Eqs. (A6) and (A7) into Eq. (A4), we obtain:

$$w_N(x,t)=\sum_{n=0}^{N}\frac{(T-t)^n}{n!}A^nc(x)+\sum_{n=0}^{N}\frac{(T-t)^{n+1}}{(n+1)!}\left(A+B\right)^n b(x,0).$$

The following proposition provides an upper bound to the approximation error for any given $N\geq 0$:

Proposition A.4. *Assume that the two functions $c(x)$ and $b(x,t)$ both belong to $D\left(A^{N+1}\right)$ and $t\mapsto b(x,t)$ is $N+1$ times differentiable. Then the approximation error satisfies:*

$$|w(x,t)-w_N(x,t)|\leq E_N(x)\,\frac{(T-t)^{N+1}}{(N+1)!},\ \text{for all}\ (x,t)\in\mathbb{R}^d\times[0,T],$$

where

$$E_N(x)=\sup_{0\leq s\leq T}\left|A^{N+1}U(s)\,b(x,s)\right|+\sup_{0\leq s\leq T}\left|A^{N+1}U(s)\,c(x)\right|+\sup_{0\leq s\leq T}\left|B^{N+1}U(s)\,b(x,s)\right|.$$

Next, we establish conditions under which the error bound established in Proposition a.4. vanishes as $N\to\infty$. Intuitively, this result will go through if the power expansion in Eq. (A5) is valid. If the operator A was bounded, $\|A\|<\infty$, then the expansion would trivially hold. However, the infinitesimal operator is unbounded and, instead, we have to impose additional restrictions to verify the validity of the expansion. We impose restrictions in terms of the operator's spectrum and resolvent so as to ensure that A is a so-called analytic operator. In turn, these restrictions imply that the power expansion is valid. We have:

Proposition A.5. *Assume:*

(i) *For some $\delta,\omega>0$ and $M\in\left(e^{-1},\infty\right)$:*

$$\sigma(A)\subset\bar{\sigma}(A)\equiv\{\lambda\in\mathbb{C}:|\arg(\lambda-\omega)|>\pi/2+\delta\},$$

and $\|R_\lambda\|\leq M/|\lambda|$ for $\lambda\in C\backslash\bar{\sigma}(A)$.

(ii) *The functions $b(\cdot,\bar{\tau})$ and $c(\cdot)$ both lie in $U(\bar{\tau})H$ for some $\bar{\tau}>0$, i.e., there exists $\phi_b,\phi_c\in H$ such that*

$$U(\bar{\tau})\,\phi_b(x)=b(x,\bar{\tau})\quad\text{and}\quad U(\bar{\tau})\,\phi_c(x)=c(x).$$

Moreover, $t\mapsto b(x,t)$ is analytic for all x.

Then for all $|t-T|<\bar{\tau}/(Me)$, with M and $\bar{\tau}$ as given in (i) and (ii):

$$\|w_N(\cdot,t)\to w(\cdot,t)\|_{\mathcal{H}}\to 0,\quad N\to\infty.$$

Finally, note that the previous results only relate to time-homogeneous diffusions. It would be of interest to derive results that also hold for time-inhomogeneous diffusions, where drift and diffusion functions vary over time t. Heuristically, this task is tantamount to analyzing systems such as,

$$-\frac{\partial w(x,t)}{\partial t} = A(t)\, w(x,t) + b(x,t),$$

where the linear operator $A(t)$ is, now, time-inhomogenous. There are still very few foundational results on the analyticity of this class of operators [for a few preliminary results, see Chapter 5 in Pazy (1983)]. We have been unable to study how the previous propositions hold within such a more general setting.

A.2. Proofs

The regularity conditions underlying Theorem 1 are:

(A.3) The two solutions, $w(x,t)$ and $w_0(x,t)$, exist and belong to $\mathcal{C}^{2,1}\left(\mathbb{R}^d \times [0,T]\right)$. Furthermore, for some $C, q > 0$:

$$|w(x,t)| + |w_0(x,t)| \le C\left(1 + \|x\|^q\right),$$

for all $x \in \mathbb{R}^d$ and $t \in [0,T]$.

(A.4) The functions μ, μ_0, σ, and σ_0 are time-homogeneous, continuous, and satisfy, for some $C > 0$:

$$\|\mu(x)\| + \|\mu_0(x)\| + \|\sigma(x)\| + \|\sigma_0(x)\| \le C\left(1 + \|x\|\right),$$

for all $x \in \mathbb{R}^d$ and $t \in [0,T]$.

(A.5) The function $R(x) \ge 0$, is continuous and satisfies the same growth condition as w and w_0.

Proof of Theorem 1. Since w and w_0 are well-defined solutions to their partial differential equations, the difference, $\Delta w = w - w_0$, is a well-defined solution to the partial differential equation (15). We then need to verify that the conditions for the Feynman-Kac formula to hold are satisfied. We use the conditions of Karatzas and Shreve (1991, Theorem 5.7.6): First, given condition (A.3), we have that $\Delta w(x,t)$ belongs to $\mathcal{C}^{2,1}\left(\mathbb{R}^d \times [0,T]\right)$ and satisfies $|\Delta w(x,t)| \le C\left(1 + \|x\|^q\right)$. Second, $\delta(x,t)$ and $R(x)$ are continuous and satisfy the same growth condition as Δw, due to conditions (A.4)-(A.5). Finally, the drift and diffusion terms μ and σ satisfy the necessary continuity and growth conditions. All the conditions in Karatzas and Shreve (1991, Theorem 5.7.6) are therefore met. ■

Proof of Proposition A.1. This is a direct consequence of Proposition A.4, since under the conditions, d and δ clearly are in the domain of L. Moreover, with A, b and c defined as in Eqs. (A2)-(A3),

$$A^{N+1} U(s)\, \delta(x,s) = \mathbb{E}\left[A^{N+1} \delta(x(s), s)\, | x(0) = x\right],$$

and similarly for the other term of $E_N(x)$ defined in Proposition A.4. This establishes the stated error bound. Finally, note that under the polynomial bounds, $\left\|A^{N+1} c(x(s), s)\right\| \le C_N\left(\|x(s)\|^{q_N} + 1\right)$ for some $C_N > 0$ and $q_N \ge 1$, and we then apply Friedman (1975, Theorems 5.2.2-5.2.3) to obtain

$$\mathbb{E}\left[\left|A^{N+1}\delta(x(s), s)\right| |x(0) = x\right] \le C_N\left(\mathbb{E}\left[\|x(s)\|^{q_N} |x(0) = z\right] + 1\right) \le \left(1 + \|x\|^{q_N}]\right) e^{c_N s},$$

for some constants $c_N, q_N > 0$. Similarly for the other term in $E_N(x)$. ■

Proof of Proposition A.2. It follows from Schaumburg (2004, Lemma 2.2) that L satisfies (A.1) under the two conditions stated in the proposition. ■

Proof of Proposition A.3. The result will follow from Proposition A.5., once we ascertain that that (A.1)-(A.2) imply (i)-(ii) of that Proposition. It is easily seen that, given the form of $U(t)$ for the choice of A given in Eq. (A2), condition (A.2) implies (ii). To verify (i), we apply Pazy (1983, Theorem 3.2.1) which will yield the desired result if we can show that the domain of L is contained in that of the operator F, defined as $F\phi(x,t) = R(x)\phi(x)$, with $\mathcal{D}(L) \subset \mathcal{D}(F)$, and that,

$$\|F\phi\|_{\mathcal{H}} \leq c_1 \|L\phi\|_{\mathcal{H}} + c_2 \|\phi\|_{\mathcal{H}},$$

for some constants c_1 and c_2. But clearly, $\mathcal{D}(F)$ contains all twice-differentiable functions and the above inequality follows by the fact that $\sup_x |R(x)| < \infty$. ■

Proof of Proposition A.4. By definition,

$$U(t)\phi(x,t) = \phi(x,t) + \int_0^t AU(s)\phi(x,s)\,ds.$$

Using this identity iteratively, we obtain

$$\begin{aligned}
U(t)\phi(x,t) &= \phi(x,t) + \int_0^t AU(t_1)\phi(x,t)\,dt_1 \\
&= \phi(x,t) + \int_0^t A\left[\phi(x,t) + \int_0^{t_1} U(t_2)\phi(x,t_2)\right] dt_2 dt_1 \\
&= \phi(x,t) + tA\phi(x,t) + \int_0^t \int_0^{t_1} AU(t_2)\phi(x,t_2)\,dt_2 dt_1 \\
&\vdots \\
&= \sum_{n=0}^{N} \frac{t^n}{n!} A^n \phi(x,t) + E_N(x,t),
\end{aligned}$$

where

$$E_N(x,t) = \frac{1}{N!} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{N+1}} A^{N+1} U(t_{N+1})\phi(x,t_{N+1})\,dt_{N+1} \cdots dt_1.$$

The approximation error $E_N(x,t)$ is bounded by:

$$\begin{aligned}
|E_N(x,t)| &= \frac{1}{N!} \left| \int_0^t \int_0^{t_1} \cdots \int_0^{t_{N+1}} A^{N+1} U(t_{N+1})\phi(x,t_{N+1})\,dt_{N+1} \cdots dt_1 \right| \\
&\leq \frac{1}{N!} \int_0^t (t-s)^N \left| A^{N+1} U(s)\phi(x,s) \right| ds \\
&\leq \sup_s \left| A^{N+1} U(s)\phi(x,s) \right| \times \frac{1}{(N+1)!} t^{N+1}.
\end{aligned}$$

Next, by an N-th order Taylor expansion of b, there exists $\bar{s} \in [0,s]$ such that

$$b(x,s) - \sum_{k=0}^{N} \frac{t^k}{k!} B^k b(x,0) = \frac{t^{N+1}}{(N+1)} B^{N+1} b(x,\bar{s}).$$

Using these results,

$$
\begin{aligned}
&|w(x,t) - w_N(x,t)| \\
&\leq |[U - U_N](T-t)\,c(x)| + \int_0^{T-t} |[U-U_N](s)\,b(x,s)|\,ds + \int_0^{T-t} |U(s)[b - b_N](x,s)|\,ds \\
&\leq \frac{(T-t)^{N+1}}{(N+1)!} \sup_s \left|A^{N+1} U(s)\,c(x)\right| + \frac{(T-t)^{N+2}}{(N+2)!} \sup_s \left|A^{N+1} U(s)\,b(x,s)\right| \\
&+ \frac{(T-t)^{N+2}}{(N+2)!} \sup_s \left|B^{N+1} U(s)\,b(x,s)\right|. \quad \blacksquare
\end{aligned}
$$

Proof of Proposition A.5. We apply Pazy (1983, Theorem 2.5.2) to obtain that the range of $U(t)$ is dense in $\mathcal{D}(A^\infty)$ and, hence, in $\mathcal{H}$ under (i). Proposition A.4. supplies an upper bound to the approximation, for a given N. By the same arguments as in Schaumburg (2004, Proof of Theorem 2.1), we obtain that

$$
\left\|(T-t)^{N+1} A^{N+1} U(s)\,c\right\| \to 0, \qquad \left\|(T-t)^{N+1} A^{N+1} U(s)\,b(\cdot,s)\right\| \to 0,
$$

as $N \to \infty$ for all $(T-t) < \bar{\tau}/(Me)$. Moreover, we have that for an analytical function, $(T-t)^{N+1} B^{N+1} b(\cdot,s) \to 0$. By dominated convergence, then,

$$
\left\|(T-t)^{N+1} U(s)\,B^{N+1} b(\cdot,s)\right\| \to 0.
$$

Hence, the bound goes to zero as $N \to \infty$. $\blacksquare$

B. Equivalence between moment and density expansions

We prove the equality stated in Eq. (27). In the process, we also obtain a direct representation of the difference between the conditional densities of the true and the auxiliary model, the "transition discrepancy." First, note that the two transition densities solve the backward Kolmogorov equation:

$$
Lp(y,T|x,t) = 0, \qquad L_0 p_0(y,T|x,t) = 0,
$$

with boundary conditions $p(y,T|x,T) = p_0(y,T|x,T) = \mathrm{Dir}(y-x)$, where $\mathrm{Dir}(\cdot)$ is Dirac's delta function. Using the arguments in Section 3, it is easily seen that the transition discrepancy, Δp, is solution to:

$$
L\Delta p(y,T|x,t) + \tilde{\delta}(y,T|x,t) = 0,
$$

with boundary condition $\Delta p(y,T|x,T) = 0$, where the adjustment term $\tilde{\delta}(y,T|x,t)$ is given by:

$$
\tilde{\delta}(y,T|x,t) = (L - L_0)\,p_0(y,T|x,t) = \Delta\mu(x,t)\,\frac{\partial p_0(y,T|x,t)}{\partial x} + \frac{1}{2}\Delta\sigma^2(x,t)\,\frac{\partial^2 p_0(y,T|x,t)}{\partial x^2}.
$$

By the Feynman-Kac representation theorem,

$$
\Delta p(y,T|x,t) = \int_t^T \mathbb{E}_{x,t}[\tilde{\delta}(y,T|x(s),s)]ds. \tag{B1}
$$

Substituting the right-hand side of Eq. (B1) back into Eq. (26),

$$\begin{aligned} w(x,t) &= w_0(x,t) + \int_{\mathbb{R}^d} b(y)\, \Delta p(y,T|x,t)\, dy \\ &= w_0(x,t) + \int_t^T \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(y)\, \tilde{\delta}(y,T|z,s)\, p(z,s|x,t)\, dy dz \right] ds. \end{aligned} \tag{B2}$$

Finally, using that

$$\frac{\partial^k w_0(x,t)}{\partial x^k} = \int_{\mathbb{R}^d} b(y)\, \frac{\partial^k p_0(y,T|x,t)}{\partial x^k} dy, \quad k \geq 0,$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(y)\, \tilde{\delta}(y,T|z,s)\, p(z,s|x,t)\, dy dz &= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} b(y)\, \frac{\partial p_0(y,T|z,s)}{\partial z} dy \right] \Delta\mu(z,s)\, p(z,s|x,t)\, dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} b(y)\, \frac{\partial^2 p_0(y,T|z,s)}{\partial z^2} dy \right] \Delta\sigma^2(z,t)\, p(z,s|x,t)\, dz \\ &= \int_{\mathbb{R}^d} \Delta\mu(z,s)\, \frac{\partial w_0(z,s)}{\partial x} p(z,s|x,t)\, dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \Delta\sigma^2(z,t)\, \frac{\partial w_0^2(z,s)}{\partial x^2} p(z,s|x,t)\, dz \\ &= \mathbb{E}_{x,t}\left[\delta(x(s),s) \right], \end{aligned}$$

where δ is as in Eq. (17). Note that this representation of the transition discrepancy, in terms of a conditional moment, gives rise to an alternative approximation scheme. Precisely, the right-hand side of Eq. (B1) can be approximated by

$$\Delta p_N(y,T|x,t) = \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} L\tilde{\delta}(y,T|x,t).$$

Plugging the previous expression into the integral in Eq. (B2), we obtain

$$\tilde{w}_N(x,t) = w_0(x,t) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \int_{\mathbb{R}^d} b(y)\, L^n \tilde{\delta}(y,T|x,t)\, dy.$$

However, as noted in the main text, this type of approximation involves the computation of N d-dimensional Riemann integrals, $\int_{\mathbb{R}^d} b(y)\, L^n \tilde{\delta}(y,T|x,t)\, dy$, $n = 1, \cdots, N$.

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Figures

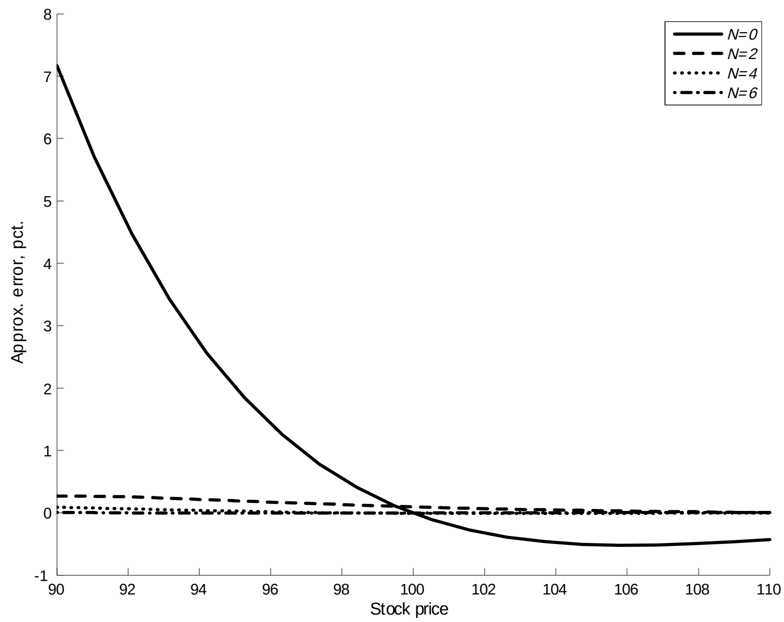

Fig. 1. Percentage errors from approximating option prices predicted by the CEV model in Eq. (28) using Black-Scholes (1973) model as auxiliary device with N corrective terms. The strike price is $K = 100$, time-to-maturity is three months, and parameter values are $\sigma_{\text{cev}} = 0.10 \cdot K^{1-\gamma}$, $\gamma = \frac{1}{2}$, $r = 5\%$. For each stock price level x, the Black-Scholes (1973) volatility is set equal to $\sigma_{\text{cev}} x^{\gamma-1}$.

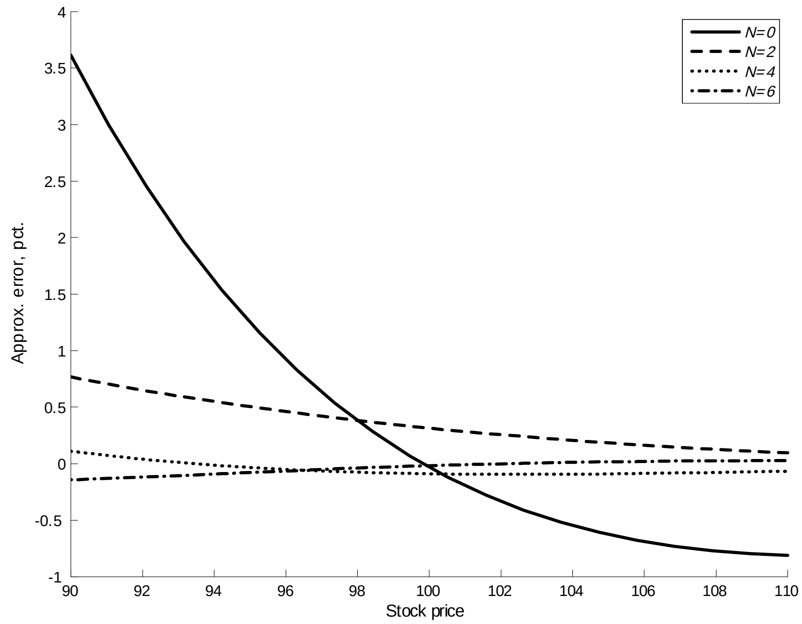

Fig. 2. Percentage errors from approximating option prices predicted by the CEV model in Eq. (28) using Black-Scholes (1973) model as auxiliary device with N corrective terms. The strike price is $K = 100$, time-to-maturity is one year, and parameter values are $\sigma_{\text{cev}} = 0.10 \cdot K^{1-\gamma}$, $\gamma = \frac{1}{2}$, $r = 5\%$. For each stock price level x, the Black-Scholes (1973) volatility is set equal to $\sigma_{\text{cev}} x^{\gamma-1}$.

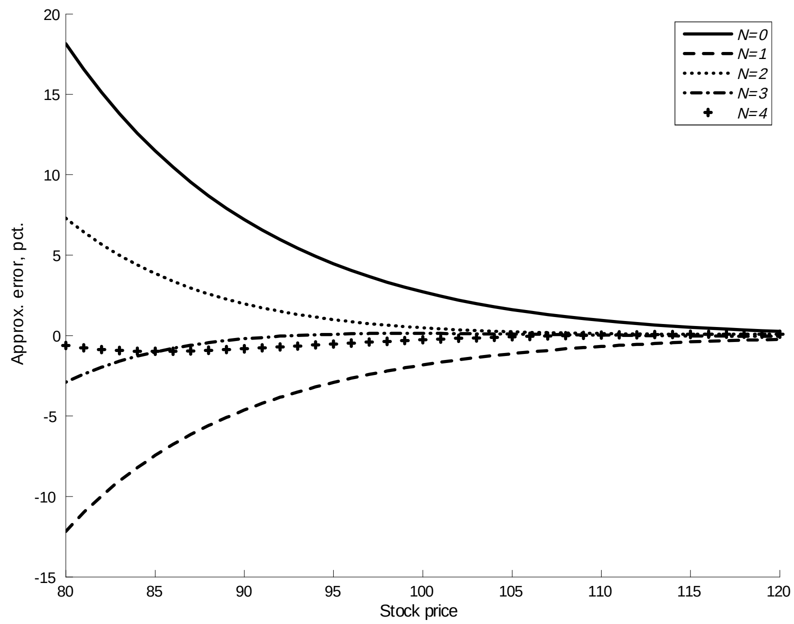

Fig. 3. Percentage errors from approximating option prices predicted by the Heston (1993) model in Eqs. (1) and (2), with $\xi = \frac{1}{2}$ using Black-Scholes (1973) model as auxiliary device with N corrective terms. The strike price is $K = 100$, time-to-maturity is one year, parameter values are $\kappa = 2$, $\alpha = 0.04$, $\omega = 0.10$, $\rho = -0.5$, $r = 10\%$, and the current value of volatility is such that $v = 0.05$. The Black-Scholes (1973) volatility is set equal to $\sqrt{0.05}$.

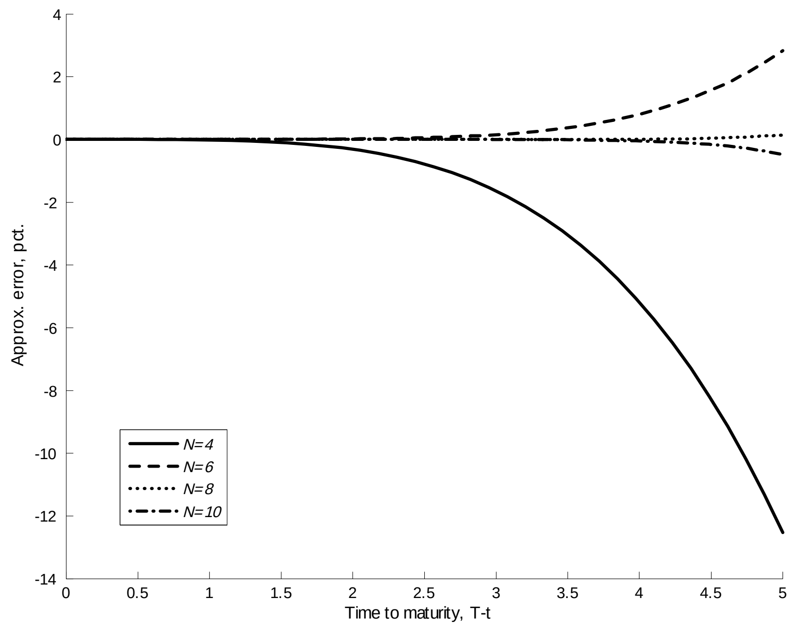

Fig. 4. Percentage errors from approximating bond prices predicted by the CIR (1985) model in Eq. (33) using the CIR model as auxiliary device but with pay-off function equal to zero. Parameter values are $\alpha = 0.06$, $\beta = 0.10$, and $\sigma = 0.12$, and the current level of the short-term rate is 10%.

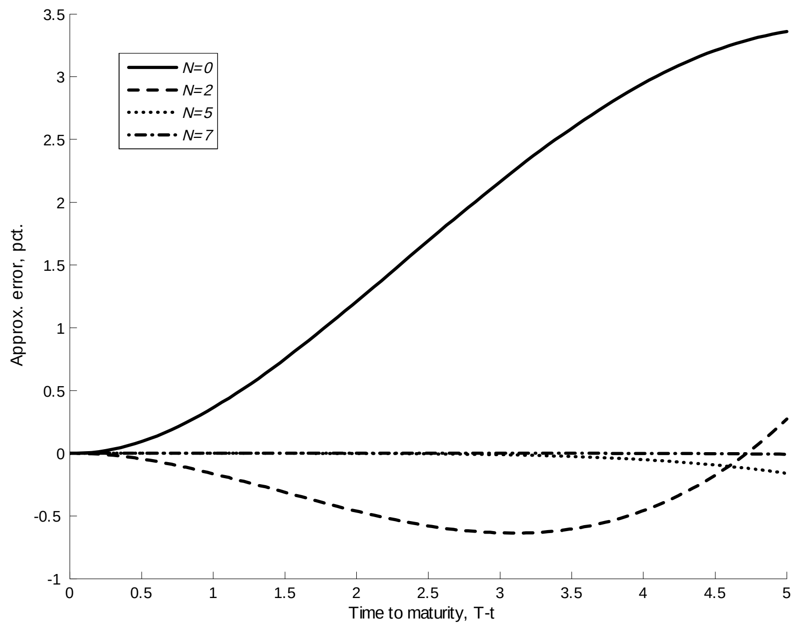

Fig. 5. Percentage errors from approximating the bond price predicted by the CIR model in Eq. (33) using the Vasicek (1973) model in Eq. (34) as auxiliary device with N corrective terms. Parameter values in Eqs. (33) and (34) are $\alpha = \alpha_0 = 0.06$, $\beta = \beta_0 = 0.10$, $\sigma = 0.12$, $\sigma_0 = \sigma\sqrt{0.10}$, and the current level of the short-term rate is 10%.

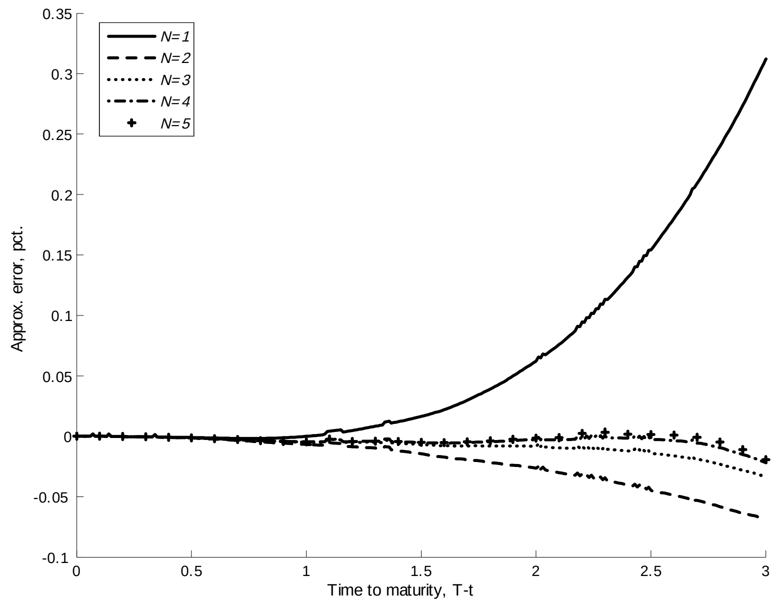

Fig. 6. Percentage errors from approximating the bond price predicted by the non-affine two-factor model in Eqs. (38) using the Vasicek (1973) model in Eq. (34) as auxiliary device with N corrective terms. Parameter values in Eqs. (34) and (38) are $\beta = \beta_0 = 0.11$, $\alpha = \alpha_0 = 0.07$, $\kappa = 0.38$, $\mu = 0.07$, $\omega = 0.81$, $\rho = 0.10$.and $\sigma_0^2 = \sigma^2 r$, and the current levels of the short-term rate and volatility are 6% and 0.50%, respectively.

Tables

Table 1
Comparison of Heston (1993) model option prices under alternative computation methods

This table compares option prices for the Heston (1993) model computed through the Fourier transform in Madan and Carr (1999), the asymptotic expansion in Yang (2006) and, finally, the expansion in this paper based on the Black-Scholes (1973) model and four leading terms. Options have a strike price of $K = 1,000$, time to expiration equals one month ($T - t = 1/12$), and the parameter values in Eqs. (1) and (2) are set equal to $\kappa = 0.1465$, $\alpha = 0.5172$, $\omega = 0.5786$, $\xi = 1/2$, $\rho = -0.0243$, as in Bollerslev and Zhou (2002), and $r = 0$. Panel A provides option prices when the initial value of the instantaneous variance equals $v(t) = \alpha = 0.5172$. Panel B provides prices when the initial value of the underlying stock price equals $S(t) = 1,000$. The columns labeled %Diff provide percentage price differences taken against the Fourier transform.

Panel A

	Fourier transform	Yang expansion		This paper expansion	
S	Price	Price	%Diff	Price	%Diff
950	57.8425	57.6761	-0.2876	57.8449	0.0042
960	62.3711	62.1980	-0.2776	62.3738	0.0043
970	67.1005	66.9218	-0.2662	67.1033	0.0042
980	72.0291	71.8463	-0.2538	72.0321	0.0042
990	77.1553	76.9696	-0.2407	77.1584	0.0040
1000	82.4766	82.2893	-0.2271	82.4797	0.0038
1010	87.9903	87.8028	-0.2131	87.9934	0.0036
1020	93.6933	93.5069	-0.1990	93.6964	0.0033
1030	99.5822	99.3980	-0.1850	99.5852	0.0030
1040	105.6532	105.4725	-0.1711	105.6560	0.0027
1050	111.9021	111.7260	-0.1574	111.9048	0.0023

Panel B

	Fourier transform	Yang expansion		This paper expansion	
v	Price	Price	%Diff	Price	%Diff
0.1	36.4488	36.0241	-1.1652	36.4854	0.1004
0.2	51.4125	51.1122	-0.5841	51.4255	0.0253
0.3	62.8997	62.6543	-0.3901	62.9068	0.0113
0.4	72.5792	72.3665	-0.2931	72.5838	0.0063
0.5	81.1007	80.9102	-0.2348	81.1040	0.0041
0.6	88.7981	88.6241	-0.1960	88.8006	0.0028
0.7	95.8702	95.7089	-0.1682	95.8721	0.0021
0.8	102.4465	102.2955	-0.1474	102.4481	0.0016
0.9	108.6171	108.4746	-0.1312	108.6184	0.0013
0.10	114.4477	114.3124	-0.1182	114.4488	0.0010
0.11	119.9878	119.8587	-0.1076	119.9888	0.0008

Table 2
Comparison of non-affine option prices under alternative computation methods

This table compares option prices for the CEV model in Eqs. (1) and (2), computed through Monte Carlo integration, the asymptotic expansion in Yang (2006) and, finally, the expansion in this paper based on the Black-Scholes (1973) model and four leading terms. Options have a strike price of $K = 1,000$, time to expiration equals one month ($T - t = 1/12$), and the parameter values in Eqs. (1) and (2) are set equal to $\kappa = 0.1465$, $\alpha = 0.5172$, $\omega = 0.5786$, $\xi = 0.6$, $\rho = -0.0243$, and $r = 0$. Panel A provides option prices when the initial value of the instantaneous variance equals $v(t) = \alpha = 0.5172$. Panel B provides prices when the initial value of the underlying stock price equals $S(t) = 1,000$. The columns labeled %Diff provide percentage price differences taken against Monte Carlo integration.

Panel A

S	Monte Carlo Price	Yang expansion Price	Yang expansion %Diff	This paper expansion Price	This paper expansion %Diff
950	57.1354	57.3729	0.4157	57.6931	0.9761
960	61.5947	61.8847	0.4709	62.2185	1.0128
970	66.2564	66.6008	0.5197	66.9456	1.0402
980	71.1264	71.5198	0.5532	71.8731	1.0498
990	76.1974	76.6400	0.5809	76.9990	1.0521
1000	81.4727	81.9591	0.5969	82.3211	1.0413
1010	86.9598	87.4741	0.5914	87.8366	1.0083
1020	92.6641	93.1821	0.5589	93.5424	0.9478
1030	98.5726	99.0793	0.5140	99.4349	0.8748
1040	104.6802	105.1618	0.4601	105.5104	0.7931
1050	110.9633	111.4253	0.4164	111.7646	0.7221

Panel B

v	Monte Carlo Price	Yang expansion Price	Yang expansion %Diff	This paper expansion Price	This paper expansion %Diff
0.1	36.0006	35.9221	-0.2182	36.4457	1.2364
0.2	50.7456	50.8765	0.2578	51.3208	1.1334
0.3	62.0950	62.3683	0.4402	62.7745	1.0943
0.4	71.6717	72.0543	0.5338	72.4363	1.0668
0.5	80.1100	80.5821	0.5893	80.9467	1.0445
0.6	87.7368	88.2854	0.6252	88.6366	1.0256
0.7	94.7477	95.3627	0.6491	95.7032	1.0085
0.8	101.2699	101.9438	0.6654	102.2754	0.9929
0.9	107.3927	108.1187	0.6760	108.4427	0.9777
0.10	113.1820	113.9532	0.6814	114.2706	0.9618
0.11	118.6858	119.4969	0.6834	119.8085	0.9460

Table 3
Approximation of Greeks for the Heston (1993) model

This table provides option price sensitivities with respect to changes in the underlying asset price (Panels A and B) and with respect to the instantaneous variance (Panel C), for the Heston (1993) model, using both the Fourier transform in Madan and Carr (1999) and the asset price expansion in this paper based on the Black-Scholes (1973) model and four leading terms. Options have a strike price of $K = 1,000$, time to expiration equals one month ($T - t = 1/12$), and the parameter values in Eqs. (1) and (2) are set equal to $\kappa = 0.1465$, $\alpha = 0.5172$, $\omega = 0.5786$, $\xi = 1/2$, $\rho = -0.0243$, as in Bollerslev and Zhou (2002), and $r = 0$. Panels A.1, B.1, and C.1 provide option prices when the initial value of the instantaneous variance equals $v(t) = \alpha = 0.5172$. Panels A.2, B.2 and C.2 provide prices when the initial value of the underlying stock price equals $S(t) = 1,000$. The columns labeled %Diff provide percentage differences taken against the quantities of interest computed through Fourier transforms.

Panel A.1

	Fourier transform	This paper expansion	
S	$\partial C/\partial S$ (%)	$\partial C/\partial S$ (%)	%Diff
950	44.2794	44.2819	0.0058
960	46.2918	46.2940	0.0047
970	48.2928	48.2945	0.0035
980	50.2776	50.2788	0.0024
990	52.2414	52.2421	0.0013
1000	54.1800	54.1801	0.0003
1010	56.0893	56.0890	-0.0006
1020	57.9657	57.9649	-0.0014
1030	59.8058	59.8046	-0.0021
1040	61.6066	61.6049	-0.0027
1050	63.3654	63.3633	-0.0033

Panel A.2

	Fourier transform	This paper expansion	
v	$\partial C/\partial S$ (%)	$\partial C/\partial S$ (%)	%Diff
0.1	51.9512	51.9534	0.0042
0.2	52.6614	52.6621	0.0015
0.3	53.2189	53.2193	0.0008
0.4	53.6929	53.6932	0.0005
0.5	54.1121	54.1123	0.0004
0.6	54.4920	54.4921	0.0003
0.7	54.8416	54.8418	0.0002
0.8	55.1673	55.1674	0.0002
0.9	55.4732	55.4733	0.0001
1.0	55.7625	55.7626	0.0001
1.1	56.0376	56.0377	0.0001

Panel B.1

	Fourier transform	This paper expansion	
S	$\partial^2 C/\partial S^2$ (%)	$\partial^2 C/\partial S^2$ (%)	%Diff
950	0.2016	0.2016	-2.4215
960	0.2008	0.2007	-2.6204
970	0.1994	0.1993	-2.7031
980	0.1975	0.1975	-2.7526
990	0.1952	0.1951	-2.6755
1000	0.1925	0.1924	-2.6690
1010	0.1893	0.1893	-2.6500
1020	0.1859	0.1858	-2.5888
1030	0.1821	0.1820	-2.5088
1040	0.1780	0.1780	-2.3893
1050	0.1737	0.1737	-2.2348

Panel B.2

	Fourier transform	This paper expansion	
v	$\partial^2 C/\partial S^2$ (%)	$\partial^2 C/\partial S^2$ (%)	%Diff
0.1	0.4241	0.4434	4.5522
0.2	0.2944	0.3118	5.9181
0.3	0.2542	0.2538	-0.1611
0.4	0.2245	0.2193	-2.3250
0.5	0.2012	0.1957	-2.7038
0.6	0.1831	0.1784	-2.5737
0.7	0.1686	0.1649	-2.1681
0.8	0.1568	0.1541	-1.7277
0.9	0.1468	0.1451	-1.1852
1.0	0.1387	0.1375	-0.8827
1.1	0.1318	0.1309	-0.6568

Panel C.1

	Fourier transform	This paper expansion	
S	$\partial C/\partial v$ (%)	$\partial C/\partial v$ (%)	%Diff
950	74.9945	74.9679	-0.0354
960	76.2501	76.2212	-0.0378
970	77.3079	77.2847	-0.0300
980	78.1815	78.1563	-0.0322
990	78.8766	78.8354	-0.0522
1000	79.3500	79.3229	-0.0341
1010	79.6521	79.6212	-0.0388
1020	79.7443	79.7336	-0.0134
1030	79.6526	79.6651	0.0157
1040	79.4439	79.4213	-0.0285
1050	79.0269	79.0090	-0.0227

Panel C.2

	Fourier transform	This paper expansion	
v	$\partial C/\partial v$ (%)	$\partial C/\partial v$ (%)	%Diff
0.1	180.6213	181.1785	0.3085
0.2	127.8147	127.9010	0.0675
0.3	104.3381	104.3475	0.0091
0.4	90.3166	90.2924	-0.0268
0.5	80.7070	80.6884	-0.0231
0.6	73.6120	73.5900	-0.0299
0.7	68.0870	68.0663	-0.0304
0.8	63.9556	63.6085	-0.5427
0.9	60.0878	59.9118	-0.2928
1.0	56.8783	56.7810	-0.1711
1.1	54.1400	54.0845	-0.1024

Table 4
Approximation of Greeks for non-affine models

This table provides option price sensitivities with respect to changes in the underlying asset price (Panels A) and with respect to the instantaneous variance (Panels B), for the CEV model in Eqs. (1) and (2), using both Monte Carlo integration and the asset price expansion in this paper based on the Black-Scholes (1973) model and four leading terms. Options have a strike price of $K = 1,000$, time to expiration equals one month $(T - t = 1/12)$, and the parameter values in Eqs. (1) and (2) are set equal to $\kappa = 0.1465$, $\alpha = 0.5172$, $\omega = 0.5786$, $\xi = 0.6$, $\rho = -0.0243$, and $r = 0$. Panels A.1 and B.1 provide option prices when the initial value of the instantaneous variance equals $v(t) = \alpha = 0.5172$. Panels A.2 and B.2 provide prices when the initial value of the underlying stock price equals $S(t) = 1,000$. The columns labeled %Diff provide percentage differences taken against the quantities of interest computed through Monte Carlo integration.

Panel A.1

	Monte Carlo	This paper expansion	
S	$\partial C/\partial S$ (%)	$\partial C/\partial S$ (%)	%Diff
950	43.5844	44.2419	1.5085
960	45.5741	46.2644	1.5147
970	47.6381	48.2756	1.3383
980	49.8072	50.2706	0.9304
990	51.6445	52.2445	1.1618
1000	53.8656	54.1930	0.6265
1010	55.9119	56.1119	0.3578
1020	58.0716	58.9976	-0.1276
1030	60.0880	59.8464	-0.4020
1040	62.0285	61.6554	-0.6015
1050	63.6870	63.4218	-0.4163

Panel A.2

	Monte Carlo	This paper expansion	
v	$\partial C/\partial S$ (%)	$\partial C/\partial S$ (%)	%Diff
0.1	51.9501	51.9568	0.0131
0.2	52.5276	52.6729	0.2767
0.3	52.9595	53.2317	0.5140
0.4	53.3255	53.7060	0.7135
0.5	53.8027	54.1252	0.5995
0.6	54.0987	54.5049	0.7508
0.7	54.4164	54.8543	0.8047
0.8	54.7429	55.1798	0.7981
0.9	54.9393	55.4854	0.9941
1.0	55.2099	55.7745	1.0227
1.1	55.4269	56.0494	1.1230

Panel B.1

	Monte Carlo	This paper expansion	
S	$\partial C/\partial v$ (%)	$\partial C/\partial v$ (%)	%Diff
950	74.2766	74.9072	0.8489
960	75.5218	76.1723	0.8614
970	76.5901	77.2457	0.8561
980	77.4442	78.1252	0.8793
990	78.1237	78.8099	0.8784
1000	78.5703	79.3008	0.9298
1010	78.8212	79.6002	0.9883
1020	78.8205	79.7115	1.1305
1030	78.6555	79.6396	1.2512
1040	78.3327	79.3905	1.3505
1050	77.9749	78.9711	1.2776

Panel B.2

	Monte Carlo	This paper expansion	
v	$\partial C/\partial v$ (%)	$\partial C/\partial v$ (%)	%Diff
0.1	177.4286	180.7478	1.8707
0.2	126.1482	127.7828	1.2958
0.3	103.1555	104.2902	1.1000
0.4	89.3784	90.2579	0.9841
0.5	79.9090	80.6650	0.9460
0.6	72.9385	73.5728	0.8696
0.7	67.4934	68.0529	0.8290
0.8	63.0958	63.5975	0.7952
0.9	59.4809	59.9025	0.7089
1.0	56.3929	56.7729	0.6739
1.1	53.7493	54.0773	0.6103

Table 5
Approximation of multifactor non-affine term structure models

This table compares the bond price computed through the expansion in this paper based on the Vasicek (1977) model and four leading terms, against that computed through Monte Carlo integration, for the non-affine two-factor model in Eqs. (38). Parameter values are: $\beta = 0.11$, $\alpha = 0.07$, $\kappa = 0.38$, $\mu = 0.07$, $\omega = 0.81$, $\rho = 0.10$. The four panels report prices computed for maturities equal to six months, and one, two, three, and five years. The columns labeled %Diff provide percentage price differences taken against Monte Carlo integration.

1/2 year

	$\sigma^2 = 0.50\%$				$r = 6\%$		
	Monte Carlo	This paper expansion			Monte Carlo	This paper expansion	
r	Price	Price	%Diff	σ^2	Price	Price	%Diff
4%	0.9798	0.9797	-0.0010	0.40%	0.9702	0.9702	-0.0008
5%	0.9750	0.9750	-0.0011	0.45%	0.9703	0.9703	-0.0009
6%	0.9703	0.9703	-0.0012	0.50%	0.9703	0.9703	-0.0009
7%	0.9656	0.9656	-0.0013	0.55%	0.9703	0.9703	-0.0010
8%	0.9609	0.9609	-0.0014	0.60%	0.9703	0.9703	-0.0010

1 year

	$\sigma^2 = 0.50\%$				$r = 6\%$		
	Monte Carlo	This paper expansion			Monte Carlo	This paper expansion	
r	Price	Price	%Diff	σ^2	Price	Price	%Diff
4%	0.9591	0.9591	-0.0041	0.40%	0.9411	0.9411	-0.0043
5%	0.9501	0.9501	-0.0045	0.45%	0.9411	0.9411	-0.0045
6%	0.9412	0.9411	-0.0048	0.50%	0.9412	0.9411	-0.0048
7%	0.9323	0.9323	-0.0050	0.55%	0.9412	0.9411	-0.0050
8%	0.9235	0.9235	-0.0053	0.60%	0.9412	0.9411	-0.0052

2 years

	$\sigma^2 = 0.50\%$				$r = 6\%$		
	Monte Carlo	This paper expansion			Monte Carlo	This paper expansion	
r	Price	Price	%Diff	σ^2	Price	Price	%Diff
4%	0.9170	0.9169	-0.0084	0.40%	0.8847	0.8847	-0.0081
5%	0.9008	0.9007	-0.0084	0.45%	0.8848	0.8847	-0.0082
6%	0.8848	0.8847	-0.0083	0.50%	0.8848	0.8847	-0.0083
7%	0.8691	0.8691	-0.0079	0.55%	0.8848	0.8848	-0.0085
8%	0.8537	0.8537	-0.0076	0.60%	0.8849	0.8848	-0.0086

3 years

	$\sigma^2 = 0.50\%$				$r = 6\%$		
	Monte Carlo	This paper expansion			Monte Carlo	This paper expansion	
r	Price	Price	%Diff	σ^2	Price	Price	%Diff
4%	0.8744	0.8741	-0.0333	0.40%	0.8310	0.8308	-0.0331
5%	0.8525	0.8523	-0.0338	0.45%	0.8312	0.8309	-0.0330
6%	0.8313	0.8310	-0.0336	0.50%	0.8313	0.8310	-0.0336
7%	0.8105	0.8102	-0.0326	0.55%	0.8314	0.8311	-0.0355
8%	0.7902	0.7900	-0.0319	0.60%	0.8315	0.8312	-0.0364

5 years

	$\sigma^2 = 0.50\%$				$r = 6\%$		
	Monte Carlo	This paper expansion			Monte Carlo	This paper expansion	
r	Price	Price	%Diff	σ^2	Price	Price	%Diff
4%	0.7914	0.7843	-0.8938	0.40%	0.7337	0.7270	-0.9124
5%	0.7623	0.7547	-1.0045	0.45%	0.7340	0.7266	-0.9966
6%	0.7344	0.7262	-1.1166	0.50%	0.7344	0.7262	-1.1166
7%	0.7074	0.6989	-1.1991	0.55%	0.7347	0.7257	-1.2326
8%	0.6814	0.6726	-1.2885	0.60%	0.7351	0.7250	-1.3732