Abstract: We consider common value second-price auctions when individuals have correlation capacities; i.e., when they have ambiguity over the joint information structures generating the signals for players, but put some bounds on the correlation between their information and others'. We show that such ambiguity can affect equilibrium bids in non-trivial ways; high and low types decrease their bids while intermediate types may increase their bids, compared to the canonical model. The results differ from the case of ambiguity about the prior, and from a cursed-equilibrium analysis.

1 Introduction

Common-value auctions are typically analyzed under the assumption of conditionally-independent private information held by the bidders on their common valuations. In reality however, information sources may be correlated: Newspapers may receive information from related sources, experts may derive predictions from similar data sources, and so on. A recent literature in behavioural economics has shown, experimentally and empirically, that individuals may fail to appreciate the correlation in their information sources.\(^2\)

Naturally, in common value auctions, individuals need to condition their valuation on the event of winning the auction, and hence on the information content of others’ signals in this case. It is therefore important to derive a framework to consider how individuals

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behave when they are uncertain about the relation between theirs and others’ information. In this paper we analyze common value auctions under a framework developed in Levy and Razin (2016). The paper models the capacity of individuals to consider correlation between information sources, by assuming that individuals have ambiguity over the possible joint information structures governing theirs and other players’ signals. In particular, given their capacity, individuals only consider the set of joint information structures that have some bounded degree of correlation.

Specifically, we characterise the correlation capacity by a single parameter, \( a \), which bounds the degree of pointwise mutual information between information sources. Consider two individuals, 1 and 2, each receiving a signal, \( s \) and \( s' \), respectively. Let \( f(s, s' | \omega) \) denote the joint distribution of the signals conditional on a state \( \omega \), and \( f_1(s | \omega) \) and \( f_2(s' | \omega) \) denote the marginal distributions of \( s \) and \( s' \) conditional on \( \omega \). The (exponential) pointwise mutual information (PMI) is defined in the Information Theory literature as \( \frac{f(s, s' | \omega)}{f_1(s | \omega)f_2(s' | \omega)} \). We assume that individuals’ ambiguity over joint information structures is restricted to those for which \( \frac{1}{a} \leq \frac{f(s, s' | \omega)}{f_1(s | \omega)f_2(s' | \omega)} \leq a \) for some finite parameter \( a \geq 1 \). The higher is \( a \), the higher is the correlation capacity. We show in Levy and Razin (2016) that this restriction provides a meaningful way to constrain the set of ambiguous beliefs, and specifically, that the higher the correlation capacity, the greater is the ambiguity faced by the individual.

We present a simple model of a second price-auction. While each player knows the marginal information structures determining the signals, he has ambiguity about the joint information structure generating these signals. Therefore, when an individual receives a signal and contemplates what strategy to play, he takes into consideration the scenarios in which he wins and considers all feasible information structures with PMIs that are bounded by \( a \) for any state and vector of signals. We assume that individuals have ambiguity aversion and that they consider the worst case scenario when comparing possible actions (as in Gilboa and Schmeidler 1989).

We analyze the equilibria in the game for any \( a \geq 1 \), and compare the equilibria in our model with correlation capacities to those in the canonical case (when signals are conditionally independent and individuals are aware of it). We show that the set of equilibrium bids changes, in a non-trivial ways. Specifically, while the high and low types typically decrease their bids, intermediate types tend to increase their bids. Therefore, the winner’s curse is mitigated for the low and high types and exacerbated for intermediate types.

In the model, ambiguity over information structures is exogenous but ambiguity over the state of the world is endogenous, in the sense that it arises only when conditioning

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on the other’s signal, and is affected by strategic behaviour.\textsuperscript{4} For this reason, ambiguity aversion does not simply imply that individuals all lower their bids. While this is true for the very low and high types, incentives are more intricate for intermediate types and depend more subtly on equilibrium behaviour. When we analyze the gains of the seller, we find the sufficient conditions for a negative effect overall when players have ambiguity and correlation capacities.

We also show that when individuals have more standard (exogenous) ambiguity about the state then all bids are lower than in the canonical model. The reason for this different result rests on the endogenous nature of ambiguity in our model, that depends on expected equilibrium behaviour of the opponent. In contrast, exogenous ambiguity about the state does not depend on strategic reasoning.

2 Related literature

Our paper is related to a recent literature on ambiguity and auctions. As far as we know, models of ambiguity have been applied in the past only to private value auctions. Specifically, Bose et al (2006) analyze the optimal auction mechanisms for private value auctions with ambiguity over other bidders’ valuations. Salo and Weber (1995) show how ambiguity aversion in a private values auction translates to higher bids. Specifically, they use non additive probabilities over the players’ distribution of signals. Higher bids arise as individuals underestimate their winning probabilities. Lo (1998) shows that the first price auction dominates the second price auction in some environments. He uses a multiple priors approach and shows that equilibrium bids are simply determined as if all players hold the worst case prior (in our analysis players with different signals use different beliefs). Chen et al (2007) show in experiments that bids are lower in the presence of ambiguity in first and second price auctions with independent private values.

Bose and Renou (2014) study how principals can use ambiguous mechanisms to implement social welfare functions that are not attainable under unambiguous mechanisms. In particular they construct ambiguous communication mechanisms between the agents and a moderator resulting with agents updating to sets of beliefs, similarly to what happens in our model when players condition their beliefs.

Bergemann, Brooks and Morris (2015) consider private values auctions and study the

\textsuperscript{4}This is also related to the notion of dilation introduced in Seidenfeld and Wasserman (1993). Seidenfeld and Wasserman (1993) focus on lower and upper probability bounds for probability events. Dilation is defined as a situation in which the probability bounds of an event A are strictly within the probability bounds for the event in which A is conditional on B. When we compare an individual’s private belief to the set of beliefs arising when conditioning in the other’s signal, sometime dilation occurs.
set of achievable utilities when considering, as modelers, the set of different feasible information structures. Our analysis is different as in our approach it is the economic agents, rather than the modeler, who spans the possible information structures. In addition, we restrict the set of possible information structures using the notion of pointwise mutual information. We show how this shifts equilibrium behaviour in a non-trivial way.

There is a recent literature in behavioural economics considering common value auctions and the winner’s curse. Specifically, the “cursed-equilibrium” concept of Eyster and Rabin (2005) allows for individuals to not fully appreciate the information content of the actions of other bidders.\(^5\) They analyze second-price auctions and show that in large auctions the average winning bid exceeds the average value of the object.\(^6\) Applying a cursed equilibrium solution to our model yields a different result to that of the ambiguity we assume. Specifically, while in their analysis low types increase their bid relative to the canonical model, in our analysis they decrease it.

Most of the experimental literature (see Kagel and Levin 2001), show overbidding (which happens in our model for some types). The experiments also highlight how bidders are more prone to the winner’s curse in auctions with a higher number of bidders.

More generally, there is a recent literature looking at how individuals perceive correlation across information sources. In Levy and Razin (2016) we advocate the use of correlation capacities as a comparative measure allowing to predict how individuals with different perceptions of correlation will behave. We show for example that an individual with lower capacity for correlation will exhibit more risky behaviour and how the number of investors in a market will affect risky and cautious behaviour. Ellis and Piccione (2016) use an axiomatic approach to represent decision makers affected by the complexity of correlations among the consequences of feasible actions. The key difference between the approaches is that in our model the agents are aware that they do not know the exact correlation structure.

3 The Model

The state of the world is \(v \in \{0, 1\}\), with an equal prior. Each individual receives a signal \(s^i\) about the state of the world. The marginal distributions determining the signals given the state of the world, are known to the players, are anonymous, and depend on the state symmetrically. Specifically, \(g_0(s)\) is a decreasing function, \(g_1(s)\) is an increasing function. Hence \(G_0(s)\) is concave and \(G_1(s)\) is convex. For simplicity, let \(g_0(s) = g_1(1 - s)\), so that

\(^5\)See Jehiel (2005) for a related concept.

\(^6\)They also show an environment with two bidders in which the seller’s revenue decreases with cursed individuals.
co-occurrence of words or symbols. It can also be written as how much information one word or symbol provides about the other, or to measure the

\[
G_0(s) = 1 - G_1(1 - s).
\]

Note that FOSD is satisfied so that \( G_0(s) > G_1(s) \) for all interior \( s \), and hence MLRP is satisfied too. Let \( s_0 < 0.5 \) be the median of \( G_0 \) and \( s_1 > 0.5 \) the median of \( G_1 \).

Individuals have ambiguity over the following set of joint distributions per state \( v \in \{0, 1\} \), which is constructed using the Morgenstern transformation:

\[
f_v(s) = [1 + \lambda_v(2G_v(s_1) - 1)(2G_v(s_2) - 1)]g_v(s_1)g_v(s_2).
\]

(1)

For this to be a distribution, for any \( v \) we need \( |\lambda_v| \leq 1 \). Note that when \( \lambda_v > 0 \) we have positive correlation of signals in state \( v \) while when \( \lambda_v < 0 \) we have negative correlation. When signals are conditionally independent, we have \( \lambda_v = 0 \) for all \( v \).

We will compare the results of the model above to the “canonical model”. By this we mean a model with independent information sources, so that \( g_v(s) \) per player are conditionally independent on the state \( v \), and there is no added ambiguity.

**Remark 1: No ambiguity absent strategic concerns:** At the interim stage, following the receipt of the signal \( s^i \), and without conditioning on equilibrium behaviour, individual \( i \) has a unique belief on the state of the world, amounting to \( \frac{g_1(s^i)}{g_1(s^i) + g_0(s^i)} \). The knowledge that the other player had received a signal as well is immaterial as given the marginal distributions, the law of iterated expectations would imply the same belief for all joint information structures considered. Thus, ambiguity over joint information structures does not necessarily lead to ambiguity over the state of the world.

**Capacity for correlation:** We use the approach of Levy and Razin (2016) and assume ambiguity over joint information structures, but constrain the level of ambiguity by using correlation capacity \( a \). Specifically, we capture correlation capacity by using the pointwise mutual information formula: Let \( f(x_1, x_2) \) be a joint probability distribution of random variables \( \tilde{x}_1, \tilde{x}_2 \), with marginal distributions \( f_i(.) \). The pointwise mutual information (PMI) at \( (x_1, x_2) \) is

\[
\ln \left[ \frac{f(x_1, x_2)}{f_1(x_1)f_2(x_2)} \right] = h(x_1) - h(x_1|x_2)
\]

where \( h(x_1) = -\log_2 \Pr(X_1 = x_1) \) is the self information (entropy) of \( x_1 \) and \( h(x_1|x_2) \) is the conditional information.

Summing over the PMIs, we can derive the well known measure of mutual information,

\[
MI(X_1, X_2) = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} f(x_1, x_2) \ln \left[ \frac{f(x_1, x_2)}{f_1(x_1)f_2(x_2)} \right] = H(X_1) - H(X_1|X_2),
\]

which can
be shown to be always non-negative as it equals the amount of uncertainty about $X_1$ which is removed by knowing $X_2$. We can also express mutual information by using the definition of Kullback-Leibler divergence between the joint distribution and the product of the marginals:

$$MI(X_1, X_2) = D_{KL}(f(x_1, x_2) | f_1(x_1) f_2(x_2)),$$

and it can therefore capture how far from independence individuals believe their information structures are. For our purposes, the local concept of the PMI is a more suitable concept than the MI, as we are looking at ex-post rationalisations given some set of signals.\(^7\)

The concept of the PMI is closely related to standard measures of correlation and specifically it implies a bound on the concordance between information structures. The most common measure of concordance is Spearman’s rank correlation coefficient or Spearman’s $\rho$, a nonparametric measure of statistical dependence between two variables. It assesses how well the relationship between the variables can be described using a monotonic function. A perfect Spearman correlation of $+1$ or $-1$ occurs when each of the variables is a perfect monotonic function of the other. In Levy and Razin (2016) we show that there is a $0 < \bar{\rho} < 1$ such that any joint information structure with bounded PMIs has a Spearman’s $\rho$ in $[-\bar{\rho}, \bar{\rho}]$. This also implies that we can put bounds directly on the copula.\(^8\)

Let $ePMI$ be the exponent of the PMI, i.e., $\frac{f(x_1, x_2)}{f_1(x_1) f_2(x_2)}$. We assume that individuals consider joint information structures that satisfy the $ePMI$ constraint at any point:

**Assumption A1: (Correlation capacity constraints)** Individual $i$ is characterised by a parameter $\alpha_i = a$, $1 \leq a < \infty$, and only considers joint information structures described in 1 with

$$\lambda_v \in \left[\frac{1}{a} - 1, 1 - \frac{1}{a}\right] \text{ for } v \in \{0, 1\}.$$  

Let $\lambda_{\min} = \frac{1}{a} - 1$, $\lambda_{\max} = 1 - \frac{1}{a}$.

The assumption allows us to have a one-parameter characterization of an individual’s capacity for correlation across information structures. Note that a joint information structure which satisfies independence would have $\lambda_v = 0$ at any point; whenever a joint information structure does not satisfy independence then the ePMI is less than 0 for some $(s_1, s_2, v)$, and is greater than 0 for some $(s'_1, s'_2, v')$, which implies that perceiving the ePMI at 0 is in some sense the simplest possibility and hence is always in the set. Moreover, it is impossible to consider only priors/information structures with ePMI that is only

\(^7\)The PMI therefore does not distinguish between rare or frequent events.

\(^8\)One alternative specification would be to make the bounds assumption directly on the copula, so that $C(u, v)$ is bounded, where $C(u, v)$ is the copula of $u, v$. This is implied by our bounds on the PMI. Nelsen (2006) also shows that $\rho^{\text{Spearman}} = 12 \int \int [C(u, v) - uv]dudv$; this implies that the distance between the relevant copula and the product copula (independence) is closely related to notions of concordance.
higher (lower) than 0. It is also easy to see from the above that the higher is \(a\), the more joint information structures can be considered as a higher \(a\) will allow for more positive correlation across information sources if \(\lambda_v > 0\), and for more negative correlation across information sources if \(\lambda_v < 0\).

Discussion of the main assumption: The key assumption here is ambiguity over joint information structures on which we place bounds that depend on correlation capacities. The methodology behind assumption A1 is similar to the rational inattention model in Sims (2003) which allows individuals to consider models of the world with finite Shannon capacity. We think about \(a\) as a parameter describing the cognitive capacity of an individual.

The ambiguity over joint information structures will play a role here as in the common-value auction game, a player conditions his valuation on the event in which he wins (which reveals information about the other’s signal and thus about the state of the world through the different joint information structures). Moreover, the probability of this event also depends on the joint distribution of signals.

In Levy and Razin (2016) in which we analyse non-strategic environments in which individuals simply exchange beliefs with one another, we show that a higher \(a\), a greater capacity for correlation, implies a larger set of beliefs about the state of the world in a set inclusion sense. This implies that greater capacity for correlation results in greater ambiguity. This will arise here too. While individuals do not exchange beliefs, they have to consider each others’ information, given the equilibrium strategy. Thus note that ambiguity here is endogenous as it would depend on equilibrium behaviour.

Still, as we show, individuals with a higher \(a\) would end up with a larger set of feasible joint information structures or greater ambiguity. With ambiguity aversion, one intuition is that individuals with greater capacity for correlation will behave more conservatively and will therefore bid less. We show later that this is not always the case.

An equilibrium is denoted by a pair of bidding strategies for the two players, \((b^1(s^1), b^2(s^2))\), and a symmetric equilibrium has \(b^1(.) = b^2(.) \equiv b(.)\). We consider max-min behaviour. Specifically, in equilibrium, given an observed signal, a bidding strategy maximizes the utility of the individual under the worst case scenario (that is, when nature chooses the worst information structure from the set described above).

4 Analysis

We now continue to analyze the equilibrium behaviour of players in the second price auction with bounded ambiguity. We will show that ambiguity over the state of the
world arises when strategic considerations are involved, and how it affects equilibrium behaviour.

Let us first write the utility of a player per each bid $b$. This is

$$U(s^1, b) = \min_{\lambda} \frac{1}{2} \left( \int_0^z (1 - b(s')) f_1(s^1, s') ds' - \int_0^z b(s') f_0(s^1, s') ds' \right)$$

where $b(s')$ is the bid used by the other player and $z = b^{-1}(b)$. Thus per each bid $b$, each player minimizes his utility by choosing a vector $\lambda$, given the strategy of the other player. Recall that $s_v$, for $v \in \{0, 1\}$, is the median of the cdf $G_v()$.

**Lemma 1:** Consider an equilibrium in which $b(s)$ is increasing. Let $\lambda^*_v(s)$ denote the information structure which minimizes the utility of the player for each $s$. Then:

(i) $(\lambda^*_0, \lambda^*_1) = (\lambda_{\text{max}}, \lambda_{\text{min}})$ for all $s < s_0$.
(ii) $(\lambda^*_0, \lambda^*_1) = (\lambda_{\text{min}}, \lambda_{\text{min}})$ for all $s \in [s_0, \min\{\hat{s}, s_1\}]$.
(iii) $(\lambda^*_0, \lambda^*_1) = (\lambda_{\text{min}}, \lambda_{\text{max}})$ in $[s_1, \hat{s}]$ if $s_1 < \hat{s}$ and $(\lambda_0, \lambda_1) = (\lambda_{\text{max}}, \lambda_{\text{min}})$ in $[\hat{s}, s_1]$ otherwise.
(iv) $(\lambda^*_0, \lambda^*_1) = (\lambda_{\text{max}}, \lambda_{\text{max}})$ for all $s > \max\{s_1, \hat{s}\}$, and $\hat{s} < 1$ satisfies

$$\int_0^{\hat{s}} b(s') g_0(s')(2G_0(s') - 1) ds' = 0.$$

**Remark 2:** The importance of ambiguity: Note that the assumption of ambiguity is important here. Specifically, the characterization if equilibrium behaviour (when it exists) is not equivalent to a model in which the individuals simply start with some unique worst case beliefs, as each type uses a different worst case belief to justify their best response. That is, $\lambda^*_v(s)$ changes with $s$, so the behaviour as described cannot be rationalized with a unique a priori $\lambda$.

We can now characterise the equilibrium. As expected, bids in equilibrium in the second-price auction will equal the expectations of the player given his signal and that the other player had received the same signal. The expectations however will depend for each $b(s)$, on the chosen vector $\lambda^*(s)$.

**Proposition 1:** When $a$ is not too high, there exists a symmetric equilibrium in which

$$b(s, \lambda^*) = E^\lambda^*(v|s, s) = \frac{[1 + \lambda^*_i(2G_1(s) - 1)^2]g_1^2(s)}{[1 + \lambda^*_i(2G_1(s) - 1)^2]g_1^2(s) + [1 + \lambda^*_0(2G_0(s) - 1)^2]g_0^2(s)}.$$

Overbidding arises in equilibrium when $\hat{s} > \frac{1}{2}$, for types in $[0.5, \hat{s}]$. For all other types, underbidding arises.
As standard, equilibrium bids will reflect the perceived valuation of a type \( s \) given that her opponent receives \( s \) as well, but now based on the correlation structure that minimises her utility at this signal.

To minimize her utility for each bid, a player needs to consider how correlation affects both the probability of her winning, and the price she will pay, at each state. Very low types minimize their utility at state 0 by postulating positive correlation to maximize the price they pay, while in state 1, they postulate negative correlation to minimize the probability of winning. Very high types postulate maximum positive correlation at each state as the likelihood of them winning is high and this maximizes the price they pay. This encourages both very low and very high type to underbid compared to the canonical model.

For intermediate types solving this trade-off is more subtle. Consider for example a type just above a half and assume that \( 0.5 < \hat{s} < s_1 \). As this type has very little information, she minimizes her utility by reducing the probability of winning. Negative correlation maximizes the variance of the other’s bids, and thus the probability that she loses. Still a signal just above a half implies that state 1 is more likely, which convinces this type to overbid, as conditional on winning he is likely to face a lower type given the negative correlation he perceives.

**Proof of Proposition 1 and Lemma 1:**

We first show in Claims 1-3 how players choose \( \lambda^* \) to minimize their utility given each \( s \), when the bid of the other player is weakly increasing in \( s' \). We then show that the bidding function described above, for the \( \lambda \)'s chosen, is an equilibrium.

Define:

\[
I_1(s) = \int_0^s (1 - b(s'))g_1(s')g_1(s)(2G_1(s) - 1)(2G_1(s') - 1)ds',
\]
\[
I_0(s) = -\int_0^s b(s')g_0(s')g_0(s)(2G_0(s) - 1)(2G_0(s') - 1)ds'.
\]

Thus:

**Claim 1:** In equilibrium, \( \lambda^*_v = \lambda_{\min} \) (\( \lambda_{\max} \)) iff \( I_v(s) > (\leq) 0 \).

\( I_v(s) \) is the derivative of the expected utility with respect to \( \lambda_v \). Given max-min behaviour, the statement follows.

**Claim 2:** (i) \( I_1(s) > 0 \) for \( s < s_1 \), \( I_1(s) < 0 \) for all \( s > s_1 \); (ii) \( I_0(s) < 0 \) for \( s < s_0 \), \( I_0(s) > 0 \) for all \( s \in (s_0, \hat{s}) \), \( I_0(s) < 0 \) for all \( s > \hat{s} \).

**Proof of Claim 2:**
(i) $I_1(s)$: This function must be strictly positive for $s < s_1$ as $(2G_1(s) - 1)(2G_1(s') - 1) > 0$ for $s, s' < s_1$. Note that $I_1(s_1) = 0$, and that

$$
\left. \frac{\partial I_1(s)}{\partial s} \right|_{s = s_1} = \left. \frac{\partial g_1(s)(2G_1(s) - 1)}{\partial s} \right|_{s = s_1} \int_0^{s_1} (1 - b(s'))g_1(s')(2G_1(s') - 1)ds' \\
= 2(g_1(s_1))^2 \int_0^{s_1} (1 - b(s'))g_1(s')(2G_1(s') - 1)ds' < 0
$$

More generally:

$$
\frac{\partial I_1(s)}{\partial s} = (g_1'(s)(2G_1(s) - 1) + 2(g_1(s))^2) \int_0^{s} (1 - b(s'))g_1(s')(2G_1(s') - 1)ds' \\
+ (1 - b(s))g_1(s)(2G_1(s) - 1)(2G_1(s) - 1) \\
= \left( \frac{g_1'(s)}{g_1(s)} + \frac{2(g_1(s))}{(2G_1(s) - 1)} \right) I_1(s) + 2(g_1(s))^2(1 - b(s))(2G_1(s) - 1)(2G_1(s) - 1)
$$

So whenever $I_1(s) > 0$ and $s > s_1$ we have that $\frac{\partial I_1(s)}{\partial s} > 0$ as $g_1(s)$ is increasing and $2(g_1(s))^2(1 - b(s))(2G_1(s) - 1)(2G_1(s) - 1)) > 0$. So now it suffices to check that $I_1(1) < 0$:

$$
I_1(1) = g_1(1) \int_0^1 (1 - b(s'))g_1(s')(2G_1(s') - 1)ds' \\
= g_1(1) \int_0^{s_1} (1 - b(s'))g_1(s')(2G_1(s') - 1)ds' + g_1(1) \int_{s_1}^1 (1 - b(s))g_1(s')(2G_1(s') - 1)ds' \\
< g_1(1) \int_0^{s_1} (1 - b(s_1))g_1(s')(2G_1(s') - 1)ds' + g_1(1) \int_{s_1}^1 (1 - b(s_1))g_1(s')(2G_1(s') - 1)ds' \\
= g_1(1) \int_0^{s_1} (1 - b(s_1))g_1(s')(2G_1(s') - 1)ds' = 0,
$$

where the last inequality follows as $b(s')$ is increasing, $(2G_1(s') - 1) > 0$ ($< 0$) whenever $> s_1$ ($s < s_1$). The last equality follows from $\int_0^1 g_1(s')(2G_1(s') - 1)ds' = 0$.

(ii) $I_0(s)$: This function must be strictly negative for $s < s_0$ as $(2G_0(s) - 1)(2G_0(s') - 1) > 0$ for $s, s' < s_0$. Note that $I_0(s_0) = 0$. Moreover,

$$
\left. \frac{\partial I_0(s)}{\partial s} \right|_{s = s_0} = -\left. \frac{\partial g_0(s)(2G_0(s) - 1)}{\partial s} \right|_{s = s_0} \int_0^{s_0} b(s')g_0(s')(2G_0(s') - 1)ds' \\
- b(s')g_0(s')(2G_0(s_0) - 1)(2G_0(s') - 1) \\
= -2(g_0(s_0))^2 \int_0^{s_0} b(s')g_0(s')(2G_0(s') - 1)ds' > 0
$$

So $I_0(s) < 0$ for $s \gtrsim s_0$. Note that $\int_0^s b(s')g_0(s')(2G_0(s') - 1)ds'$ is decreasing for $s > s_0$. 

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Thus if \( I_0(1) < 0 \), we have the result. But
\[
|I_0(1)| = g_0(1) \int_0^1 b(s')g_0(s')(2G_0(s') - 1)ds'
> g_0(1) \int_0^{s_0} b(s_0)g_0(s')(2G_0(s') - 1)ds'
+ g_0(1) \int_{s_0}^1 b(s_0)g_0(s')(2G_0(s') - 1)ds'
= g_0(1)b(s_0) \int_0^1 g_0(s')(2G_0(s') - 1)ds' = 0.
\]
Thus we know there exists \( \hat{s} < 1 \) such that:
\[
\int_0^{\hat{s}} b(s')g_0(s')(2G_0(s') - 1)ds' = 0,
\]
and we can conclude that \( I_0(s) > 0 \) for \( s \in (s_0, \hat{s}) \) and that \( I_0(s) < 0 \) for \( s > \hat{s} \).

Consider now the bidding function \( E^{x^*}(v|s, s) \). Note that overbidding, compared to the canonical model, arises when
\[
\frac{[1 + \lambda_1(2G_1(s) - 1)^2]g_1^2(s)}{[1 + \lambda_1(2G_1(s) - 1)^2]g_1^2(s) + [1 + \lambda_0(2G_0(s) - 1)^2]g_0^2(s)} > \frac{g_1^2(s)}{g_1^2(s) + g_0^2(s)}
\]
which holds if and only if:
\[
\frac{[1 + \lambda_1(2G_1(s') - 1)^2]}{[1 + \lambda_0(2G_0(s') - 1)^2]} > 1.
\]
We then have:

**Claim 3:** When \( b(s) = E^{x^*}(v|s, s) \), a necessary condition for overbidding compared to the canonical model is \( \hat{s} > 0.5 \), that is:
\[
\int_0^{0.5} b(s')g_0(s')(2G_0(s') - 1)ds' < 0
\]
If this holds, there is overbidding in the region \([s_0, \hat{s}]\), and underbidding for any other \( s \). Otherwise, all types underbid compared to the canonical model.

**Proof of Claim 3:** Given Claims 1 and 2, we can then deduce that the different values of \( \lambda^*_0 \) in equilibrium and consider when overbidding/underbidding arises compared to the canonical model when the bidding function is as described in the Proposition.

(i) \( (\lambda_0, \lambda_1) = (\lambda_{\text{max}}, \lambda_{\text{min}}) \) for all \( s < s_0 \). As a result, if this is an equilibrium, we would have underbidding as
\[
\frac{[1 + \lambda_{\text{min}}(2G_1(s') - 1)^2]}{[1 + \lambda_{\text{max}}(2G_0(s') - 1)^2]} < 1,
\]
which is indeed the case as \( \lambda_{\text{min}} < 0 < \lambda_{\text{max}} \).

b. \((\lambda_0, \lambda_1) = (\lambda_{\text{min}}, \lambda_{\text{min}})\) for all \(s \in [s_0, \min\{\hat{s}, s_1\}]\). We have underbidding iff:

\[
\frac{[1 + \lambda_{\text{min}}(2G_1(s') - 1)]}{[1 + \lambda_{\text{min}}(2G_0(s') - 1)]} < 1
\]

If \(\min\{\hat{s}, s_1\} > 0.5\), then we would have overbidding because in the region above 0.5, as \((2G_1(0.5) - 1)^2 = (2G_0(0.5) - 1)^2\) by symmetry, but because of convexity (concavity) of \(G_1\) (\(G_0\)), the fraction would be greater than 1, as we would have \((2G_1(s') - 1)^2 < (2G_0(s') - 1)^2\) just above 0.5.

c. \((\lambda_0, \lambda_1) = (\lambda_{\text{max}}, \lambda_{\text{max}})\) for all \(s > \max\{s_1, \hat{s}\}\). In this case we also have underbidding as \([1 + \lambda_{\text{max}}(2G_1(s') - 1)] < [1 + \lambda_{\text{max}}(2G_0(s') - 1)]\), because \(\frac{1}{2} < G_1(s') < G_0(s')\).

d. If \(0.5 < s_1 < \hat{s}\) : in the region \([s_1, \hat{s}]\) we have \((\lambda_0, \lambda_1) = (\lambda_{\text{min}}, \lambda_{\text{max}})\). In this case we have overbidding as:

\[
\frac{[1 + \lambda_{\text{max}}(2G_1(s') - 1)]}{[1 + \lambda_{\text{min}}(2G_0(s') - 1)]} > 1
\]

For this we need \(s_1 < \hat{s}\), implying that \(0.5 < \hat{s}\).

e. if \(\hat{s} < s_1\) : Then we have \((\lambda_0, \lambda_1) = (\lambda_{\text{max}}, \lambda_{\text{min}})\) in this region between the two values. Then we have underbidding as:

\[
\frac{[1 + \lambda_{\text{min}}(2G_1(s') - 1)]}{[1 + \lambda_{\text{max}}(2G_0(s') - 1)]} < 1.
\]

Thus the structure of the equilibrium is therefore as above. So for overbidding we need:

\[
\begin{align*}
\int_0^{0.5} b(s')g_0(s')(2G_0(s') - 1)ds' \\
= \int_0^{s_0} \frac{[1 + \lambda_{\text{min}}(2G_1(s') - 1)]g_1^2(s')}{{[1 + \lambda_{\text{min}}(2G_1(s') - 1)]g_1^2(s') + [1 + \lambda_{\text{max}}(2G_0(s') - 1)]g_0^2(s')}g_0(s')(2G_0(s') - 1)ds' \\
&+ \int_{s_0}^{0.5} \frac{[1 + \lambda_{\text{min}}(2G_1(s') - 1)]g_1^2(s')}{{[1 + \lambda_{\text{min}}(2G_1(s') - 1)]g_1^2(s') + [1 + \lambda_{\text{min}}(2G_0(s') - 1)]g_0^2(s')}g_0(s')(2G_0(s') - 1)ds' \\
< 0
\end{align*}
\]

which is analogous to what is in the Proposition. Finally we need to show that the construction above is an equilibrium:

**Claim 4:** The bidding function \(b(s')\) defined above with the values of \(\lambda^*(s)\) described above consists a symmetric equilibrium when \(a\) is low enough.

**Proof of Claim 4:** We now show that given the above it is optimal, wlog, for Player 1 to choose \(b(s)\) at \(s\), when player 2 uses \(b(s')\) and \(\lambda^*(s)\) as defined above.

Let \(\hat{\lambda}\) equal \(\lambda^*(s)\) and consider the virtual utility:
\[ \hat{U}(s, z) = \int_0^z (E^{\hat{\lambda}(s)}(v|s, s') - b(s'))dF^{\hat{\lambda}(s)}(s, s') \]
\[ = \frac{1}{2}(\int_0^z ((1 - b(s'))f_1(\hat{\lambda}, s, s') - b(s')f_0(\hat{\lambda}, s, s'))ds' \]

This is not player 1’s utility as it is evaluated at \( \hat{\lambda} \) for all \( s' \). However note that when \( z = s \), then the integrand is zero. To see that the integrand equals 0 note that, as \( \hat{\lambda} = \lambda^*(s) \),
\[ (1 - b(s))f_1(\hat{\lambda}, s, s) = b(s)f_0(\hat{\lambda}, s, s) \]
iff
\[ [1 + \lambda_0^*(2G_0(s) - 1)^2]g_0^2(s)[1 + \lambda_1^*(2G_1(s) - 1)(2G_1(s) - 1)]g_1(s)g_1(s) \]
\[ = [1 + \lambda_0^*(2G_0(s) - 1)^2]g_1^2(s)[1 + \lambda_1^*(2G_0(s) - 1)(2G_0(s) - 1)]g_0(s)g_0(s) \]
which holds.

Moreover as we now show the first order condition w.r.t. \( s' \) is zero, the second order condition evaluated at this point is negative, thus \( z = s \) is a maximum. To see this, suppose that we have a \( z \) for which \( \hat{U}(s, z) = 0 \). Taking a second derivative w.r.t. \( z \) we get: \(-b'(z)(f_1(\hat{\lambda}, s, z) + f_0(\hat{\lambda}, s, z)) + (1 - b(z))f_1'(\hat{\lambda}, s, z) - b(z)f_0'(\hat{\lambda}, s, z)\). As \( \hat{U}(s, z) = 0 \), this implies that \((1 - b(z)) = b(z)f_0(\hat{\lambda}, s, z) / f_1(\hat{\lambda}, s, z)\), and thus the second order derivative at that \( z \) is
\[ -b'(z)(f_1(\hat{\lambda}, s, z) + f_0(\hat{\lambda}, s, z)) + b(z)f_0(\hat{\lambda}, s, z) / f_1(\hat{\lambda}, s, z) f_1'(\hat{\lambda}, s, z) - b(z)f_0'(\hat{\lambda}, s, z) \]
\[ = -b'(z)(f_1(\hat{\lambda}, s, z) + f_0(\hat{\lambda}, s, z)) + b(z)(f_0'(\hat{\lambda}, s, z) f_1'(\hat{\lambda}, s, z) - f_0'(\hat{\lambda}, s, z)) \]

Note that the first element is always negative. The second element is negative iff:
\[ g_1'(z)g_1(s)(1+\lambda_0^*(2G_1(z)-1)(2G_1(s)-1))g_1(z)g_1(s)\lambda_2g_1(z)(2G_1(z)-1) \]
\[ g_0'(z)g_0(s)(1+\lambda_0^*(2G_0(z)-1)(2G_0(s)-1))g_0(z)g_0(s)\lambda_02g_0(z)(2G_0(z)-1) \]
\[ < g_1'(z)g_1(s)(1+\lambda_1^*(2G_1(z)-1)(2G_1(s)-1))g_1(z)g_1(s)\lambda_2g_1(z)(2G_1(z)-1) \]
\[ g_0'(z)g_0(s)(1+\lambda_0^*(2G_0(z)-1)(2G_0(s)-1))g_0(z)g_0(s)\lambda_02g_0(z)(2G_0(z)-1) \]

Note that when \( \hat{\lambda} \) is small enough, this is always the case as the LHS is negative. Thus a solution to the first order condition is unique.

But the above implies that player 1 can achieve this utility above and cannot improve upon it when using other bids \( z \neq s \).

So we know that the player bids until the integrand gets negative, so, written differently, until \( E^{\hat{\lambda}(s)}(v|s, s) = b(s) \), which gives us the equilibrium bidding function.\( \blacksquare \)

This completes the proof of Proposition 1.\( \blacksquare \)

5 Discussion

In this Section we discuss the relation between our results to other approaches taken in the literature and stress the importance of the specific type of ambiguity that we use.
5.1 Comparison to exogenous ambiguity

We now show that our equilibrium characterisation differs from that of a model in which there is some exogenous ambiguity on the state of the world, through the prior for example. Specifically, assume now that the players believe that their information is (conditionally) independent, with marginals \( g_1(s), g_0(s) \) as above, and that the probability the state is 1 is \( p \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \). Thus ambiguity here is exogenous and does not depend on equilibrium strategies. In this case, as we show, all types would see the worst case scenario as the lowest prior on the state. Thus all bids would be reduced.

**Proposition 2:** In the independent information model with exogenous ambiguity over the prior, all bids are lower than in the case of no ambiguity. Specifically, in a symmetric equilibrium, each bids

\[
b(s) = \frac{(\frac{1}{2} - \varepsilon)g_1^2(s)}{(\frac{1}{2} - \varepsilon)g_1^2(s) + (\frac{\varepsilon}{2} + \varepsilon)g_0^2(s)}
\]

**Proof of Proposition 2:** Suppose that player 2 is using this strategy and consider player 1. For each bid he needs to choose \( \varepsilon \) in order to minimize his expected utility. That is

\[
U(s, z) = \min_{p \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]} \Pr(v = 1|s, p)G_1(z)(1 - E(b(s')|v = 1, s' < z))
- \Pr(v = 0|s, p)G_0(z)E(b(s')|v = 0, s' < z)
= \min_{p \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]} \frac{pg_1(s)}{pg_1(s) + (1 - p)g_0(s)}G_1(z)[1 - \frac{1}{G_1(z)} \int_0^z b(s')g_1(s')ds']
- \frac{(1 - p)g_0(s)}{pg_1(s) + (1 - p)g_0(s)} \int_0^z b(s')g_0(s')ds'
\]

which is minimized by setting \( p = \frac{1}{2} - \varepsilon \).

Expected utility is then:

\[
\int_0^z \frac{(\frac{1}{2} - \varepsilon)g_1(s)}{(\frac{1}{2} - \varepsilon)g_1(s) + (\frac{\varepsilon}{2} + \varepsilon)g_0(s)}(1 - b(s'))g_1(s')
- \frac{(\frac{1}{2} - \varepsilon)g_0(s)}{(\frac{1}{2} - \varepsilon)g_1(s) + (\frac{\varepsilon}{2} + \varepsilon)g_0(s)}b(s')g_0(s')ds',
\]

which can be written as:

\[
\int_0^z \frac{(\frac{1}{2} - \varepsilon)g_1(s)}{(\frac{1}{2} - \varepsilon)g_1(s) + (\frac{\varepsilon}{2} + \varepsilon)g_0(s)(1 - \frac{(\frac{1}{2} - \varepsilon)g_1^2(s')}{(\frac{1}{2} - \varepsilon)g_1^2(s') + (\frac{\varepsilon}{2} + \varepsilon)g_0^2(s)})g_1(s')}
- \frac{(\frac{1}{2} - \varepsilon)g_0(s)}{(\frac{1}{2} - \varepsilon)g_1(s) + (\frac{\varepsilon}{2} + \varepsilon)g_0(s)(1 - \frac{(\frac{1}{2} - \varepsilon)g_1^2(s')}{(\frac{1}{2} - \varepsilon)g_1^2(s') + (\frac{\varepsilon}{2} + \varepsilon)g_0^2(s)})g_0(s')ds'.
\]
When $s' = s$, the integrand is zero. By MLRP, the integrand is increasing in $s$. Thus, for any $s' < s$ the integrand is positive, and for any $s' > s$, the integrand is negative, which implies that $z = s$ is optimal. This completes the proof. ■

5.2 Seller’s revenue

Obviously the seller’s revenues decrease when there is ambiguity over the prior as described in Proposition 2. What happens with ambiguity over information sources? The result below applies to the case in which $\lambda$ is not too large:

**Proposition 3:** When $\hat{s} < 0.5$, or when $w(y)$ is decreasing over $[1 - \hat{s}, \hat{s}]$, for

$$w(y) \equiv (1 - G_1(y))g_1(y) + (1 - G_0(y))g_0(y),$$

then the seller’s revenue decreases in $a$.

Of course in the case in which there is always underbidding so that the sufficient condition from Proposition 1 does not hold, then the seller’s revenues decrease with $a$. This arises also when there is overbidding in some region. Specifically if this region has a low enough probability, revenues decrease.

**Proof of Proposition 3:** Consider the case when $\hat{s} > s_1$. Let

$$w(s') = (1 - G_1(s'))g_1(s') + (1 - G_0(s'))g_0(s')$$

The seller’s revenues can be written as:

$$R(a) = \int_0^{s_0} b(s', \lambda_{\text{max}}, \lambda_{\text{min}})w(s')ds' + \int_{s_0}^{s_1} b(s', \lambda_{\text{min}}, \lambda_{\text{min}})w(s')ds' + \int_{s_1}^{\hat{s}} b(s', \lambda_{\text{min}}, \lambda_{\text{max}})w(s')ds' + \int_{\hat{s}}^1 b(s', \lambda_{\text{max}}, \lambda_{\text{max}})w(s')ds'$$

The derivative w.r.t. $a$ is:

$$\frac{\partial R(a)}{\partial a} = \int_0^{s_0} \frac{\partial}{\partial a} b(s', \lambda_{\text{max}}, \lambda_{\text{min}})w(s')ds' + \int_{s_0}^{s_1} \frac{\partial}{\partial a} b(s', \lambda_{\text{min}}, \lambda_{\text{min}})w(s')ds' + \int_{s_1}^{\hat{s}} \frac{\partial}{\partial a} b(s', \lambda_{\text{min}}, \lambda_{\text{max}})w(s')ds' + \int_{\hat{s}}^1 \frac{\partial}{\partial a} b(s', \lambda_{\text{max}}, \lambda_{\text{max}})w(s')ds' + \frac{\partial}{\partial a} (b(\hat{s}, \lambda_{\text{min}}, \lambda_{\text{max}}) - b(\hat{s}, \lambda_{\text{max}}, \lambda_{\text{max}}))w(\hat{s})$$
We note that
\[
\frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} = g_1(s')^2 g_0(s')^2 \frac{[-(2G_1(s') - 1)^2 - (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2}
\]
\[
\frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} = g_1(s')^2 g_0(s')^2 \frac{[-(2G_1(s') - 1)^2 + (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2}
\]
\[
\frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max})|_{a=1} = g_1(s')^2 g_0(s')^2 \frac{[(2G_1(s') - 1)^2 + (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2}
\]
\[
\frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} = g_1(s')^2 g_0(s')^2 \frac{[(2G_1(s') - 1)^2 - (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2}
\]

So that:
\[
\frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} = -\frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max})|_{a=1} = -\frac{\partial}{\partial a} b(1 - s', \lambda_{\min}, \lambda_{\max})|_{a=1}
\]
\[
\frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} = -\frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max})|_{a=1} = -\frac{\partial}{\partial a} b(1 - s', \lambda_{\min}, \lambda_{\min})|_{a=1}
\]

And therefore we can write \( \frac{\partial R(a)}{\partial a} \big|_{a=1} \) as:
\[
\frac{\partial R(a)}{\partial a} \big|_{a=1} = -\int_0^{1-\tilde{s}} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max}) w(s') ds' - \int_{1-\tilde{s}}^{s_0} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max}) [w(s') - w(1 - s')] ds' +
\]
\[
-\int_{s_0}^{0.5} \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\max}) w(s') ds' - \int_{0.5}^{1} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\min}) w(s') ds'.
\]

Thus, a sufficient condition for revenue to be decreasing in \( a \) is \( w(s') \) decreasing over \([\tilde{s}, 1 - \tilde{s}]\).\[\Box\]

### 5.3 Comparison to a “cursed-equilibrium”.

Our equilibrium characterisation also differs from the one arising in a cursed-equilibrium à la Eyster and Rabin (2005). In their framework, individuals do not realize that others’ strategies relate to others’ information. Specifically, bids then equal \( E(v|s) \) rather than \( E(v|s, s) \). In our model this implies that for types above some cutoff \( \bar{s} \) will underbid (compared with the canonical model) and types below \( \bar{s} \) will overbid.\(^9\) The intuition is simple: Low types who are cursed will mistakenly take into consideration that they win against relatively low types, and high types will not take into consideration that at the margin they win against high types. Thus, the two models differ conceptually and in terms of their predictions.

\(^9\)Specifically overbidding (underbidding) arises whenever \( g_1(s) > (\leq) g_1(s) \). As \( g_1() \) is increasing and \( g_0() \) is decreasing, there is a unique such \( \bar{s} \).
References


