Common-Value Auctions with Ambiguity over Correlation

Krittanai Laohakunakorn, Gilat Levy and Ronny Razin, LSE[1]

Abstract: We consider common value auctions when individuals have ambiguity over the joint information structures generating the signals for players. This implies that ambiguity interacts with strategic effects as individuals condition their behaviour on their opponent's equilibrium bid and hence their signal. We show that compared to the canonical model, both in the first-price and second-price auctions, low types underbid and high types sometimes overbid. Therefore, the winner's curse is mitigated for low types and potentially exacerbated for high types. We also show that these results differ from a model with "standard" ambiguity about the prior over the state. Finally, we characterize the optimal auction and show that the optimal revenue decreases with this type of ambiguity. A novel feature that arises in the optimal mechanism is that the seller only partially insures the high type against ambiguity.

1 Introduction

In auctions as in many other strategic situations, individuals often have a good understanding of their own private information but they might know less about others' information sources. For example, they might worry that they do not understand well the correlation between their own information and that of other players they are engaged with. Common-value auctions are typically analyzed under the assumption of conditionally-independent private information, with the bidders aware of this fact. However, bidders may believe that their information sources might be correlated, as would be the case when there are common factors generating their private signals.[2] In auctions, bidders condition their valuation on the event of winning, and therefore have to consider the endogenous

[1]This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No SEC-C413.

[2]We therefore consider sophisticated individuals who entertain the possibility that such correlation might exist. Recent literature has also looked at the opposite possibility, that information sources may be correlated and naive individuals may not be aware of this. See Ortoleva and Snowberg (2015), Glaeser and Sunstein (2009) and Levy and Razin (2015a, 2015b), Eyster and Weizsacker (2011), Kallir and Sonsino (2009) and Enke and Zimmermann (2013).

information they learn about others' signals. In this sense it is important to analyze how individuals behave when they are uncertain about the relation between theirs and others' information.

In this paper we analyze common-value auctions when individuals have ambiguity over the joint information structures generating signals for the auction participants. Specifically, we assume that individuals know the marginal information structure of each bidder but they may believe that their information sources are correlated to a degree, and have ambiguity over the possible correlation scenarios.

To model the ambiguity over the joint information structure, we follow Levy and Razin (2017) and use a single parameter, a, to bound the degree of *pointwise mutual information* of the information structures. This formulation allows us to analyse a range of levels of ambiguity and to get arbitrarily close to the canonical model.

Specifically, consider two individuals, 1 and 2, each receiving a signal, s and s', respectively. Let $q(s, s'|\omega)$ denote a joint probability of the signals conditional on a state ω, and $q_1(s|\omega)$ and $q_2(s'|\omega)$ denote the marginal probabilities of s and s' conditional on ω. The (exponential) pointwise mutual information (PMI) is defined in the Information Theory literature as $\frac{q(s,s'|\omega)}{q_1(s|\omega)q_2(s'|\omega)}$.[3] We assume that individuals' ambiguity is over the set of joint information structures satisfying $\frac{1}{a} \leq \frac{q(s,s'|\omega)}{q_1(s|\omega)q_2(s'|\omega)} \leq a$ for some finite parameter $a \geq 1$.[4] The higher is a, the larger is the set of correlation scenarios considered. When $a = 1$ we are back to the standard model with conditionally independent signals and no ambiguity.

We analyze a simple model with two possible valuations and two signals (the model is extended to continuous signals in Section 5). When an individual receives a signal and contemplates what strategy to play, she takes into consideration the scenarios in which she wins and considers all feasible information structures with PMIs that are bounded by a for any state and vector of signals. We assume that individuals have ambiguity aversion and that they consider the worst-case scenario when comparing possible actions (as in Gilboa and Schmeidler 1989).

We analyze the equilibria in the second-price and first-price auctions. In the model, ambiguity over information structures is exogenous but ambiguity over the state of the world is endogenous, and depends on the strategic interaction. For this reason, ambiguity aversion does not simply imply that individuals lower their bids. Indeed our key result is that low types underbid and high types may overbid as compared to the standard model. Therefore, the winner's curse is mitigated for the low type and is sometimes exacerbated

[3] See Church and Hanks (1991).

[4] Levy and Razin (2017) show that this restriction provides a meaningful way to constrain the set of ambiguous beliefs, and specifically, that the higher the "correlation capacity" a, the greater is the ambiguity over the state of the world faced by the individual.

for the high type.

The above result is due to the interaction of ambiguity over correlation structures with strategic reasoning. In equilibrium, the low type's worst-case scenario is that the value is the lowest possible when both players have received low signals (which is the only case in which she can win). The high type sometimes minimizes her utility by believing that the value is the highest possible when both players have received high signals, inducing her to bid higher.

In contrast, if one considers exogenous ambiguity about the state, such as standard ambiguity over the prior, the results are different. We show that in this case all bids are lower than in the canonical model. In addition, exogenous ambiguity implies that the modeler can simply assume that individuals share a single worst-case belief to start with. In contrast, as we saw above, when ambiguity is about correlation, different types use different correlation scenarios to minimize their utilities.

Next we study the seller's revenue. First we show that in the first and second price auctions, the seller's revenue is decreasing in the ambiguity about the information structure. Although bids increase for the high types, the overall effect of ambiguity on the seller is negative. We also show that the first-price auction yields a higher revenue to the seller compared with the second-price auction. The intuition for this result stems from the fact that in the second-price auction individuals condition their bids on more information implying that the ambiguity is more pronounced.

We then turn to consider the optimal auction in the face of ambiguity over correlation structures. We characterise the optimal mechanism in this environment assuming that the true distribution of signals is independent. When a is small, in the optimal mechanism, the good is always allocated to the player with the highest signal. As in the standard model, the seller makes side bets with the low type. Since the high type is more likely to win when the other player has received a low signal, she minimizes her utility by believing that the other player is likely to have received a high signal in the good state. This implies that the high type underestimates the probability of the other player receiving a low signal. The seller exploits this by asking the high type to pay more when the other player receives a low signal. As a result, the seller only partially insures the high type against ambiguity.

When a is large and the signal is not very informative, the seller finds it optimal to fully insure the buyers against the ambiguity, so that the allocation of the good does not depend on their signals. As a result, the high type earns positive rents in equilibrium. Finally, we show that the seller's revenue in the optimal mechanism is decreasing in the amount of ambiguity, as we found in both the first and second price auctions.

Our paper is related to a recent literature on ambiguity and auctions. As far as we know, our paper is the first to analyse ambiguity in common-value auctions. For private-value auctions, Salo and Weber (1995) show how ambiguity aversion translates to higher bids as individuals underestimate their winning probabilities.[5] Bose et al (2006) analyze optimal auction mechanisms for private-value auctions with ambiguity over other bidders' valuations. They show that the seller will fully insure the buyers against ambiguity. We show that in some cases only partial insurance arises. Lo (1998) shows that the first-price auction dominates the second-price auction in some environments. He uses a multiple priors approach and shows that equilibrium bids are simply determined as if all players hold the worst-case prior. In our analysis players with different signals use different beliefs.

Bose and Renou (2014) study how principals can use ambiguous mechanisms to implement social welfare functions that are not attainable under unambiguous mechanisms. In particular, they construct ambiguous communication mechanisms between the agents and a moderator resulting with agents updating to sets of beliefs. Finally, Bergemann, Brooks and Morris (2015) consider private values auctions and study the set of achievable utilities when considering, as modelers, the set of different feasible information structures. Our analysis is different as in our approach it is the economic agents, rather than the modeler, who span the possible information structures. In addition, we restrict the set of possible information structures using the notion of pointwise mutual information. We show how this shifts equilibrium behaviour in a non-trivial way.

2 The Model

Consider a common-value auction with two bidders (1 and 2), two possible common valuations $v \in \{0, 1\}$ and a uniform prior. Each individual receives one of two signals: l or h. In Section 5 we consider continuous information structures.

Let $q(s|v)$ denote the marginal conditional probability of receiving signal $s \in \{l, h\}$ given state $v \in \{0, 1\}$. We assume that the (marginal) probability of receiving the signal l in state 0, or the signal h in state 1, is $q > \frac{1}{2}$ i.e., $q(s = l|v = 0) = q(s = h|v = 1) = q$.[6] Let $q(s_1, s_2|v)$ denote the joint conditional probability of player 1 receiving signal $s_1 \in \{l, h\}$ and player 2 receiving $s_2 \in \{l, h\}$ given state $v \in \{0, 1\}$.

We assume that the true joint probability distribution, $q(s_1, s_2|v)$, satisfies conditional independence, so that $q(s_1, s_2|v) = q(s_1|v)q(s_2|v)$. However, while individuals know the true marginal probability distribution generating both their signals, they have ambiguity over the set of joint information structures. Thus, individuals perceive the following family

[5] Chen et al (2007) show in experiments that bids are lower in the presence of ambiguity in first and second-price auctions with independent private values.

[6] The analysis can be extended to non-symmetric marginal probability distributions.

of information structures:[7]

Table 1: joint information structures

$v=0$	l	h	$v=1$	l	h
l	α_0	$q-\alpha_0$	l	α_1	$1-q-\alpha_1$,
h	$q-\alpha_0$	$1-2q+\alpha_0$	h	$1-q-\alpha_1$	$2q-1+\alpha_1$

Under independence, $\alpha_0 = q^2$ and $\alpha_1 = (1-q)^2$. However, in our models α_0 and α_1 are the parameters over which there is ambiguity, as we define below.

Remark 1: *No ambiguity absent strategic concerns*: At the interim stage, having received the signal $s = l$, and without conditioning on equilibrium behaviour, individual i has a unique belief that the state of the world is 1, which equals $(1-q)$. The knowledge that the other player had received a signal as well is immaterial as given the marginal distributions, the law of iterated expectations would imply the *same belief* for all joint information structures considered. Thus, ambiguity over joint information structures does not necessarily lead to a set of beliefs.[8]

To consider different levels of ambiguity, we use the notion of *pointwise mutual information.* Specifically, we consider information structures which satisfy the following:

$$\frac{1}{a} \leq \frac{q(s_1, s_2|v)}{q(s_1|w)q(s_2|v)} \leq a, \ \forall \mathbf{s} = (s_1, s_2) \in \{l, h\}^2 \text{ and } \forall v \in \{0, 1\},$$

This formulation allows a simple one-parameter characterisation of the extent of ambiguity. When $a = 1$, only information structures which are conditionally independent are considered. Note that for general a, the ePMI constraints imply:

$$\underline{\alpha}_0(a) \leq \alpha_0 \leq \overline{\alpha}_0(a)$$
$$\underline{\alpha}_1(a) \leq \alpha_1 \leq \overline{\alpha}_1(a)$$

[7]The table describes an information structure so for each state, all cell entries are non-negative and all entries sum up to one.

[8]This is related to the dilation principle explored in Seidenfeld and Wasserman (1993), where more information can create ambiguity.

where:

$$\underline{\alpha}_0(a) = \frac{1}{a}(1-q)^2 + 2q - 1$$
$$\underline{\alpha}_1(a) = \frac{1}{a}(1-q)^2$$
$$\overline{\alpha}_0(a) = \begin{cases} a(1-q)^2 + 2q - 1 & a \leq \frac{q}{1-q} \\ q - \frac{1}{a}q(1-q) & a > \frac{q}{1-q} \end{cases}$$
$$\overline{\alpha}_1(a) = \begin{cases} a(1-q)^2 & a \leq \frac{q}{1-q} \\ 1 - q - \frac{1}{a}q(1-q) & a > \frac{q}{1-q} \end{cases}$$

It is easy to see that the higher is a, the larger is the set of possible information structures that are considered by bidders. Still, when no confusion occurs, we will omit the dependence of $\underline{\alpha}_0$, $\overline{\alpha}_0$, $\underline{\alpha}_1$, and $\overline{\alpha}_1$ on a.

The ambiguity over joint information structures will play an important role in common-value auctions as a player conditions her valuation on the event in which she wins. This reveals endogenous information about the other's signal and thus about the state of the world. Thus, ambiguity about the state of the world arises here endogenously as it would depend on equilibrium behaviour. Moreover, the probability of this event (of winning) also depends on the joint distribution of signals.

An equilibrium is denoted by a pair of bidding strategies for the two players, $(b^1(s^1), b^2(s^2))$, and a symmetric equilibrium has $b^1(.) = b^2(.) \equiv b(.)$. We consider max-min behaviour. Specifically, in equilibrium, given an observed signal, a bidding strategy maximizes the utility of the individual under the worst-case information structure.

For the remainder of this section we provide theoretical background on pointwise mutual information as a measure of correlation. The analysis continues in Section 3.

Pointwise mutual information: Let $f(x_1, x_2)$ be a joint probability distribution of random variables $\tilde{x}_1, \tilde{x}_2$, with marginal distributions $f_i(.)$. The pointwise mutual information (PMI) at (x_1, x_2) is $\ln[\frac{f(x_1,x_2)}{f_1(x_1)f_2(x_2)}]$. PMI was suggested by Church and Hanks (1991) and is used in information theory and text categorization or coding, to understand how much information one word or symbol provides about the other, or to measure the co-occurrence of words or symbols. It can also be written as

$$\ln[\frac{f(x_1, x_2)}{f_1(x_1)f_2(x_2)}] = h(x_1) - h(x_1|x_2)$$

where $h(x_1) = -\log_2 \Pr(X_1 = x_1)$ is the self information (entropy) of x_1 and $h(x_1|x_2)$ is the conditional information.

Summing over the PMIs, we can derive the well known measure of mutual information, $MI(X_1, X_2) = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} f(x_1, x_2) \ln[\frac{f(x_1,x_2)}{f_1(x_1)f_2(x_2)}] = H(X_1) - H(X_1|X_2)$, which can be shown to be always non-negative as it equals the amount of uncertainty about X_1 which is removed by knowing X_2. We can also express mutual information by using the definition of Kullback-Leibler divergence between the joint distribution and the product of the marginals:

$$MI(X_1, X_2) = D_{KL}(f(x_1, x_2)|f_1(x_1)f_2(x_2)),$$

and it can therefore capture how far from independence individuals believe their information structures are. For our purposes, the local concept of the PMI is a more suitable concept than the MI, as we are looking at *ex-post* rationalisations given some set of signals.[9]

The concept of the PMI is closely related to standard measures of correlation and specifically it implies a bound on the *concordance* between information structures. The most common measure of concordance is Spearman's rank correlation coefficient or Spearman's ρ, a nonparametric measure of statistical dependence between two variables. It assesses how well the relationship between the variables can be described using a monotonic function. A perfect Spearman correlation of +1 or -1 occurs when each of the variables is a perfect monotonic function of the other. In Levy and Razin (2017) we show that there is a $0 < \bar{\rho} < 1$ such that any joint information structure with bounded PMIs has a Spearman's ρ in $[-\bar{\rho}, \bar{\rho}]$. As can be seen above, we simply use the ePMI, the exponent of the PMI, i.e., $\frac{f(x_1,x_2)}{f_1(x_1)f_2(x_2)}$.

Note that a joint information structure which satisfies independence would have $a = 1$ at any point; whenever a joint information structure does not satisfy independence then the ePMI is less than 1 for some (s_1, s_2, v), and is greater than 1 for some (s'_1, s'_2, v'), which implies that perceiving the ePMI at 1 is always in the set around which ambiguity is constructed.[10]

3 Second-price auction

We analyze the equilibria for the second-price auction. The results for the first-price auction are similar (see Section 4).

As standard in the second-price auction, bids will equal the expected valuation, conditional on both players receiving the same signal. But given the ambiguity about information structures, this means that different types might use different information structures

[9] The PMI therefore does not distinguish between rare or frequent events.

[10] It is impossible to consider only priors/information structures with ePMI that is only higher (lower) than 1.

to compute this expectation. Given some information structure (α'_0, α'_1), we have,

$$E_{\alpha'_0,\alpha'_1}(v|l,l) = \frac{\alpha'_1}{\alpha'_1 + \alpha'_0}, \ E_{\alpha'_0,\alpha'_1}(v|h,h) = \frac{2q-1+\alpha'_1}{\alpha'_1 + \alpha'_0}$$

Recall that $\bar{\alpha}_v, \underline{\alpha}_v$ are the maximum and minimum values respectively of α_v. We now characterise the equilibria (for proof see Appendix):

Proposition 1. *The unique symmetric pure-strategy equilibrium satisfies:*

1. *The low type bids $b_a(l) = E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)$, a bid that decreases in a;*
2. *For all $a \leq \frac{q}{1-q}$, there exist cutoffs $\underline{q}, \bar{q}$, with $0.5 < \underline{q} < \bar{q} < 1$, where:*
   (a) *For $q \in (0.5, \underline{q})$, the high type bids $b_a(h) = E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)$, a bid that increases with a.*
   (b) *For $q \in (\bar{q}, 1)$, the high type bids $b_a(h) = E_{(\bar{\alpha}_0,\bar{\alpha}_1)}(v|h,h)$, a bid that decreases with a.*
   (c) For $q \in (\underline{q}, \bar{q})$, *the high type bids $b_a(h) = E_{(\alpha_0,\bar{\alpha}_1)}(v|h,h)$ for some α_0 satisfying $E_{(\alpha_0,\bar{\alpha}_1)}(v|h,h) = 2E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)$, a bid that decreases with a.*
3. *For all $a \geq \bar{a}(q) \geq \frac{q}{1-q}$, the high type bids $b_a(h) = E_{(\bar{\alpha}_0,\bar{\alpha}_1)}(v|h,h)$.*[11]

Remember that when $a = 1$ the model becomes the standard model with no ambiguity. Therefore, the Proposition implies that compared to the standard model the ambiguity about correlation induces the low type to underbid and the high type to sometimes overbid. In other words, ambiguity over information structures affects the distribution of bids in a non-trivial way.

To see how the result is derived, consider the low type. Her expected utility when $b = b_a(l)$ is

$$\min_{(\alpha_0,\alpha_1)} \frac{1}{2} \Pr(l|l,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|l,l) - b_a(l)),$$

where

$$\Pr(l|l,(\alpha_0,\alpha_1)) = \alpha_0 + \alpha_1$$

and thus $\Pr(l|l,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|l,l) - b_a(l)) = \alpha_1(1 - b_a(l)) - \alpha_0 b_a(l)$, which implies that what minimizes utility is $(\bar{\alpha}_0, \underline{\alpha}_1)$. As $E_{\alpha'_0,\alpha'_1}(v|l,l) = \frac{\alpha'_1}{\alpha'_1+\alpha'_0}$, this belief minimizes the valuation of the good conditional on both players receiving low signals, the only event in

[11] When q is not too small, $\bar{a}(q) = \frac{q}{1-q}$. When q is sufficiently close to 0.5, $\bar{a}(q) > \frac{q}{1-q}$ and symmetric pure-strategy equilibria may not exist in the region $[\frac{q}{1-q}, \bar{a}(q)]$.

which she can win the good. Note that as the bid of the low type is *lower* compared with the canonical case, her expected utility (computed for the true information structure) will be *higher.*

Consider now the high type. Her expected utility when $b = b_a(h)$ is

$$\min_{(\alpha_0,\alpha_1)} \Pr(l|h,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|l,h) - b_a(l)) + \\ (1/2)\Pr(h|h,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|h,h) - b_a(h)).$$

Note that:

$$\begin{aligned} \Pr(l|h,(\alpha_0,\alpha_1))E_{(\alpha_0,\alpha_1)}(v|l,h) &= 1 - q - \alpha_1 \\ \frac{1}{2}\Pr(h|h,(\alpha_0,\alpha_1))E_{(\alpha_0,\alpha_1)}(v|h,h) &= \frac{1}{2}(2q - 1 + \alpha_1) \end{aligned}$$

and that

$$\begin{aligned} \Pr(l|h,(\alpha_0,\alpha_1))b_a(l) &= (1 - \alpha_1 - \alpha_0)b_a(l), \\ \frac{1}{2}\Pr(h|h,(\alpha_0,\alpha_1))b_a(h) &= \frac{1}{2}(\alpha_1 + \alpha_0)b_a(h). \end{aligned}$$

Thus, increasing α_0 has the effect of increasing expected payment by $\frac{b_a(h)}{2}$ and decreasing it by $b_a(l)$. For a that is not too high, when q is low enough, then $\frac{b_a(h)}{2} = \frac{E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)}{2} < b_a(l) = E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)$, which implies that the effect of increasing α_0 is positive and hence to minimize utility, the high type has to minimize α_0. Increasing α_1 has a similar effect in terms of the expected payment, however, it also has a negative effect in terms of the expected valuation of the good. Specifically, the negative effect is in the order of $\frac{1}{2}$, which is greater than the positive effect of $b_a(l) - \frac{b_a(h)}{2} < \frac{b_a(l)}{2} < \frac{1}{2}$. Thus to minimize utility one has to maximize α_1.

Intuitively, in this case of low enough q, the high type's worst-case scenario is $(\underline{\alpha}_0, \bar{\alpha}_1)$, and she ends up maximizing $E_{\alpha_0',\alpha_1'}(v|h,h) = \frac{2q-1+\alpha_1'}{\alpha_1'+\alpha_0'}$, the value of the good conditional on winning against the high type. This implies that her bid is *higher* than in the canonical model. In the other case, when q is high, or when a is high, the equilibrium will satisfy $\frac{b_a(h)}{2} > b_a(l)$. This implies that the belief that minimises utility is $(\bar{\alpha}_0, \bar{\alpha}_1)$. This belief maximizes the probability of encountering the high type, which induces her to *lower* her bid compared with the canonical model.

As we show, even when the high type increases her bid, compared to the case of no ambiguity, her expected utility evaluated at the true joint probability distribution will be higher compared to the canonical model. Thus, considering the utility of bidders and the seller, we have:

Proposition 2: *(i) The utility of both the high and the low type is higher under ambiguity; (ii) The seller's revenue decreases with ambiguity.*

We have already seen that the low type pays a lower bid than in the canonical model. This is also the case sometimes for the high type. When the high types increases her bid, we show in the Appendix that the size of the increase in the bid exactly offsets the reduction in the bid of the low type. As the high type is more likely to pay the low type's bid for low values of q, for which such equilibrium holds, her utility overall increases even in this case. Finally, we can show that the seller's revenue decreases with a in all equilibria described above. This can already be gleaned from the fact that both the high and the low type gain a higher rent compared with the canonical model. It is also intuitive as a higher level of ambiguity implies that individuals are more likely to consider the worst-case scenario.

It is instructive to consider the equilibrium in the limit, where all information structures are feasible, and so ambiguity is very large. It is the limit of the equilibria constructed for high a in Proposition 1 above. Specifically, we show in the Appendix that in this case, $b_a(l) = 0$ and $b_a(h) = q$.[12] These bids are the lowest among all equilibria and the seller's revenue is therefore substantially lower compared with the canonical model.

Remark 2: *The endogeneity of ambiguity:* Note that ambiguity over the state is affected endogenously in this model. Specifically, the characterization is not equivalent to a model in which the individuals simply start with some unique worst-case belief. Each type (l or h) uses a different worst-case belief to justify her best response and this worst-case scenario depends on others' strategies. In Section 5 we consider the case of exogenous ambiguity over the prior. This type of ambiguity does not depend on the strategic behaviour of others. As we show, this implies that bids are uniformly lower, so that also the high type under-bids.

4 Optimal auctions

In this Section, we characterise the optimal mechanism. In the absence of ambiguity, full surplus extraction is possible.[13] For example, consider the mechanism which always gives the good to player 1 and charges player 2 nothing. Player 2 has a weak incentive to reveal her signal, and if the mechanism punishes player 1 sufficiently harshly when the reported signals do not match, player 1 will also have an incentive to tell the truth, since their signals are positively correlated. The individual rationality constraint can be made to bind by rewarding player 1 when the reported signals do match; in this way the seller can fully extract surplus.

[12] This is supported by the low type believing $\alpha_0 = 2q - 1$ and $\alpha_1 = 0$, and the high type believing $\alpha_0 = q$ and $\alpha_1 = 1 - q$.

[13] See Crémer and McLean (1988).

However, this mechanism cannot extract all the surplus when the buyers are ambiguity averse about the joint distribution of signals. This is because the elicitation will no longer be costless: the expected transfers are larger under the buyer's distribution than the under the seller's (true) distribution.

Note that it is easy to establish that the second-price auction is not optimal. In particular, for a small enough a, we show:

Proposition 3: *Revenue is higher in the first-price than in the second-price auction.*[14]

In the second-price auction, an individual's payment depends on the other's bid and as a result, there are more elements in her utility in which her beliefs play a role. For the case of no ambiguity, this implies that she conditions her behaviour on more information, which increases the seller's revenue. For the case of ambiguity, this implies that individuals have more possibilities to condition on their worst-case beliefs, which decreases the seller's revenue.[15]

The key issue when considering optimal auctions under ambiguity is the level of insurance provided by the seller to the bidders. Under independent private values, Bose et al (2006) show that the optimal mechanism fully insures the bidders against ambiguity. However, in a common-value setting full insurance has implications for the allocation as well as the transfers (i.e. fixing an allocation rule, in general it is not possible to fully insure the buyers against ambiguity by only adjusting transfers). Full insurance implies that both the probability of winning the good and the transfers must not depend on the signal (type) of the other player. Moreover, it can be shown that in any mechanism that allocates the good with probability 1, full insurance implies that for each player, the probability of winning must be the same for both types. Thus, the revenue from such a mechanism is at most $1 - q$, whereas the revenue from the mechanism described above converges to $\frac{1}{2}$ as the ambiguity becomes small. This implies that when a is small, full insurance is not optimal in a common-value setting.

In this Section we show that when a is small, in the optimal mechanism, the seller incentivizes the high type to tell the truth by making side bets with the low type. Unlike in the standard model, this elicitation is not costless as the low type's worst-case belief minimizes correlation between the signals. If the two players receive different signals, the seller allocates the object to the high type; this implies that ambiguity will be important but it also slackens the incentive constraint of the high type, and the seller is able to

[14] In the Appendix we characterise the equilibrium in the first-price auction for a sufficiently low a; the results are similar to those of the second-price (that is, overbidding can also arise).

[15] With private values and ambiguity over the prior, Lo (1998) shows that the first-price auction dominates the second-price auction in some environments.

partially insure the high type against this ambiguity. On the other hand, when a is large and the signal is not very informative, full insurance is optimal.

4.1 Seller's Problem

A direct mechanism (x, t) is an allocation rule $x : \{l, h\}^2 \mapsto [0, 1]^2$ and a transfer rule $t : \{l, h\}^2 \mapsto R^2$. Let $U_i^{\boldsymbol{\alpha}}(s_i', s_i)$ be i's utility from reporting s_i' when i's signal is s_i, given that the information structure is $\boldsymbol{\alpha} = (\alpha_0, \alpha_1)$. A direct mechanism is maxmin incentive compatible if for all $s_i \in S_i$:

$$\min_{\boldsymbol{\alpha}} U_i^{\boldsymbol{\alpha}}(s_i, s_i) \geq \min_{\boldsymbol{\alpha}} U_i^{\boldsymbol{\alpha}}(s_i', s_i)$$

for all $s_i' \in S_i$. The revelation principle applies in this setting as long as we make the following assumption:

No-hedging: The utility from playing the mixed strategy $\sigma_i \in \Delta S_i$ is $E^{\sigma_i} \min_{\boldsymbol{\alpha}} U_i^{\boldsymbol{\alpha}}(s_i', s_i)$ (as opposed to $\min_{\boldsymbol{\alpha}} E^{\sigma_i} U_i^{\boldsymbol{\alpha}}(s_i', s_i)$).

This assumption is standard in the literature on mechanism design with maxmin agents (see for example, Bose et al 2006 or Wolitzky 2016). In what follows, we restrict attention to maxmin incentive compatible direct mechanisms.

The seller's problem is:

$$\max_{x_i, t_i} \frac{1}{2}\left(q^2 + (1-q)^2\right)\left[\sum_{i=1}^{2} t_i(l,l) + t_i(h,h)\right] + q(1-q)\left[\sum_{i=1}^{2} t_i(l,h) + t_i(h,l)\right]$$

subject to incentive and participation constraints:

$$\min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1)t_i(l,l) - (1-\alpha_0-\alpha_1)t_i(l,h)$$
$$\geq \min_{\alpha_0,\alpha_1} \alpha_1 x_i(h,l) + (1-q-\alpha_1)x_i(h,h) - (\alpha_0+\alpha_1)t_i(h,l) - (1-\alpha_0-\alpha_1)t_i(h,h)$$

$$\min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(h,l) + (2q-1+\alpha_1)x_i(h,h) - (1-\alpha_0-\alpha_1)t_i(h,l) - (\alpha_0+\alpha_1)t_i(h,h)$$
$$\geq \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(l,l) + (2q-1+\alpha_1)x_i(l,h) - (1-\alpha_0-\alpha_1)t_i(l,l) - (\alpha_0+\alpha_1)t_i(l,h)$$

$$\min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1)t_i(l,l) - (1-\alpha_0-\alpha_1)t_i(l,h) \geq 0$$

$$\min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(h,l) + (2q-1+\alpha_1)x_i(h,h) - (1-\alpha_0-\alpha_1)t_i(h,l) - (\alpha_0+\alpha_1)t_i(h,h) \geq 0$$

4.2 Analysis

For given a, the optimal mechanism will depend on two cutoff values of q, which we now define. Let $q^*(a)$ be the solution to $q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1=\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_0+\overline{\alpha}_1-1$ that lies between $\frac{1}{2}$ and 1, and let $q^{**}(a)$ be the solution to $q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1=3q-2+\underline{\alpha}_1+\overline{\alpha}_1$ that lies between $\frac{1}{2}$ and 1. We derive the explicit expressions for $q^*(a), q^{**}(a)$ in the Appendix. For $1<a<\infty$, $q^*(a)<q^{**}(a)$. We then have:

Proposition 4.

(i) When $q \leq q^*(a)$, an optimal mechanism allocates the good with equal probability for each player disregarding their type, and for a transfer of $\frac{1}{2}(1-q)$. The revenue of the seller is $1-q$, and the high type earns positive rents.

(ii) When $q \geq q^{**}(a)$, the optimal mechanism allocates the good to the high type and with equal probability to each player if both are of the same type. Transfers are such that the high type is partially insured, the seller makes side bets with the low type, and the buyers earn no rents.

(iii) When $q^*(a) < q < q^{**}(a)$, the optimal mechanism allocates the good to the high type and with equal probability to each player if both are of the same type. Transfers are such that the high type is partially insured, but there are no side bets with the low type, and the high type earns positive rents.[16]

(iv) As $a \to \infty$, both $q^*(a)$ and $q^{**}(a)$ converge to $\frac{1}{2}(3-\sqrt{3})$, and as $a \to 1$, both $q^*(a)$ and $q^{**}(a)$ converge to $\frac{1}{2}$.

When $q \leq q^*(a)$, an implementation of the optimal mechanism is for the seller to first choose each buyer with equal probability, and then sell to the chosen buyer at price $1-q$. Since the decision to sell is not based on the signal realisation, the good is worth $1-q$ to the low type and q to the high type. Thus, the high type earns positive rents in equilibrium. Note that this mechanism is efficient, and that given the seller's design, ambiguity is not relevant or does not arise in equilibrium.

When $q \geq q^{**}(a)$, the participation constraint of the high type is binding. The seller engages in a side bet with the low type to prevent the high type from deviating. Unlike in the classical case, side bets are costly to the seller, so the seller uses the smallest bet that is sufficient to prevent the high type from deviating. To reduce this cost further, the seller allocates the good to the high type when the players receive different signals, which generates endogenous ambiguity over the expected value of the good. The seller is able

[16] In case (ii) the revenue of the seller is $\frac{1}{2}\left(1-(q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1)\frac{\underline{\alpha}_0+\overline{\alpha}_0-q}{\overline{\alpha}_0+\overline{\alpha}_1+\underline{\alpha}_0+\underline{\alpha}_1-1}\right)$, and in case (iii) the revenue of the seller is $\frac{1}{2}\left(\underline{\alpha}_0+2\underline{\alpha}_1+\overline{\alpha}_1+q-q^2-(1-q)^2\right)$. All explicit expressions for the optimal transfers are derived in the Appendix.

to partially insure the high type by asking her to pay more when the other player has received a low signal. In this case, the expected payment from the high type is:

$$T_i^h = \frac{1}{2}(1-(1-q)^2) - \frac{1}{2}(q^2 - \underline{\alpha}_0).$$

The expected payment from the low type is:

$$T_i^l = \frac{\underline{\alpha}_1}{2} - \left[q^2 + (1-q)^2 - \underline{\alpha}_0 - \underline{\alpha}_1\right] \frac{1 - q - \overline{\alpha}_1 - \underline{\alpha}_1}{2(\overline{\alpha}_0 + \overline{\alpha}_1 + \underline{\alpha}_0 + \underline{\alpha}_1 - 1)}.$$

The low type chooses α_0 and α_1 both to minimise the perceived surplus from winning the object and to maximise the perceived value of the transfers. Note that in the optimal mechanism, the belief of the high type regarding α_1 is irrelevant. When a converges to 1, only this case remains.

When $q^*(a) < q < q^{**}(a)$, it is optimal to allocate the good to the high type when the players receive different signals, but it is not optimal to conduct side bets with the low type. Instead, the high type earns positive rents in equilibrium in order to satisfy the incentive constraint. This interval shrinks as a goes to either 1 or ∞.

Note that as $a \to 1$, $q^*(a), q^{**}(a) \to \frac{1}{2}$, so for small a, both participation constraints are binding. The intuition is as follows. If the participation constraint of the high type is slack, the seller can achieve a first order increase in revenue by increasing the payment of the high type. In order to ensure that the incentive constraint is not violated, the seller can increase $t_i(l,h)$ and decrease $t_i(l,l)$ in such a way that keeps the low type indifferent, but lowers the high type's utility from deviating. Since the low type may have different beliefs to the seller, these changes in transfers may decrease the seller's revenue; however, as the ambiguity becomes small, this fall in revenue converges to zero. On the other hand, the increase in revenue from increasing the payment of the high type is fixed.

5 Discussion and Extensions

To conclude we discuss some extensions and related models. Importantly, we show how the analysis differs in the case of exogenous ambiguity about the prior.

5.1 Comparison to exogenous ambiguity

We now show that our equilibrium characterisation differs from that of a model in which there is some exogenous ambiguity about the state of the world. Specifically, assume now that the players believe that their information is (conditionally) independent, and that the probability that the state is 1 is in $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$. The information structure they consider is therefore unique, common knowledge, and given by:

$v=0$	s^l	s^h	$v=1$	s^l	s^h
s^l	q^2	$q(1-q)$	s^l	$(1-q)^2$	$q(1-q)$
s^h	$q(1-q)$	$1-2q+q^2$	s^h	$q(1-q)$	$2q-1+(1-q)^2$

In this model, ambiguity is exogenous and does not depend on equilibrium strategies. In this case, as we show, both types would consider the same worst-case scenario which is the lowest possible prior about the state. As a result both the minimum and the maximum bids are smaller compared to the model with no ambiguity.

Proposition 5. *In the independent information model with exogenous ambiguity over the prior, individuals behave as if the prior is $\frac{1}{2}-\varepsilon$, and thus both the highest and the lowest bids are lower, both in the second-price and in the first-price auction, compared with the canonical model.*

5.2 Comparison to Crémer-McLean

Crémer and McLean (1988) show that some of the conclusions from the analysis of optimal auctions with independent private values are not robust. For example, since surplus extraction is possible when signals are correlated, the optimal mechanism is efficient and leaves no rents to the buyers. Proposition 4 shows that these results continue to hold for a close to 1.[17] On the other hand, when a is large, it is possible for buyers to earn positive rents in the optimal mechanism. Note that in this environment, it is always possible to fully extract rent (see Renou 2015); however, the preceding discussion implies that rent extraction is not necessarily optimal.

5.3 Continuous signals

In Appendix B we show that the results of Section 3 are robust to the case of a continuum of signals. Specifically, we assume that signals are drawn from [0,1], with marginals $g(s|v)$ that satisfy the MLRP, and individuals believe that the joint distribution is the F-G-M copula, that is:

$$f_v(\mathbf{s}) = [1+\lambda_v(2G_v(s_1)-1)(2G_v(s_2)-1)]g_v(s_1)g_v(s_2), \quad (1)$$

and consider the values of $\lambda_v \in [\frac{1}{a}-1, 1-\frac{1}{a}]$ that satisfy the ePMI constraints. We analyze a second-price auction and show that it is still the case that some types over bid (in this

[17] The set of optimal mechanisms when $a=1$ is large. As $a \to 1$, the optimal mechanism described in the third part of Proposition 4, which is the unique symmetric mechanism when a is close to 1, converges to an optimal mechanism for the case when $a=1$.

case, these are types with a high but not too high signal), and that the seller's revenue decreases with a.[18]

6 Appendix

6.1 Appendix A: Omitted Proofs

Proof of Proposition 1:

Standard arguments will imply that $b_a(l) = E_{(\alpha_0,\alpha_1)}(v|l,l)$ for some (α_0,α_1) and $b_a(h) = E_{(\alpha_0',\alpha_1')}[v|h,h]$ for some (α_0',α_1'). We will consider monotone equilibria so that $b_a(l) < b_a(h)$.

Under the no hedging condition, deviations to mixed strategies will have lower utility, and thus equilibria are easier to sustain. We use this to characterize equilibria for large values of a. Of course all equilibria derived under no "no hedging" will remain equilibria under "no hedging".

Consider first the low type. For any bid $b \in [b_a(l), b_a(h))$, we have:

$$\min_{(\alpha_0,\alpha_1)} \rho Pr(l|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|l,l] - b_a(l)\right)$$
$$= \min_{\alpha_0,\alpha_1} \rho\alpha_1 - \rho(\alpha_1+\alpha_0)b_a(l)$$

where $\rho = 1$ if $b > b_a(l)$ and $\frac{1}{2}$ otherwise. This is minimised by $(\bar{\alpha}_0, \underline{\alpha}_1)$. Thus the conjectured equilibrium bid is $b_a(l) = E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)$. This will be the case for all equilibria considered.

Equilibrium with over-bidding for the high type:

Consider now the high type. Consider the case of an equilibrium that satisfies $b_a(l) > \frac{1}{2}b_a(h)$. Bidding $b = b_a(h)$ yields:

$$\min_{(\alpha_0,\alpha_1)} Pr(l|h,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|l,h] - b_a(l)\right) + \frac{1}{2}Pr(h|h,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|h,h] - b_a(h)\right) \tag{2}$$

where the optimal (α_0,α_1) is the same as the one that solves

$$\min_{\alpha_0,\alpha_1} -\alpha_0 \left(\frac{b_a(h)}{2} - b_a(l)\right) - \alpha_1 \left([1-b_a(l)] - \frac{[1-b_a(h)]}{2}\right)$$

Since by assumption $b_a(l) > \frac{1}{2}b_a(h)$, the payoff is minimised by $(\underline{\alpha}_0, \bar{\alpha}_1)$. Thus the conjectured equilibrium bid for the high type is $b_a(h) = E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$. Note that $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h] =$

[18] To insure existence in the continuous case, we consider small values of a.

$\frac{2q-1+a(1-q)^2}{a(1-q)^2+\frac{1}{a}(1-q)^2+2q-1}$ is increasing in a. The equilibrium will hold then only if $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l) > \frac{1}{2}E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$.

Note that the equilibrium payoff will be $Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h]-b_a(l)\right)$. This has to be non negative and thus under $(\underline{\alpha}_0,\bar{\alpha}_1)$, we must have $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|l,h] \geq b_a(l)$.

We now consider deviations.

For the low type, the payoff from any mixed strategy is:

$$\min_{(\alpha_0,\alpha_1)} \rho_0 Pr(l|l,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,l]-b_a(l)\right)+$$
$$\rho_1 Pr(h|l,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,l]-b_a(h)\right)$$

where $0 \leq \rho_1 \leq \rho_0 \leq 1$. Under the information structure $(\bar{\alpha}_0,\underline{\alpha}_1)$, the first term is 0. Note that $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|h,l] < b_a(h)$ is a necessary and sufficient condition for no deviation.

In that case, players bid $b = b_a(l)$, use $(\bar{\alpha}_0,\underline{\alpha}_1)$ as the information structure, and standard arguments imply that the equilibrium payoff is 0.

Let us now consider the high type. As long as the other player is playing the equilibrium (pure) strategy, the payoff from any mixed strategy is:

$$\min_{(\alpha_0,\alpha_1)} \rho_0 Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h]-b_a(l)\right)+$$
$$\rho_1 Pr(h|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,h]-b_a(h)\right),$$

where $0 \leq \rho_1 \leq \rho_0 \leq 1$. Under the information structure $(\underline{\alpha}_0,\bar{\alpha}_1)$, $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h] = b_a(h)$, which implies that the payoff from the deviation is at most $Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))\left(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|l,h]-b_a(l)\right)$, which is the equilibrium payoff. Thus, it is not profitable to deviate to any mixed strategy.

Bringing together all the conditions, we now have:

$E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|l,h] > E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]$

$E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|h,l] < E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$

$E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] > \frac{1}{2}E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$

For $a \leq \frac{q}{1-q}$, these conditions are:

(1) $\frac{1-q-a(1-q)^2}{1-(\frac{1}{a}(1-q)^2+2q-1)-a(1-q)^2} - \frac{\frac{1}{a}(1-q)^2}{\frac{1}{a}(1-q)^2+a(1-q)^2+2q-1} > 0$

(2) $\frac{2q-1+a(1-q)^2}{a(1-q)^2+\frac{1}{a}(1-q)^2+2q-1} - \frac{1-q-\frac{1}{a}(1-q)^2}{1-(a(1-q)^2+2q-1)-\frac{1}{a}(1-q)^2} > 0$

(3) $\frac{\frac{1}{a}(1-q)^2}{\frac{1}{a}(1-q)^2+a(1-q)^2+2q-1} - \frac{1}{2}\frac{2q-1+a(1-q)^2}{a(1-q)^2+\frac{1}{a}(1-q)^2+2q-1} > 0$

Condition (1) and (2) are satisfied for all q, while condition (3) is satisfied for all $q < \underline{q}(a)$.

For $a \geq \frac{q}{1-q}$, condition (3), now $\frac{\frac{1}{a}(1-q)^2}{\frac{1}{a}(1-q)^2+q-\frac{1}{a}q(1-q)} - \frac{1}{2}\frac{2q-1+1-q-\frac{1}{a}q(1-q)}{1-q-\frac{1}{a}q(1-q)+\frac{1}{a}(1-q)^2+2q-1} > 0$, is not satisfied for a which is above a cutoff $\bar{a}$. Note that allowing for no hedging will not affect the existence of this equilibrium for high a.

Equilibria with under-bidding for the high type:

Next consider the case $b_a(l) < \frac{1}{2}b_a(h)$. Consider the high type, and assume that the other player is playing the equilibrium strategy $(b_a(l), b_a(h))$.

Bidding $b = b_a(h)$ yields:

$$\min_{(\alpha_0,\alpha_1)} Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h] - b_a(l)\right) + \frac{1}{2}Pr(h|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,h] - b_a(h)\right)$$

which is like solving

$$\begin{aligned} & \min_{(\alpha_0,\alpha_1)} -\alpha_1 + (\alpha_1+\alpha_0)b_a(l) + \frac{1}{2}\alpha_1 - \frac{1}{2}(\alpha_1+\alpha_0)b_a(h) \\ = & \min_{(\alpha_0,\alpha_1)} -\alpha_1(\frac{1}{2} - b_a(l) + \frac{1}{2}b_a(h)) + \alpha_0(b_a(l) - \frac{1}{2}b_a(h)) \end{aligned}$$

For $b_a(l) < \frac{1}{2}b_a(h)$, this is minimised by $(\bar{\alpha}_0, \bar{\alpha}_1)$.

The equilibrium payoff is then $Pr(l|h,(\bar{\alpha}_0,\bar{\alpha}_1))\left(E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|l,h] - b_a(l)\right)$.

We now consider deviations. Let us consider first the high type. The payoff from any mixed strategy is:

$$\min_{(\alpha_0,\alpha_1)} \rho_0 Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h] - b_a(l)\right) + \rho_1 Pr(h|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,h] - b_a(h)\right),$$

where $0 \leq \rho_1 \leq \rho_0 \leq 1$. Under the information structure $(\bar{\alpha}_0, \bar{\alpha}_1)$, $E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h] = b_a(h)$. Note that this bid decreases with a.

Note that in equilibrium we must have $\left(E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|l,h] - E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]\right) \geq 0$, and that the equilibrium maximises the probability of winning against the low type.

Consider now the low type. Under no "no hedging", we have that the payoff from any mixed strategy is:

$$\min_{(\alpha_0,\alpha_1)} \rho_0 Pr(l|l,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,l] - b_a(l)\right) + \rho_1 Pr(h|l,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,l] - b_a(h)\right)$$

Under the information structure $(\bar{\alpha}_0, \underline{\alpha}_1)$, the first term is 0. Thus a necessary and sufficient condition for the low type not to deviate is $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|h,l] < E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$.

The equilibrium conditions as described above are therefore:

(4) $E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|l,h] > E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]$

(5) $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|h,l] < E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$

(6) $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] < \frac{1}{2}E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$.

Conditions (4) and (5) are satisfied for $a \leq \frac{q}{1-q}$, while condition (6) is satisfied for $q > \bar{q}(a)$.

To consider $a > \frac{q}{1-q}$, consider deviations of the low type under the "no hedging" condition. Her utility from a mixed strategy which wins against the low type only with probability β and with probability $1-\beta$ wins against the low type with probability 1 as well as against the high type with probability $\frac{1}{2}$ is:

$$\beta \min_{(\alpha_0,\alpha_1)} Pr(l|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|l,l] - b_a(l)\right)$$
$$+(1-\beta) \min_{(\alpha_0,\alpha_1)} (Pr(l|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|l,l] - b_a(l)\right) +$$
$$\frac{1}{2}Pr(h|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|h,l] - b_a(h)\right))$$

Note that $\arg\min_{(\alpha_0,\alpha_1)} Pr(l|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|l,l] - b_a(l)\right)$ is $(\overline{\alpha}_0, \underline{\alpha}_1)$, and thus this part of the utility is 0, and that

$$\begin{aligned} &\arg\min_{(\alpha_0,\alpha_1)} \Pr(l|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|l,l] - b_a(l)\right) + \frac{1}{2}Pr(h|l,(\alpha_0,\alpha_1)) \left(E_{(\alpha_0,\alpha_1)}[v|h,l] - b_a(h)\right) \\ = \; &(\underline{\alpha}_0, \underline{\alpha}_1) \end{aligned}$$

A necessary and sufficient condition under the no hedging is for the above utility to be lower than 0, the equilibrium utility.

Thus, for $a > \frac{q}{1-q}$, the equilibrium conditions are:

(4) $E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|l,h] > E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]$

(5*) $\Pr(l|l,(\underline{\alpha}_0,\underline{\alpha}_1)) \left(E_{\underline{\alpha}_0,\underline{\alpha}_1}[v|l,l] - b_a(l)\right) + \frac{1}{2}Pr(h|l,(\underline{\alpha}_0,\underline{\alpha}_1)) \left(E_{(\underline{\alpha}_0,\underline{\alpha}_1)}[v|h,l] - b_a(h)\right) <$
0

(6) $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] < \frac{1}{2}E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$.

This equilibrium exists when $a > \bar{a}(q) \geq \frac{q}{1-q}$, where $\bar{a}(q) > \frac{q}{1-q}$ for a low enough q but $\bar{a}(q) = \frac{q}{1-q}$ otherwise.

Note also that the equilibrium converges to the equilibrium in the limit where all information structures are allowed. To see the limit equilibrium, suppose that $b_a(l) = 0$. For the low type we minimize α_1 at 0 and set $\alpha_0 = 2q-1$ (which she is indifferent to) and hence $E(v|l,l) = 0$. We are therefore in the case in which $b_a(l) < \frac{1}{2}b_a(h)$ and hence the high type uses $\alpha_0 = q$ and $\alpha_1 = 1-q$. As a result, $b_a(h) = q = E(v|h) < E(v|h,h)$. This yields to the seller the lowest revenue.

Finally, consider the case $b_l(a) = \frac{1}{2}b_h(a)$. We will show that this equilibrium holds for $a < \frac{q}{1-q}$, for values $\underline{q}(a) < q < \bar{q}(a)$.

Let a and q satisfy: $\frac{1}{2}E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h] < E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] < \frac{1}{2}E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$

Consider the high type, and assume that the other player is playing the equilibrium strategy $(b_a(l), b_a(h))$.

Bidding $b = b_a(h)$ yields:

$$\min_{(\alpha_0,\alpha_1)} Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h]-b_a(l)\right)+\frac{1}{2}Pr(h|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,h]-b_a(h)\right)$$

Since $b_a(l) = \frac{1}{2}b_a(h)$, both $(\underline{\alpha}_0, \bar{\alpha}_1)$ and $(\bar{\alpha}_0, \bar{\alpha}_1)$ achieve the minimum payoff.

The payoff from any mixed strategy is:

$$\begin{aligned}&\min_{(\alpha_0,\alpha_1)} \rho_0 Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h]-b_a(l)\right)+\\ &\rho_1 Pr(h|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,h]-b_a(h)\right)\end{aligned}$$

Since $Pr_{(\alpha_0,\alpha_1)}(l|h)\left(E_{(\alpha_0,\alpha_1)}[v|l,h]-b_a(l)\right) > 0$ for any (α_0,α_1) and increasing ρ_0 relaxes the constraint on ρ_1, it is without loss to set $\rho_0 = 1$. Using the fact that $b_a(l) = \frac{1}{2}b_a(h)$, the payoff becomes:

$$\begin{aligned}&\min_{(\alpha_0,\alpha_1)} Pr(l|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,h]-b_a(l)\right)+\\ &\rho_1 Pr(h|h,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,h]-2b_a(l)\right) \qquad (3)\\ &=\min_{\alpha_0,\alpha_1} 1-b_a(l)-q+\rho_1(2q-1)+\alpha_0 b_a(l)(1-2\rho_1)+\alpha_1\left(\rho_1[1-2b_a(l)]-[1-b_a(l)]\right)\end{aligned}$$

The payoff is minimised by $(\underline{\alpha}_0, \bar{\alpha}_1)$ when $\rho_1 \leq \frac{1}{2}$ and $(\bar{\alpha}_0, \bar{\alpha}_1)$ when $\rho_1 \geq \frac{1}{2}$.

Suppose that $\rho_1 > \frac{1}{2}$. Then under $(\bar{\alpha}_0, \bar{\alpha}_1)$, the payoff is lower than when $\rho_1 = \frac{1}{2}$, since $E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h] < b_a(h)$. If $\rho_1 < \frac{1}{2}$, then under $(\underline{\alpha}_0, \bar{\alpha}_1)$, the payoff is lower than when $\rho_1 = \frac{1}{2}$, since $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h] > b_a(h)$. Thus, for $\rho_1 \neq \frac{1}{2}$, the payoff must be lower than when $\rho_1 = \frac{1}{2}$, which is the equilibrium payoff.

Now consider the low type. As before, the equilibrium payoff is 0. The payoff from any mixed strategy is:

$$\begin{aligned}&\min_{(\alpha_0,\alpha_1)} \rho_0 Pr(l|l,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|l,l]-b_a(l)\right)+\\ &\rho_1 Pr(h|l,(\alpha_0,\alpha_1))\left(E_{(\alpha_0,\alpha_1)}[v|h,l]-b_a(h)\right)\end{aligned}$$

Under the information structure $(\bar{\alpha}_0, \underline{\alpha}_1)$, the first term is 0 and the second term is negative if $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|h,l] < b_a(h)$.

So for this to hold we need $\frac{1}{2}E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h] < E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] < \frac{1}{2}E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$ and $\frac{1}{2}E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|h,l] < E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]$, which is satisfied for the range of qs considered.

We show that the seller's revenue decreases in a in the proof of Proposition A below.■

Proposition A: First-price auction: *In equilibrium, the minimum bid is* $b_a(l) = E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)$ *and the maximum bid* $b_a(h)$ *increases in* a. *(ii) The expected payment of the high type decreases in* a *for low* q *and increases in* a *for high* q. *(iii) The seller's* *revenue decreases in a. (iv) Revenue is higher in the first-price rather than the second-price auction.*

Proofs of Proposition A, Proposition 2 and Proposition 3: We first characterize the equilibria in the first-price auction, show that the maximum bid increases in a, and that the minimum bid decreases in a, for a close to 1.

Consider a low type. For any $b_a(l)$, this type's expected utility is perceived as

$$\begin{aligned} & \min_{\alpha_0,\alpha_1}(\alpha_0+\alpha_1)\left(\frac{\alpha_0}{\alpha_0+\alpha_1}\underline{v}+\frac{\alpha_1}{\alpha_0+\alpha_1}\bar{v}-b_a(l)\right) \\ = \; & \min_{\alpha_0,\alpha_1}\alpha_0(\underline{v}-b_a(l))+\alpha_1(\bar{v}-b_a(l)) \end{aligned}$$

which (assuming $b_a(l)>\underline{v}$) is resolved by setting α_0 to be the highest possible value and α_1 to be the lowest possible value, given the ePMI constraints. Therefore for a sufficiently close to 1, the solution is $(\bar{\alpha}_0,\underline{\alpha}_1)$.

Note that the low type cannot have any rent as in the standard model, and thus we set

$$b_a(l)=E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)=\frac{\frac{1}{a}(1-q)^2}{\frac{1}{a}(1-q)^2+a(1-q)^2+2q-1}<\frac{(1-q)^2}{(1-q)^2+q^2}$$

Taking a derivative of $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)$ with respect to a, it is straightforward to see that it is negative. Thus the bid of the low type decreases with a. We will establish later that this type will not want to use any other bid given the behaviour of the high type.

Now let us consider the high type. Wlog we can consider a mixed strategy with support on $[b_a(l),\bar{b}_a(h)]$, as bidding less than $b_a(l)$ will provide a zero utility.

First let us consider a bid just above $b_a(l)$ which allows the individual to win against the low type only. We then need to solve the following,

$$\begin{aligned} & \min_{(\alpha_0,\alpha_1)}\Pr(l|h,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|h,l)-b_a(l)) \\ = \; & \min_{\alpha_0,\alpha_1}(q-\alpha_0)(\underline{v}-b_a(l))+(1-q-\alpha_1)(\bar{v}-b_a(l)), \end{aligned}$$

which, as $b_a(l)>\underline{v}$, yields the need to maximize α_1 and to minimize α_0. The solution is $(\underline{\alpha}_0,\bar{\alpha}_1)$. Note that this bid provides a utility of $\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,l)-E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l))$, and that $\bar{\alpha}_0+\underline{\alpha}_1=\underline{\alpha}_0+\bar{\alpha}_1$.

We now consider the highest bid in the support, $\bar{b}_a(h)$. Such bid implies winning for sure and thus unambiguous gain of $E(v|h)$. To be indifferent, this bid has to satisfy

$$\begin{aligned} & E(v|h)-\bar{b}_a(h) \\ = \; & \Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,l)-E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)) \end{aligned}$$

Thus:

$$\begin{aligned}
\bar{b}_a(h) &= \Pr(h|h,(\underline{\alpha}_0,\bar{\alpha}_1))E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)+\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)\\
&= 2q-1+\bar{\alpha}_1+(1-\underline{\alpha}_0-\bar{\alpha}_1)\frac{\underline{\alpha}_1}{\bar{\alpha}_0+\underline{\alpha}_1}\\
&= 2q-1+\bar{\alpha}_1+\frac{\underline{\alpha}_1}{\bar{\alpha}_0+\underline{\alpha}_1}-\underline{\alpha}_1\\
&= 2q-1+a(1-q)^2+\frac{\frac{1}{a}(1-q)^2}{a(1-q)^2+2q-1+\frac{1}{a}(1-q)^2}-\frac{1}{a}(1-q)^2\\
&= 2q-1+(1-q)^2(a-\frac{1}{a}+\frac{\frac{1}{a}}{(a+\frac{1}{a}-1)(1-q)^2+q^2})
\end{aligned}$$

Note that the derivative of $a-\frac{1}{a}+\frac{\frac{1}{a}}{(\frac{1}{a}+a-1)(1-q)^2+q^2}$, evaluated at $a=1$, is $\frac{(2q-1)^2}{2q^2-2q+1}>0$. Thus the maximum bid increases in a.

We now continue to characterize the equilibrium distribution. Let us consider the worst case scenario in terms of utility for some distribution $F(b)$ with density $f(b)$. The expected utility is

$$\begin{aligned}
&\int_b f(b)[\Pr(l|h,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|h,l)-b)+\\
&\Pr(h|h,(\alpha_0,\alpha_1))F(b)(E_{(\alpha_0,\alpha_1)}(v|h,h)-b)]db\\
=\ &\int_b f(b)[E_{(\alpha_0,\alpha_1)}(v|h)-b-\\
&(1-F(b))\Pr(h|h,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|h,h)-b)]db
\end{aligned}$$

To choose the information structure to minimize utility, we maximise

$$\begin{aligned}
&\Pr(h|h,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|h,h)-b)\\
=\ &(2q-1)(\bar{v}-\underline{v})+(\alpha_0(\underline{v}-b)+\alpha_1(\bar{v}-b))
\end{aligned}$$

and the solution is therefore, for all b in $[\underline{v},\bar{v}]$, to maximize α_1 and to minimize α_0. $F(b)$ is simply characterized by using the indifference condition and so: $(\underline{\alpha}_0,\bar{\alpha}_1)$

$$\begin{aligned}
&\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,l)-b)+\\
&\Pr(h|h,(\underline{\alpha}_0,\bar{\alpha}_1))F(b)(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)-b)\\
=\ &\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,l)-b_a(l))
\end{aligned}$$

implying that

$$F_a(b)=\frac{\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))(b-b_a(l))}{\Pr(h|h,(\underline{\alpha}_0,\bar{\alpha}_1))(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)-b)}.$$

We complete the equilibrium characterization by showing that given the strategy of the high type, the low type will not deviate.

For the low type, bidding any b above $b_a(l)$, we choose the belief to minimize expected utility:

$$\begin{aligned} & \min_{\alpha_1,\alpha_0} \Pr(l|l,(\alpha_0,\alpha_1))(E_{(\alpha_0,\alpha_1)}(v|l,l)-b) + \\ & \Pr(h|l,(\alpha_0,\alpha_1))F_a(b)(E_{(\alpha_0,\alpha_1)}(v|l,h)-b) \\ = & \min_{\alpha_0,\alpha_1} (\underline{v}-b)(\alpha_0(1-F_a(b))+F_a(b)q)+(\bar{v}-b)(\alpha_1(1-F_a(b))+F_a(b)(1-q)) \end{aligned}$$

As $F_a(b) \leq 1$, the solution is $(\underline{\alpha}_1, \bar{\alpha}_0)$.

This gives us a utility of $\Pr(l|l,(\bar{\alpha}_0,\underline{\alpha}_1))(E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l)-b)+$
$\Pr(h|l,(\bar{\alpha}_0,\underline{\alpha}_1))\frac{\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))(b-E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l))}{\Pr(h|h,(\underline{\alpha}_0,\bar{\alpha}_1))(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)-b)}(E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,h)-b) =$
$\Pr(h|l,(\bar{\alpha}_0,\underline{\alpha}_1))(b-E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,l))(\frac{\Pr(l|h,(\underline{\alpha}_0,\bar{\alpha}_1))}{\Pr(h|h,(\underline{\alpha}_0,\bar{\alpha}_1))}\frac{(E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,h)-b)}{(E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)-b)}-\frac{\Pr(l|l,(\bar{\alpha}_0,\underline{\alpha}_1))}{\Pr(h|l,(\bar{\alpha}_0,\underline{\alpha}_1))})$. Note that $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}(v|l,h) = \frac{1-q-\frac{1}{a}(1-q)^2}{1-\bar{\alpha}_0-\underline{\alpha}_1} < \frac{2q-1+a(1-q)^2}{\underline{\alpha}_0+\bar{\alpha}_1} = E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h)$, for a sufficiently close to 1, and that $\frac{\Pr_{(\underline{\alpha}_0,\bar{\alpha}_1)}(l|h)}{\Pr_{(\underline{\alpha}_0,\bar{\alpha}_1)}(h|h)} = \frac{1-\underline{\alpha}_0-\bar{\alpha}_1}{\underline{\alpha}_0+\bar{\alpha}_1} < \frac{(\bar{\alpha}_0+\underline{\alpha}_1)}{1-(\bar{\alpha}_0+\underline{\alpha}_1)} = \frac{\Pr(l|l,(\bar{\alpha}_0,\underline{\alpha}_1))}{\Pr(h|l,(\bar{\alpha}_0,\underline{\alpha}_1))}$, as $\bar{\alpha}_0+\underline{\alpha}_1 = \bar{\alpha}_1+\underline{\alpha}_0 > \frac{1}{2}$ for all a. Thus the utility is negative and the low type does not deviate.

We now proceed to consider the payoff of the seller.

Expected payment to seller, Π, is given by the linear combination of receiving the bid of the low type (when both are l), the expected bid of the high type (when only one is h), and the maximum bid of the two h types:

$$\Pi = \Pr(l,l)E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] + 2\Pr(l,h)E[b_a(h)] + \Pr(h,h)E[\max_{i=1,2} b_a^i(h)]$$

For expositional purposes we write this as

$$\Pi = \frac{1}{2}\gamma y + (1-\gamma)\int_y^{\alpha x+(1-\alpha)y} bf_a(b)db + \frac{\gamma}{2}\int_y^{\alpha x+(1-\alpha)y} b2f(b)F_a(b)db,$$

where:

$\gamma = \Pr(l|l)$, according to the true (independent) information structure, $\alpha = \Pr(l|l,(\underline{\alpha}_0,\bar{\alpha}_1))$ according to the belief of the high bidder, $(\underline{\alpha}_0,\bar{\alpha}_1)$, $x = E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$, $y = E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] = b_a(l)$. We therefore also have $\bar{b}_a(h) = \alpha x + (1-\alpha)y$, $F_a(b) = \frac{1-\alpha}{\alpha}\frac{b-y}{x-b}$ and $f_a(b) = \frac{1}{\alpha}\frac{1-\alpha}{(x-b)^2}(x-y)$.

We start by some preliminary results:

Fact 1

$$\frac{\partial x}{\partial a} = -\frac{\partial y}{\partial a} > 0$$

Proof of Fact 1:

Note that $x = E_{(\underline{\alpha}_0,\bar{\alpha}_1)}(v|h,h) = \frac{2q-1+a(1-q)^2}{q^2-(1-\frac{1}{a})(1-q)^2+a(1-q)^2}$

$\frac{\partial x}{\partial a} = \frac{\partial}{\partial a}(\frac{2q-1+a(1-q)^2}{q^2-(1-\frac{1}{a})(1-q)^2+a(1-q)^2}) = (q-1)^2\frac{2a+2q+2aq^2-4aq-1}{(a^2q^2-2a^2q+a^2+2aq-a+q^2-2q+1)^2} > 0$

$\frac{\partial y}{\partial a} = \frac{\partial}{\partial a} \frac{\frac{1}{a}(1-q)^2}{2q-1+a(1-q)^2+\frac{1}{a}(1-q)^2} = -(q-1)^2 \frac{2a+2q+2aq^2-4aq-1}{(a^2q^2-2a^2q+a^2+2aq-a+q^2-2q+1)^2}.$■

Note that this fact proves Proposition 2. For the case 2b of Proposition 1, for low q and low a, $\Pr(l|h) > \frac{1}{2}\Pr(h|h)$, $\frac{\partial x}{\partial a} = -\frac{\partial y}{\partial a}$, and thus overall average bid for the high type decreases. In all other cases, the high type reduces her bid with a. Finally the bid of the low type is always lower than in the canonical model. This implies also that the seller's revenue decreases for all a.

Fact 2

$$\frac{\partial \alpha}{\partial a}_{|a=1} = 0$$

Proof of Fact 2:

$\alpha = \Pr_{(\underline{\alpha}_0, \bar{\alpha}_1)}(l|l) = \underline{\alpha}_0 + \bar{\alpha}_1 = a(1-q)^2 + \frac{1}{a}(1-q)^2 + 2q - 1$

$\frac{\partial \alpha}{\partial a}_{|a=1} = (1-q)^2 \frac{\partial(a+\frac{1}{a})}{\partial a}_{|a=1} = (1-q)^2(1-\frac{1}{a^2})_{|a=1} = 0.$■

Fact 3 The bid of the low type is decreasing in a.

Proof of Fact 3: Follows from Fact 1.■

Fact 4 (i)

$$E[b_a(h)] = x(1 + \frac{1-\alpha}{\alpha}\ln(1-\alpha)) - y\frac{1-\alpha}{\alpha}\ln(1-\alpha).$$

(ii) At $a = 1$, the expected bid of the high type decreases in a for low q and increases in a for high q.

Proof of Fact 4:

(i) Note that $\int \frac{b}{(x-b)^2}db = \frac{1}{x-b}\left(x - b\ln(x-b) + x\ln(x-b)\right) = \frac{x-(b-x)\ln(x-b)}{x-b}$, therefore, $\int_y^{\alpha x+(1-\alpha)y} \frac{b}{(b-x)^2}db = \frac{\alpha x}{(1-\alpha)(x-y)} + \ln(1-\alpha)$.

Hence

$E[b_a(h)] = \frac{1-\alpha}{\alpha}(x-y)\int_y^{\alpha x+(1-\alpha)y} \frac{b}{(b-x)^2}db = x(1+\frac{1-\alpha}{\alpha}\ln(1-\alpha)) - y\frac{1-\alpha}{\alpha}\ln(1-\alpha).$

(ii) As $\frac{\partial \alpha}{\partial a}_{|a=1} = 0$ and $\frac{\partial x}{\partial a}_{|a=1} = -\frac{\partial y}{\partial a}_{|a=1}$, $\frac{\partial E[b_a(h)]}{\partial a}_{|a=1} = \frac{\partial x}{\partial a}_{|a=1}(1+2\frac{1-\alpha}{\alpha}\ln(1-\alpha))_{|a=1} = \frac{\partial x}{\partial a}_{|a=1}(1+2\frac{1-2(1-q)^2-2q+1}{2(1-q)^2+2q-1}\ln(1-2(1-q)^2-2q+1))$.

For $q > \frac{1}{2}$, the expression $(1+2\frac{1-2(1-q)^2-2q+1}{2(1-q)^2+2q-1}\ln(1-2(1-q)^2-2q+1))$ is strictly increasing, negative for $q < q^*$ and positive for $q > q^*$ for some $q^* \in (0.5, 1)$. As $\frac{\partial x}{\partial a}_{|a=1} > 0$, we are done.■

Fact 5 (i)

$$E[\max_{i=1,2} b_a^i(h)] = (x-y)\,2((\frac{1-\alpha}{\alpha})^2\ln(1-\alpha) - \frac{1-\alpha}{\alpha}) + x$$

(ii) The expectation of the maximal bid when both are high types decreases in a for low q and increases in a for high q.

Proof of Fact 5:

(i) $\int \frac{b(b-y)}{(x-b)^3} db = \frac{1}{2(x-b)^2} \left(2b^2 \ln(x-b) + 2x^2 \ln(x-b) - 4bx + 2by - xy + 3x^2 - 4bx \ln(x-b)\right) = -\ln(x-b) - \frac{-4bx+2by-xy+3x^2}{2(x-b)^2}$. Therefore, $\int_y^{\alpha x+(1-\alpha)y} \frac{b(b-y)}{(x-b)^3} db = -\ln(1-\alpha) - \frac{\alpha(2x-2y-3x\alpha+2y\alpha)}{2(1-\alpha)^2(x-y)}$.

Hence

$E[\max_{i=1,2} b_a^i(h)] = 2(\frac{1-\alpha}{\alpha})^2 (x-y) \int_y^{\alpha x+(1-\alpha)y} \frac{b(b-y)}{(x-b)^3} db = -2(\frac{1-\alpha}{\alpha})^2 (x-y) (\ln(1-\alpha) + \frac{\alpha(2x-2y-3x\alpha+2y\alpha)}{2(1-\alpha)^2(x-y)}) = -(x-y)\, 2((\frac{1-\alpha}{\alpha})^2 \ln(1-\alpha) + \frac{1-\alpha}{\alpha}) + x$.

(ii) Recalling that $\frac{\partial x}{\partial a} = -\frac{\partial y}{\partial a}$and that $\frac{\partial \alpha}{\partial a} = 0$ we have,

$\frac{\partial E[\max_{i=1,2} b_a^i(h)]}{\partial a} = \frac{\partial x}{\partial a}(-4(\frac{1-\alpha}{\alpha})^2 \ln(1-\alpha) - 4\frac{1-\alpha}{\alpha} + 1)$, and

$\frac{\partial E[\max_{i=1,2} b_a^i(h)]}{\partial a}_{|a=1} = \frac{\partial x}{\partial a}(-4(\frac{2q(1-q)}{q^2+(1-q)^2})^2 \ln(1-\alpha) - 4\frac{2q(1-q)}{q^2+(1-q)^2} + 1)$

For the expression $(-4(\frac{2q(1-q)}{q^2+(1-q)^2})^2 \ln(1-2q(1-q)) - 4\frac{2q(1-q)}{q^2+(1-q)^2} + 1)$ there is a $\bar{q} \in (0.5, 1)$ such that the expression is negative for $q < \bar{q}$ and positive for $q > \bar{q}$. As $\frac{\partial x}{\partial a} > 0$, we are done.■

Given the above facts we can write the profit function as:

$$\begin{aligned}
\Pi &= \frac{1}{2}\gamma y + (1-\gamma)x + \frac{1-\alpha}{\alpha}(x-y)(1-\gamma)\ln(1-\alpha) \\
&\quad + -\frac{1-\alpha}{\alpha}^2 (x-y)\,\gamma \ln(1-\alpha) - \frac{1}{\alpha}\gamma(\frac{2x-2y-3x\alpha+2y\alpha}{2}) \\
&= x(\frac{\gamma}{2} + \frac{\alpha-\gamma}{\alpha}(\frac{1-\alpha}{\alpha}\ln(1-\alpha)+1)) + y(1 - \frac{\gamma}{2} - \frac{\alpha-\gamma}{\alpha}(\frac{1-\alpha}{\alpha}\ln(1-\alpha)+1))
\end{aligned}$$

Taking a derivative with respect to a, recalling that $(d\alpha/da)_{|a=1} = 0$ and that $\frac{\partial x}{\partial a} = -\frac{\partial y}{\partial a}$ we get,

$$\begin{aligned}
\frac{\partial \Pi}{\partial a}_{|a=1} &= \frac{\partial x}{\partial a}(\gamma - 1 + 2\frac{\alpha-\gamma}{\alpha}(\frac{1-\alpha}{\alpha}\ln(1-\alpha)+1)) \\
&= \frac{\partial x}{\partial a}(\gamma - 1) < 0.\blacksquare
\end{aligned}$$

Finally we consider the different revenue of the seller in the first and second price auction. We also show that the profits of the seller are higher in the first-price auction.

In the second price auction, the low type always bids $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]$, and the high type bids at most $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$ (either $b_a(h) = E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$ for low a and q, where $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h] > E_{(\bar{\alpha}_0,\bar{\alpha}_1)}[v|h,h] = b_a(h)$ for higher q and a, or where $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h] > 2E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l] = b_a(h)$). Let $\bar{R}^{SPA}$ be the revenue from a virtual auction where the low type bids $E_{(\bar{\alpha}_0,\underline{\alpha}_1)}[v|l,l]$ and the high type bids $E_{(\underline{\alpha}_0,\bar{\alpha}_1)}[v|h,h]$. Then the actual revenue in the second price auction must be weakly less than $\bar{R}^{SPA}$.

The seller's revenue in the second price auction satisfies:

$$R^{SPA} \leq \bar{R}^{SPA} = x\left(\frac{\gamma}{2}\right) + y\left(1 - \frac{\gamma}{2}\right)$$

where $x = E_{(\underline{\alpha}_0, \bar{\alpha}_1)}[v|h,h]$, $y = E_{(\bar{\alpha}_0, \underline{\alpha}_1)}[v|l,l]$, and $\gamma = Pr(l|l)$. The seller's revenue in the first price auction is:

$$R^{FPA} = x\left[\frac{\gamma}{2} + \frac{\alpha-\gamma}{\alpha}\left(\frac{1-\alpha}{\alpha}\ln(1-\alpha)+1\right)\right] + y\left[1-\frac{\gamma}{2}-\frac{\alpha-\gamma}{\alpha}\left(\frac{1-\alpha}{\alpha}\ln(1-\alpha)+1\right)\right]$$

where $\alpha = Pr(l|l, (\underline{\alpha}_0, \bar{\alpha}_1))$. Thus, the difference in revenue between the two auctions is:

$$R^{FPA} - R^{SPA} \geq R^{FPA} - \bar{R}^{SPA} = (x-y)\left[\frac{\alpha-\gamma}{\alpha}\left(\frac{1-\alpha}{\alpha}\ln(1-\alpha)+1\right)\right] > 0.$$

Finally, to see that R^{SPA} is decreasing in a, first note that when a and q are low:

$$\frac{\partial R_1^{SPA}}{\partial a}_{|a=1} = \frac{\partial x}{\partial a}\left(\frac{\gamma}{2}\right) + \frac{\partial y}{\partial a}\left(1-\frac{\gamma}{2}\right) = -(1-\gamma)\frac{\partial x}{\partial a} < 0$$

Let $x' = E_{(\bar{\alpha}_0 \bar{\alpha}_1)}[v|h,h]$. Then the revenue when q is high and a is high:

$$R_2^{SPA} = x'\left(\frac{\gamma}{2}\right) + y\left(1-\frac{\gamma}{2}\right)$$

Therefore:

$$\frac{\partial R_2^{SPA}}{\partial a}_{|a=1} = \frac{\partial x'}{\partial a}\left(\frac{\gamma}{2}\right) + \frac{\partial y}{\partial a}\left(1-\frac{\gamma}{2}\right) < 0$$

since $\frac{\partial x'}{\partial a} < 0$. Finally in the last case:

$$\frac{\partial R_3^{SPA}}{\partial a}_{|a=1} = \frac{\partial y}{\partial a}\left(1+\frac{\gamma}{2}\right) < 0.$$

This completes the proofs for part (ii) of Proposition 2, Proposition 3 and Proposition A.■

Proposition 4*.
(i) When $q \leq q^*(a)$, an optimal mechanism is, for $i \in \{1,2\}$:

- $x_i(l,l) = x_i(h,h) = x_i(l,h) = x_i(h,l) = \frac{1}{2}$
- $t_i(h,l) = t_i(h,h) = t_i(l,h) = t_i(l,l) = \frac{1}{2}(1-q)$.

and the revenue of the seller is $1-q$.

(ii) When $q^*(a) < q < q^{**}(a)$, an optimal mechanism is:

- $x_i(l,l) = x_i(h,h) = \frac{1}{2}$, $x_i(l,h) = 0$, and $x_i(h,l) = 1$
- $t_i(l,l) = t_i(l,h) = \frac{1}{2}\underline{\alpha}_1$
- $t_i(h,h) = \frac{q+\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1}{2}$
- $t_i(h,l) = \frac{q+\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1}{2}$

and the revenue of the seller is $\frac{1}{2}\left(\underline{\alpha}_0 + 2\underline{\alpha}_1 + \overline{\alpha}_1 + q - q^2 - (1-q)^2\right)$.

(iii) When $q \geq q^{**}(a)$*, an optimal mechanism is*:

- $x_i(l,l) = x_i(h,h) = \frac{1}{2}$, $x_i(l,h) = 0$, and $x_i(h,l) = 1$
- $t_i(h,l) = \dfrac{1}{2}(1+\underline{\alpha}_0)$
- $t_i(h,h) = \dfrac{\underline{\alpha}_0}{2}$
- $t_i(l,h) = \dfrac{\underline{\alpha}_1}{2} + (\underline{\alpha}_0 + \underline{\alpha}_1)\dfrac{1-q-\overline{\alpha}_1-\underline{\alpha}_1}{2(\overline{\alpha}_0+\overline{\alpha}_1+\underline{\alpha}_0+\underline{\alpha}_1-1)}$
- $t_i(l,l) = \dfrac{\underline{\alpha}_1}{2} - (1-\underline{\alpha}_0-\underline{\alpha}_1)\dfrac{1-q-\overline{\alpha}_1-\underline{\alpha}_1}{2(\overline{\alpha}_0+\overline{\alpha}_1+\underline{\alpha}_0+\underline{\alpha}_1-1)}$

and the revenue of the seller is $\frac{1}{2}\left(1-(q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1)\dfrac{\underline{\alpha}_0+\overline{\alpha}_0-q}{\overline{\alpha}_0+\overline{\alpha}_1+\underline{\alpha}_0+\underline{\alpha}_1-1}\right)$.

Proof of Proposition 4*: The expressions for $q^*(a)$ and $q^{**}(a)$ are as follows:

$$q^*(a) = \begin{cases} \frac{1}{2}\left(\frac{6-7a+2a^2}{3-2a+a^2} + \sqrt{\frac{-12a+17a^2-4a^3}{(3-2a+a^2)^2}}\right) & 1 < a \leq \frac{1}{2}(-1+\sqrt{13}) \\ \frac{1}{4}\left(\frac{-7+6a}{-2+a} + \sqrt{\frac{1-12a+12a^2}{(-2+a)^2}}\right) & \frac{1}{2}(-1+\sqrt{13}) < a < 2 \\ \frac{3}{5} & a = 2 \\ \frac{1}{4}\left(\frac{-7+6a}{-2+a} - \sqrt{\frac{1-12a+12a^2}{(-2+a)^2}}\right) & a > 2 \end{cases}$$

$$q^{**}(a) = \begin{cases} \frac{2-2a+a^2}{2-a+a^2} + \frac{\sqrt{\frac{-2a+3a^2-a^3)}{(2-a+a^2)^2}}}{\sqrt{2}} & 1 < a \leq \frac{1}{2}(-1+\sqrt{17}) \\ \frac{1}{2}\left(\frac{-5+3a}{-3+a} + \sqrt{\frac{1-4a+3a^2}{(-3+a)^2}}\right) & \frac{1}{2}(-1+\sqrt{17}) < a < 3 \\ \frac{5}{8} & a = 3 \\ \frac{1}{2}\left(\frac{-5+3a}{-3+a} - \sqrt{\frac{1-4a+3a^2}{(-3+a)^2}}\right) & a > 3 \end{cases}$$

We will ignore the incentive constraint of the low type and check ex post that it is satisfied. Therefore, the participation constraint of the low type must be binding.

Let U_i^h be the utility of the high type in equilibrium and U_i^l be the utility of the low type in equilibrium, that is:

$$U_i^l \equiv \min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1))t_i(l,l) - (1-\alpha_0-\alpha_1)t_i(l,h)$$
$$U_i^h \equiv \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(h,l) + (2q-1+\alpha_1)x_i(h,h) - (1-\alpha_0-\alpha_1))t_i(h,l) - (\alpha_0+\alpha_1)t_i(h,h)$$

Note that it is optimal to set $U_i^l = 0$. The incentive constraint of the high type is:

$$\min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(l,l) + (2q-1+\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1))t_i(l,h) - (1-\alpha_0-\alpha_1)t_i(l,l) \leq U_i^h$$

The participation constraint of the low type is:

$$\min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1))t_i(l,l) - (1-\alpha_0-\alpha_1)t_i(l,h) = 0$$

We can subtract the latter from the former to get:

$$\frac{(1-q-\underline{\alpha}_1-\overline{\alpha}_1)x_i(l,l) + (3q-2+\underline{\alpha}_1+\overline{\alpha}_1)x_i(l,h) - U_i^h}{\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_0+\overline{\alpha}_1-1} \leq t_i(l,h) - t_i(l,l)$$

Define:

$$\Delta t_{l,i} \equiv t_i(l,h) - t_i(l,l)$$
$$\Delta t_{h,i} \equiv t_i(h,l) - t_i(h,h)$$

We can write the expected transfers to the seller from each type as:

$$T_i^l = \min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - \left(q^2+(1-q)^2-(\alpha_0+\alpha_1)\right)\Delta t_{l,i}$$
$$T_i^h = \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(h,l) + (2q-1+\alpha_1)x_i(h,h) - \left(q^2+(1-q)^2-(\alpha_0+\alpha_1)\right)\Delta t_{h,i} - U_i^h$$

Let α_0^l, α_1^l and α_0^h, α_1^h be solutions to these minimization problems. The seller chooses $x_i(l,l) \in [0, \frac{1}{2}]$, $x_i(h,h) \in [0, \frac{1}{2}]$, $x_i(l,h) \in [0,1]$, $x_i(h,l) \in [0, 1-x_i(l,h)]$, $\Delta t_{l,i} \in R$, $\Delta t_{h,i} \in R$, and $U_i^h \geq 0$ to maximize $T_i^l + T_i^h$ subject to:

$$\frac{(1-q-\underline{\alpha}_1-\overline{\alpha}_1)x_i(l,l) + (3q-2+\underline{\alpha}_1+\overline{\alpha}_1)x_i(l,h) - U_i^h}{\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_0+\overline{\alpha}_1-1} \leq \Delta t_{l,i}$$

Clearly, it is optimal to set $x_i(h,h) = \frac{1}{2}$ and $x_i(h,l) = 1 - x_i(l,h)$. Thus, the seller's problem is:

$$\max_{x_i(l,l),x_i(l,h),\Delta t_{l,i},\Delta t_{h,i},U_i^h} \Bigg\{ \min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - \left(q^2+(1-q)^2-(\alpha_0+\alpha_1)\right)\Delta t_{l,i}$$
$$+ \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)(1-x_i(h,l)) + \frac{1}{2}(2q-1+\alpha_1) - \left(q^2+(1-q)^2-(\alpha_0+\alpha_1)\right)\Delta t_{h,i} - U_i^h \Bigg\}$$

subject to:

$$\frac{(1-q-\underline{\alpha}_1-\overline{\alpha}_1)x_i(l,l)+(3q-2+\underline{\alpha}_1+\overline{\alpha}_1)x_i(l,h)-U_i^h}{\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_0+\overline{\alpha}_1-1} \leq \Delta t_{l,i}$$
$$0 \leq x_i(l,l) \leq \frac{1}{2}$$
$$0 \leq x_i(l,h) \leq 1$$

Define:

$$\Delta t^*_{l,i}(x_i(l,l),x_i(l,h),U_i^h) \equiv \max\left\{0, \frac{(1-q-\underline{\alpha}_1-\overline{\alpha}_1)x_i(l,l)+(3q-2+\underline{\alpha}_1+\overline{\alpha}_1)x_i(l,h)-U_i^h}{\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_0+\overline{\alpha}_1-1},\right.$$
$$\left. x_i(l,h)-x_i(l,l)\right\}$$

We show that $\Delta t_{l,i} = \Delta t^*_{l,i}(x_i(l,l),x_i(l,h),U_i^h)$ is optimal. Note that $\Delta t_{l,i} < 0$ implies $\alpha_0^l = \overline{\alpha}_0$, which implies that $q^2+(1-q)^2-\alpha_0^l-\alpha_1^l < 0$, so it is profitable to increase $\Delta t_{l,i}$ (which also slackens IC_h). Similarly when $\Delta t_{l,i} < x_i(l,h)-x_i(l,l)$, $\alpha_1^l = \overline{\alpha}_1$, which implies that $q^2+(1-q)^2-\alpha_0^l-\alpha_1^l < 0$, so it is profitable to increase $\Delta t_{l,i}$. If $\Delta t_{l,i} > x_i(l,h)-x_i(l,l) \geq 0$, then it is profitable to decrease $\Delta t_{l,i}$, which is possible if $\Delta t_{l,i} > \frac{(1-q-\underline{\alpha}_1-\overline{\alpha}_1)x_i(l,l)+(3q-2+\underline{\alpha}_1+\overline{\alpha}_1)x_i(l,h)-U_i^h}{\underline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_0+\overline{\alpha}_1-1}$.

Define:

$$\Delta t^*_{h,i}(x_i(l,h)) \equiv \max\left\{0, \frac{1}{2}-x_i(l,h)\right\}$$

Similarly, it is optimal to set $\Delta t_{h,i} = \Delta t^*_{h,i}(x_i(l,h))$. Thus, the problem becomes:

$$\max_{x_i(l,l),x_i(l,h),U_i^h} R(x_i(l,l),x_i(l,h),U_i^h) = \min_{\alpha_1} \left\{\alpha_1 x_i(l,l)+(1-q-\alpha_1)x_i(l,h)\right.$$
$$\left. -\left(q^2+(1-q)^2-(\underline{\alpha}_0+\alpha_1)\right)\Delta t^*_{l,i}(x_i(l,l),x_i(l,h),U_i^h)\right\}$$
$$+\min_{\alpha_0,\alpha_1}\left\{(1-q-\alpha_1)(1-x_i(l,h))+\frac{2q-1+\alpha_1}{2}\right.$$
$$\left. -\left(q^2+(1-q)^2-(\alpha_0+\alpha_1)\right)\Delta t^*_{h,i}(x_i(l,h))-U_i^h\right\}$$

Now we show that $x_i(l,l)=\frac{1}{2}$ is optimal. Note that:

$$\frac{\partial R}{\partial x_i(l,l)} = \begin{cases} \underline{\alpha}_1 - \frac{(q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1)(1-q-\underline{\alpha}_1-\overline{\alpha}_1)}{\underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1} > 0 & x_i(l,l) > \max\left\{x_i(l,h), \frac{-x_i(l,h)(3q-2+\underline{\alpha}_1+\overline{\alpha}_1)+U_i^h}{1-q-\underline{\alpha}_1-\overline{\alpha}_1}\right\} \\ \underline{\alpha}_1 > 0 & x_i(l,h) < x_i(l,l) < \frac{-x_i(l,h)(3q-2+\underline{\alpha}_1+\overline{\alpha}_1)+U_i^h}{1-q-\underline{\alpha}_1-\overline{\alpha}_1} \\ q^2+(1-q)^2-\underline{\alpha}_0 > 0 & \frac{-x_i(l,h)(3q-1-\underline{\alpha}_0-\overline{\alpha}_0)+U_i^h}{\underline{\alpha}_0+\overline{\alpha}_0-q} < x_i(l,l) < x_i(l,h) \\ \overline{\alpha}_1 - \frac{(q^2+(1-q)^2-\underline{\alpha}_0-\overline{\alpha}_1)(1-q-\underline{\alpha}_1-\overline{\alpha}_1)}{\underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1} > 0 & x_i(l,l) < \min\left\{x_i(l,h), \frac{-x_i(l,h)(3q-1-\underline{\alpha}_0-\overline{\alpha}_0)+U_i^h}{\underline{\alpha}_0+\overline{\alpha}_0-q}\right\} \end{cases}$$

Thus, the problem becomes:

$$\begin{aligned}\max_{x_i(l,h),U_i^h} \ R(x_i(l,h),U_i^h) = \min_{\alpha_1} \Bigg\{ & \frac{\alpha_1}{2} + (1-q-\alpha_1)x_i(l,h) \\ & - \left(q^2 + (1-q)^2 - (\underline{\alpha}_0 + \alpha_1)\right) \Delta t^*_{l,i}(x_i(l,h),U_i^h) \Bigg\} \\ & + \min_{\alpha_1} \Bigg\{ (1-q-\alpha_1)(1-x_i(l,h)) + \frac{2q-1+\alpha_1}{2} \\ & - \left(q^2 + (1-q)^2 - (\underline{\alpha}_0 + \alpha_1)\right) \Delta t^*_{h,i}(x_i(l,h)) - U_i^h \Bigg\}\end{aligned}$$

Now we find the optimal $x_i(l,h)$ as a function of U_i^h:

$$\frac{\partial R(x_i(l,h),U_i^h)}{\partial x_i(l,h)} = \begin{cases} -\frac{(q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1)(3q-2+\underline{\alpha}_1+\overline{\alpha}_1)}{\underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1} < 0 & x_i(l,h) > \max\left\{\frac{-\frac{1}{2}(\underline{\alpha}_0+\overline{\alpha}_0-q)+U_i^h}{3q-1-\underline{\alpha}_0-\overline{\alpha}_0}, \frac{1}{2}\right\} \\ -(q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1) < 0 & \frac{1}{2} < x_i(l,h) < \frac{-\frac{1}{2}(\underline{\alpha}_0+\overline{\alpha}_0-q)+U_i^h}{3q-1-\underline{\alpha}_0-\overline{\alpha}_0} \\ -\frac{(q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1)(3q-1-\underline{\alpha}_0-\overline{\alpha}_0)}{\underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1} < 0 & \frac{-\frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1)+U_i^h}{3q-2+\underline{\alpha}_1+\overline{\alpha}_1} < x_i(l,h) < \frac{1}{2} \\ q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1 > 0 & x_i(l,h) < \min\left\{\frac{1}{2}, \frac{-\frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1)+U_i^h}{3q-2+\underline{\alpha}_1+\overline{\alpha}_1}\right\} \end{cases}$$

Therefore:

$$x_i^*(l,h)(U^h) = \begin{cases} 0 & U_i^h \leq \frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1) \\ \frac{-\frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1)+U_i^h}{3q-2+\underline{\alpha}_1+\overline{\alpha}_1} & \frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1) < U_i^h < \frac{1}{2}(2q-1) \\ \frac{1}{2} & U_i^h \geq \frac{1}{2}(2q-1) \end{cases}$$

We can now write the problem just in terms of U_i^h:

$$\begin{aligned}\max_{U_i^h} \ R(U_i^h) = & \frac{\underline{\alpha}_1}{2} + (1-q-\underline{\alpha}_1)x_i^*(l,h)(U_i^h) - \left(q^2+(1-q)^2-(\underline{\alpha}_0+\underline{\alpha}_1)\right)\Delta t^*_{l,i}(U_i^h) \\ & + (1-q-\underline{\alpha}_1)(1-x_i^*(l,h)(U_i^h)) + \frac{2q-1+\underline{\alpha}_1}{2} \\ & - \left(q^2+(1-q)^2-(\underline{\alpha}_0+\underline{\alpha}_1)\right)\Delta t^*_{h,i}(x^*(l,h)(U_i^h)) - U_i^h\end{aligned}$$

$$\frac{\partial R(U_i^h)}{\partial U_i^h} = \begin{cases} \frac{q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1}{\underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1} - 1 & U_i^h < \frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1) \\ \frac{q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1}{3q-2+\underline{\alpha}_1+\underline{\alpha}_1} - 1 & \frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1) < U_i^h < \frac{1}{2}(2q-1) \\ -1 & U_i^h > \frac{1}{2}(2q-1) \end{cases}$$

Thus,

$$U^h = \begin{cases} 0 & q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1 \leq \underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1 \\ \frac{1}{2}(1-q-\underline{\alpha}_1-\overline{\alpha}_1) & \underline{\alpha}_0+\overline{\alpha}_0+\underline{\alpha}_1+\overline{\alpha}_1-1 < q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1 < 3q-2+\underline{\alpha}_1+\underline{\alpha}_1 \\ \frac{1}{2}(2q-1) & q^2+(1-q)^2-\underline{\alpha}_0-\underline{\alpha}_1 \geq 3q-2+\underline{\alpha}_1+\underline{\alpha}_1 \end{cases}$$

This is equivalent to:

$$U^h = \begin{cases} 0 & q \geq q^{**}(a) \\ \frac{1}{2}(1 - q - \underline{\alpha}_1 - \overline{\alpha}_1) & q^*(a) < q < q^{**}(a) \\ \frac{1}{2}(2q - 1) & q \leq q^*(a) \end{cases}$$

Thus, in the optimal symmetric mechanism, $x_i(l, l) = x_i(h, h) = \frac{1}{2}$, and:

$$x_i(l, h) = \begin{cases} 0 & q \geq q^{**}(a) \\ \frac{1}{2} & q^*(a) < q < q^{**}(a) \\ \frac{1}{2} & q \leq q^*(a) \end{cases}$$

$$\Delta t_{l,i} = \begin{cases} \frac{1 - q - \underline{\alpha}_1 - \overline{\alpha}_1}{2(\underline{\alpha}_0 + \underline{\alpha}_1 + \overline{\alpha}_0 + \overline{\alpha}_1 - 1)} & q \geq q^{**}(a) \\ 0 & q^*(a) < q < q^{**}(a) \\ 0 & q \leq q^*(a) \end{cases}$$

$$\Delta t_{h,i} = \begin{cases} \frac{1}{2} & q \geq q^{**}(a) \\ \frac{1}{2} & q^*(a) < q < q^{**}(a) \\ 0 & q \leq q^*(a) \end{cases}$$

To recover the transfers $t_i(l, l)$, $t_i(l, h)$, $t_i(h, l)$, and $t_i(h, h)$, use:

$$\min_{\alpha_0, \alpha_1} \; \alpha_1 x_i(l, l) + (1 - q - \alpha_1) x_i(l, h) - (\alpha_0 + \alpha_1)) t_i(l, l) - (1 - \alpha_0 - \alpha_1)(t_i(l, l) + \Delta t_{l,i}) = 0$$

$$\min_{\alpha_0, \alpha_1} \Big\{ (1 - q - \alpha_1) x_i(h, l) + (2q - 1 + \alpha_1) x_i(h, h)$$
$$- (1 - \alpha_0 - \alpha_1))(t_i(h, h) + \Delta t_{h,i}) - (\alpha_0 + \alpha_1) t_i(h, h) \Big\} = U_i^h$$

Now we show that the optimal symmetric mechanism is fully optimal. Suppose that there exists an asymmetric mechanism (x, t) that is optimal. Define:

$$\overline{x}(\cdot, \cdot) \equiv \frac{1}{2} x_1(\cdot, \cdot) + \frac{1}{2} x_2(\cdot, \cdot)$$
$$\overline{t}(\cdot, \cdot) \equiv \frac{1}{2} t_1(\cdot, \cdot) + \frac{1}{2} t_2(\cdot, \cdot)$$

Consider the following symmetric mechanism:

$$x_i'(\cdot,\cdot) = \overline{x}(\cdot,\cdot)$$

$$t_i'(l,\cdot) = \overline{t}(l,\cdot) + \min_{\alpha_0,\alpha_1}\Big\{\alpha_1\overline{x}(l,l) + (1-q-\alpha_1)\overline{x}(l,h)$$

$$- (\alpha_0+\alpha_1)\overline{t}(l,l) - (1-\alpha_0-\alpha_1)\overline{t}(l,h) - \frac{1}{2}\sum_{i=1}^{2} U_i^l\Big\}$$

$$t_i'(h,\cdot) = \overline{t}(h,\cdot) + \min_{\alpha_0,\alpha_1}\Big\{(1-q-\alpha_1)\overline{x}(h,l) + (2q-1+\alpha_1)\overline{x}(h,h)$$

$$- (1-\alpha_0+\alpha_1)\overline{t}(h,l) - (\alpha_0+\alpha_1)\overline{t}(h,h) - \frac{1}{2}\sum_{i=1}^{2} U_i^h\Big\}$$

By construction, the high type gets $\frac{1}{2}\sum_{i=1}^{2} U_i^h$ in equilibrium and the low type gets $\frac{1}{2}\sum_{i=1}^{2} U_i^l$ in equilibrium; therefore both participation constraints are satisfied.

Define $\Delta\overline{x}_l \equiv \overline{x}(l,h) - \overline{x}(l,l)$, $\Delta\overline{t}_l \equiv \overline{t}(l,h) - \overline{t}(l,l)$, $\Delta x_{l,i} \equiv x_i(l,h) - x_i(l,l)$, and $\Delta t_{l,i} \equiv t_i(l,h) - t_i(l,l)$. To see that IC_h is satisfied, first note that:

$$\min_{\alpha_0,\alpha_1} (1-q-\alpha_1)\overline{x}(l,l) + (2q-1+\alpha_1)\overline{x}(l,h) - (1-\alpha_0-\alpha_1)\overline{t}(l,l) - (\alpha_0+\alpha_1)\overline{t}(l,h)$$

$$= \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(l,l) + (2q-1+\alpha_1)x_i(l,h) - (1-\alpha_0-\alpha_1)t_i(l,l) - (\alpha_0+\alpha_1)t_i(l,h)$$

$$+ \min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_l - (\alpha_0+\alpha_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2}\min_{\alpha_0,\alpha_1} \alpha_1\Delta x_{l,i} - (\alpha_0+\alpha_1)\Delta t_{l,i}$$

and by definition:

$$\frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1)t_i(l,l) - (1-\alpha_0-\alpha_1)t_i(l,h) - U_i^l = 0$$

Therefore:

$$\min_{\alpha_0,\alpha_1} (1-q-\alpha_1)\overline{x}(l,l) + (2q-1+\alpha_1)\overline{x}(l,h) - (1-\alpha_0-\alpha_1)\overline{t}(l,l) - (\alpha_0+\alpha_1)\overline{t}(l,h)$$

$$- \left(\min_{\alpha_0,\alpha_1} \alpha_1\overline{x}(l,l) + (1-q-\alpha_1)\overline{x}(l,h) - (\alpha_0+\alpha_1)\overline{t}(l,l) - (1-\alpha_0-\alpha_1)\overline{t}(l,h) \right) + \frac{1}{2}\sum_{i=1}^{2} U_i^l$$

$$= \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(l,l) + (2q-1+\alpha_1)x_i(l,h) - (1-\alpha_0-\alpha_1)t_i(l,l) - (\alpha_0+\alpha_1)t_i(l,h)$$

$$+ \min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_l - (\alpha_0+\alpha_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} \alpha_1\Delta x_{l,i} - (\alpha_0+\alpha_1)\Delta t_{l,i}$$

$$+ \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} \alpha_1 x_i(l,l) + (1-q-\alpha_1)x_i(l,h) - (\alpha_0+\alpha_1)t_i(l,l) - (1-\alpha_0-\alpha_1)t_i(l,h) - U_i^l$$

$$- \left(\min_{\alpha_0,\alpha_1} \alpha_1\overline{x}(l,l) + (1-q-\alpha_1)\overline{x}(l,h) - (\alpha_0+\alpha_1)\overline{t}(l,l) - (1-\alpha_0-\alpha_1)\overline{t}(l,h) \right) + \frac{1}{2}\sum_{i=1}^{2} U_i^l$$

$$= \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} (1-q-\alpha_1)x_i(l,l) + (2q-1+\alpha_1)x_i(l,h) - (1-\alpha_0-\alpha_1)t_i(l,l) - (\alpha_0+\alpha_1)t_i(l,h)$$

$$+ \min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_l - (\alpha_0+\alpha_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} \alpha_1\Delta x_{l,i} - (\alpha_0+\alpha_1)\Delta t_{l,i}$$

$$+ \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} -\alpha_1\Delta x_{l,i} + (\alpha_0+\alpha_1)\Delta t_{l,i} - \min_{\alpha_0,\alpha_1} -\alpha_1\Delta\overline{x}_l + (\alpha_0+\alpha_1)\Delta\overline{t}_l \leq \frac{1}{2}\sum_{i=1}^{2} U_i^h$$

The last inequality follows because:

$$\min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_l - (\alpha_0+\alpha_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} \alpha_1\Delta x_{l,i} - (\alpha_0+\alpha_1)\Delta t_{l,i}$$

$$+ \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} -\alpha_1\Delta x_{l,i} + (\alpha_0+\alpha_1)\Delta t_{l,i} - \min_{\alpha_0,\alpha_1} -\alpha_1\Delta\overline{x}_l + (\alpha_0+\alpha_1)\Delta\overline{t}_l$$

$$= \min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_l - (\alpha_0+\alpha_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2} \min_{\alpha_0,\alpha_1} \alpha_1\Delta x_{l,i} - (\alpha_0+\alpha_1)\Delta t_{l,i}$$

$$+ \max_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_l - (\alpha_0+\alpha_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2} \max_{\alpha_0,\alpha_1} \alpha_1\Delta x_{l,i} - (\alpha_0+\alpha_1)\Delta t_{l,i}$$

$$= (\underline{\alpha}_1 + \overline{\alpha}_1)\Delta\overline{x}_l - (\underline{\alpha}_0 + \overline{\alpha}_0 + \underline{\alpha}_1 + \overline{\alpha}_1)\Delta\overline{t}_l - \frac{1}{2}\sum_{i=1}^{2}(\underline{\alpha}_1 + \overline{\alpha}_1)\Delta x_{l,i} - (\underline{\alpha}_0 + \overline{\alpha}_0 + \underline{\alpha}_1 + \overline{\alpha}_1)\Delta t_{l,i} = 0$$

The proof that IC_l is satisfied is analogous. Define $\Delta\overline{x}_h \equiv \overline{x}(h,l) - \overline{x}(h,h)$, $\Delta\overline{t}_h \equiv$

$\bar{t}(h,l) - \bar{t}(h,h)$, $\Delta x_{h,i} \equiv x_i(h,l) - x_i(h,h)$, and $\Delta t_{h,i} \equiv t_i(h,l) - t_i(h,h)$. Then:

$$\min_{\alpha_0,\alpha_1} \ \alpha_1\overline{x}(h,l) + (1-q-\alpha_1)\overline{x}(h,h) - (\alpha_0+\alpha_1)\bar{t}(h,l) - (1-\alpha_0-\alpha_1)\bar{t}(h,h)$$
$$-\left(\min_{\alpha_0,\alpha_1} \ (1-q-\alpha_1)\overline{x}(h,l) + (2q-1+\alpha_1)\overline{x}(h,h) - (1-\alpha_0-\alpha_1)\bar{t}(h,l) - (\alpha_0+\alpha_1)\bar{t}(h,h)\right)$$
$$+\frac{1}{2}\sum_{i=1}^{2} U_i^h$$
$$=\frac{1}{2}\sum_{i=1}^{2} \ \min_{\alpha_0,\alpha_1} \ \alpha_1 x_i(h,l) + (1-q-\alpha_1)x_i(h,h) - (\alpha_0+\alpha_1)t_i(h,l) - (1-\alpha_0-\alpha_1)t_i(h,h)$$
$$+\min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_h - (\alpha_0+\alpha_1)\Delta\bar{t}_h - \frac{1}{2}\sum_{i=1}^{2}\min_{\alpha_0,\alpha_1}\alpha_1\Delta x_{h,i} - (\alpha_0+\alpha_1)\Delta t_{h,i}$$
$$+\frac{1}{2}\sum_{i=1}^{2} \ \left(\min_{\alpha_0,\alpha_1} \ (1-q-\alpha_1)x_i(h,l) + (2q-1+\alpha_1)x_i(h,h) - (1-\alpha_0-\alpha_1)t_i(h,l) - (\alpha_0+\alpha_1)t_i(h,h)\right.$$
$$\left. - U_i^h\right)$$
$$-\left(\min_{\alpha_0,\alpha_1} \ (1-q-\alpha_1)\overline{x}(h,l) + (2q-1+\alpha_1)\overline{x}(h,h) - (1-\alpha_0-\alpha_1)\bar{t}(h,l) - (\alpha_0+\alpha_1)\bar{t}(h,h)\right)$$
$$+\frac{1}{2}\sum_{i=1}^{2} U_i^h$$
$$=\frac{1}{2}\sum_{i=1}^{2} \ \min_{\alpha_0,\alpha_1} \ \alpha_1 x_i(h,l) + (1-q-\alpha_1)x_i(h,h) - (\alpha_0+\alpha_1)t_i(h,l) - (1-\alpha_0-\alpha_1)t_i(h,h)$$
$$+\min_{\alpha_0,\alpha_1} \alpha_1\Delta\overline{x}_h - (\alpha_0+\alpha_1)\Delta\bar{t}_h - \frac{1}{2}\sum_{i=1}^{2}\min_{\alpha_0,\alpha_1}\alpha_1\Delta x_{h,i} - (\alpha_0+\alpha_1)\Delta t_{h,i}$$
$$+\frac{1}{2}\sum_{i=1}^{2} \ \min_{\alpha_0,\alpha_1} -\alpha_1\Delta x_{h,i} + (\alpha_0+\alpha_1)\Delta t_{h,i} - \min_{\alpha_0,\alpha_1} \ -\alpha_1\Delta\overline{x}_h + (\alpha_0+\alpha_1)\Delta\bar{t}_h \leq \frac{1}{2}\sum_{i=1}^{2} U_i^l$$

Finally, note that $\frac{1}{2}\sum_{i=1}^{2} t_i'(l,l) \geq \bar{t}(l,l)$, $\frac{1}{2}\sum_{i=1}^{2} t_i'(l,h) \geq \bar{t}(l,h)$, $\frac{1}{2}\sum_{i=1}^{2} t_i'(h,l) \geq \bar{t}(h,l)$, and $\frac{1}{2}\sum_{i=1}^{2} t_i'(h,h) \geq \bar{t}(h,h)$, so the symmetric mechanism (x',t') is incentive compatible and yields weakly greater revenue to the seller than (x,t).■

Proof of Proposition 5: First, we consider first-price auctions. Let us above let us conjecture the minimum bid $b_\varepsilon(l)$ for the low type and a distribution for h on $[b_\varepsilon(l), \bar{b}_\varepsilon(h)]$. Let us first compute the bid of the low type. We aim to minimize for any b the utility from winning upon the other receiving the low signal, in other words:

$$\begin{aligned} & \min_{p\in[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]} \Pr_{\varepsilon}(l|l)(E_{\varepsilon}(v|l,l)-b_{\varepsilon}(l)) \\ = & \min_{p\in[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]} (1-p)q^2(-b_{\varepsilon}(l))+p(1-q)^2(1-b_{\varepsilon}(l)) \end{aligned}$$

Assuming $b_{\varepsilon}(l) > \underline{v} = 0$ we then have $p = \frac{1}{2} - \varepsilon$. Given that the rent for the low type would be 0 in equilibrium, we then set

$$b_{\varepsilon}(l) = E_{\varepsilon}(v|l,l) = \frac{(\frac{1}{2}-\varepsilon)(1-q)^2}{(\frac{1}{2}+\varepsilon)q^2+(\frac{1}{2}-\varepsilon)(1-q)^2},$$

and thus the bid is lower than in the case in which there is a unique prior $\frac{1}{2}$.

Consider now the strategy of the high type. If he uses some $F(b)$, we have:

$$\begin{aligned} & \int_b f(b)[\Pr_{\varepsilon}(l|h)(E_{\varepsilon}(v|h,l)-b)+\Pr_{\varepsilon}(h|h)F(b)(E_{\varepsilon}(v|h,h)-b)]db \\ = & \int_b f(b)[E_{\varepsilon}(v|h)-b-(1-F(b))\Pr_{\varepsilon}(h|h)(E_{\varepsilon}(v|h,h)-b)]db \end{aligned}$$

To choose the information structure to minimize utility, we

$$\min_p E_{\varepsilon}(v|h)-(1-F(b))\Pr_{\varepsilon}(h|h)(E_{\varepsilon}(v|h,h)-b)$$

The above equals

$$\min_p \frac{pq(1-(1-F(b))q)}{pq+(1-p)(1-q)}+(1-F(b))\frac{pq^2+(1-p)(1-q)^2}{pq+(1-p)(1-q)}b$$

For any $F(b) \leq 1$ of the other bidder, this decreases in p and hence we choose the lowest $p = \frac{1}{2} - \varepsilon$.

Let us look at $\bar{b}_{\varepsilon}(h) = \Pr_{\varepsilon}(l|h)E_{\varepsilon}(v|l,l) + \Pr_{\varepsilon}(h|h)E_{\varepsilon}(v|h,h)$. Note that $E_{\varepsilon}(v|h,h) < E(v|h,h)$, $E_{\varepsilon}(v|l,l) < E(v|l,l)$, and that $\Pr_{\varepsilon}(h|h) < \Pr(h|h) = \frac{pq^2+(1-p)(1-q)^2}{pq+(1-p)(1-q)}$. We therefore have a uniformly lower value. It is straightforward to show that $F_{\varepsilon}(b) = \frac{\Pr_{\varepsilon}(l|h)}{\Pr_{\varepsilon}(h|h)}\frac{b-E_{\varepsilon}(v|l,l)}{E_{\varepsilon}(v|h,h)-b}$.

To complete the equilibrium characterization note that the low type will not deviate for a bid b if

$$\min_{\varepsilon}(\Pr_{\varepsilon}(l|l)(E_{\varepsilon}(v|l,l)-b)+\Pr_{\varepsilon}(h|l)F_{\varepsilon}(b)(E_{\varepsilon}(v|l,h)-b)) < 0$$

Note that $\Pr_{\varepsilon}(l|l)(E_{\varepsilon}(v|l,l)-b)+\Pr_{\varepsilon}(h|l)F_{\varepsilon}(b)(E_{\varepsilon}(v|l,h)-b)) = \frac{p(1-q)^2+(1-p)q^2}{p(1-q)+(1-p)q}(\frac{p(1-q)^2}{p(1-q)^2+(1-p)q^2} - b) + \frac{q(1-q)}{p(1-q)+(1-p)q}F_{\varepsilon}(b)(p-b))$ increases in p for all $F_{\varepsilon}(b) \leq 1$ and thus again we choose the lowest $p = \frac{1}{2} - \varepsilon$.

Given the same belief $p = \frac{1}{2} - \varepsilon$, we have that

$$\begin{aligned} & \Pr_{\varepsilon}(l|l)(E_{\varepsilon}(v|l,l) - b) + \Pr_{\varepsilon}(h|l)F_{\varepsilon}(b)(E_{\varepsilon}(v|l,h) - b) \\ = & \Pr_{\varepsilon}(l|l)(E_{\varepsilon}(v|l,l) - b) + \Pr_{\varepsilon}(h|l)\frac{\Pr_{\varepsilon}(l|h)}{\Pr_{\varepsilon}(h|h)}\frac{b - E_{\varepsilon}(v|l,l)}{E_{\varepsilon}(v|h,h) - b}(E_{\varepsilon}(v|l,h) - b) \\ = & (b - E_{\varepsilon}(v|l,l)(-\Pr_{\varepsilon}(l|l) + \Pr_{\varepsilon}(h|l)\frac{\Pr_{\varepsilon}(l|h)}{\Pr_{\varepsilon}(h|h)}\frac{E_{\varepsilon}(v|l,h) - b}{E_{\varepsilon}(v|h,h) - b}) < 0 \end{aligned}$$

as $\frac{E_{\varepsilon}(v|l,h)-b}{E_{\varepsilon}(v|h,h)-b} < 1$ and $\frac{\Pr_{\varepsilon}(l|l)}{\Pr_{\varepsilon}(h|l)} > \frac{\Pr_{\varepsilon}(l|h)}{\Pr_{\varepsilon}(h|h)}$ by the MLRP which is satisfied here. Finally note that $F_{\varepsilon}(b) = \frac{\Pr_{\varepsilon}(l|h)}{\Pr_{\varepsilon}(h|h)}\frac{b-E_{\varepsilon}(v|l,l)}{E_{\varepsilon}(v|h,h)-b} \geq F(b)$ which is computed in the canonical case with no ambiguity, so $F(b)$ first order stochastically dominates $F_{\varepsilon}(b)$ for the same support. This implies that the seller's revenue would be lower.

Now consider second-price auction. We need to show that

$$b_{\varepsilon}(l) = \frac{(\frac{1}{2} - \varepsilon)(1-q)^2}{(\frac{1}{2} + \varepsilon)q^2 + (\frac{1}{2} - \varepsilon)(1-q)^2}$$

and

$$b_{\varepsilon}(h) = \frac{(\frac{1}{2} - \varepsilon)q^2}{(\frac{1}{2} + \varepsilon)(1-q)^2 + (\frac{1}{2} - \varepsilon)q^2}$$

constitute an equilibrium strategy. Suppose that player 2 is following the equilibrium strategy. Then the utility to player 1 with signal l from bidding $b_{\varepsilon}(l)$ is:

$$\begin{aligned} & \min_{p} \Pr_{\varepsilon}(1|l)\left(\frac{1-q}{2}(1 - b_{\varepsilon}(l))\right) - \Pr_{\varepsilon}(0|l)\left(\frac{q}{2}b_{\varepsilon}(l)\right) \\ & \min_{p} \frac{p(1-q)}{p(1-q) + (1-p)q}\left(\frac{1-q}{2}(1 - b_{\varepsilon}(l))\right) - \frac{(1-p)q}{p(1-q) + (1-p)q}\left(\frac{q}{2}b_{\varepsilon}(l)\right) \end{aligned}$$

which is minimized by $p = 1 - \varepsilon$. The utility to player 1 with signal h from bidding $b_{\varepsilon}(h)$ is:

$$\begin{aligned} & \min_{p} \Pr_{\varepsilon}(1|h)\left((1-q)(1 - b_{\varepsilon}(l)) + \frac{q}{2}(1 - b_{\varepsilon}(h))\right) - \Pr_{\varepsilon}(0|h)\left(qb_{\varepsilon}(l) + \frac{(1-q)}{2}b_{\varepsilon}(h)\right) \\ & \min_{p} \left\{ \frac{pq}{pq + (1-p)(1-q)}\left((1-q)(1 - b_{\varepsilon}(l)) + \frac{q}{2}(1 - b_{\varepsilon}(h))\right) \right. \\ & \quad \left. - \frac{(1-p)(1-q)}{pq + (1-p)(1-q)}\left(qb_{\varepsilon}(l) + \frac{(1-q)}{2}b_{\varepsilon}(h)\right) \right\} \end{aligned}$$

which is minimized by $p = 1 - \varepsilon$. Since $(b_{\varepsilon}(l), b_{\varepsilon}(h))$ is an equilibrium strategy profile in the standard model with prior $1 - \varepsilon$, there is no profitable deviation fixing $p = 1 - \varepsilon$.

However, this also implies that there no profitable deviation when the buyers consider the set of priors $p \in [1-\varepsilon, 1+\varepsilon]$, since every deviation is not profitable at least for the belief $p = 1-\varepsilon$.■

6.2 Appendix B: The continuous model

The state of the world is $v \in \{0,1\}$, with an equal prior. Each individual receives a signal $s^i \in [0,1]$ about the state of the world. The marginal distributions determining the signals given the state of the world, are known to the players, are anonymous, and depend on the state symmetrically. Specifically, $g_0(s)$ is a decreasing function, $g_1(s)$ is an increasing function. Hence $G_0(s)$ is concave and $G_1(s)$ is convex. For simplicity, let $g_0(s) = g_1(1-s)$, so that $G_0(s) = 1 - G_1(1-s)$. Note that FOSD is satisfied so that $G_0(s) > G_1(s)$ for all interior s, and hence MLRP is satisfied too. Let $s_0 < 0.5$ be the median of G_0 and $s_1 > 0.5$ the median of G_1.

Individuals have ambiguity over a set of joint distributions per state $v \in \{0,1\}$. We use a simple set of joint distributions, the F-G-M transformation, which was introduced by Morgenstern in 1956. Specifically, given $g_v(s)$, we have:

$$f_v(\mathbf{s}) = [1 + \lambda_v(2G_v(s_1) - 1)(2G_v(s_2) - 1)]g_v(s_1)g_v(s_2). \tag{4}$$

For this to be a distribution, for any v we need $|\lambda_v| \leq 1$, which implies that the highest correlation coefficient in this family is $\frac{1}{3}$ in absolute value.[19] Note that when $\lambda_v > 0$ we have positive correlation of signals in state v while when $\lambda_v < 0$ we have negative correlation. When signals are conditionally independent, we have $\lambda_v = 0$ for all v. Adding ePMI constraints, we then have:

$$\lambda_v \in [\frac{1}{a} - 1, 1 - \frac{1}{a}] \text{ for } v \in \{0,1\}.$$

Let us first write the utility of a player per each bid b. This is

$$U(s^1, b) = \min_{\boldsymbol{\lambda}} \frac{1}{2}\left(\int_0^z (1 - b(s'))f_1(s^1, s')ds' - \int_0^z b(s')f_0(s^1, s')ds'\right)$$

where $b(s')$ is the bid used by the other player and $z = b^{-1}(b)$. Thus per each bid b, each player minimizes his utility by choosing a vector $\boldsymbol{\lambda}$, given the strategy of the other player. Recall that s_v, for $v \in \{0,1\}$, is the median of the cdf $G_v()$.

Lemma B1: *Consider an equilibrium in which $b(s)$ is increasing. Let $\boldsymbol{\lambda}_v^*(s)$ denote the information structure which minimizes the utility of the player for each s. Then:*

[19] See Schucany et al (1978).

(i) $(\lambda_0^*, \lambda_1^*) = (\lambda_{\max}, \lambda_{\min})$ *for all* $s < s_0$.

(ii) $(\lambda_0^*, \lambda_1^*) = (\lambda_{\min}, \lambda_{\min})$ *for all* $s \in [s_0, \min\{\hat{s}, s_1\}]$.

(iii) $(\lambda_0^*, \lambda_1^*) = (\lambda_{\min}, \lambda_{\max})$ *in* $[s_1, \hat{s}]$ *if* $s_1 < \hat{s}$ *and* $(\lambda_0, \lambda_1) = (\lambda_{\max}, \lambda_{\min})$ *in* $[\hat{s}, s_1]$ *otherwise.*

(iv) $(\lambda_0^*, \lambda_1^*) = (\lambda_{\max}, \lambda_{\max})$ *for all* $s > \max\{s_1, \hat{s}\}$,

and $\hat{s} < 1$ *satisfies*

$$\int_0^{\hat{s}} b(s')g_0(s')(2G_0(s') - 1))ds' = 0.$$

That is, $\boldsymbol{\lambda}^*(s)$ changes with s, so the behaviour as described cannot be rationalized with a unique a priori $\boldsymbol{\lambda}$.

We can now characterise the equilibrium. As expected, bids in equilibrium in the second-price auction will equal the expectations of the player given his signal and that the other player had received the same signal. The expectations however will depend for each $b(s)$, on the chosen vector $\boldsymbol{\lambda}^*(s)$.

Proposition B1: *When* a *is not too high, there exists a symmetric equilibrium in which*

$$b(s, \boldsymbol{\lambda}^*) = E^{\boldsymbol{\lambda}^*}(v|s, s) = \frac{[1 + \lambda_1^*(2G_1(s) - 1)^2]g_1^2(s)}{[1 + \lambda_1^*(2G_1(s) - 1)^2]g_1^2(s) + [1 + \lambda_0^*(2G_0(s) - 1)^2]g_0^2(s)}.$$

Overbidding arises in equilibrium when $\hat{s} > \frac{1}{2}$, *for types in* $[0.5, \hat{s}]$. *For all other types, underbidding arises. When* $\hat{s} < 0.5$, *or when* $w(y)$ *is decreasing over* $[1 - \hat{s}, \hat{s}]$, *for*

$$w(y) \equiv (1 - G_1(y))g_1(y) + (1 - G_0(y))g_0(y),$$

then the seller's revenue decreases in a.

Proofs of Lemma B1 and Proposition B1: We first show in Claims 1-3 how players choose $\boldsymbol{\lambda}^*$ to minimize their utility given each s, when the bid of the other player is weakly increasing in s'. We then show that the bidding function described above, for the $\boldsymbol{\lambda}$s chosen, is an equilibrium.

Define:

$$\begin{aligned} I_1(s) &= \int_0^s (1 - b(s'))g_1(s')g_1(s)(2G_1(s) - 1)(2G_1(s') - 1)ds' \\ I_0(s) &= -\int_0^s b(s')g_0(s')g_0(s)(2G_0(s) - 1)(2G_0(s') - 1))ds', \end{aligned}$$

Thus:

Claim 1: In equilibrium, $\lambda_v^* = \lambda_{\min}$ $(\lambda_{\max})$ iff $I_v(s) > (<)0$.

$I_v(s)$ is the derivative of the expected utility with respect to λ_v. Given max-min behaviour, the statement follows.■

Claim 2: (i) $I_1(s) > 0$ for $s < s_1$, $I_1(s) < 0$ for all $s > s_1$; (ii) $I_0(s) < 0$ for $s < s_0$, $I_0(s) > 0$ for all $s \in (s_0, \hat{s})$, $I_0(s) < 0$ for all $s > \hat{s}$.

Proof of Claim 2:

(i) $I_1(s)$: This function must be strictly positive for $s < s_1$ as $(2G_1(s)-1)(2G_1(s')-1) > 0$ for $s, s' < s_1$. Note that $I_1(s_1) = 0$, and that

$$\begin{aligned}\frac{\partial I_1(s)}{\partial s}|_{s=s_1} &= \frac{\partial g_1(s)(2G_1(s)-1)}{\partial s}|_{s=s_1} \int_0^{s_1} (1-b(s'))g_1(s')(2G_1(s')-1)ds' \\ &= 2(g_1(s_1))^2 \int_0^{s_1} (1-b(s'))g_1(s')(2G_1(s')-1)ds' < 0\end{aligned}$$

More generally:

$$\begin{aligned}\frac{\partial I_1(s)}{\partial s} &= (g_1'(s)(2G_1(s)-1) + 2(g_1(s))^2) \int_0^{s} (1-b(s'))g_1(s')(2G_1(s')-1)ds' \\ &\quad +(1-b(s))g_1(s)g_1(s)(2G_1(s)-1)(2G_1(s)-1) \\ &= (\frac{g_1'(s)}{g_1(s)} + \frac{2(g_1(s))}{(2G_1(s)-1)})I_1(s) + 2(g_1(s))^2)(1-b(s))(2G_1(s)-1)(2G_1(s)-1))\end{aligned}$$

So whenever $I_1(s) > 0$ and $s > s_1$ we have that $\frac{\partial I_1(s)}{\partial s} > 0$ as $g_1(s)$ is increasing and $2(g_1(s))^2)(1-b(s))(2G_1(s)-1)(2G_1(s)-1)) > 0$. So now it suffices to check that $I_1(1) < 0$:

$$\begin{aligned}I_1(1) &= g_1(1) \int_0^1 (1-b(s'))g_1(s')(2G_1(s')-1)ds' \\ &= g_1(1) \int_0^{s_1} (1-b(s'))g_1(s')(2G_1(s')-1)ds' + g_1(1) \int_{s_1}^1 (1-b(s))g_1(s')(2G_1(s')-1)ds' \\ &< g_1(1) \int_0^{s_1} (1-b(s_1))g_1(s')(2G_1(s')-1)ds' + g_1(1) \int_{s_1}^1 (1-b(s_1))g_1(s')(2G_1(s')-1)ds' \\ &= g_1(1) \int_0^{s_1} (1-b(s_1))g_1(s')(2G_1(s')-1)ds' = 0,\end{aligned}$$

where the last inequality follows as $b(s')$ is increasing, $(2G_1(s')-1) > 0$ (< 0) whenever $s > s_1$ $(s < s_1)$. The last equality follows from $\int_0^1 g_1(s')(2G_1(s')-1)ds' = 0$.

(ii) $I_0(s)$: This function must be strictly negative for $s < s_0$ as $(2G_0(s)-1)(2G_0(s')-1) > 0$ for $s, s' < s_0$. Note that $I_0(s_0) = 0$. Moreover,

$$\frac{\partial I_0(s)}{\partial s}|_{s=s_0} = -\frac{\partial g_0(s)(2G_0(s)-1)}{\partial s}|_{s=s_0} \int_0^{s_0} b(s')g_0(s')(2G_0(s')-1)ds'$$
$$-b(s')g_0(s')g_0(s_0)(2G_0(s_0)-1)(2G_0(s')-1))$$
$$= -2(g_0(s_0))^2 \int_0^{s_0} b(s')g_0(s')(2G_0(s')-1)ds' > 0$$

So $I_0(s) < 0$ for $s \gtrsim s_0$. Note that $-\int_0^s b(s')g_0(s')(2G_0(s')-1))ds'$ is decreasing for $s > s_0$. Thus if $I_0(1) < 0$, we have the result. But

$$|I_0(1)| = g_0(1) \int_0^1 b(s')g_0(s')(2G_0(s')-1)ds'$$
$$> g_0(1) \int_0^{s_0} b(s_0)g_0(s')(2G_0(s')-1)ds'$$
$$+g_0(1) \int_{s_0}^1 b(s_0)g_0(s')(2G_0(s')-1)ds'$$
$$= g_0(1)b(s_0) \int_0^1 g_0(s')(2G_0(s')-1)ds' = 0.$$

Thus we know there exists $\hat{s} < 1$ such that:

$$\int_0^{\hat{s}} b(s')g_0(s')(2G_0(s')-1))ds' = 0,$$

and we can conclude that $I_0(s) > 0$ for $s \in (s_0, \hat{s})$ and that $I_0(s) < 0$ for $s > \hat{s}$.■

Consider now the bidding function $E^{\boldsymbol{\lambda}^*}(v|s,s)$. Note that overbidding, compared to the canonical model, arises when

$$\frac{[1+\lambda_1(2G_1(s)-1)^2]g_1^2(s)}{[1+\lambda_1(2G_1(s)-1)^2]g_1^2(s)+[1+\lambda_1(2G_0(s)-1)^2]g_0^2(s)} > \frac{g_1^2(s)}{g_1^2(s)+g_0^2(s)}$$

which holds if an only if:

$$\frac{[1+\lambda_1(2G_1(s')-1)^2]}{[1+\lambda_0(2G_0(s')-1)^2]} > 1.$$

We then have:

Claim 3: When $b(s) = E^{\boldsymbol{\lambda}^*}(v|s,s)$, a necessary condition for overbidding compared to the canonical model is $\hat{s} > 0.5$, that is:

$$\int_0^{0.5} b(s')g_0(s')(2G_0(s')-1)ds' < 0$$

If this holds, there is overbidding in the region $[s_0, \hat{s}]$, and underbidding for any other s. Otherwise, all types underbid compared to the canonical model.

Proof of Claim 3: Given Claims 1 and 2, we can then deduce the different values of λ_v^* in equilibrium and consider when overbidding/underbidding arises compared to the canonical model when the bidding function is as described in the Proposition.

(i) $(\lambda_0, \lambda_1) = (\lambda_{\max}, \lambda_{\min})$ for all $s < s_0$. As a result, if this is an equilibrium, we would have underbidding as

$$\frac{[1 + \lambda_{\min}(2G_1(s') - 1)^2]}{[1 + \lambda_{\max}(2G_0(s') - 1)^2]} < 1,$$

which is indeed the case as $\lambda_{\min} < 0 < \lambda_{\max}$.

b. $(\lambda_0, \lambda_1) = (\lambda_{\min}, \lambda_{\min})$ for all $s \in [s_0, \min\{\hat{s}, s_1\}]$. We have underbidding iff:

$$\frac{[1 + \lambda_{\min}(2G_1(s') - 1)^2]}{[1 + \lambda_{\min}(2G_0(s') - 1)^2]} < 1$$

If $\min\{\hat{s}, s_1\} > 0.5$, then we would have overbidding because in the region above 0.5, as $(2G_1(0.5) - 1)^2 = (2G_0(0.5) - 1)^2$ by symmetry, but because of convexity (concavity) of G_1 (G_0), the fraction would be greater than 1, as we would have $(2G_1(s') - 1)^2 < (2G_0(s') - 1)^2$ just above 0.5.

c. $(\lambda_0, \lambda_1) = (\lambda_{\max}, \lambda_{\max})$ for all $s > \max\{s_1, \hat{s}\}$. In this case we also have underbidding as $[1 + \lambda_{\max}(2G_1(s') - 1)^2] < [1 + \lambda_{\max}(2G_0(s') - 1)^2]$, because $\frac{1}{2} < G_1(s') < G_0(s')$.

d. If $0.5 < s_1 < \hat{s}$: in the region $[s_1, \hat{s}]$ we have $(\lambda_0, \lambda_1) = (\lambda_{\min}, \lambda_{\max})$. In this case we have overbidding as:

$$\frac{[1 + \lambda_{\max}(2G_1(s') - 1)^2]}{[1 + \lambda_{\min}(2G_0(s') - 1)^2]} > 1$$

For this we need $s_1 < \hat{s}$, implying that $0.5 < \hat{s}$.

e. if $\hat{s} < s_1$: Then we have $(\lambda_0, \lambda_1) = (\lambda_{\max}, \lambda_{\min})$ in this region between the two values. Then we have underbidding as:

$$\frac{[1 + \lambda_{\min}(2G_1(s') - 1)^2]}{[1 + \lambda_{\max}(2G_0(s') - 1)^2]} < 1. \blacksquare$$

Thus the structure of the equilibrium is therefore as above. So for overbidding we need:

$$\begin{aligned} & \int_0^{0.5} b(s') g_0(s')(2G_0(s') - 1) ds' \\ = & \int_0^{s_0} \frac{[1 + \lambda_{\min}(2G_1(s') - 1)^2] g_1^2(s')}{[1 + \lambda_{\min}(2G_1(s') - 1)^2] g_1^2(s') + [1 + \lambda_{\max}(2G_0(s') - 1)^2] g_0^2(s')} g_0(s')(2G_0(s') - 1) ds' \\ & + \int_{s_0}^{0.5} \frac{[1 + \lambda_{\min}(2G_1(s') - 1)^2] g_1^2(s')}{[1 + \lambda_{\min}(2G_1(s') - 1)^2] g_1^2(s') + [1 + \lambda_{\min}(2G_0(s') - 1)^2] g_0^2(s')} g_0(s')(2G_0(s') - 1) ds' \\ < & \ 0 \end{aligned}$$

which is analogous to what is in the Proposition. Finally we need to show that the construction above is an equilibrium:

Claim 4: The bidding function $b(s')$ defined above with the values of $\boldsymbol{\lambda}^*(s)$ described above consists a symmetric equilibrium when a is low enough.

Proof of Claim 4: We now show that given the above it is optimal, wlog, for player 1 to choose $b(s)$ at s, when player 2 uses $b(s')$ and $\boldsymbol{\lambda}^*(s)$ as defined above.

Let $\hat{\boldsymbol{\lambda}}$ equal $\boldsymbol{\lambda}^*(s)$ and consider the virtual utility:

$$\begin{aligned}\hat{U}(s,z) &= \int_0^z (E^{\hat{\boldsymbol{\lambda}}(s)}(v|s,s') - b(s'))dF^{\hat{\boldsymbol{\lambda}}(s)}(s,s') \\ &= \frac{1}{2}(\int_0^z ((1-b(s'))f_1(\hat{\boldsymbol{\lambda}},s,s') - b(s')f_0(\hat{\boldsymbol{\lambda}},s,s'))ds'\end{aligned}$$

This is not player 1's utility as it is evaluated at $\hat{\boldsymbol{\lambda}}$ for all s'. However note that when $z = s$, then the integrand is zero. To see that the integrand equals 0 note that, as $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*(s)$,

$$(1-b(s))f_1(\hat{\boldsymbol{\lambda}},s,s) = b(s)f_0(\hat{\boldsymbol{\lambda}},s,s)$$

iff

$$\begin{aligned}&[1+\lambda_0^*(2G_0(s)-1)^2]g_0^2(s)[1+\lambda_1^*(2G_1(s)-1)(2G_1(s)-1)]g_1(s)g_1(s) \\ =\ &[1+\lambda_1^*(2G_1(s)-1)^2]g_1^2(s)[1+\lambda_0^*(2G_0(s)-1)(2G_0(s)-1)]g_0(s)g_0(s)\end{aligned}$$

which holds.

Moreover as we now show the first order condition w.r.t. s' is zero, the second order condition evaluated at this point is negative, thus $z = s$ is a maximum. To see this, suppose that we have a z for which $\hat{U}(s,z) = 0$. Taking a second derivative w.r.t. z we get: $-b'(z)(f_1(\hat{\boldsymbol{\lambda}},s,z) + f_0(\hat{\boldsymbol{\lambda}},s,z)) + (1-b(z))f_1'(\hat{\boldsymbol{\lambda}},s,z) - b(z)f_0'(\hat{\boldsymbol{\lambda}},s,z)$. As $\hat{U}(s,z) = 0$, this implies that $(1-b(z)) = \frac{b(z)f_0(\hat{\boldsymbol{\lambda}},s,z)}{f_1(\hat{\boldsymbol{\lambda}},s,z)}$, and thus the second order derivative at that z is

$$\begin{aligned}&-b'(z)(f_1(\hat{\boldsymbol{\lambda}},s,z) + f_0(\hat{\boldsymbol{\lambda}},s,z)) + \frac{b(z)f_0(\hat{\boldsymbol{\lambda}},s,z)}{f_1(\hat{\boldsymbol{\lambda}},s,z)}f_1'(\hat{\boldsymbol{\lambda}},s,z) - b(z)f_0'(\hat{\boldsymbol{\lambda}},s,z) \\ =\ &-b'(z)(f_1(\hat{\boldsymbol{\lambda}},s,z) + f_0(\hat{\boldsymbol{\lambda}},s,z)) + b(z)(\frac{f_0(\hat{\boldsymbol{\lambda}},s,z)}{f_1(\hat{\boldsymbol{\lambda}},s,z)}f_1'(\hat{\boldsymbol{\lambda}},s,z) - f_0'(\hat{\boldsymbol{\lambda}},s,z))\end{aligned}$$

Note that the first element is always negative. The second element is negative iff: $\frac{g_1'(z)g_1(s)(1+\hat{\boldsymbol{\lambda}}_1(2G_1(z)-1)(2G_1(s)-1))+g_1(z)g_1(s)\hat{\boldsymbol{\lambda}}_1 2g_1(z)(2G_1(s)-1))}{g_0'(z)g_0(s)(1+\hat{\boldsymbol{\lambda}}_0(2G_0(z)-1)(2G_0(s)-1))+g_0(z)g_0(s)\hat{\boldsymbol{\lambda}}_0 2g_0(z)(2G_0(s)-1))} < \frac{g_1(z)g_1(s)(1+\hat{\boldsymbol{\lambda}}_1(2G_1(z)-1)(2G_1(s)-1))}{g_0(z)g_0(s)(1+\hat{\boldsymbol{\lambda}}_0(2G_0(z)-1)(2G_0(s)-1))}$

Note that when $\hat{\boldsymbol{\lambda}}$ is small enough, this is always the case as the LHS is negative. Thus a solution to the first order condition is unique.

But the above implies that player 1 can achieve this utility above and cannot improve upon it when using other bids $z \neq s$.

So we know that the player bids until the integrand gets negative, so, written differently, until $E^{\hat{\boldsymbol{\lambda}}(s)}(v|s,s) = b(s)$, which gives us the equilibrium bidding function.

We now consider the seller's revenue and show they decrease in a, under the sufficient condition identified. Consider the case when $\hat{s} > s_1$. Let

$$w(s') = (1 - G_1(s'))g_1(s') + (1 - G_0(s'))g_0(s')$$

The seller's revenue can be written as:

$$\begin{aligned} R(a) &= \int_0^{s_0} b(s', \lambda_{\max}, \lambda_{\min}) w(s') ds' + \int_{s_0}^{s_1} b(s', \lambda_{\min}, \lambda_{\min}) w(s') ds' + \\ & \int_{s_1}^{\hat{s}} b(s', \lambda_{\min}, \lambda_{\max}) w(s') ds' + \int_{\hat{s}}^{1} b(s', \lambda_{\max}, \lambda_{\max}) w(s') ds' \end{aligned}$$

The derivative w.r.t. a is:

$$\begin{aligned} \frac{\partial R(a)}{\partial a} &= \int_0^{s_0} \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min}) w(s') ds' + \int_{s_0}^{s_1} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\min}) w(s') ds' + \\ & \int_{s_1}^{\hat{s}} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max}) w(s') ds' + \int_{\hat{s}}^{1} \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\max}) w(s') ds' + \\ & \frac{\partial \hat{s}}{\partial a} (b(\hat{s}, \lambda_{\min}, \lambda_{\max}) - b(\hat{s}, \lambda_{\max}, \lambda_{\max})) w(\hat{s}) \end{aligned}$$

We note that

$$\begin{aligned} \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} &= \frac{g_1(s')^2 g_0(s')^2 [-(2G_1(s') - 1)^2 - (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2} \\ \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\min})|_{a=1} &= \frac{g_1(s')^2 g_0(s')^2 [-(2G_1(s') - 1)^2 + (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2} \\ \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max})|_{a=1} &= \frac{g_1(s')^2 g_0(s')^2 [(2G_1(s') - 1)^2 + (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2} \\ \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\max})|_{a=1} &= \frac{g_1(s')^2 g_0(s')^2 [(2G_1(s') - 1)^2 - (2G_0(s') - 1)^2]}{(g_1(s')^2 + g_0(s')^2)^2} \end{aligned}$$

So that:

$$\begin{aligned} \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\min})|_{a=1} &= -\frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max})|_{a=1} = -\frac{\partial}{\partial a} b(1 - s', \lambda_{\min}, \lambda_{\max})|_{a=1} \\ \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\max})|_{a=1} &= -\frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\min})|_{a=1} = \frac{\partial}{\partial a} b(1 - s', \lambda_{\min}, \lambda_{\min})|_{a=1} \end{aligned}$$

And therefore we can write $\frac{\partial R(a)}{\partial a}|_{a=1}$ as:

$$\begin{aligned} \frac{\partial R(a)}{\partial a}|_{a=1} &= -\int_0^{1-\hat{s}} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max}) w(s') ds' - \int_{1-\hat{s}}^{s_0} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\max}) [w(s') - w(1 - s')] ds' \\ & - \int_{s_0}^{0.5} \frac{\partial}{\partial a} b(s', \lambda_{\max}, \lambda_{\max}) [w(s') - w(1 - s')] ds' - \int_{\hat{s}}^{1} \frac{\partial}{\partial a} b(s', \lambda_{\min}, \lambda_{\min}) w(s') ds'. \end{aligned}$$

Thus, a sufficient condition for revenue to be decreasing in a is $w(s')$ decreasing over $[\hat{s}, 1 - \hat{s}]$.■

References

[1] Bergemann, D., B. Brooks and S. Morris (2015), First Price Auctions with General Information Structures: Implications for Bidding and Revenue, mimeo.

[2] Bose, S., E. Ozdenoren, and A. Pape (2006), Optimal auctions with ambiguity, Theoretical Economics 1, 411–438.

[3] Bose, S. and Renou, L. (2014), Mechanism Design With Ambiguous Communication Devices. Econometrica, 82: 1853–1872.

[4] Chen, Y. P. Katušcák and E. Ozdenoren (2007). Sealed bid auctions with ambiguity: Theory and experiments. Journal of Economic Theory 136, 513-535.

[5] Church, K.W and P. Hanks (1991), Word association norms, mutual information, and lexicography. Comput. Linguist. 16 (1): 22–29.

[6] Crémer, J. and R. P. McLean (1988). Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions. Econometrica, 56: 1247–1258.

[7] Enke, B. and F. Zimmerman. (2013). Correlation neglect in belief formation. mimeo.

[8] Eyster, E. and G. Weizsäcker. (2011). Correlation Neglect in Financial Decision-Making. Discussion Papers of DIW Berlin 1104.

[9] Gilboa, I. and D. Schmeidler (1989), Maxmin expected utility with non-unique prior, Journal of Mathematical Economics 18, 141-153.

[10] Glaeser, E. and Cass R. Sunstein (2009), Extremism and social learning. Journal of Legal Analysis, Volume 1(1), 262 - 324.

[11] Kallir, I. and Sonsino, D. (2009). The Perception of Correlation in Investment Decisions. Southern Economic Journal 75 (4): 1045-66.

[12] Levy, G. and R. Razin (2017). Combining Forecasts: Why Decision Makers Neglect Correlation. mimeo.

[13] Levy, G. and R. Razin. (2015a). Correlation neglect, voting behaviour and information aggregation. American Economic Review 105: 1634-1645.

[14] Levy, G. and R. Razin. (2015b). Does polarisation of opinions lead to polarisation of platforms? The case of correlation neglect, Quarterly Journal of Political Science, Volume 10, Issue 3.

[15] Lo, K. C., (1998). Sealed bid auctions with uncertainty averse bidders. Economic Theory 12, 1–20.

[16] Ortoleva, P. and E. Snowberg. (2015). Overconfidence in political economy. *American Economic Review,* 105: 504-535.

[17] Renou, L. (2015). Rent Extraction and Uncertainty. mimeo.

[18] Salo, A. S., Weber, M., (1995). Ambiguity aversion in first-price sealed-bid auctions. Journal of Risk and Uncertainty 11, 123–137.

[19] Seidenfeld, T. and L. Wasserman (1993). Dilation for Sets of Probabilities. *Annals of Statistics*. 21 (1993), no. 3, 1139–1154.

[20] Wolitzky, A. (2016). Mechanism design with maxmin agents: Theory and an application to bilateral trade. Theoretical Economics 11: 971–1004.