

Dynamic legislative decision making when interest groups control the agenda

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Abstract: We consider dynamic decision making in a legislature, in which in each period legislators vote between the status quo (previous period's policy) and a new bill. However, the agenda formation process is captured by interest groups, that is, the new bill on the agenda is determined by an all-pay auction among these groups. We show that convergence to the median voter of the legislature arises if interest groups are patient enough but not necessarily otherwise. We characterize the bound on the speed of convergence in a family of stationary equilibria in which policy bounces between right-wing and left-wing policies. We also show that convergence may be faster if organized interest groups represent only one side of the policy space, e.g., when only business and not consumer interests are organized.

1 Introduction

The involvement of special interests and lobby groups in politics is well documented in the economics and political science literature.³ In the United States these groups participate in almost all stages of the political process, e.g., formulating bills, taking part in committee hearings and coordinating majorities in the ballot.

A recent trend is for interest groups to explicitly write down model bills to be introduced in state and federal legislatures.⁴ A report drafted by the Centre for Media and Democracy documents the activities of the American Legislative Exchange Council (ALEC), “a conservative association,

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³For a survey of this literature see Grossman and Helpman (2001).

⁴For some recent press coverage see the Op-Ed by Paul Krugman, “Lobbyists, Guns and Money”, New York times, March 25, 2012 and Editorial, “The Big Money Behind State Laws”, New York Times, February 12, 2012.

funded almost entirely by big business interests, that brings together some 300 corporate lobbyists and 2000 legislators to write model legislation that helps corporate bottom lines. ALEC is a corporate *bill mill*".⁵ According to this report, over 50 bills drawn by ALEC have been introduced in the Virginia General Assembly; at least three of ALEC bills were requested to be introduced by the Virginia Governor Bob McDonnell. According to ALEC's self reported 2010 Legislative Scorecard, 826 pieces of ALEC legislation were introduced in statehouses around the country in 2009 and 115 were enacted into law.

Another example of an interest group that is active in forming the agenda is the drug lobby, PhRMA. In 2005, 26 states were considering legislation guaranteeing provision of cheaper drugs, including Washington state, California, Ohio and Maine, where the latter two states have already approved discount plans for those with low-income. As a response, PhRMA negotiated in California both with Governor Arnold Schwarzenegger and with Democratic legislators to develop an alternative voluntary discount plan to be brought forward to the legislature, "in order to avert a ballot battle".⁶ Lobbying over the agenda is therefore considered to be cost effective compared with lobbying for votes in the actual ballot; it might be easier to influence a handful of agenda setters rather than a majority in the legislature.

In this paper we analyse how the capture of the agenda by interest groups affects dynamic policy choices in a legislature. We analyse an infinite dynamic game in which in each period, the Congress or state legislature (the floor) votes whether to accept a new bill or to retain the status quo (previous period's policy) for that period. In keeping with previous literature we model the capture of the agenda by lobbyists as an all-pay-auction game among interest groups.⁷ Specifically, the lobby that exerts more efforts or invests more resources places his favourite bill on the agenda. To isolate this particular effect, we assume that lobbyists do not influence the actual vote between the new bill and the status quo.

⁵From <http://www.progressva.org/alec/ProgressVA%20-%20ALEC%20in%20Virginia.pdf>.

⁶http://www.sourcewatch.org/index.php?title=Pharmaceutical_Research_and_Manufacturers_of_America. PhRMA's role in influencing the legislation on prescription drugs under medicare is also well documented. See Geer *et al* (2010).

⁷See Becker (1983). In contrast with this literature, we assume that the lobbyists compete in an all-pay auction only for the *right* to put the policy on the agenda. See also Austen-Smith (1995) where interest groups compete for access to politicians.

The ability of the floor to maintain the status quo allows the floor to retain some authority over decision making; the dynamic path of policy changes will therefore be a result of the interaction of this formal authority of the floor and the real authority of lobbyists over the agenda. One possibility is that the competition among extreme and strong interests on both sides (of the median) of the floor will imply that policy forever oscillates among the floor’s less favourable policies. This may arise as extreme lobbyists worry more about other extreme policies and this high intensity of preferences will induce them to bid more aggressively. On the other hand, to maintain their policies in the future, extremists will have to constantly exert pressure to fend off competition from other more moderate groups whereas once a moderate policy wins, future policies will remain moderate. Thus, an alternative intuition is that the authority of the floor over voting will allow policies to gravitate towards the ideal policy of the floor. How often policy “jumps” towards the floor and the magnitude of the jump, might also depend on the parameters of the model; these are the set of interest groups that are organised and the patience of the interest groups (or alternatively, the frequency of voting in the legislature).

We show that in all subgame perfect equilibria, when interest groups are sufficiently patient, policies must converge to the ideal policy of the floor. This arises as the popularity advantage of the moderates, which implies that they have to compete less often to maintain their policies, is manifested when the future is important. However, when the interest groups are relatively impatient, divergence may arise in equilibrium and policies might bounce forever between extreme positions. In this case, the higher intensity to win of extremists keeps the moderates from entering the competition.

To understand the dynamics of policy processes, we also characterize a family of stationary equilibria with the property that policies converge to the floor while switching between both sides of the floor. In these “ping pong” equilibria, in any period, a lobbyist representing the status quo from e.g., the left of the floor, competes in the all-pay-auction against a more moderate lobbyist from the right of the floor. As in the case of drug policy described above, where PhRMA was mobilized into action once laws that provided cheap drugs had passed, these equilibria have the intuitive feature that once one side wins, the other side is motivated to provide an opposite policy, albeit a more moderate response to appease the floor. We therefore provide a novel explanation to the cyclical nature of policy making. Our explanation is based on the dynamic competition for

influence in legislatures.⁸

We show that in these stationary equilibria convergence is faster (i.e., policy “jumps” towards the median are larger and more frequent) when the interests groups are more patient, and that the rate of convergence is bounded from above and from below. If too fast, extremists will enter the competition and block the policies from moving towards the floor. If too slow, moderates will enter and hasten convergence. In the long run, these equilibria are characterized by the interest groups on both sides fighting essentially to maintain policies very similar to the moderate status quo. This might coincide with the observation that in many lobbying battles, keeping the status quo or policies close to it is already seen as a victory.⁹

We next examine the effect of changing the set of organised groups on the outcome. In particular we consider what happens when interest groups only on one side of the floor are organized. Surprisingly, we find that convergence might be faster in this case. The key intuition is that lobbyists with views close to the floor are now both moderates and extremists; they therefore enjoy both a popularity advantage in the eyes of the floor, and a relatively high intensity to win due to their polarized views. This allows for quicker and greater jumps of policy towards the floor. Asymmetry in representation (e.g., organization or resources) by interest groups is therefore not necessarily detrimental to the floor.

Our paper contributes to two strands of the literature on decision making in the legislature. The first strand considers dynamic legislative decision making with endogenous agenda, and excludes interest groups. This literature focuses mainly on legislative bargaining, and assumes that each member of the legislature is recognized to make a proposal with an exogenous positive probability (see Baron and Ferejohn 1989, Baron 1996, Cho and Duggan 2009). Specifically, Baron (1996) and Cho and Duggan (2009) show that policies will converge to the median of the legislature. We show that convergence to the median can arise even when the agenda is controlled by lobbyists, and when recognition probabilities are endogenous. In contrast we show that when interests are

⁸For a survey on the cyclical nature of policy in legislatures see L. C, Dodd, Chapter 5 in Weisberg (1986). More recently Alesina and Rosenthal (1989) have explained policy cycles by the changes in voting behaviour by voters. Rogoff (1990) has modelled cycles in economic policy that are due to the signalling efforts of politicians before re-election. Rajan and Zingales (2003)’s empirical analysis is suggestive of how policy reversals in the financial sector are due to interest group influence and changing economic factors.

⁹See Baumgartner and Leech (2009).

relatively impatient divergence might arise. More related is Deng and Fong (2011), who allow the proposer to be determined by an all-pay auction. They focus on one particular equilibrium in which only the extremists participate in bidding.¹⁰

The second strand of the literature focuses on how interest groups can directly affect the actual vote or decisions of politicians.¹¹ Diermeier and Myerson (1999) analyse how institutional requirements in legislatures such as supermajority may arise in response to vote-buying interest groups. Groseclose and Snyder (1996) derive the relative cost of buying supermajorities or minimal winning ones, while Dekel, Jackson and Wollinsky (2009) analyse the effects of the budget constraints of lobbyists. While vote buying typically results in policies diverging from the legislature's median, in our model, in which lobbyists are only allowed to control the agenda, convergence to the median legislator can arise.

Our paper also contributes to a recent literature on all-pay auctions with negative externalities.¹² The literature typically focuses on equilibria with only two active players and on a static game, while our characterization in Section 4 pertains to a continuum of players in a dynamic game. Recently, Klose and Kovenock (2012) show, in a one-stage game, how extremists' higher intensity to win implies that they tend to participate in equilibria; our paper shows how in a dynamic game, moderates may gain the upper hand.

The assumption that the floor votes on each bill relative to the status quo means that policies will move only in the direction of the preferences of the floor, implying irreversibility in the process of policy implementation. We discuss in Section 5.3 how convergence will not necessarily arise if lobbying can occur at the voting stage as well so that the policy process becomes reversible. Our paper is therefore related to a recent literature on public good games with irreversibility.¹³ Battaglini *et al* (2011) show that contributions in such a game converge to the highest possible compared to public good games with reversibility of investments (when depreciation approaches

¹⁰Other models with endogenous agenda formation are Board and Zwiebel (2005), Yildirim (2007), Evans (1997), Barbera and Coelho (2009) and Copic and Katz (2007).

¹¹Other papers consider the effect of lobbying on politicians' actions through expert information transmission. See for example Austen-Smith and Banks (2002).

¹²For a survey of this literature, see Jehiel and Moldovanu (2006).

¹³For other games with irreversibility see Admati and Perry (1991) and Lockwood and Thomas (2002). Also related are models of concession games (see Compte and Jehiel 2004 or Watson 2002).

zero). Marx and Matthews (2000) analyse a dynamic contribution game where players can only observe the aggregate contribution at each period. They show that when players are patient enough the public project can be completed and efficiency may be achieved.

The remainder of the paper is organized as follows. In the next Section we present the model. We characterize a family of stationary equilibria and consider a modification to a one-sided policy space, in Section 3. Section 4 contains the general results on convergence and divergence, whereas Section 5 considers some extensions and robustness to some of the assumptions. All proofs that are not in the text are relegated to the Appendix.

2 The Model

Below we construct a dynamic model of decision making in legislatures. In every period, interest groups compete in an influence game to place one new policy on the agenda. Following this competition, the floor votes between the status quo policy and the new policy on the agenda. The winning policy in the voting stage is then implemented for the duration of that period and becomes the status quo for the next period.

Voting in the Legislature (policy determination): In each period, $t \in \{1, 2, \dots\}$ a policy y^t has to be chosen. The policy is implemented for the duration of this period and becomes the status quo for period $t + 1$.

The policy y^t is chosen as follows. The floor votes between the status quo and a new bill which is determined by the winner in an all-pay auction, described below. We model the floor as a unitary player, representing the median legislator. We further assume that the floor votes myopically at each period. Thus, the floor will approve the new bill only if it prefers it to the status quo. This assumption simplifies our analysis in Section 3 but is not essential for our general results. It can also be motivated by the observation that representatives are constrained by their constituents' preferences which might be more myopic compared with organized lobby groups. In Section 5.2 we explain how the main results are robust to the case of a strategic floor.

To fix ideas, assume that the floor has symmetric single-peaked preferences over y^t with ideal policy at 0, specifically let $u_f(y^t) = -|y^t|$.

Lobbying (agenda formation): There is a continuum of interest groups with ideal policies

x_i on $[-1, 1]$.¹⁴ Each interest group has single-peaked preferences on an implemented policy y^t , with a utility function $u(x_i, y^t)$ which we fix to be $-|x_i - y^t|$ (our general results, in Theorems 1 and 2, as well as robustness shown in the extensions in Section 5, hold for any negatively interdependent utilities as long as one can order players according to some metric of moderates and extremists). Note that our setup implicitly assumes that all interests are organized and are potentially involved in the influence game. We relax this assumption in Section 3.2 where we consider the case in which groups from only one side of the floor are organized.

We model the process of agenda formation, i.e., the introduction of a new bill, as an influence game with an all-pay auction.¹⁵ At any period t , each player can invest some $b_i^t \geq 0$. The player that places the maximal bid wins the influence game; his ideal policy competes then against the status quo policy. If several players place the maximum bid then an equiprobability lottery will determine who can push his policy against the status quo.¹⁶ For simplicity, we ignore budget constraints, which implies that the specific distribution of interests along $[-1, 1]$ does not play a role.¹⁷ Note that naturally, as in any model which has costly political activism, free riding will arise. That is, interest groups may count on others with similar preferences to invest resources, which will result in a multiplicity of equilibria.

To simplify, we assume that only players that represent policies that are not more extreme than the current status quo (relative to the floor's ideal policy at 0) can participate in the influence game at every period. In terms of the actual policies that can potentially be implemented, this is a neutral assumption as if a player whose policy is more extreme than the status quo wins, his policy will lose to the status quo in the voting stage. This assumption, beyond eliminating some equilibria that arise due to free riding, is helpful as -together with the continuum of ideal policies- it allows us to construct stationary equilibria. To see why, note that with these two assumptions, our model has a nested scaled structure. Starting from any status quo, all subgames are strategically

¹⁴The symmetry of the policy space is helpful for our results in Section 3 but is not important for the qualitative nature of our results. The assumption that there exists a lobby with the same ideal policy as the floor's is also not important, and such lobby can be replaced in our analysis by the one closest to the floor.

¹⁵The results can be generalized to other contest functions, such as the simple Tullock function. See Levy and Razin (2012).

¹⁶The all-pay auction is well defined only when a finite set of players participate in bidding and we henceforth focus on equilibria in which this is the case.

¹⁷See Che and Gale (1998) for a model of interest groups with asymmetric budget constraints.

equivalent (up to a reassignment of players and scaling of payoffs). We will elaborate more on this feature and make use of it in Section 3.

Finally note that for clarity of exposition, we assume that the winning player can only push his own ideal policy. Our results are robust to an alternative assumption in which players choose policies strategically (see Section 6.3 in the Appendix).

Utilities: Let $\rho \in (0, 1)$ represent the length of a period during which a policy is implemented until the next voting opportunity. For simplicity we use $1 - \rho$ as the discount rate between periods.¹⁸ Let \tilde{y}^t be the random variable describing the policy chosen at t , and therefore the status quo for $t + 1$ and fix $\tilde{y}^0 = -1$. Let \tilde{b}_i^t be the random variable describing the bid chosen at time t by player i . The expected utility of player i from the game given his bids and others' behaviour is:

$$\sum_{t=1}^{\infty} (1 - \rho)^{t-1} E[-\rho|x_i - \tilde{y}^t| - \tilde{b}_i^t].$$

Equilibria: Players' strategies in every period in which they can bid are distributions over bids. We will focus on Subgame Perfect Equilibria; for simplicity, we assume that players do not observe the bids others place at each stage (but only the winner of the competition). Existence of a Subgame Perfect Equilibria is not guaranteed due to the (non-continuous) all-pay auction mechanism. However, we will construct SPE in Sections 3 and 4, rendering our characterization results meaningful.

Before moving on to the analysis, we discuss some of the issues that arise in all-pay auctions with interdependent utilities.

All-pay auctions with negative externalities: Let histories record the players who have won the all-pay auction up to date t . Let V_i^h be the continuation utility of player i following some history $h = (i^1, ..i^k)$. Let u_{ij}^h be the utility of player i when player j wins right after history h , abstracting from the possible payments made by player i right after this history. That is, $u_{ij}^h = -\rho|x_i - x_j| + (1 - \rho)V_i^{(h,j)}$. Let $w_{ij}^h \equiv u_{ii}^h - u_{ij}^h$ denote the willingness to win of player i against player j in history h . That is:

$$w_{ij}^h = \rho|x_i - x_j| + (1 - \rho)(V_i^{(h,i)} - V_i^{(h,j)}) \quad (1)$$

¹⁸The model has discrete periods with continuous time flows of payments between periods. An alternative more standard formulation is the exponential discounting of utilities; we show in the appendix that this will yield equivalent analysis.

When only two players are involved in equilibrium, continuation utilities aside, the willingness to win is standard - as in games without negative externalities - in the sense that no present equilibrium strategies are involved in its computation.

When more than two players are active however, the definition is more subtle. Let A denote the set of active players in equilibrium, i.e., players who use strictly positive bids with a strictly positive probability mass. Let $\bar{H}_i(\tilde{b})$ be the expected probability that some player i wins the all-pay auction given the potentially random vector of equilibrium bids, \tilde{b} . The equilibrium willingness to win of a player i in history h , denoted w_i^h , is then defined as:

$$w_i^h \equiv \lim_{\varepsilon \rightarrow 0} \sum_{j \in A, j \neq i} \bar{H}_j(\varepsilon, \tilde{b}_{-i}) w_{ij}^h. \quad (2)$$

Unlike the standard definition e.g., (1), this definition measures the *equilibrium* willingness to win, i.e., it depends on current equilibrium behaviour. The reason for the limit in this definition stems from the possibility that all other active players place an atom on zero.

For the case of two active players in each period, we show in the appendix (Lemma 1) that some of the standard results of static private values all-pay auctions hold. In particular, the two active players i and j use uniform distributions over the support $[0, \min\{w_{ij}^h, w_{ji}^h\}]$, where an atom on zero can be placed only by one player. Moreover, non active players' willingness to win must be smaller than $\min\{w_{ij}^h, w_{ji}^h\}$. When we compute such equilibria (see Section 3) we can use (1) to compute the active players strategies.¹⁹ For the purpose of our general results however, when equilibria involve multiple active players, whenever we use the term willingness to win, it refers to (2).

3 The rate of convergence of policies to the floor

In this section we establish the existence of a family of simple stationary equilibria in which policy converges to the floor's ideal policy. It also allows us to illustrate the constraints that are imposed on the rate of convergence. In particular, convergence has to be slow enough so as to keep extremists from investing too much in pushing their policies on the agenda. On the other hand, convergence

¹⁹We still have to use (2) to compute the willingness to win of non active players, although in this case the limit in the definition is not necessary as an atom is placed by only one active player.

has to be fast enough so as to restrain moderates from participating in the influence game too early.

We conclude this section by carrying out comparative statics on the set of interest groups that are organized. In particular, we compare our model to a scenario in which only groups on one side of the floor are organized. We show that in the latter model, convergence to the floor is faster.

3.1 “Ping-pong” stationary equilibria

The family of equilibria we construct, termed γ -equilibria, exhibit a “ping-pong” property, so that in every period an interest group representing the status quo, say left of the floor, will compete against a more moderate group from the right of the floor. Once the interest on the right wins, it will compete against a more moderate player from the left, and so on. This process will continue in such fashion with the oscillating policies moving closer and closer to the floor. What is helpful for the construction of the equilibrium is the nested structure of the model in which starting from any status quo subgames are strategically equivalent. That is, a game with a status quo -1 , where all available policies and players are in $[-1,1]$, is strategically equivalent to a subgame with a status quo $-x$ (or x), where all available policies and players are in $[-x, x]$.

Specifically, the equilibrium strategies (on and off equilibrium path) are as follows. Consider some status quo $-x$ where $x > 0$ and let $\gamma < 1$. As long as $-x$ is the status quo, at each period, two players with ideal policies $-x$ and γx will be drawn at random from the continuum of players with such ideal policies to bid against each other. Once γx becomes the status quo, at every period two players at γx and $-\gamma^2 x$ are drawn to compete, and so on.²⁰ We show that such an equilibrium exists for any $\rho \in (0, 1)$ for some values of γ :

PROPOSITION 1: *A γ -equilibrium exists iff $\gamma \in [\frac{1}{3}, \frac{1+\rho}{3(1-\rho)}]$. In this equilibrium, after each history with status quo x , the policy moves to a new policy $-\gamma x$ with probability $\frac{1}{2}$.*

Thus, for some values of γ , such an equilibrium can hold for any ρ . Note that the winning

²⁰Only two players are active at each period, which enables us to explicitly solve for the all-pay auction bidding functions. This is now standard in the literature on multiplayer all-pay auctions with negative externalities (see Klose and Kovenock 2011 and Alcalde and Dahm 2010). Note also that a player expects to be active only once. If he wins, with probability one an identical player will be drawn to represent the same policy in the next period. This feature simplifies the calculations but this family of equilibria can be similarly sustained without it.

probability of each player -the status quo or the more moderate one- does not depend on ρ , whereas the upper bound on γ is increasing in ρ . Therefore, in this family of equilibria, the rate of convergence is faster when ρ is lower (so that players are more patient).

The intuition for the above result is as follows. The constraints on γ , which measures the policy jumps towards the median, stem from the tension between the popularity advantage of moderates versus the intensity of preferences of extremists. Lobbyists with extreme ideal policies have high intensity to win and thus if convergence is too quick (or γ too low), will deviate and destabilize the equilibrium.

Moderates on the other hand, and specifically the lobbyist at 0, will step in if convergence is too slow (γ too high). This imposes an upper bound on γ which depends on ρ ; the popularity advantage of moderates looms larger when the future is important as in this case extremists have to win many competitions to maintain their policies for longer while moderates need only to win once. Thus, as ρ tends to zero, the only sustainable γ 's are in a neighbourhood of $\frac{1}{3}$ and convergence is the fastest possible. When ρ is sufficiently large, the popularity advantage of moderates is weak so that the willingness to win of extreme players is enough to deter the player at 0 from entry which implies that the upper bound on γ is not binding. In that case, and specifically whenever $\rho > \frac{1}{2}$, one can find equilibria in which policies bounce very closely to the two extremes, 1 and -1 , so that convergence is very slow.

Note that in each period, the competing players from each side of the median are on somewhat equal footing; neither will pay again, and all future outcomes are in between their ideal policies. This implies that in each period policy either becomes more moderate or stays the same with an equal probability.

In the remainder of this subsection we derive the equilibrium strategies and constraints on γ , which illustrate how we make use of strategic equivalence and the nested feature of the model. Readers who are less interested in the technical details of deriving these equilibria can skip to Section 3.2, where we modify the model to consider a one-sided policy space.

Equilibrium bidding strategies: Our model has a nested feature which implies that all subgames are strategically equivalent to the initial game. That is, we can find a mapping to scale the strategies (and continuation utilities) so that equilibria in the initial game could be translated

into equilibria in the subgames. Given the strategic equivalence, it is then enough to characterize the behaviour of active players (and the constraints imposed by non active players) in the γ -equilibrium only for the initial game $[-1, 1]$ with a status quo -1 .

Specifically, the mapping of the equilibrium strategies in the γ -equilibrium will satisfy the following:

1) *Symmetry*: The strategies in a subgame with status quo $x > 0$ played on $[-x, x]$ are symmetric (with respect to 0) to the strategies in a subgame with status quo $-x$ played on $[-x, x]$.

2) *Scale-stationarity*: If a player with ideal policy x' uses the strategy $F(b)$ on some support $C \subset [0, \infty)$ in the game $[-x, x]$ and status quo $-x$, then there exists a player $\gamma x'$ who uses $F_\gamma(\cdot)$ with support γC in the game $[-\gamma x, \gamma x]$ with a status quo $-\gamma x$ such that $F_\gamma(\gamma b) = F(b)$ for all $b \in C$, and vice versa.

In what follows, when no confusion arises, we identify a player i with the policy x_i that he represents, and a history with the status quo policy (this is sufficient by scale-stationarity). Hence V_x^z denotes the continuation utility of a player with ideal policy x when the status quo is z .

We focus now on the active players -1 and γ . Note that for any state $z \in [-1, \gamma]$, implemented policies will be (weakly) in between the two players at -1 and γ . This implies, by the linearity of utilities, that for any status quo z , $V_{-1}^z + V_\gamma^z = -(1 + \gamma)$, as combined future losses are simply the total distance between the players (neither will pay again in the future). We then have (for $z = 1, -\gamma$) :

$$V_{-1}^{-1} - V_{-1}^\gamma = V_\gamma^\gamma - V_\gamma^{-1},$$

which implies, by (1), that

$$\begin{aligned} w_{-1, \gamma}^{-1} &= \rho(1 + \gamma) + (1 - \rho)(V_{-1}^{-1} - V_{-1}^\gamma) \\ &= \rho(1 + \gamma) + (1 - \rho)(V_\gamma^\gamma - V_\gamma^{-1}) = w_{\gamma, -1}^{-1} \end{aligned}$$

As their willingness to win is equal, by Lemma 1 in the appendix, their strategies would consist of uniform distributions on $[0, w_{-1, \gamma}^{-1}]$, and each will win with probability 1/2. Intuitively, the players are on somewhat equal footing even though one is more moderate as the game will proceed in a symmetric and analogous “ping pong” fashion once any player wins.

We now need to compute the maximum bid, which is their willingness to win. Given that each

player wins with probability $1/2$ at any stage, and does not expect to pay any more, we have:

$$\begin{aligned} V_{-1}^{-1} &= -\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \left(\frac{1}{2} V_{-1}^{\gamma} + \frac{1}{2} V_{-1}^{-1} \right) \implies \\ V_{-1}^{-1} &= \frac{-\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \frac{1}{2} V_{-1}^{\gamma}}{1 - \frac{1}{2}(1 - \rho)}. \end{aligned}$$

By scale-stationarity (and linearity), we have that

$$V_{\gamma}^{\gamma} = \gamma V_{-1}^{-1}.$$

Using this and $V_{-1}^{\gamma} = -(1 + \gamma) - V_{\gamma}^{\gamma}$, we have:

$$\begin{aligned} V_{\gamma}^{\gamma} &= \gamma V_{-1}^{-1} = \gamma \frac{-\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \frac{1}{2} (-(1 + \gamma) - V_{\gamma}^{\gamma})}{1 - \frac{1}{2}(1 - \rho)} \implies \\ V_{\gamma}^{\gamma} &= -\frac{\gamma(1 + \gamma)}{\gamma + \rho - \gamma\rho + 1} \end{aligned}$$

which allows us to solve for the other continuation values:

$$\begin{aligned} V_{-1}^{\gamma} &= -1 - \gamma - V_{\gamma}^{\gamma} = (1 + \gamma) \left(\frac{-\rho + \gamma\rho - 1}{\gamma + \rho - \gamma\rho + 1} \right); \\ V_{-1}^{-1} &= -\frac{\gamma + 1}{\gamma + \rho - \gamma\rho + 1}; \end{aligned}$$

Plugging into the willingness to win, we can compute the maximum bid:

$$\begin{aligned} w_{-1,\gamma}^{-1} &= w_{\gamma,-1}^{-1} = \rho(1 + \gamma) + (1 - \rho) \left(\rho \frac{1 - \gamma^2}{\gamma + \rho - \gamma\rho + 1} \right) \\ &= 2\rho \frac{\gamma + 1}{\gamma + \rho - \gamma\rho + 1}. \end{aligned}$$

The constraints on γ : Given the equilibrium bids of the active players, we need all other players to remain inactive, i.e., prefer to bid 0. By Lemma 1 in the Appendix, this implies that the willingness to win of all other players (as defined in (2)) has to be lower than $w_{-1,\gamma}^{-1}$.²¹

Intuitively, the most extreme non-active player (at 1) and the most moderate one (at 0) have the highest incentive to deviate. Among all players who are more moderate than the participating players, the player representing the ideal policy of the floor has the most to gain from deviating and winning, specifically when ρ is relatively low, as once he wins his ideal policy will remain for

²¹This is true conditional on $w_{i,-1}^{-1} + w_{i,\gamma}^{-1} > 0$ for a non active player i , which indeed holds.

good. This player might deviate if the game converges too slowly (γ is too high). To compute his constraint, note that:

$$V_0^{-1} = -\rho\left(\frac{1}{2} + \frac{1}{2}\gamma\right) + (1 - \rho)\left(\frac{1}{2}V_0^{-1} + \frac{1}{2}V_0^\gamma\right)$$

where by scale-stationarity, $V_0^\gamma = \gamma V_0^{-1}$. Plugging in the above, we have:

$$\begin{aligned} V_0^{-1} &= -\frac{\rho(\gamma + 1)}{\rho - \gamma + \gamma\rho + 1} \text{ and:} \\ w_0^{-1} &= \frac{\rho(\gamma + 1)}{\rho - \gamma + \gamma\rho + 1} \end{aligned}$$

This player will not deviate if

$$\begin{aligned} w_0^{-1} &= \frac{\rho(\gamma + 1)}{\rho - \gamma + \gamma\rho + 1} \leq \frac{2\rho(\gamma + 1)}{\gamma + \rho - \gamma\rho + 1} = w_{-1,\gamma}^{-1} \Leftrightarrow \\ \gamma &\leq \frac{1 + \rho}{3(1 - \rho)}. \end{aligned}$$

Thus γ has to be sufficiently small. For $\rho > 0.5$ the constraint does not bind as $\frac{1+\rho}{3(1-\rho)} > 1$. Moreover, $\frac{1+\rho}{3(1-\rho)}$ is increasing in ρ and converges to $\frac{1}{3}$ as $\rho \rightarrow 0$.

Suppose now that γ is too low, for example close to zero so that the policy will converge very quickly to the floor. The willingness to win of the active players (at -1 and almost 0) will be, for all ρ , at most one. On the other hand, for the extremist on the other side (at 1), the willingness to win should be at least one, so he might deviate when γ is too low. To compute the constraint imposed on γ by the extremist (arbitrarily close to 1) being non-active, we use the fact that $V_\gamma^\gamma = V_1^\gamma + 1 - \gamma$ and that $V_1^{-1} + V_{-1}^{-1} = -2$, to get:

$$\begin{aligned} w_1^{-1} &= \rho\left(\frac{1}{2}(1 - \gamma) + 1\right) + (1 - \rho)\left(V_1^1 - \frac{1}{2}V_1^\gamma - \frac{1}{2}V_1^{-1}\right) \\ &= \rho\left(\frac{1}{2}(1 - \gamma) + 1\right) + (1 - \rho)\left(V_{-1}^{-1} - \frac{1}{2}(V_\gamma^\gamma - (1 - \gamma)) - \frac{1}{2}(-2 - V_{-1}^{-1})\right) \\ &= \rho\frac{3 - \gamma}{\gamma + \rho - \gamma\rho + 1} \end{aligned}$$

This player will therefore not deviate if

$$\begin{aligned} w_1^{-1} &= \rho\frac{3 - \gamma}{\gamma + \rho - \gamma\rho + 1} \leq 2\rho\frac{\gamma + 1}{\gamma + \rho - \gamma\rho + 1} = w_{-1,\gamma}^{-1} \Leftrightarrow \\ \gamma &\geq \frac{1}{3} \end{aligned}$$

Thus, policies have to be close enough to the player at 1, in other words, γ cannot be too low. Note that this constraint is not sensitive to ρ ; compared with the willingness to win of the player

at -1, the deviation only affects the short term, as in the long term, both players have the same willingness to win.

Finally, we show in the appendix that the constraints for the players at 1 and 0 are the only binding ones, i.e., once they are satisfied all other players do not deviate as well. This completes the characterization of this family of γ -equilibria.

3.2 One-sided lobbying

The above analysis captures a situation in which all interests are organised and can potentially exert influence on the agenda. In this section we compare such an environment to one in which only a subset of the interests are organized. Consider for example lobbying over financial regulation; it might be more reasonable that some interests, those of the financial sector, are more organised or have disproportionately more resources compared with non-institutional investors or the general public.

To this effect, suppose that the organized interest groups only span the ideal policy space $[0, 1]$, where we maintain the assumption that the floor is at 0, and let $\tilde{y}^0 = 1$. We consider again γ -equilibria as above (with two active players). Thus, the game starts with a player in 1 competing against a player in γ , and such two players are drawn to compete until γ wins, and then two players in γ and γ^2 compete and so on. Note that by strategic equivalence the γ -equilibrium will be exactly the same if we consider for example the policy space as before, $[-1, 1]$, but instead shift the ideal policy of the median of the floor to -1, so that the floor sides with one of the the extreme lobbyist.

The following proposition characterizes the γ -equilibria in this environment:

PROPOSITION 2: *In the one-sided lobbying game, for all ρ , a γ -equilibrium has to satisfy: (i) $\gamma \leq \gamma(\rho) < \frac{1}{3}$, with $\gamma(\rho)$ increasing in ρ ; (ii) in each period the more moderate player is (weakly) more likely to win. Moreover, for all ρ , there exists a γ -equilibrium with $\gamma = 0$, i.e., where 0 and 1 are the only competing policies in any history.*

Compared with two-sided lobbying, for all ρ , convergence is faster in the set of stationary γ -equilibria in the one-sided lobbying. Specifically, there exists an equilibrium in which $\gamma = 0$, i.e., a player representing the floor competes right from the start, and he wins with a probability greater than a half; in this equilibrium convergence is faster than in all equilibria identified in Proposition

1. But this also holds when we compare among the two sets of γ -equilibria. Thus, when the game is dynamic and interest groups capture the agenda, the fact that policies cannot oscillate implies faster convergence.

The intuition for the result is as follows. In the two-sided model γ was constrained to be sufficiently high (to appease extremists) and sufficiently low (to cater to moderates). In this one-sided model, there is no constraint on γ to be sufficiently high, and moreover, the constraint that γ has to be sufficiently low is more binding. The result that there is no constraint on γ to be sufficiently high arises from the non-existence of extremists on the other side, which reduces the willingness to win of the existing extreme players who now face a smaller threat of unattractive policies. The result that γ has to be even lower than in the two-sided game is driven by the fact that the most moderate lobbyist, the one at 0, does not only enjoy a popularity advantage compared to the floor but also has a relatively high intensity to win as *de facto*, among the set of organized lobbyists, he is an extremist. This implies that policies have to move quicker in his direction.

Moreover, at each period, the more moderate lobbyist wins with a higher probability. To see why, note that in the two-sided game, the future policies are between the ideal policies of the competing players, sometimes moving in the direction of the moderate player and sometimes in the direction of the extreme player. In the one-sided model however the future policies move away from both players but are always closer to the more moderate lobbyist. This implies that in each period the more moderate player is more advantageous and he thus wins with a probability that is (weakly) greater than a half.

4 General results

The analysis of the γ -equilibria has illustrated a relation between the patience of the interest groups and the speed of convergence. When ρ is large, the binding constraint is that of the extremist which implies that convergence cannot be too fast whereas when ρ is small, the binding constraint is that of the moderate implying that convergence cannot be too slow.

In this section we generalise these ideas to the set of all subgame perfect equilibria (this set of equilibria is naturally large and may include multiple bidders at each stage). In particular, we show that when ρ is small, the popularity advantage of moderates is strong enough and implies

that convergence to the floor always arises in equilibrium. On the other hand, when ρ is large, the intensity of preferences of extremists implies that some polarization must arise in equilibrium; policies cannot converge too fast, and moreover, there exist equilibria with divergence.

4.1 Convergence

Let $\{\tilde{y}^t\}_{t=1}^\infty$ denote the stochastic process of policy in the game, given the equilibrium strategies. Let G^t be the distribution of policies at period t . Denote the ergodic distribution over the state space by G^∞ . Focusing on a small ρ allows us to characterize the incentives of the players in terms of G^∞ and in turn characterize the equilibrium's ergodic distribution over policy. Finally, $B_\varepsilon(0)$ is the ε -ball or interval of policies around the ideal policy of the floor. We then have:

Theorem 1 $\forall \varepsilon, \delta > 0, \exists \rho^* > 0$, such that $\forall \rho < \rho^*, G^\infty(B_\varepsilon(0)) > 1 - \delta$.

For small enough ρ , all equilibria must involve convergence to a small neighbourhood of the floor, or in other words, the ergodic distribution centres on 0.

The gist of the proof is as follows. It is easy to see that if convergence does not arise, as policies can only progress towards the floor, they must converge to some other policy $x > 0$ or to two policies, x and $-x$. Thus the willingness to win of the player representing the floor, at 0, is substantial (of order x). We therefore have to show that the bids of the other active players, or alternatively their willingness to win, is of a smaller order. This will imply that the player at 0 will deviate so that equilibria without convergence cannot be sustained.

For a small ρ , the willingness to win of active players is related to how withdrawing their bid will affect the ergodic distribution, or how it will affect their own future payments. We then focus on an equilibrium that yields an ergodic probability of converging to policy x that is close to the supremum of all such equilibrium probabilities; by doing so we bound the possible changes in the ergodic distribution that result from a deviation. Future payments are also bounded (and of order ρ) as once policies become more moderate (which happens in finite time in expectations) players will not bid any more. This allows us to show that the willingness to win of active players is strictly smaller than x , implying that the moderate player at 0 will have an incentive to enter the competition and win for sure. This ensures convergence in any subgame perfect equilibrium for a small enough ρ .

4.2 Divergence

When ρ is large enough (i.e., players are impatient or the instances of voting in the legislature are infrequent), lobbyists from all policy positions are on a more equal footing as their popularity in the eyes of the floor has a reduced role. On the other hand, intensity of preferences plays a more important role. We now show that this yields two related results. First, there exist equilibria in which policies do not converge to the floor's ideal policy. Second, equilibria have to exhibit some degree of polarization.

Theorem 2 *In every period, starting from some status quo $z \in \{-x, x\}$ for some $x > 0$: (i) Divergence: For all $\rho > \frac{1}{2}$ and $x \geq \frac{1}{2\rho}$, there exist equilibria in which $\text{Support}G^\infty = \{-x, x\}$. (ii) Polarization: For all $|x| > \delta > 0$, there exists an $\varepsilon > 0$, and a $\rho^* < 1$, such that for all $\rho > \rho^*$, the probability that the implemented policy is in some interval of length $|x| - \delta$ is less than $1 - \varepsilon$.*

Theorem 1 has established that when ρ is sufficiently small, all equilibria involve convergence to the floor, and Proposition 1 has established that equilibria with convergence exist for all ρ . However, for high ρ , there also exist equilibria with divergence. The reason is that for high ρ , the willingness to win of active players in these divergent equilibria is large enough, and their high bids will keep the most moderate player out of the competition. For example, in the limit when $\rho = 1$, for x close to 1, the willingness to win of the two active players will be close to two, and each will win with an equal probability. The willingness to win of all other players is close to one which implies that they stay inactive.

While the first result shows how extremists can price out moderates, the second result shows how, when ρ is high, moderates cannot price out extremists. Specifically, it identifies some degree of polarization and puts more structure on the nature of all equilibria (even those with convergence). For example, it implies that there cannot be an immediate convergence to the median, as policies cannot arise in an interval around the median with too high a probability. It also implies that policies cannot be concentrated only on one side of the floor and must include, with some probability, policies from the other side.

5 Discussion

In this section we discuss the implications of the main assumptions of the model.

5.1 Strategic choice of policies by the interest groups

We have assumed that once an interest group wins the all-pay auction, it has to put forward its own ideal policy. An alternative assumption is that interest groups strategically choose the policies they wish to put forward. Note that such a change doesn't affect Theorem 1, as its proof relied on the median lobbyist entering the competition if convergence does not arise, and the median lobbyist would always suggest his ideal policy.

In Section 6.3 in the Appendix we present this alternative model and prove that all the other results hold as well. Intuitively, a similar trade-off arises: whenever in our model it is attractive for an extreme player to fight and win the right to offer a policy, it will also be attractive for him to offer his own ideal policy, otherwise he will just hasten policy convergence. Thus both the equilibrium construction of Section 3 and the general results of Section 4 hold in this alternative model.

5.2 Strategic floor

We have assumed for simplicity that the floor, modelled as a unitary median voter, is myopic. The implication of this assumption was that only policies that were more moderate than the status quo were chosen. In turn this implied monotonicity or irreversibility in the policy process, a feature that came into play in some of the proofs.

An alternative assumption would be to allow the floor's median to be strategic. In this case, the policy process need not be monotone, as the floor might be willing to choose a policy that is more extreme than the status quo if it anticipates that future policies would be more moderate.

However, we can prove that Theorem 1 holds in this environment as well. To see why, note that the proof of Theorem 1 relies on the fact that, due to the monotonicity of the policy process, non convergence to the median implies that policies converge to some policy $x \neq 0$ or to two policies, x and $-x$. When the floor is strategic the policy process might cycle between several policies. But this cannot be the case, as when the process arrives at the most moderate policy in this cycle, the floor would never vote against this policy; any future stream of policies would always be weakly

dominated to keeping this current policy. As a result, the policy process can still only converge to some policy $x \neq 0$ or to two policies, x and $-x$. In other words, allowing the floor to be strategic will endogenize the monotonicity of the policy process. We can then use the same proof and our result will still hold. Moreover, it is easily shown that stationary equilibria of Section 3 are also equilibria in this modified model.

5.3 Interest groups' capture of voting

The legislative process includes an agenda formation stage and a voting stage. The focus in this paper is on the capture of the agenda by special interest groups. The implication of this is the tension between intensity of preferences and the electability of proposals. On a technical level, giving the floor the authority at the voting stage implies some structure on the possible policy processes, as discussed above.

To be sure, special interests also invest resources in trying to affect congressional votes. An interesting question arises as to what would happen if both the agenda and the voting stages were captured. In the context of our framework a simple way to model this is to assume that at any period, the policy that wins the all-pay auction is implemented so that the floor has no role in the decision process.

A model without any role for the floor disentangles the dynamic link between the periods. For example in a Markov Perfect Equilibrium, players will behave *as if* $\rho = 1$, as there are no state variables affecting utility to condition behaviour on. It is then easy to show that polarization and divergence as in Theorem 2 will hold *for all* ρ ; specifically, there is an equilibrium in which in every period, two extremists, at -1 and 1, compete for the winning position. Thus, Theorem 1 does not hold and convergence is not guaranteed, even if the lobbyists themselves are very patient.

5.4 Letting all players bid

We have assumed that players with more extreme positions than the status quo cannot participate in the bidding. The main appeal of this assumption is that it allows for the scale-stationarity described above, as a symmetric scaling of the policy space, such as $[-\gamma x, \gamma x]$, is strategically equivalent to the original policy space $[-x, x]$, if we exclude the players in $[-x, -\gamma x]$ and $[\gamma x, x]$. A second reason for

ignoring these players is that it simplifies the proof of Theorem 1. Specifically, players expect that when they lose to a more moderate player they will not bid in the future and this limits possible equilibrium punishments. We can relax this assumption and derive a version of Theorem 1 under other assumptions such as symmetry or anonymity in bidding.

Allowing players with ideal policies that are more extreme than the status quo to bid implies a potentially bigger role for free riding in the all-pay auction. If they do bid and win, they will lose to the status quo and thus de facto, they might bid to help maintain the status quo against other policies. This may encourage the player who represents the status quo to enter the competition in the first place as the burden of repetitive bidding might be shared by others. Note that, as in any other model of costly political activism, free riding is already an inherent feature in our model, as players who share the same position, or are close to active players, enjoy their efforts without exerting their own.

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6 Appendix

6.1 Preliminaries

We start with a note about the utility specification. We present an alternative model of discounting and show that it is equivalent to what we assume in the paper. Let r be the discount rate and $1/k$ the length of each period. The utility function can be written as $\sum_{t=1}^{\infty} e^{-\frac{rt}{k}} (E[-\int_0^{\frac{1}{k}} e^{-\frac{rx}{k}} dx |x_i - \tilde{y}^t| - \tilde{b}_i^t])$. In this formulation the equivalent of ρ , or the weight on period t payoffs vis a vis bids, is $\beta(k, r) = \int_0^{\frac{1}{k}} e^{-\frac{rx}{k}} dx$. The equivalent of $1 - \rho$, or the discount factor on period t , is $\gamma(k, r) = e^{-\frac{rt}{k}}$. For any $\rho > 0$, there exist (k, r) such that the two models are equivalent.

We now present a useful Lemma for the case of two active players.

Lemma 1: *Suppose that in some history only two players i and j are active. Let w_{ij}^h be the second highest (standard) willingness to win of an active player after history h . Let F_i^h be player i 's distribution function over bids. Then*

$$F_j^h(b) = \frac{b}{w_{ij}^h}; \quad F_i^h(b) = \frac{w_{ji}^h - w_{ij}^h + b}{w_{ji}^h} \text{ for all } b \in [0, w_{ij}^h]$$

and for any other non-active player k for whom $w_{ki}^h + w_{kj}^h > 0$, it has to be that $w_k^h \leq w_{ij}^h$.

Proof of Lemma 1: The shape of the bidding functions follows from standard analysis in the literature; see Hillman and Riley (1989) and Baye et al (1996). Note that the first order condition for player i is:

$$f_j(b)w_{ij}^h = 1.$$

For some non-active player k , any utility maximizing bid must satisfy the first order condition $f_i^h F_j^h w_{ki}^h + f_j^h F_i^h w_{kj}^h - 1 = 0$. The second order condition, using the first order condition above, is $f_i^h f_j^h (w_{ki}^h + w_{kj}^h) \geq 0$. Hence utility maximizing bids are either 0 or the maximum bid which is w_{ij}^h . So for player k not to enter, we must have that his utility from a bid of zero is higher than the utility from the maximum bid, which implies the condition in the Lemma. ■

6.2 Proofs of results in the text

Proof of Proposition 1: We complete the proof from Section 3. We now show that the constraints for 1 and 0 not to deviate are the only binding constraints.

First consider $x \in [\gamma, 1]$:

$$\begin{aligned} w_x^{-1} &= \rho \frac{1}{2} (2x + 1 - \gamma) + (1 - \rho)(V_x^x - V_x^{-1}) \\ w_1^{-1} &= \rho \frac{1}{2} (2 + 1 - \gamma) + (1 - \rho)(V_1^1 - V_1^{-1}) \end{aligned}$$

Therefore,

$$\begin{aligned} w_1^{-1} - w_x^{-1} &= \rho(1 - x) + (1 - \rho)(V_1^1 - V_x^x - V_1^{-1} + V_x^{-1}) = \\ &\quad \rho(1 - x) + (1 - \rho)(V_1^1 - xV_1^1 - V_1^{-1} + V_x^{-1}) \\ &\quad \rho(1 - x) + (1 - \rho)(1 - x)V_1^1 - (1 - \rho)(V_1^{-1} - V_x^{-1}) \end{aligned}$$

and as

$$\begin{aligned} V_1^{-1} - V_x^{-1} &= -(1 - x) \text{ and} \\ V_1^1 &= V_{-1}^{-1} = -\frac{\gamma + 1}{\gamma + 1 + \rho(1 - \gamma)} > -1 \end{aligned}$$

we have that

$$w_1^{-1} - w_x^{-1} = (1 - \rho)(1 - x)V_1^1 + (1 - x) > 0$$

so a player with ideal policy at x will not want to deviate whenever a player with ideal policy

at 1 would not. Now consider $x \in [-1, -\gamma^2]$:

$$\begin{aligned}
w_x^{-1} &= \rho \frac{1}{2}(1 + \gamma) + (1 - \rho)(V_x^x - V_x^{-1}) \\
w_{-1}^{-1} &= \rho \frac{1}{2}(1 + \gamma) + (1 - \rho)(V_{-1}^{-1} - V_{-1}^\gamma) \\
w_{-1}^{-1} - w_x^{-1} &= (1 - \rho)(V_{-1}^{-1} - V_{-1}^\gamma + V_x^{-1} - V_x^x) \\
&= (1 - \rho)(V_{-1}^{-1} - V_{-1}^\gamma + \frac{-\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \frac{1}{2}(V_{-1}^\gamma + (1 + x))}{1 - (1 - \rho) \frac{1}{2}} - |x|V_{-1}^{-1})
\end{aligned}$$

When $x = -1$ it is easy to verify that the above is strictly positive. For other values of $x \in (-1, -\gamma^2]$, note that the derivative with respect to x does not depend on x , and hence the binding constraint will be either at -1 (which holds) or at $-\gamma^2$, which we do below.

Suppose then that $x > 0$ and that $x \in [\gamma^{i+2}, \gamma^i]$ for some odd $i \geq 1$.

$$\begin{aligned}
V_0^{-1} &= \frac{-\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \frac{1}{2} V_0^\gamma}{1 - (1 - \rho) \frac{1}{2}} = \frac{-\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \frac{1}{2} \left(\frac{-\rho \frac{1}{2}(\gamma + \gamma^2) + (1 - \rho) \frac{1}{2} V_0^{-\gamma^2}}{1 - (1 - \rho) \frac{1}{2}} \right)}{1 - (1 - \rho) \frac{1}{2}} = \\
&= \frac{-\rho \frac{1}{2}(1 + \gamma) + (1 - \rho) \frac{1}{2} \left(\frac{-\rho \frac{1}{2}(\gamma + \gamma^2) + (1 - \rho) \frac{1}{2} \left(\frac{-\rho \frac{1}{2}(\gamma^2 + \gamma^3) + (1 - \rho) \frac{1}{2} V_0^{\gamma^3}}{1 - (1 - \rho) \frac{1}{2}} \right)}{1 - (1 - \rho) \frac{1}{2}} \right)}{1 - (1 - \rho) \frac{1}{2}} \\
&= \sum_{j=0}^i \frac{-\rho \frac{1}{2}}{1 - (1 - \rho) \frac{1}{2}} z^j (\gamma^j + \gamma^{j+1}) + z^{i+1} V_0^{(-1)^i \gamma^{i+1}}
\end{aligned}$$

where $z = \frac{(1 - \rho) \frac{1}{2}}{1 - (1 - \rho) \frac{1}{2}}$. Similarly,

$$V_x^{-1} = \sum_{j=0}^i \frac{-\rho \frac{1}{2}}{1 - (1 - \rho) \frac{1}{2}} z^j (\gamma^j + \gamma^{j+1}) + z^{i+1} V_x^{(-1)^i \gamma^{i+1}}$$

So that

$$V_0^{-1} - V_x^{-1} = z^{i+1} (V_0^{(-1)^i \gamma^{i+1}} - V_x^{(-1)^i \gamma^{i+1}})$$

But

$$\begin{aligned}
V_0^{(-1)^i \gamma^{i+1}} &= \sum_{j=i+1}^{\infty} \frac{-\rho}{1 - (1 - \rho) \frac{1}{2}} z^{j-i-1} \frac{1}{2} (\gamma^j + \gamma^{j+1}) \\
V_x^{(-1)^i \gamma^{i+1}} &= \sum_{j=i+1}^{\infty} \frac{-\rho}{1 - (1 - \rho) \frac{1}{2}} z^{j-i-1} \left(x + \frac{1}{2} (-1)^j (\gamma^j - \gamma^{j+1}) \right)
\end{aligned}$$

And therefore,

$$V_0^{-1} - V_x^{-1} = \left(\sum_{j=i+1}^{\infty} \frac{-\rho}{1 - (1 - \rho) \frac{1}{2}} z^j (-x + I_{\text{odd}}(j) \gamma^j + I_{\text{even}}(j) \gamma^{j+1}) \right)$$

$I_{odd}(j)$ and $I_{even}(j)$ are dummy variables with value one if j is odd (even) and zero otherwise.

Note that

$$V_x^x = xV_{-1}^{-1} = \sum_{j=0}^{\infty} \frac{-\rho}{1 - (1-\rho)^{\frac{1}{2}}} z^j (x + \frac{1}{2}(-1)^j(\gamma^{j+1} - \gamma^j)x)$$

Therefore,

$$\begin{aligned} & w_0^{-1} - w_x^{-1} \\ &= (1-\rho)(-V_x^x - (V_0^{-1} - V_x^{-1})) \\ &= (1-\rho) \sum_{j=0}^{i+1} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j (x + \frac{1}{2}(-1)^j(\gamma^{j+1} - \gamma^j)x) + \\ & \quad (1-\rho) \left(\sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j [(x + \frac{1}{2}(-1)^j(\gamma^{j+1} - \gamma^j)x) + (-x + I_{odd}(j)\gamma^j + I_{even}(j)\gamma^{j+1})] \right) \\ &> 0 \end{aligned}$$

Where the last inequality arises as, when j is even,

$$\begin{aligned} & \frac{1}{2}(-1)^j(\gamma^{j+1} - \gamma^j)x + I_{odd}(j)\gamma^j + I_{even}(j)\gamma^{j+1} \\ &= \frac{1}{2}(\gamma^{j+1} - \gamma^j)x + \gamma^{j+1} = \gamma^{j+1}(1 + \frac{1}{2}x) - \gamma^j \frac{1}{2}x > 0 \end{aligned}$$

as $x < \gamma$, and when j is odd

$$\frac{1}{2}(\gamma^j - \gamma^{j+1})x + \gamma^j = \gamma^j(1 + \frac{1}{2}x) - \gamma^{j+1} \frac{1}{2}x > 0$$

and so

$$w_0^{-1} - w_x^{-1} > 0$$

So that x will not deviate if 0 does not.

Suppose that $x < 0$ and that $x \in [-\gamma^i, -\gamma^{i+2}]$ for some even $i \geq 2$.

$$\begin{aligned} V_0^{-1} &= \frac{-\rho \frac{1}{2}(1+\gamma) + (1-\rho) \frac{1}{2} V_0^\gamma}{1 - (1-\rho)^{\frac{1}{2}}} = \frac{-\rho \frac{1}{2}(1+\gamma) + (1-\rho) \frac{1}{2} \left(\frac{-\rho \frac{1}{2}(\gamma+\gamma^2) + (1-\rho) \frac{1}{2} V_0^{-\gamma^2}}{1 - (1-\rho)^{\frac{1}{2}}} \right)}{1 - (1-\rho)^{\frac{1}{2}}} = \\ & \quad \frac{-\rho \frac{1}{2}(1+\gamma) + (1-\rho) \frac{1}{2} \left(\frac{-\rho \frac{1}{2}(\gamma+\gamma^2) + (1-\rho) \frac{1}{2} \left(\frac{-\rho \frac{1}{2}(\gamma^2+\gamma^3) + (1-\rho) \frac{1}{2} V_0^{\gamma^3}}{1 - (1-\rho)^{\frac{1}{2}}} \right)}{1 - (1-\rho)^{\frac{1}{2}}} \right)}{1 - (1-\rho)^{\frac{1}{2}}} \\ &= \sum_{j=0}^i \frac{-\rho \frac{1}{2}}{1 - (1-\rho)^{\frac{1}{2}}} z^j (\gamma^j + \gamma^{j+1}) + z^{i+1} V_0^{(-1)^i \gamma^{i+1}} \end{aligned}$$

where $z = \frac{(1-\rho)^{\frac{1}{2}}}{1-(1-\rho)^{\frac{1}{2}}}$. Similarly,

$$V_x^{-1} = \sum_{j=0}^i \frac{-\rho^{\frac{1}{2}}}{1-(1-\rho)^{\frac{1}{2}}} z^j (\gamma^j + \gamma^{j+1}) + z^{i+1} V_x^{(-1)^i \gamma^{i+1}}$$

So that

$$V_0^{-1} - V_x^{-1} = z^{i+1} (V_0^{(-1)^i \gamma^{i+1}} - V_x^{(-1)^i \gamma^{i+1}})$$

But

$$\begin{aligned} V_0^{(-1)^i \gamma^{i+1}} &= - \sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \frac{1}{2} (\gamma^j + \gamma^{j+1}) \\ V_x^{(-1)^i \gamma^{i+1}} &= - \sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left(-x + \frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) \right) \end{aligned}$$

and we have,

$$\begin{aligned} V_0^{-1} - V_x^{-1} &= - \sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left(\frac{1}{2} (\gamma^j + \gamma^{j+1}) - \left(-x + \frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) \right) \right) = \\ &= - \sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j (x + I_{\text{even}}(j) \gamma^j + I_{\text{odd}}(j) \gamma^{j+1}) \end{aligned}$$

where again,

$$V_x^x = |x| V_{-1}^{-1} = \sum_{j=0}^{\infty} \frac{-\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left(|x| + \frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) |x| \right)$$

Therefore,

$$\begin{aligned} &w_0^{-1} - w_x^{-1} \\ &= (1-\rho) (-V_x^x - (V_0^{-1} - V_x^{-1})) \\ &= (1-\rho) \sum_{j=0}^{i+1} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left(|x| + \frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) |x| \right) + \\ &= (1-\rho) \left(\sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left[\left(|x| + \frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) |x| \right) + (x + I_{\text{even}}(j) \gamma^j + I_{\text{odd}}(j) \gamma^{j+1}) \right] \right) \\ &= (1-\rho) \sum_{j=0}^{i+1} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left(x + \frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) x \right) + \\ &= (1-\rho) \frac{1}{2} \left(\sum_{j=i+1}^{\infty} \frac{\rho}{1-(1-\rho)^{\frac{1}{2}}} z^j \left[\frac{1}{2} (-1)^j (\gamma^{j+1} - \gamma^j) |x| + I_{\text{even}}(j) \gamma^j + I_{\text{odd}}(j) \gamma^{j+1} \right] \right) \\ &> 0 \end{aligned}$$

Where the last inequality follows as, when j is even,

$$\begin{aligned} & \frac{1}{2}(-1)^j(\gamma^{j+1} - \gamma^j)|x| + I_{\text{even}}(j)\gamma^j + I_{\text{odd}}(j)\gamma^{j+1} \\ &= \frac{1}{2}(\gamma^{j+1} - \gamma^j)|x| + \gamma^j > 0 \end{aligned}$$

and when j is odd,

$$\begin{aligned} & \frac{1}{2}(-1)^j(\gamma^{j+1} - \gamma^j)|x| + I_{\text{even}}(j)\gamma^j + I_{\text{odd}}(j)\gamma^{j+1} \\ &= \frac{1}{2}(\gamma^j - \gamma^{j+1})|x| + \gamma^{j+1} > 0 \end{aligned}$$

implying that

$$w_0^{-1} - w_x^{-1} > 0$$

So that x will not deviate if 0 doesn't. ■

Proof of Proposition 2: (i) Compute first the equilibrium play between γ and 1. Assume that if there is an atom, it is on player 1. Thus his continuation utility is from losing. We have:

$$\begin{aligned} w_{1,\gamma}^1 &= \rho(1 - \gamma) + (1 - \rho)(V_1^1 - V_1^\gamma) \\ V_1^1 &= -\rho(1 - \gamma) + (1 - \rho)V_1^\gamma \\ V_1^\gamma &= V_\gamma^\gamma - (1 - \gamma) = \gamma V_1^1 - (1 - \gamma) \Rightarrow \\ V_1^1 &= \frac{-(1 - \gamma)}{1 - \gamma(1 - \rho)} \Rightarrow \\ w_{1,\gamma}^1 &= \rho(1 - \gamma) + (1 - \rho)\gamma\rho\frac{1 - \gamma}{\gamma\rho - \gamma + 1} \end{aligned}$$

For player 1, and for player γ (assuming that the measure of the atom is α) :

$$\begin{aligned} V_\gamma^1 &= -\rho(1 - \gamma)(1 - \alpha) + (1 - \rho)(\alpha\gamma\frac{-(1 - \gamma)}{1 - \gamma(1 - \rho)} + (1 - \alpha)V_\gamma^1) \Rightarrow \\ V_\gamma^1 &= \frac{-\rho(1 - \gamma)(1 - \alpha) + (1 - \rho)(\alpha\gamma\frac{-(1 - \gamma)}{1 - \gamma(1 - \rho)})}{1 - (1 - \rho)(1 - \alpha)} \Rightarrow \\ w_{\gamma,1}^1 &= \rho(1 - \gamma) + (1 - \rho)(\rho\frac{(\gamma - 1)^2}{\gamma\rho - \gamma + 1}) \end{aligned}$$

:

By Lemma 1, we have

$$\begin{aligned} \alpha &= 1 - \frac{\rho(1 - \gamma) + (1 - \rho)\gamma\rho\frac{1 - \gamma}{\gamma\rho - \gamma + 1}}{\rho(1 - \gamma) + (1 - \rho)(\rho\frac{(\gamma - 1)^2}{\gamma\rho - \gamma + 1})} \Rightarrow \\ \alpha &= \frac{2\gamma + \rho - 2\gamma\rho - 1}{2\gamma + \rho - 2\gamma\rho - 2} \end{aligned}$$

Note that for α to be positive we need $\gamma < 0.5$. Now consider the willingness to win of player 0. Let $\delta = \alpha + 0.5(1 - \alpha)$.

$$\begin{aligned} w_0^1 &= \rho(\delta\gamma + (1 - \delta)) + (1 - \rho)(-\delta\gamma V_0^1 - (1 - \delta)V_0^1) \\ V_0^1 &= -\rho(\delta\gamma + (1 - \delta)) + (1 - \rho)(\delta\gamma V_0^1 + (1 - \delta)V_0^1) \\ V_0^1 &= \frac{-\rho(\delta\gamma + (1 - \delta))}{1 - (1 - \rho)(\delta\gamma + (1 - \delta))} \Rightarrow \\ w_0^1 &= \rho(\delta\gamma + (1 - \delta)) + (1 - \rho) \frac{-\rho(\delta\gamma + (1 - \delta))}{1 - (1 - \rho)(\delta\gamma + (1 - \delta))} (-\delta\gamma - (1 - \delta)) \end{aligned}$$

We have that $w_0^1 < w_{1,\gamma}^1$ iff $\gamma < \gamma(\rho) = \frac{1}{8\rho-8} \left(2\rho + \sqrt{-4\rho + 4\rho^2 + 9} - 5 \right) < \frac{1}{3}$ for all ρ , so that for all $\gamma < \gamma(\rho)$ this holds and otherwise it is negative.

Finally, one can conjecture that the atom is on γ . Solving this in the same way we find that this is not feasible.

(ii) Suppose that $\gamma = 0$ so the median fights with the most extreme position, as long as the extreme wins, continue. If there is a deviation so that another player x wins, play continues in the same way i.e., , the median competes with x . We look at the game at $[0, 1]$.

$$\begin{aligned} w_{1,0}^1 &= \rho + (1 - \rho)(V_1^1 + 1) \\ w_{0,1}^1 &= \rho + (1 - \rho)(-V_0^1) \end{aligned}$$

Suppose the atom is on player 1. Then $V_1^1 = -\rho + (1 - \rho)(-1) = -1$ which implies that $w_{1,0}^1 = \rho$. On the other hand, given that the measure of the atom is α , the continuation utility of player 0 at bid 0 is

$$\begin{aligned} V_0^1 &= \frac{-\rho(1 - \alpha)}{1 - (1 - \rho)(1 - \alpha)} \rightarrow \\ w_0^1 &= \rho + (1 - \rho) \left(\frac{\rho(1 - \alpha)}{1 - (1 - \rho)(1 - \alpha)} \right) = \frac{\rho}{\alpha + \rho - \alpha\rho} \end{aligned}$$

By Lemma 1:

$$\alpha = 1 - \frac{\rho}{\frac{\rho}{\alpha + \rho - \alpha\rho}} \rightarrow \alpha = \frac{1 - \rho}{2 - \rho}$$

Let $\delta = \alpha + (1 - \alpha)0.5$ denote the probability that player 0 wins. For any other player γ , we

have:

$$\begin{aligned}
V_\gamma^1 &= -\rho(\delta\gamma + (1-\delta)(1-\gamma)) + (1-\rho)(\delta(-\gamma) + (1-\delta)V_\gamma^1) \Rightarrow \\
V_\gamma^1 &= \frac{3\gamma + \rho - 3\gamma\rho}{\rho - 3} \Rightarrow \\
w_\gamma^1 &= \frac{\rho}{3-\rho}(\gamma - \gamma\rho + 1)
\end{aligned}$$

However, for all ρ, γ , we have that $w_\gamma^1 < \rho = w_{1,0}^1$. ■

Proof of Theorem 1: We start with the following Lemma.

Lemma 2: *Suppose some history and a player i who wins in equilibrium with a probability less than some $\varepsilon > 0$. Let $E(b_i)$ be player i 's expected bid. Then, $E(b_i) < 2\varepsilon$.*

Proof of Lemma 2: For any bid b_i in the support of player i we have,

$$\begin{aligned}
\bar{H}_i(b_i, \tilde{b}_{-i})u_{ii} + \sum_{j \neq i} \bar{H}_j(b_i, \tilde{b}_{-i})u_{ij} - b_i &\geq \lim_{\varepsilon \rightarrow 0} \bar{H}_i(\varepsilon, \tilde{b}_{-i})u_{ii} + \sum_{j \neq i} \lim_{\varepsilon \rightarrow 0} \bar{H}_j(\varepsilon, \tilde{b}_{-i})u_{ij} \\
\Leftrightarrow b &\leq \sum_{j \neq i} (\lim_{\varepsilon \rightarrow 0} \bar{H}_j(\varepsilon, \tilde{b}_{-i}) - \bar{H}_j(b, \tilde{b}_{-i}))w_{ij}
\end{aligned}$$

Note that the worst IR outcome for a player is -2 and the best possible outcome is zero.

Therefore, as $\lim_{\varepsilon \rightarrow 0} \bar{H}_j(\varepsilon, \tilde{b}_{-i}) \geq \bar{H}_j(b, \tilde{b}_{-i})$ the above implies that,

$$b \leq 2 \sum_{j \neq i} (\lim_{\varepsilon \rightarrow 0} \bar{H}_j(\varepsilon, \tilde{b}_{-i}) - \bar{H}_j(b, \tilde{b}_{-i})) = 2(\bar{H}_i(b_i, \tilde{b}_{-i}) - \lim_{\varepsilon \rightarrow 0} \bar{H}_i(\varepsilon, \tilde{b}_{-i}))$$

Taking an integral on both sides with respect to bids, we get,

$$E(b_i) \leq 2\varepsilon. \square$$

Suppose that the theorem is false, so that there exists $\varepsilon > 0$ and $\delta > 0$ and a sequence of equilibria, such that $\forall \rho, G^\infty(B_\varepsilon(0)) < 1 - \delta$.

Lemma 3: *If the ergodic distribution has a mass $\delta > 0$ on some policy $x \neq 0$, then for any $\nu, \eta > 0$ there exists a history after which $G_h^\infty(B_\eta(x) \bigcup B_\eta(-x)) > 1 - \nu$.*

Proof of Lemma 3: Assume that the ergodic has a mass $\delta > 0$ on some policy $x \neq 0$. Note that for any $\eta > 0$, $G_h^\infty(B_\eta(x) \bigcup B_\eta(-x)) = \lim_{t \rightarrow \infty} \Pr(\tilde{y}_t \in B_\eta(x) \cup B_\eta(-x))$. For some $\nu, \eta > 0$, and any history of the game, assume that $G_h^\infty(B_\eta(x) \bigcup B_\eta(-x)) < 1 - \nu$. This implies that at any history $\Pr(\tilde{y}_t \in B_\eta(x) \cup B_\eta(-x)) < 1 - \nu$ for any t large enough. But this implies, by the monotonicity of the process that $G_h^\infty(B_\eta(x) \bigcup B_\eta(-x)) = 0$ for any history, which is a contradiction. □

The Lemma implies that there exists an $x^* > 0$ such that for any ρ , and a corresponding sequence of equilibria, $Support G^\infty = \{-x, x\}$ where $x \geq x^*$ and that $G^\infty(0) < 1 - \delta$.

Let $\bar{\alpha}$ be the supremum of the ergodic probability mass on $x \geq x^*$, among all histories and all ρ . Note that by the symmetry of the game, looking at $x > 0$ is without loss of generality.

Let $\varepsilon > 0$ and choose an equilibrium with an ergodic probability mass on $x > 0$, α , such that $\frac{1}{\varepsilon}|\bar{\alpha} - \alpha| < \frac{x}{4}$. Let us consider first a player i who wins in equilibrium with a probability that is less than ε . Let $I_\varepsilon = \{i \text{ s.t. } \Pr(i \text{ wins}) \leq \varepsilon\}$.

Lemma 4: $\Pr(\forall i \in I_\varepsilon, b_i \leq \frac{x}{2}) \geq 1 - \frac{4\varepsilon}{x}$.

Proof of Lemma 4: Let $\gamma_i = \Pr(b_i \leq \frac{x}{2})$, i.e., the probability that i 's equilibrium bid is less than or equal to the value $\frac{x}{2}$.

$$\Pr(\forall i \in I_\varepsilon, b_i \leq \frac{x}{2}) = \prod_{i \in I_\varepsilon} \gamma_i$$

Note that $\gamma_i = 1 - \Pr(b_i > \frac{x}{2})$. We now find an upper bound for $\Pr(b_i > \frac{x}{2})$. By Lemma 2, we know that for player $i \in I_\varepsilon$, $E(b_i) \leq 2\varepsilon$. The upper bound will be achieved by putting all mass on $\frac{x}{2}$ and the rest on zero constrained by satisfying $E(b_i) \leq 2\varepsilon$. This implies that,

$$\begin{aligned} \Pr(b_i > \frac{x}{2}) \frac{x}{2} &\leq 2\varepsilon \Leftrightarrow \\ \Pr(b_i > \frac{x}{2}) &\leq \frac{4\varepsilon}{x} \Leftrightarrow \\ \gamma_i &\geq 1 - \frac{4\varepsilon}{x}. \end{aligned}$$

We now find a lower bound for $\prod_{i \in I_\varepsilon} \gamma_i$. For any finite set I_ε there is a corresponding vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{|I_\varepsilon|})$ with $\varepsilon_i \leq \varepsilon$ which corresponds to the winning probabilities of all elements in I_ε . Our problem is to minimize $\prod_{i \in I_\varepsilon} \gamma_i$ by choosing I_ε and ε with the constraint that $\sum_{i \in I_\varepsilon} \varepsilon_i \leq 1$. As γ_i are decreasing in ε we can set $\sum_{i \in I_\varepsilon} \varepsilon_i = 1$.

We first fix I_ε and minimise by choosing ε . Note that $\sum_{i \in I_\varepsilon} \gamma_i \geq |I_\varepsilon| - \frac{4}{x}$ which is constant for this minimization problem. Suppose the minimizer has two players with $\varepsilon_i > \varepsilon_j > 0$. Consider decreasing ε_j by $\eta > 0$ and moving this mass to player i . The difference in $\sum_{i \in I_\varepsilon} \gamma_i$ before and after the move will be,

$$\Delta(\eta) = -C\left(\frac{16}{x^2}\eta^2 + \frac{16}{x^2}\eta(\varepsilon_i - \varepsilon_j)\right) < 0$$

and therefore this contradicts ε being a solution. As a result, in the solution all but one element of

ε will be zero and the strictly positive element will be ε . Therefore, for any I_ε ,

$$\prod_{i \in I_\varepsilon} \gamma_i \geq (1 - \frac{4\varepsilon}{x})$$

Moreover, since the solution doesn't depend on the choice of I_ε this holds more generally. \square

We now consider those players that win with a probability that is larger than ε . Let \bar{I}_ε be the set of these players and note that $|\bar{I}_\varepsilon| \leq \frac{1}{\varepsilon}$.

Order the n players in \bar{I}_ε according to their distance from the median, where we denote by player 1 the most extreme and by player n the closest to the median. First, consider player 1. We now calculate his willingness to pay above $\frac{x}{2}$, i.e., as ρ is small, we calculate $V_1^{h,1} - \tilde{V}_1^h(b_1 = \frac{x}{2})$ where $\tilde{V}_1^h(b_1 = \frac{x}{2})$ is the continuation payoff given that player 1 has chosen a bid of $\frac{x}{2}$, and excluding the payment of this bid.

Note that if player 1 loses, he will not pay any more bids in the future. Therefore, neglecting to consider future payments (when he wins) only increases the amount we calculate. We therefore only consider the effects of his actions on the policy outcomes in the future, and in particular, due to small ρ , at the ergodic distribution on policies. By the definition of ε and of α we know that these effects are smaller than $\frac{x}{4}\varepsilon(1 - \varepsilon)$. To see this note that given α , the furthest ergodic probability away from α , denoted by β , will satisfy,

$$\begin{aligned} (1 - \varepsilon)\bar{\alpha} + \varepsilon\beta &= \alpha \Leftrightarrow \\ |\beta - \alpha| &= (1 - \varepsilon)|\alpha - \bar{\alpha}| \leq \frac{x}{4}\varepsilon(1 - \varepsilon) \end{aligned}$$

Therefore we know that

$$V_1^{h,1} - \tilde{V}_1^h(b_1 = \frac{x}{2}) \leq \frac{x}{4}\varepsilon(1 - \varepsilon) + \frac{4\varepsilon}{x}2.$$

Now focus on player 2 and consider his willingness to pay above $\frac{x}{2} + \frac{x}{4}\varepsilon(1 - \varepsilon) + \frac{4\varepsilon}{x}2$. Following the same arguments as above we have that,

$$V_2^{h,2} - \tilde{V}_2^h(b_2 = \frac{x}{2} + \frac{x}{4}\varepsilon(1 - \varepsilon) + \frac{4\varepsilon}{x}2) \leq \frac{x}{4}\varepsilon(1 - \varepsilon) + \frac{4\varepsilon}{x}2.$$

Continuing this process with all players in \bar{I}_ε we get for any $i \in \bar{I}_\varepsilon$, the highest bid will be:

$$\frac{x}{2} + |\bar{I}_\varepsilon|(\frac{x}{4}\varepsilon(1 - \varepsilon) + \frac{4\varepsilon}{x}2)$$

If $|\bar{I}_\varepsilon|$ is finite, the highest bid converges to $\frac{x}{2}$ as ε goes to zero. Can $|\bar{I}_\varepsilon|$ be infinite? for there to be infinite active players the probabilities of winning must be going down fast enough, at least in the rate of $\frac{1}{n^2}$. But the maximal number of players above ε when the probabilities are decreasing according to $\frac{1}{n^2}$ is $\sqrt{\frac{1}{\varepsilon}}$ so that $|\bar{I}_\varepsilon| \lesssim \sqrt{\frac{1}{\varepsilon}}$ and therefore, as ε goes to zero,

$$\frac{x}{2} + |\bar{I}_\varepsilon| \left(\frac{x}{4} \varepsilon + \frac{4\varepsilon}{x} 2 \right) \simeq \frac{x}{2}$$

But now we have a contradiction. By Lemma 4, if the player representing the median of the floor, at 0, bids $\frac{x}{2} + |\bar{I}_\varepsilon| \left(\frac{x}{4} \varepsilon + \frac{4\varepsilon}{x} 2 \right)$, he wins at least with probability $1 - \frac{4\varepsilon}{x}$ gaining almost $\frac{x}{2}$ gaining a benefit, vis a vis equilibrium payoff, of almost $\frac{x}{2}$. ■

Proof of Theorem 2: (i) Consider the following equilibrium construction when $z = -1$. At each period only two players are bidding; Let ε be small enough so that $\sum_{i=1}^{\infty} \varepsilon^i \ll 1$. First 1 plays against $-1 + \varepsilon$, then, if $-1 + \varepsilon$ wins he bids against $1 - \varepsilon - \varepsilon^2$ and so on. Assume that a player who participates expects to do it only once, and whenever he wins someone else with the same preferences is randomly drawn to represent this policy. It will be enough to compute continuation values and willingnesses to win for $\varepsilon \rightarrow 0$.

First for the willingnesses to pay of both players who are bidding:

$$\begin{aligned} w_{1,-1}^1 &= \rho 2 + (1 - \rho)(V_1^1 - V_1^{-1}) \\ w_{-1,1}^1 &= \rho 2 + (1 - \rho)(V_{-1}^{-1} - V_{-1}^1). \end{aligned}$$

Conjecture a symmetric equilibrium in which each player wins with probability half; in that case $V_1^1 = V_1^{-1} = V_{-1}^{-1} = V_{-1}^1 = -1$ and hence $w_{1,-1}^1 = w_{-1,1}^1 = 2\rho$ which indeed confirms that each player wins with probability half. To complete the specification of the equilibrium suppose that if a player deviates, we revert to the γ -equilibrium, in which this player x starts by participating and competes against $-\gamma x$.

The utility of the median in this equilibrium (and hence his willingness to win) is 1 and so, if $\rho > \frac{1}{2}$, the median does not deviate. To see that other non-participating players do not deviate, note that any other player x gains in this equilibrium -1 , and will never gain more than 0 at any continuation utility thus the willingness to win of all other players is less than 1, and hence it is sufficient to check the median's willingness to deviate.

Note now that if instead of players 1,-1 we start with some $x, -x$, the analysis is exactly the same but we also have to consider whether extreme players want to deviate. Note that for any x , the utility of the player at 1 is -1, and if he deviates and wins it is at most 0. On the other hand, the highest payment in equilibrium is $2\rho x$. Thus if $2\rho x > 1$, the extreme players (and surely others less extreme) do not deviate which gives us the condition on x .

(ii) Consider $\rho = 1$ (this would hold for smaller ρ with continuity). We prove the result for a general history after which the status quo is some x . Suppose to the contrary so that there exists a δ and an interval of size $|x| - \delta$, such that for any ε , the probability that the winning outcome is in this interval is greater than $1 - \varepsilon$. Denote this interval by $I(\delta)$ and choose a sequence of $\varepsilon \rightarrow 0$.

First note that for all individuals who represent policies outside this interval, their joint winning probability is ε . Denote the set of these individuals by $I_\varepsilon(\delta)$.

We can then repeat Lemma 2 in Theorem 1 to show that for each such individual, their expected bid is less than 2ε . We can then repeat Lemma 4 with a slight modification to show that $\Pr(\forall i \in I_\varepsilon(\delta), b_i \leq |x| - \delta) \geq 1 - \frac{2\varepsilon}{|x| - \delta}$.

Now consider the willingness to pay above $|x| - \delta$ of some active individual in $I(\delta)$. Against players in $I(\delta)$ he is not willing to pay more than $|x| - \delta$, whereas against players in $I_\varepsilon(\delta)$ he is willing to pay at most $\frac{2\varepsilon}{|x| - \delta}(2)$ as -2 is the worst possible payment in the game. Thus no player in this interval will pay more than $|x| - \delta + \frac{2\varepsilon}{|x| - \delta}(2)$.

Now consider the most extreme individual who is furthest away from the expected outcome in $I(\delta)$. If he places a bid $|x| - \delta + \frac{2\varepsilon}{|x| - \delta}(2)$, his expected utility is at least $\frac{2\varepsilon}{|x| - \delta}(-2) - |x| + \delta - \frac{2\varepsilon}{|x| - \delta}(2)$ whereas in equilibrium his utility is worse than $(1 - \varepsilon)(-|x|)$. As δ is fixed and $\varepsilon \rightarrow 0$, this bid represents a profitable gain. ■

6.3 Strategic choice of policies

In this Appendix we analyse the case in which players can choose strategically which policies to espouse after winning the all-pay auction. We show how Propositions 1 and 2 and Theorems 1 and 2 are robust to this extension.

Consider the following modification of the model. Assume that players choose bids for the all-pay auction first. The winner of the all-pay auction can choose any policy in $[-1, 1]$. The floor then votes between the chosen policy and the status quo (note that the results below will also hold if we

assume that bids and policies are chosen simultaneously).

Note that Lemma 1 would hold in the same way when we consider some more general willingness to win.

Proof of Proposition 1: We maintain the same strategies as in the γ -equilibrium in the text.²² Note that for the active players, the calculation of continuation values is the same as before as the equilibrium strategies are the same. We have to check though whether an active player wants to deviate and announce a different policy.

Note that if someone deviates and says x , the off-equilibrium path is the same as on-equilibrium path starting from x .

Consider the player at -1 saying any other policy $-x$, for $x > 0$. Note that deviating to any policy x will be dominated by deviating to $-x$. Conditional on winning the bid, his utility in equilibrium is $(1-\rho)V_{-1}^{-1}$. Otherwise his utility is $-\rho(1-x) + (1-\rho)V_{-1}^{-x}$, where $V_{-1}^{-x} = V_{-x}^{-x} - (1-x)$. We therefore need:

$$\begin{aligned} (1-\rho)V_{-1}^{-1} &> -\rho(1-x) + (1-\rho)(xV_{-1}^{-1} - (1-x)) \Leftrightarrow \\ (1-\rho)V_{-1}^{-1}(1-x) &> -(1-x) \Leftrightarrow \\ (1-\rho)\frac{\gamma+1}{\gamma+\rho-\gamma\rho+1} &< 1 \text{ which holds.} \end{aligned}$$

Note that if do we the same holds for the player at γ , i.e., this player does not want to announce a more moderate policy.

We need to check that γ does not want to announce a more extreme policy. In equilibrium his utility is $(1-\rho)V_{\gamma}^{\gamma} = (1-\rho)\gamma V_{-1}^{-1}$. Suppose γ chooses $x > \gamma$. His utility will be $-\rho(x-\gamma) + (1-\rho)V_{\gamma}^x$, where

$$\begin{aligned} V_{\gamma}^x &= \frac{-\rho(0.5(x-\gamma)+0.5(\gamma+\gamma x))+(1-\rho)0.5V_{\gamma}^{-\gamma x}}{1-0.5(1-\rho)} \\ V_{\gamma}^{-\gamma x} &= \frac{-\rho(0.5(\gamma-\gamma^2 x)+0.5(\gamma+\gamma x))+(1-\rho)0.5V_{\gamma}^{\gamma^2 x}}{1-0.5(1-\rho)}, \text{ implying that} \\ V_{\gamma}^x &= \frac{-\rho(0.5(x-\gamma)+0.5(\gamma+\gamma x))+(1-\rho)0.5\frac{-\rho(0.5(\gamma-\gamma^2 x)+0.5(\gamma+\gamma x))+(1-\rho)0.5(\gamma^2 x V_{-1}^{-1} - (\gamma-\gamma^2 x))}{1-0.5(1-\rho)}}{1-0.5(1-\rho)} \end{aligned}$$

Then we need,

$$(1-\rho)\gamma V_{-1}^{-1} > -\rho(x-\gamma)$$

²²Note that strategies to choose policies can also be scaled to a degree $\gamma < 1$ to maintain the scale-stationarity.

$$+(1-\rho)\left(\frac{-\rho(0.5(x-\gamma)+0.5(\gamma+\gamma x))+ (1-\rho)0.5\frac{-\rho(0.5(\gamma-\gamma^2x)+0.5(\gamma+\gamma x))+ (1-\rho)0.5(\gamma^2xV_{-1}^{-1}-(\gamma-\gamma^2x))}{1-0.5(1-\rho)}}{1-0.5(1-\rho)}\right) \Leftrightarrow$$

$$2\frac{\rho}{\rho+1}\frac{(x-\gamma)(2\gamma+\rho-2\gamma\rho+1)}{\gamma+\rho-\gamma\rho+1} > 0,$$

which holds. Given the above, by strategic equivalence, each player who considers deviating will announce his ideal policy. Thus, the constraints we were using before are sufficient. In particular the median's constraint (where the median will surely state his ideal policy) is sufficient for all more moderate ones and the constraint on 1 is sufficient for all more extreme ones. Thus the result is exactly as we have. ■

Proof of Proposition 2: (i) The first part of the proof computes equilibrium behaviour between 1 and γ , conditional on them stating their positions. If the median deviates, surely he states his own position and thus his constraint is binding if this is an equilibrium. Thus this will hold as before. So let us check that γ and 1 want to state their positions and not something else.

If the player at 1 says some $x \in [0, 1]$, the game will continue with x and γx . Note that $V_x^x = x\frac{-(1-\gamma)}{1-\gamma(1-\rho)}$, and that

$$V_1^x = V_x^x - (1-x) = x\frac{-(1-\gamma)}{1-\gamma(1-\rho)} - (1-x)$$

$$(1-\rho)V_1^1 > (1-\rho)\left(x\frac{-(1-\gamma)}{1-\gamma(1-\rho)} - (1-x)\right)$$

For him not to deviate we need,

$$(1-\rho)V_1^1 > -\rho(1-x) + (1-\rho)V_1^x \Leftrightarrow$$

$$\frac{\rho}{\gamma\rho - \gamma + 1} > 0 \text{ which is satisfied.}$$

Now suppose γ deviates to a policy $x < \gamma$. This is the same calculation as above. Assume therefore that γ deviates to $x \in (\gamma, 1]$. To compute the constraint on γ we have, as in the original proof,

$$\alpha = \frac{2\gamma + \rho - 2\gamma\rho - 1}{2\gamma + \rho - 2\gamma\rho - 2}$$

$$\delta = \alpha + (1-\alpha)0.5$$

$$V_\gamma^x = \frac{-\rho(\delta(\gamma+\gamma x)+(1-\delta)(x-\gamma))+ (1-\rho)\delta(-\gamma(1+x)+\gamma x\frac{(1-\gamma)}{1-\gamma(1-\rho)})}{1-(1-\rho)(1-\delta)}$$

As $V_\gamma^{\gamma x} + V_{\gamma x}^{\gamma x} = -\gamma(1-x)$, we have that if he sticks to equilibrium behaviour, he gets $(1-\rho)V_\gamma^\gamma =$

$(1 - \rho)\gamma\frac{-(1-\gamma)}{1-\gamma(1-\rho)}$ and if he deviates he gets

$$-\rho(x - \gamma) + (1 - \rho)\frac{-\rho(\delta(\gamma + \gamma x) + (1 - \delta)(x - \gamma)) + (1 - \rho)\delta(-\gamma(1 + x) + \gamma x\frac{(1-\gamma)}{1-\gamma(1-\rho)})}{1 - (1 - \rho)(1 - \delta)}$$

We need to show:

$$(1 - \rho)\gamma\frac{-(1-\gamma)}{1-\gamma(1-\rho)} + \rho(x - \gamma) - (1 - \rho)\frac{-\rho(\delta(\gamma + \gamma x) + (1 - \delta)(x - \gamma)) + (1 - \rho)\delta(-\gamma(1 + x) + \gamma x\frac{(1-\gamma)}{1-\gamma(1-\rho)})}{1 - (1 - \rho)(1 - \delta)} > 0$$

The derivative of the left hand term with respect to x is

$$\frac{\rho(\gamma\delta - \gamma + \gamma\rho - \gamma\delta\rho + 1)}{(\delta + \rho - \delta\rho)(\gamma\rho - \gamma + 1)} > 0$$

This implies that the higher is x , the higher is the difference between equilibrium behaviour and the deviation. So if it is negative for some x , it will be so for the smallest x , but this is $x = \gamma$ for which it is 0, and the higher is x the better is the equilibrium behaviour. So γ would not want to deviate and announce another policy.

(ii) Suppose $\gamma = 0$ so a player representing the median fights with a player at 1 until the median positions wins. If there is a deviation so that another player x wins, play continues in the same way i.e., the median competes with x . We look at the game at $[0, 1]$. Clearly the median will state his own position. What about the extreme type? he can only say something more moderate. In the equilibrium his utility is $-(1 - \rho)$. If he states another position x , then his utility is $-\rho(1 - x) + (1 - \rho)V_1^x$. Note that $V_1^x = V_x^x - (1 - x) = xV_{-1}^{-1} - (1 - x)$ hence he has $-\rho(1 - x) + (1 - \rho)(-1)$ which is therefore worse. Thus he will not deviate.

Will any other player deviate? we know that conditional on the above, when a player deviates and wins, he would rather say his own ideal policy. Thus, the constraints we have done before would hold as well. ■

Proof of Theorem 1: The proof holds for this model exactly as in Section 6.2. ■

Proof of Theorem 2: (i) Consider the equilibrium construction we had before. Note that the median, if deviates, will state his position and thus his constraint is as before. Suppose now that the active players will consider saying something else. Suppose that upon a deviation to x , the same type of equilibrium holds so that policy is very close to $\{-x, x\}$, which will hold by strategic equivalence. Computing this game is as above. Thus the utility from a deviation will be $-\rho(1 - x) + (1 - \rho)V_1^x$ where $V_1^x = -0.5(1 - x) - 0.5(1 + x) = -1$. But today's utility is $(1 - \rho)V_1^1 = -(1 - \rho)$ which is higher. So equilibrium play continues. The remainder of the proof is as before.

(ii) If $\rho = 1$, conditional on winning, any player would state his ideal policy. Therefore, by continuity and by the original proof the result holds. ■