Lecture 2

The Lagrange sufficiency theorem

- So we have reached the conclusion that z^* is a solution to COP if and only if $(k, g(z^*))$ lies on the upper boundary of B.
- But which values of z give rise to these boundary points?
- Suppose we can draw a line with a slope q though a point $(k, v) = (h(z^*)), g(z^*))$ which lies entirely on or above the set C.
- The equation for this line is

$$v - qk = g(z^*) - qh(z^*))$$

- We can draw such a line through $(.25, .75)$ with slope $q = 1$ and a line though $(1,1)$ with slope 0.
- Recalling that C consists of all points $(h(z), g(z))$, the fact that the line lies entirely on or above the set C can be restated as

$$g(z^*) - qh(z^*) \geq g(z) - qh(z)$$

for all $z \in Z$.
- Because the line has non negative slope, it also lies above the set B, which consists of all the points in C, and all the points below and to the right of points in C.
- But this implies that $(h(z^*)), g(z^*))$ lies on the upper boundary of the set B, and as the points on the upper boundary of B are all solutions to our problem, it means that z^* solves the optimization problem for $k = h(z^*)$.
- It is crucial for this argument that the slope of the line is non negative. For example for $(4, 0)$ there is a line with a slope $-.5$. This corresponds to $z = 4$ but it is not a solution and the line does not lie above B.
- This helps to find a way to solve the problem with equality constraints. What about inequality constraints? well then, as with $k = 4$, if we take the line with $q = 0$ through for example $(4, 1)$, it indeed lies above the set B. The point $z^* = 1$ satisfies:

$$g(z^*) - 0h(z^*) \geq g(z) - 0h(z)$$

for all $z \in Z$, and as $h(z^*) < k^*$, then z^* solves the optimization problem for $k = k^* \geq 1$.
- Summarizing the argument so far, suppose k^*, q and z^* satisfy the following conditions:

$$g(z^*) - qh(z^*) \geq g(z) - qh(z) \text{ for all } z \in Z$$

$$q \geq 0$$

$$z^* \in Z$$

$$\text{either } k^* = h(z^*)$$

$$\text{or } k^* > h(z^*) \text{ and } q = 0$$

then z^* solves the COP for $k = k^*$.

- This is the lagrange sufficiency theorem. It is convenient to write it slightly different: add qk^* to both sides of the first condition we have

$$\begin{aligned} g(z^*) + q(k^* - h(z^*)) &\geq g(z) + q(k^* - h(z)) \text{ for all } z \in Z \\ q &\geq 0 \\ z^* &\in Z \text{ and } k^* \geq h(z^*) \\ q[k^* - h(z^*)] &= 0 \end{aligned}$$

- The term q is the lagrange multiplier.
- The term $g(z) + q(k^* - h(z))$is the Lagrangian.
- The condition says that z^* maximizes the lagrangian.
- Then, we have the non negativity restriction, feasibility, and the complementary slackness condition. What does it mean?

THEOREM *if for some $q \geq 0$, z^* maximizes $L(z,k^*,q)$ subject to the three conditions, it also solves COP.*

Proof: from the complementary slackness condition, $q[k^* - h(z^*)] = 0$. Thus, $g(z^*) = g(z^*) + q(k^* - h(z^*))$. By $q \geq 0$ and $k^* - h(z) > 0$ for all feasible z, then $g(z) + q(k^* - h(z)) \geq g(z)$. By maximization of L we get $g(z^*) \geq g(z)$, for all feasible z and since z^* itself is feasible, then it solves COP.

- We now solve the example of the firm ABC...

$$L = 2z^{.5} - z + q[k - z]$$

Let us use FOC, although we have to prove that we can use them and we will do so in the future. We need differentiability, concavity, and some other conditions.

$$z^{-.5} - 1 - q = 0$$

Consider the case of $q > 0$, and the case of $q = 0$.

The Lagrange multipliers: what does it mean to relax the constraint?
When are the conditions also necessary?

- The conditions that we have stated are sufficient conditions. This means that some solutions of *COP* cannot be characterized by the Lagrangian. For example, if the set *B* is not convex, then solving the Lagrangian is not necessary.
- If the objective function is concave and the constraint is convex, then *B* is convex. We will show it. Then, Lagrange conditions are also necessary.
- That is, if we find all the points that max the lagrangian, we find all the points that solve *COP*.
- With differentiability, these points are also solutions to FOCS.