

PSEUDO-MAXIMUM LIKELIHOOD ESTIMATION OF ARCH(∞) MODELS

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Strong consistency and asymptotic normality of the Gaussian pseudo-maximum likelihood estimate of the parameters in a wide class of ARCH(∞) processes are established. The conditions are shown to hold in case of exponential and hyperbolic decay in the ARCH weights, though in the latter case a faster decay rate is required for the central limit theorem than for the law of large numbers. Particular parameterizations are discussed.

1. Introduction. ARCH(∞) processes comprise a wide class of models for conditional heteroscedasticity in time series. Consider, for $t \in \mathbb{Z} = \{0, \pm 1, \dots\}$, the equations

$$(1) \quad x_t = \sigma_t \varepsilon_t,$$

$$(2) \quad \sigma_t^2 = \omega_0 + \sum_{j=1}^{\infty} \psi_{0j} x_{t-j}^2,$$

where

$$(3) \quad \omega_0 > 0, \quad \psi_{0j} > 0 \quad (j \geq 1), \quad \sum_{j=1}^{\infty} \psi_{0j} < \infty,$$

and $\{\varepsilon_t\}$ is a sequence of independent identically distributed (i.i.d.) unobservable real-valued random variables. We shall assume that a strictly stationary solution x_t to (1) and (2) exists almost surely (a.s.) under (3), and call it an ARCH(∞) process. We consider a parametric version, in which we know functions $\psi_j(\zeta)$ of the $r \times 1$ vector ζ , for $r < \infty$, such that, for some unknown ζ_0 ,

$$(4) \quad \psi_j(\zeta_0) = \psi_{0j}, \quad j \geq 1.$$

Also, ω_0 is unknown and x_t is unobservable but we observe

$$(5) \quad y_t = \mu_0 + x_t$$

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for some unknown μ_0 .

ARCH(∞) processes, extending the ARCH(m), $m < \infty$, process of Engle [11] and the GARCH(n, m) process of Bollerslev [4], were considered by Robinson [29] as a class of parametric alternatives in testing for serial independence of y_t . Empirical evidence of Whistler [35] and Ding, Granger and Engle [10] has suggested the possibility of long memory autocorrelation in the squares of financial data. Taking [contrary to the first requirement in (3)] $\omega_0 = 0$, such long memory in x_t^2 driven by (1) and (2) was considered by Robinson [29], the ψ_{0j} being the autoregressive weights of a fractionally integrated process, implying $\sum_{j=1}^{\infty} \psi_{0j} = 1$; see also Ding and Granger [9]. For such ψ_{0j} , and the same objective function as was employed to generate the tests of Robinson [29], Koulikov [20] established asymptotic statistical properties of estimates of ζ_0 . On the other hand, under our assumption $\omega_0 > 0$, Giraitis, Kokoszka and Leipus [13] found that such ψ_{0j} are inconsistent with covariance stationarity of x_t , which holds when $\sum_{j=1}^{\infty} \psi_{0j} < 1$. Finite variance of x_t implies summability of coefficients of a linear moving average in martingale differences representation of x_t^2 ; see [37]. In this paper we do not assume finite variance of x_t , but rather that x_t has a finite fractional moment of degree less than 2. The first requirement in (3) was shown by Kazakevičius and Leipus [18] to be necessary for existence of an x_t satisfying (1) and (2). The intermediate requirement in (3) is sufficient but not necessary for a.s. positivity of σ_t^2 , and is imposed here to facilitate a clearer focus on the ψ_{0j} , which decay, possibly slowly, but never vanish.

We wish to estimate the $(r + 2) \times 1$ vector $\theta_0 = (\omega_0, \mu_0, \zeta_0)'$ on the basis of observations y_t , $t = 1, \dots, T$, the prime denoting transposition. The case when μ_0 is known, for example, $\mu_0 = 0$, is covered by a simplified version of our treatment. If the y_t were instead unobserved regression errors, we have $\mu_0 = 0$, but would then need to replace x_t by residuals in what follows; the details of this extension would be relatively straightforward. Another relatively straightforward extension would cover simultaneous estimation of the regression parameters ω_0 and ζ_0 , after replacing μ_0 by a more general parametric function; as in (1), (2) and (5), efficiency gain is afforded by simultaneous estimation.

Under stronger restrictions than $\sum_{j=1}^{\infty} \psi_{0j} < 1$, Giraitis and Robinson [14] considered discrete-frequency Whittle estimation of ζ_0 , based on the squared observations y_t^2 (with μ_0 known to be zero), this being asymptotically equivalent to constrained least squares regression of y_t^2 on the y_{t-s}^2 , $s > 0$, a method employed in special cases of (2) by Engle [11] and Bollerslev [4]. In these the spectral density of y_t^2 , when it exists, has a convenient closed form. This property, along with availability of the fast Fourier transform, makes discrete-frequency Whittle estimation based on the y_t^2 a computationally attractive option for point estimation, even in very long financial time series. However, it has a number of disadvantages, as discussed by Giraitis and Robinson [14]: it is not only asymptotically inefficient under Gaussian ε_t , but never asymptotically efficient; it requires finiteness

of fourth moments of y_t for consistency and of eighth moments for asymptotic normality, which are sometimes considered unacceptable for financial data; its limit covariance matrix is relatively complicated to estimate; it is less well motivated in ARCH models than in stochastic volatility and nonlinear moving average models, such as those of Taylor [33], Robinson and Zaffaroni [30, 31], Harvey [15], Breidt, Crato and de Lima [5] and Zaffaroni [36], where the actual likelihood is computationally relatively intractable, while Whittle estimation also plays a less special role in the short-memory-in- y_t^2 ARCH models of Giraitis and Robinson [14] than in the long-memory-in- y_t^2 models of the previous five references, where it entails automatic “compensation” for possible lack of square-integrability of the spectrum of y_t^2 . Mikosch and Straumann [26] have shown that a finite fourth moment is necessary for consistency of Whittle estimates, and that convergence rates are slowed by fat tails in ε_t .

For Gaussian ε_t , a widely-used approximate maximum likelihood estimate is defined as follows. Denote by $\theta = (\omega, \mu, \zeta')$ any admissible value of θ_0 and define

$$x_t(\mu) = y_t - \mu,$$

$$\sigma_t^2(\theta) = \omega + \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}^2(\mu)$$

for $t \in \mathbb{Z}$, and

$$\bar{\sigma}_t^2(\theta) = \omega + \sum_{j=1}^{t-1} \psi_j(\zeta) x_{t-j}^2(\mu) \mathbb{1}(t \geq 2)$$

for $t \geq 1$, where $\mathbb{1}(\cdot)$ denotes the indicator function. Define also

$$q_t(\theta) = \frac{x_t^2(\mu)}{\sigma_t^2(\theta)} + \ln \sigma_t^2(\theta), \quad \bar{q}_t(\theta) = \frac{x_t^2(\mu)}{\bar{\sigma}_t^2(\theta)} + \ln \bar{\sigma}_t^2(\theta), \quad 1 \leq t \leq T,$$

$$Q_T(\theta) = T^{-1} \sum_{t=1}^T q_t(\theta), \quad \bar{Q}_T(\theta) = T^{-1} \sum_{t=1}^T \bar{q}_t(\theta),$$

$$\tilde{\theta}_T = \arg \min_{\theta \in \Theta} Q_T(\theta), \quad \hat{\theta}_T = \arg \min_{\theta \in \Theta} \bar{Q}_T(\theta),$$

where Θ is a prescribed compact subset of \mathbb{R}^{r+2} . The quantities with over-bar are introduced due to y_t being unobservable for $t \leq 0$; $\hat{\theta}_T$ is uncomputable. Because we do not assume Gaussianity in the asymptotic theory, we refer to $\hat{\theta}_T$ as a pseudo-maximum likelihood estimate (PMLE).

We establish strong consistency of $\hat{\theta}_T$ and asymptotic normality of $T^{1/2}(\hat{\theta}_T - \theta_0)$, as $T \rightarrow \infty$, for a class of $\psi_j(\zeta)$ sequences. In the case of the first property this is accomplished by first showing strong consistency of $\tilde{\theta}_T$ and then that $\hat{\theta}_T - \tilde{\theta}_T \rightarrow 0$, a.s. In the case of the second we likewise first show it for

$T^{1/2}(\tilde{\theta}_T - \theta_0)$ and then show that $\hat{\theta}_T - \tilde{\theta}_T = o_p(T^{-1/2})$, but the latter property, and thus the asymptotic normality of $T^{1/2}(\hat{\theta}_T - \theta_0)$, is achieved only under a restricted set of possible ζ_0 values, and this seems of practical concern in relation to some popular choices of the $\psi_j(\zeta)$. These results are presented in the following section, along with a description of regularity conditions and partial proof details. The structure of the proof is similar in several respects to earlier ones for the GARCH case of (2), especially that of Berkes, Horváth and Kokoszka [3]. Sections 3 and 4 apply the results to particular models.

2. Assumptions and main results. Our assumptions are as follows.

ASSUMPTION A(q), $q \geq 2$. The ε_t are i.i.d. random variables with $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$, $E|\varepsilon_0|^q < \infty$ and probability density function $f(\varepsilon)$ satisfying

$$f(\varepsilon) = O(L(|\varepsilon|^{-1})|\varepsilon|^b) \quad \text{as } \varepsilon \rightarrow 0,$$

for $b > -1$ and a function L that is slowly varying at the origin.

ASSUMPTION B. There exist $\omega_L, \omega_U, \mu_L, \mu_U$ such that $0 < \omega_L < \omega_U < \infty$, $-\infty < \mu_L < \mu_U < \infty$, and a compact set $\Upsilon \in R^r$ such that $\Theta = [\omega_L, \omega_U] \times [\mu_L, \mu_U] \times \Upsilon$.

ASSUMPTION C. θ_0 is an interior point of Θ .

ASSUMPTION D. For all $j \geq 1$,

(6) $\inf_{\zeta \in \Upsilon} \psi_j(\zeta) > 0;$

(7) $\sup_{\zeta \in \Upsilon} \psi_j(\zeta) \leq K j^{-\underline{d}-1}$ for some $\underline{d} > 0;$

(8) $\psi_{0j} \leq K \psi_{0k}$ for $1 \leq k \leq j$,

where K throughout denotes a generic, positive constant.

ASSUMPTION E. There exists a strictly stationary and ergodic solution x_t to (1) and (2), and for some

(9) $\rho \in ((\underline{d} + 1)^{-1}, 1),$

with \underline{d} as in Assumption D, we have

(10) $E|x_0|^{2\rho} < \infty.$

ASSUMPTION F(l). For all $j \geq 1$, $\psi_j(\zeta)$ has continuous k th derivative on Υ such that, with ζ_i denoting the i th element of ζ ,

$$(11) \quad \left| \frac{\partial^k \psi_j(\zeta)}{\partial \zeta_{i_1} \cdots \partial \zeta_{i_k}} \right| \leq K \psi_j(\zeta)^{1-\eta}$$

for all $\eta > 0$ and all $i_h = 1, \dots, r, h = 1, \dots, k, k \leq l$.

ASSUMPTION G. For each $\zeta \in \Upsilon$, there exist integers $j_i = j_i(\zeta), i = 1, \dots, r$, such that $1 \leq j_1(\zeta) < \dots < j_r(\zeta) < \infty$ and

$$\text{rank}\{\Psi_{(j_1, \dots, j_r)}(\zeta)\} = r,$$

where

$$\Psi_{(j_1, \dots, j_r)}(\zeta) = \{\psi_{j_1}^{(1)}(\zeta), \dots, \psi_{j_r}^{(1)}(\zeta)\}, \quad \psi_j^{(1)}(\zeta) = \frac{\partial \psi_j(\zeta)}{\partial \zeta}.$$

ASSUMPTION H. There exists

$$(12) \quad d_0 > \frac{1}{2}$$

such that

$$(13) \quad \psi_{0j} \leq K j^{-1-d_0},$$

and (10) holds for

$$(14) \quad \rho \in (4/(2d_0 + 3), 1).$$

Assumption A(q) allows some asymmetry in ε_t , but implies the less primitive condition (which does not even require existence of a density) employed in a similar context by Berkes, Horváth and Kokoszka [3]. Assumptions B and C are standard. Inequalities (7) and (13) together imply $d_0 \geq \underline{d}$, while (8) with (3) is milder than monotonicity but implies $\psi_{0j} = o(j^{-1})$ as $j \rightarrow \infty$. We take $\eta > 0$ in Assumption F(l) because $\psi_j(\zeta) < 1$ for all large enough j , by (7). Assumption G is crucial to the proof of consistency, being used in Lemmas 9 and 10 to show that in the limit θ_0 globally minimizes $Q_T(\theta)$; it also ensures nonsingularity of the matrix H_0 in Proposition 2 and Theorem 2 below. This and other assumptions are discussed in Sections 3 and 4 in connection with some parameterizations of interest.

We present asymptotic results for the uncomputable $\tilde{\theta}_T$ as propositions, those for $\hat{\theta}_T$ as theorems. All these, and the corollaries in Sections 3 and 4 and lemmas in Section 5, assume (1)–(5).

PROPOSITION 1. For some $\delta > 0$, let Assumptions A($2 + \delta$), B, C, D, E, F(1) and G hold. Then

$$\tilde{\theta}_T \rightarrow \theta_0 \quad \text{a.s. as } T \rightarrow \infty.$$

PROOF. The proof follows as in, for example, [17], Theorem 6, from uniform a.s. convergence over Θ of $Q_T(\theta)$ to $Q(\theta) = E q_0(\theta)$ established in Lemma 7, the fact that $Q_T(\hat{\theta}_T) \leq Q_T(\theta)$, and Lemma 10. \square

THEOREM 1. *For some $\delta > 0$, let Assumptions **A**($2 + \delta$), **B**, **C**, **D**, **E**, **F**(1) and **G** hold. Then*

$$(15) \quad \hat{\theta}_T \rightarrow \theta_0 \quad \text{a.s. as } T \rightarrow \infty.$$

PROOF. From Lemmas 7 and 8, $\bar{Q}_T(\theta)$ converges uniformly to $Q(\theta)$ a.s., whence the proof is as indicated for Proposition 1. \square

Denote by κ_j the j th cumulant of ε_t and introduce

$$G_0 = (2 + \kappa_4)M - 2\kappa_3(N + N') + P, \quad H_0 = M + \frac{1}{2}P,$$

where

$$M = E(\tau_0 \tau_0'), \quad N = E(\sigma_0^{-1} \tau_0) e_2', \quad P = E(\sigma_0^{-2}) e_2 e_2',$$

for $\tau_0 = \tau_0(\theta_0)$, $\tau_t(\theta) = (\partial/\partial\theta) \log \sigma_t^2(\theta)$, and e_2 the second column of the $(r + 2) \times (r + 2)$ identity matrix. In case μ_0 is known (e.g., to be zero), we omit the second row and column from M , and have instead $G_0 = (2 + \kappa_4)M$, $H_0 = M$. In case ε_t is Gaussian, $\kappa_3 = \kappa_4 = 0$, so $G_0 = 2H_0 = 2M + P$.

PROPOSITION 2. *Let Assumptions **A**(4), **B**, **C**, **D**, **E**, **F**(3) and **G** hold. Then*

$$T^{1/2}(\tilde{\theta}_T - \theta_0) \xrightarrow{d} N(0, H_0^{-1} G_0 H_0^{-1}) \quad \text{as } T \rightarrow \infty.$$

PROOF. Write

$$Q_T^{(1)}(\theta) = \frac{\partial Q_T(\theta)}{\partial \theta} = T^{-1} \sum_{t=1}^T u_t(\theta),$$

where

$$u_t(\theta) = \tau_t(\theta)(1 - \chi_t^2(\theta)) + \sigma_t^{-2}(\theta) v_t(\theta),$$

with

$$\chi_t(\theta) = \frac{x_t^2(\mu)}{\sigma_t^2(\theta)}, \quad v_t(\theta) = \frac{\partial x_t^2(\mu)}{\partial \theta} = -2x_t(\mu) e_2.$$

By the mean value theorem,

$$(16) \quad 0 = Q_T^{(1)}(\tilde{\theta}_T) = Q_T^{(1)}(\theta_0) + \tilde{H}_T(\tilde{\theta}_T - \theta_0),$$

where \tilde{H}_T has as its i th row the i th row of $H_T(\theta) = T^{-1} \sum_{t=1}^T h_t(\theta)$ evaluated at $\theta = \tilde{\theta}_T^{(i)}$, where $h_t(\theta) = (\partial^2/\partial\theta \partial\theta') Q_T(\theta)$, $\|\tilde{\theta}_T^{(i)} - \theta_0\| \leq \|\tilde{\theta}_T - \theta_0\|$, where

we define $\|A\| = \{\text{tr}(A'A)\}^{1/2}$ for any real matrix A . Now $u_t(\theta_0) = \tau_t(\theta_0)(1 - \varepsilon_t^2) - 2e_2\varepsilon_t/\sigma_t$ is, by Lemmas 2, 3 and 7, a stationary ergodic martingale difference vector with finite variance, so from Brown [6] and the Cramér–Wold device, $T^{1/2}Q_T^{(1)}(\theta_0) \rightarrow_d N(0, G_0)$ as $T \rightarrow \infty$. Finally, by Lemma 7 and Theorem 1, $\tilde{H}_T \rightarrow_p H_0$, whence the proof is completed in standard fashion. \square

Define

$$\begin{aligned} \bar{u}_t(\theta) &= \frac{\partial \bar{q}_t(\theta)}{\partial \theta}, & \bar{g}_t(\theta) &= \bar{u}_t(\theta)\bar{u}_t'(\theta), & \bar{h}_t(\theta) &= \frac{\partial^2 \bar{q}_t(\theta)}{\partial \theta \partial \theta'}, \\ \bar{G}_T(\theta) &= T^{-1} \sum_{t=1}^T \bar{g}_t(\theta), & \bar{H}_T(\theta) &= T^{-1} \sum_{t=1}^T \bar{h}_t(\theta). \end{aligned}$$

THEOREM 2. *Let Assumptions A(4), B, C, D, E, F(3), G and H hold. Then*

$$(17) \quad T^{1/2}(\hat{\theta}_T - \theta_0) \xrightarrow{d} N(0, H_0^{-1}G_0H_0^{-1}) \quad \text{as } T \rightarrow \infty,$$

and $H_0^{-1}G_0H_0^{-1}$ is strongly consistently estimated by $\bar{H}_T^{-1}(\hat{\theta}_T)\bar{G}_T(\hat{\theta}_T)\bar{H}_T^{-1}(\hat{\theta}_T)$.

PROOF. We have

$$0 = \bar{Q}_T^{(1)}(\hat{\theta}_T) = \bar{Q}_T^{(1)}(\theta_0) + \hat{H}_T(\hat{\theta}_T - \theta_0),$$

where $\bar{Q}_T^{(1)}(\theta) = (\partial/\partial\theta)Q_T(\theta)$ and \hat{H}_T has as its i th row the i th row of $\bar{H}_T(\theta)$ evaluated at $\theta = \hat{\theta}_T^{(i)}$, for $\|\hat{\theta}_T^{(i)} - \theta_0\| \leq \|\hat{\theta}_T - \theta_0\|$. Thus, from (16),

$$\hat{\theta}_T - \tilde{\theta}_T = (\tilde{H}_T^{-1} - \hat{H}_T^{-1})\bar{Q}_T^{(1)}(\theta_0) - \tilde{H}_T^{-1}\{\bar{Q}_T^{(1)}(\theta_0) - Q_T^{(1)}(\theta_0)\},$$

where the inverses exist a.s. for all sufficiently large T by Lemma 9. In view of Proposition 2 and Lemma 8, (17) follows on showing that

$$\bar{Q}_T^{(1)}(\theta_0) - Q_T^{(1)}(\theta_0) = o_p(T^{-1/2}).$$

The left-hand side can be written $(B_{1T} + B_{2T} + B_{3T})/T$, where

$$B_{1T} = \sum_{t=1}^T \varepsilon_t^2 b_{1t}, \quad B_{2T} = -\sum_{t=1}^T (\varepsilon_t^2 - 1)b_{2t}, \quad B_{3T} = -2e_2 \sum_{t=1}^T \varepsilon_t b_{3t},$$

with

$$b_{1t} = -\frac{\bar{\sigma}_t^{2(1)}(\sigma_t^2 - \bar{\sigma}_t^2)}{\bar{\sigma}_t^4}, \quad b_{2t} = \frac{\sigma_t^{2(1)}}{\sigma_t^2} - \frac{\bar{\sigma}_t^{2(1)}}{\bar{\sigma}_t^2}, \quad b_{3t} = \frac{\sigma_t^2 - \bar{\sigma}_t^2}{\bar{\sigma}_t^2 \sigma_t},$$

for $\bar{\sigma}_t^2 = \bar{\sigma}_t^2(\theta_0)$, $\sigma_t^{2(1)} = \sigma_t^{2(1)}(\theta_0)$, $\bar{\sigma}_t^{2(1)} = \bar{\sigma}_t^{2(1)}(\theta_0)$, with $\sigma_t^{2(1)}(\theta) = (\partial/\partial\theta)\sigma_t^2(\theta)$, $\bar{\sigma}_t^{2(1)}(\theta) = (\partial/\partial\theta)\bar{\sigma}_t^2(\theta)$. We show that $B_{iT} = o_p(T^{1/2})$, $i = 1, 2, 3$. For the remainder of this proof, we drop the zero subscript in ψ_{0j} .

Consider first B_{1T} . We have

$$(18) \quad \bar{\sigma}_t^{2(1)} = \left(1, -2 \sum_{j=1}^{t-1} \psi_j x_{t-j}, \sum_{j=1}^{t-1} \psi_j^{(1)} x_{t-j}^2 \right)',$$

where $\psi_j^{(1)} = \psi_j^{(1)}(\zeta_0)$. From Assumption F(1),

$$\|\bar{\sigma}_t^{2(1)}\| \leq 1 + 2 \sum_{j=1}^{t-1} \psi_j |x_{t-j}| + K \sum_{j=1}^{t-1} \psi_j^{1-\eta} x_{t-j}^2,$$

for all $\eta > 0$. Now

$$\sum_{j=1}^{t-1} \psi_j |x_{t-j}| \leq \left(\sum_{j=1}^{t-1} \psi_j x_{t-j}^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} \psi_j \right)^{1/2} \leq K \bar{\sigma}_t,$$

so since $\bar{\sigma}_t \geq \omega_L > 0$,

$$\bar{\sigma}_t^{-2} \sum_{j=1}^{t-1} \psi_j |x_{t-j}| \leq K \bar{\sigma}_t^{-1} < \infty.$$

From (8),

$$\sum_{j=1}^{t-1} \psi_j^{1-\eta} x_{t-j}^2 \leq K \psi_t^{-\eta} \bar{\sigma}_t^2.$$

It follows that

$$(19) \quad \|\bar{\sigma}_t^{2(1)}\| / \bar{\sigma}_t^2 \leq K \psi_t^{-\eta}.$$

On the other hand, by the c_r -inequality ([23], page 157) and (10),

$$(20) \quad E(\sigma_t^2 - \bar{\sigma}_t^2)^\rho \leq K \sum_{j=t}^{\infty} \psi_j^\rho E|x_{t-j}|^{2\rho} \leq K \sum_{j=t}^{\infty} \psi_j^\rho.$$

Thus, by (8) and (14),

$$(21) \quad E\|b_{1t}\|^\rho \leq K \psi_t^{-\eta\rho} \sum_{j=t}^{\infty} \psi_j^\rho \leq K \sum_{j=t}^{\infty} \psi_j^{\rho(1-\eta)} \leq K t^{1-\rho(d_0+1)(1-\eta)},$$

choosing $\eta < 1 - 1/\{\rho(d_0 + 1)\}$, which (14) enables. Applying the c_r -inequality again,

$$E\|B_{1T}\|^\rho \leq K \sum_{t=1}^T E|\varepsilon_0|^{2\rho} E\|b_{1t}\|^\rho.$$

Applying (21), this is $O(1)$ when $\rho > 2/(d_0 + 1)$, while when $\rho \leq 2/(d_0 + 1)$, we may choose η so small to bound it by

$$KT^{2-\rho(d_0+1)(1-\eta)} \leq KT^{\rho/2-\{1+2(d_0+1)(1-\eta)\}[\rho/2-2/\{1+2(d_0+1)(1-\eta)\}]} = o(T^{\rho/2}),$$

using (12) [which requires (13)] and arbitrariness of η . Thus, $B_{1T} = o_p(T^{1/2})$ by Markov's inequality.

Consider B_{2T} . By independence of ε_t and b_{2t} , by the c_r -inequality when $\rho \leq \frac{1}{2}$, and by the inequality of von Bahr and Esseen [34] and the fact that the ε_t^2 are i.i.d. with mean 1 when $\rho > \frac{1}{2}$,

$$E\|B_{2T}\|^{2\rho} \leq K \sum_{t=1}^T (E|\varepsilon_0|^{4\rho} + 1)E\|b_{2t}\|^{2\rho} \leq K \sum_{t=1}^T (E\|b_{4t}\|^{2\rho} + E\|b_{5t}\|^{2\rho}),$$

where

$$b_{4t} = \frac{\sigma_t^{2(1)} - \bar{\sigma}_t^{2(1)}}{\sigma_t^2}, \quad b_{5t} = \frac{\bar{\sigma}_t^{2(1)}(\sigma_t^2 - \bar{\sigma}_t^2)}{\bar{\sigma}_t^2 \sigma_t^2}.$$

Thus, from Assumptions F(1) and H,

$$\begin{aligned} \|b_{4t}\| &\leq \left(2 \sum_{j=t}^{\infty} \psi_j |x_{t-j}| + \sum_{j=t}^{\infty} \|\psi_j^{(1)}\| x_{t-j}^2 \right) / \sigma_t^2 \\ &\leq \sigma_t^{-2} \left[2 \left\{ \sum_{j=t}^{\infty} \psi_j \right\}^{1/2} + \left\{ \sum_{j=t}^{\infty} (\|\psi_j^{(1)}\|^2 / \psi_j) x_{t-j}^2 \right\}^{1/2} \right] \left\{ \sum_{j=t}^{\infty} \psi_j x_{t-j}^2 \right\}^{1/2} \\ &\leq K \left\{ \left(\sum_{j=t}^{\infty} j^{-d_0-1} \right)^{1/2} + \left(\sum_{j=t}^{\infty} \psi_j^{1-2\eta} x_{t-j}^2 \right)^{1/2} \right\} \\ &\leq K \left[t^{-d_0/2} + \left\{ \sum_{j=t}^{\infty} j^{-(d_0+1)(1-2\eta)} x_{t-j}^2 \right\}^{1/2} \right], \end{aligned}$$

so

$$E\|b_{4t}\|^{2\rho} \leq K t^{-\rho d_0} + K \sum_{j=t}^{\infty} j^{-(d_0+1)\rho(1-2\eta)} \leq K t^{1-(d_0+1)\rho(1-2\eta)}$$

for sufficiently small η . Thus, $\sum_{t=1}^T E\|b_{4t}\|^{2\rho}$ is $O(1)$ for $\rho > 2/(d_0 + 1)$, while for $\rho \leq 2/(d_0 + 1)$, it is bounded by

$$KT^{2-(d_0+1)\rho(1-2\eta)} \leq KT^{\rho-(d_0+2)\{\rho-2/(d_0+2)\}+2(d_0+1)\rho\eta} = o(T^\rho)$$

from (14) and arbitrariness of η . Also, $\|b_{5t}\| \leq K \|\bar{\sigma}_t^{2(1)} / \bar{\sigma}_t^2\| (\sigma_t^2 - \bar{\sigma}_t^2)^{1/2}$, so from (19) and (20) we have $E\|b_{5t}\|^{2\rho} \leq K t^{1-(d_0+1)\rho(1-2\eta)}$, and proceeding as before,

$$\sum_{t=1}^T E\|b_{5t}\|^{2\rho} = o(T^\rho),$$

and thence, $B_{2T} = o_p(T^{1/2})$.

Next,

$$E \|B_{3T}\|^{2\rho} \leq K E \left| \sum_{t=1}^T \varepsilon_t b_{3t} \right|^{2\rho} \leq K \sum_{t=1}^T E |\varepsilon_0|^{2\rho} E b_{3t}^{2\rho},$$

applying the c_r -inequality when $\rho \leq \frac{1}{2}$ and von Bahr and Esseen [34] when $\rho > \frac{1}{2}$. Now $b_{3t} \leq (\sigma_t^2 - \bar{\sigma}_t^2)^{1/2} \bar{\sigma}_t^{-2}$, so from (20),

$$\begin{aligned} E \|B_{3T}\|^{2\rho} &\leq K \sum_{t=1}^T \sum_{j=t}^{\infty} \psi_j^\rho \\ &\leq K \{ \mathbb{1}(\rho > 2/(d_0 + 1)) + (\ln T) \mathbb{1}(\rho = 2/(d_0 + 1)) \\ &\quad + T^{2-\rho(d_0+1)} \mathbb{1}(\rho < 2/(d_0 + 1)) \} \\ &= o(T^\rho), \end{aligned}$$

much as before. Thence, $B_{3T} = o_p(T^{1/2})$.

It remains to consider the last statement of the theorem, which follows on standard application of Propositions 1 and 2, Theorem 1 and Lemmas 7 and 8. \square

In earlier versions of this paper we checked the conditions in the case of GARCH(n, m) models in which the $\psi_j(\zeta)$ decay exponentially and we allow the possibility that the GARCH coefficients lie in a subspace of dimension less than $m + n$; the details are available from the authors on request. However, the literature on asymptotic theory for estimates of GARCH models is now extensive, recent references including [3, 7, 12, 16, 22, 32], along with investigations of the properties of the models themselves; see recently [2, 18, 25]. We focus instead on alternative models which have received less attention, and for which our theoretical framework is primarily intended.

We introduce the generating function

$$(22) \quad \psi(z; \zeta) = \sum_{j=1}^{\infty} \psi_j(\zeta) z^j, \quad |z| \leq 1.$$

3. Fractional GARCH models. A slowly decaying class of ARCH(∞) weights was considered by Robinson [29], Ding and Granger [9] and Koulikov [20], generated by

$$(23) \quad \psi(z; \zeta) = 1 - (1 - z)^\zeta, \quad 0 < \zeta < 1,$$

where $r = 1$ and formally

$$(24) \quad (1 - z)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j - d)}{\Gamma(-d)\Gamma(j + 1)} z^j, \quad |z| \leq 1, \quad d > 0.$$

In these references $\omega_0 = 0$ was assumed in (2), but we assume $\omega_0 > 0$ and generalize (23) as follows. Introduce the functions $a_j = a_j(\zeta)$, $b_j = b_j(\zeta)$ and, for $m \geq 1$, $n \geq 0$, $n + m \geq r$,

$$(25) \quad a(z; \zeta) = \sum_{j=1}^m a_j z^j, \quad b(z; \zeta) = 1 - \sum_{j=1}^n b_j z^j \mathbb{1}(n \geq 1);$$

and for all $\zeta \in \Upsilon$,

$$(26) \quad a_j > 0, \quad j = 1, \dots, m; \quad b_j > 0, \quad j = 1, \dots, n;$$

$$(27) \quad b(z; \zeta) \neq 0, \quad |z| \leq 1;$$

$$(28) \quad a(z; \zeta) \text{ and } b(z; \zeta) \text{ have no common zeros in } z.$$

Now take $\psi(z; \zeta)$ (22) to be given by

$$(29) \quad \psi(z; \zeta) = \frac{a(z; \zeta)\{1 - (1 - z)^d\}}{zb(z; \zeta)},$$

with $d = d(\zeta)$ satisfying

$$(30) \quad d \in (0, 1).$$

We call x_t based on (29) a fractional GARCH, FGARCH(n, d_0, m) process, for $d_0 = d(\zeta_0)$.

COROLLARY 1. *Let $\psi(z; \zeta)$ be given by (29) and (25) with $m \geq 1$, $n \geq 0$, and let d and the a_j, b_j be continuously differentiable. For some $\delta > 0$, let Assumptions A($2 + \delta$), B, C and E hold, with all $\zeta \in \Upsilon$ satisfying (26)–(28), (30) and*

$$\text{rank} \left\{ \frac{\partial}{\partial \zeta} (a_1, \dots, a_m, b_1, \dots, b_n, d) \right\} = r.$$

Then (15) is true. Let also d and the a_j, b_j be thrice continuously differentiable and $d_0 > \frac{1}{2}$. Then (17) is true.

PROOF. Denoting by c_j ($j \geq 1$) and d_j ($j \geq 0$) the coefficients of z^j in the expansions of $a(z; \zeta)/b(z; \zeta)$, $z^{-1}\{1 - (1 - z)^d\}$, respectively, we have $\psi_j(\zeta) = \sum_{k=0}^{j-1} c_{j-k} d_k$, $j \geq 1$. From [3], the c_j are bounded above and below by positive, exponentially decaying sequences when $n \geq 1$, and are all nonnegative when $n = 0$. Since the d_j are all positive, it follows that (6) holds. Also, Stirling's approximation indicates that $j^{-d-1}/K \leq d_j \leq K j^{-d-1}$, so the $\psi_j(\zeta)$ satisfy the same inequalities. Compactness of Υ , smoothness of d , and (30), imply $d(\zeta) \geq \underline{d}$, to check (7). The above argument indicates that $\psi_{0j} \leq K j^{-d_0-1} \leq K k^{-d_0-1} \leq K \psi_{0k}$ for $j > k \geq 1$, so (8) holds, and thus Assumption D. With regard to (11), note that $(\partial/\partial d)\psi(z; \zeta) = -\{a(z; \zeta)/b(z; \zeta)\}z^{-1}(1 - z)^d \ln(1 - z)$, where the coefficient of z^j in $-z^{-1}(1 - z)^d \ln(1 - z)$ is $\sum_{k=1}^j k^{-1} d_{j-k} \leq K(\ln j)j^{-d-1} \leq$

$Kj^{-(d+1)(1-\eta)} \leq K\psi_j^{1-\eta}(\zeta)$ for any $\eta > 0$. Derivatives with respect to the a_j, b_j are dominated, and higher derivatives can be dealt with similarly, to complete the checking of Assumption **F**(l). To check Assumption **G**, suppress reference to ζ in a, b, ψ and

$$\phi(z) = b(z)^{-1}\{1 - (1 - z)^d\}, \quad \gamma(z) = b(z)^{-1}a(z),$$

and note that

$$\frac{\partial\psi(z)}{\partial a_j} = z^{j-1}\phi(z), \quad j = 1, \dots, m,$$

$$\frac{\partial\psi(z)}{\partial b_j} = z^{j-1}\gamma(z)\phi(z), \quad j = 1, \dots, n,$$

$$\frac{\partial\psi(z)}{\partial d} = -\frac{\gamma(z)}{z}(1 - z)^d \log(1 - z).$$

Choose $j_i(\zeta) = i$ for $i = 1, \dots, m + n, \zeta \in \Upsilon$, leaving $j_{m+n+1}(\zeta)$ to be determined subsequently. Fix ζ and write $U = \Psi_{(j_1, \dots, j_r)}(\zeta)$, partitioning it in the ratio $m + n : 1$ and calling its (i, j) th submatrix U_{ij} . We first show that the $(m + n) \times (m + n)$ matrix U_{11} is nonsingular. Write R for the $n \times (m + n)$ matrix with (i, j) th element γ_{j-i} , and S for the $(m + n) \times (m + n)$ matrix with (i, j) th element ϕ_{j-i+1} , where $\phi_j = \gamma_j = 0$ for $j \leq 0$, and for $j > 0, \phi_j$ and γ_j are respectively given by

$$\phi(z) = \sum_{j=1}^{\infty} \phi_j z^j, \quad \gamma(z) = \sum_{j=1}^{\infty} \gamma_j z^j,$$

these series converging absolutely for $|z| \leq 1$ in view of (30). Noting that $\psi_j^{(1)}$ is given by $(\partial/\partial\zeta)\psi(z) = \sum_{j=1}^{\infty} \psi_j^{(1)} z^j$, we find that the first m rows of U_{11} can be written $(I_m, O)S$, where I_m is the m -rowed identity matrix, O is the $m \times n$ matrix of zeroes and, when $n \geq 1$ the last n rows of U_{11} can be written RS . Now S is upper-triangular with nonzero diagonal elements. Thus, for $n = 0, U_{11} = S$ is nonsingular. For $n \geq 1, U_{11}$ is nonsingular if and only if the $n \times n$ matrix R_2 having (i, j) th element γ_{m+j-i} and consisting of the last n columns of R is nonsingular. This is not so if and only if the $\gamma_j, j = m, \dots, m + n - 1$, are generated by a homogeneous linear difference equation of degree $n - 1$, that is, if there exist scalars $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$, not all zero, such that

$$\lambda_0 \gamma_j - \sum_{i=1}^{n-1} \lambda_i \gamma_{j-i} = 0, \quad j = m, \dots, m + n - 1.$$

But it follows from (25) and (27) that they are generated by the linear difference equation

$$\gamma_j - \sum_{i=1}^{n-1} b_i \gamma_{j-i} = \pi_j, \quad j = m, \dots, m + n - 1,$$

where $\pi_m = a_m + b_n \gamma_{m-n}$, $\pi_j = b_n \gamma_{j-n}$ for $j = m + 1, \dots, m + n - 1$. Since $b_n \neq 0$, the π_j are all zero if and only if $\gamma_{m-n} = -a_m/b_n$ and $\gamma_j = 0$ for $j = m + 1 - n, \dots, m - 1$. But this implies $\gamma_m = 0$ also, and thence, $\gamma_j = 0$, all $j \geq m - n + 1$. For $m \leq n$, this is inconsistent with the requirement $a_j > 0$, $j = 1, \dots, m$, and for $m > n$, it implies a has a factor b , which is inconsistent with (28). Thus, U_{11} is nonsingular when $n \geq 1$. Nonsingularity of U follows if $U_{22} \neq U_{21}U_{11}^{-1}U_{12}$. For large enough $j_{m+n+1} = j_{m+n+1}(\zeta)$, this must be true because U_{22} decays like $(\ln j_{m+n+1})j_{m+n+1}^{-d-1}$, whereas the elements of U_{12} are $O(\beta^{j_{m+n+1}})$ for some $\beta \in (0, 1)$. Thus Assumption G is true, and thence (15). Clearly (13) is true, so under the additional conditions so is Assumption H, and thence (17). \square

For $m = 1$, $n = 0$, (29) reduces to (23) when $a_1 = 1$, while when $a_1 \in (0, 1)$, it gives model (4.24) of Ding and Granger [9]. The important difference between these two cases is that the covariance stationarity condition $\psi(1; \zeta_0) < 1$ is satisfied in the second but not in the first. In general with (29), as with the GARCH model, x_t is covariance stationary when $a(1; \zeta_0) < b(1; \zeta_0)$ but not otherwise. We compare (29) with

$$(31) \quad \psi(z; \zeta) = 1 - \frac{\{1 - a(z; \zeta)\}}{b(z; \zeta)}(1 - z)^d,$$

with d again satisfying (30) and a and b again given as in (25), though we now allow $m = 0$, meaning $a(z; \zeta) \equiv 0$. Thus, with $m = n = 0$, (31) reduces to (23). ARCH(∞) models with ψ given by (31) were proposed by Baillie, Bollerslev and Mikkelsen [1] and called FIGARCH(n, d_0, m). In general, though (31) also gives hyperbolically decaying ψ_{0j} , it differs in some notable respects. Application of (26)–(28) again ensures positivity of $\psi_j(\zeta)$ in case of FIGARCH and facilitates the above proof, but sufficient conditions in FIGARCH are less apparent in general, though Baillie, Bollerslev and Mikkelsen [1] indicated that they can be obtained. Also, unlike FIGARCH, FIGARCH x_t never has finite variance.

The requirement $d_0 > \frac{1}{2}$ for the central limit theorem in Corollary 1 would also be imposed in a corresponding result for FIGARCH. This is automatically satisfied in GARCH models but if only $d_0 \in (0, \frac{1}{2}]$ in (13) is possible in the general setting of Section 3, it appears that the asymptotic bias in $\hat{\theta}_T$ is of order at least $T^{-1/2}$, whereas that for $\tilde{\theta}_T$ is always $o(T^{-1/2})$. Assumption H copes with the replacement of $\sigma_t^2(\theta)$ by $\bar{\sigma}_t^2(\theta)$, the truncation error varying inversely with d_0 . Inspection of the proof of Theorem 2 indicates that this bias problem is due to the term $H^{-1}B_{1T}$. The factor $\sigma_t^2 - \bar{\sigma}_t^2$ in b_{1t} is nonnegative, and if j^{-d_0-1} is an exact rate for ψ_{0j} , $\sigma_t^2 - \bar{\sigma}_t^2$ exceeds t^{-d_0}/K as $t \rightarrow \infty$ with probability approaching one. So far as the factor $\bar{\sigma}_t^{2(1)}/\sigma_t^4$ in b_{1t} is concerned, the second element of $\bar{\sigma}^{2(1)}$ [see (18)] has zero mean, but the first is positive, and though the $\psi_j^{(1)}$ can have elements of either sign, whenever $d_0 \leq \frac{1}{2}$ it seems unlikely that the last r elements of B_{1T} can be

$o_p(T^{1/2})$. Nor is there scope for relaxing (12) by strengthening other conditions. With regard to implications for choice of ρ , when $d_0 \geq 2\underline{d} + \frac{1}{2}$, (14) entails no restriction over (9).

Though results of Giraitis, Kokoszka and Leipus [13] indicate existence of a stationary solution of (1)–(3) when $\psi(1; \zeta_0) < 1$, Kazakevičius and Leipus [19] have questioned the existence of strictly stationary FIGARCH processes, and thus the relevance of Assumption E here. The same reservations can be expressed about FGARCH when $a(1; \zeta_0) \geq b(1; \zeta_0)$, and more generally about ARCH(∞) processes with $\psi(1; \zeta_0) \geq 1$. A sufficient condition for (10) can be deduced as follows. Recursive substitution gives

$$\sigma_t^2 \leq K + K \sum_{l=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \psi_{0j_1} \cdots \psi_{0j_l} \varepsilon_{t-j_1}^2 \varepsilon_{t-j_1-j_2}^2 \cdots \varepsilon_{t-j_1-\dots-j_l}^2 \right),$$

so by the c_r -inequality,

$$\begin{aligned} \sigma_t^{2\rho} \leq K + K \sum_{l=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \psi_{0j_1}^\rho \cdots \psi_{0j_l}^\rho |\varepsilon_{t-j_1}|^{2\rho} \right. \\ \left. \times |\varepsilon_{t-j_1-j_2}|^{2\rho} \cdots |\varepsilon_{t-j_1-\dots-j_l}|^{2\rho} \right). \end{aligned}$$

Thus, from Lemma 2,

$$E|x_t|^{2\rho} < E|\sigma_t|^{2\rho} \leq K + K \sum_{l=0}^{\infty} \left(E|\varepsilon_0|^{2\rho} \sum_{j=1}^{\infty} \psi_{0j}^\rho \right)^l.$$

The last bound is finite if and only if

$$(32) \quad E|\varepsilon_0|^{2\rho} \sum_{j=1}^{\infty} \psi_{0j}^\rho < 1.$$

Thus, (10) holds if there is a ρ satisfying (9) and (32). Recursive substitution and the c_r -inequality were also used by Nelson ([27], Corollary) to upper-bound $E|\sigma_t|^{2\rho}$ in the GARCH(1, 1) case, but he employed the simple dynamic structure available there, and (32) does not reduce to his necessary and sufficient condition.

If $\psi(1; \zeta_0) < 1$, (32) adds nothing because we already know that $Ex_0^2 < \infty$ here, but if $\psi(1; \zeta_0) \geq 1$, the second factor on the left-hand side of (32) exceeds 1 and increases with ρ ; the question is whether the first factor, which is less than 1 and decreases with ρ [due to Assumption A(q)], can over-compensate. Analytic verification of (32) for given ζ_0, ρ seems in general infeasible, and numerical verification highly problematic when the ψ_j decay slowly. However, consider the family of densities

$$(33) \quad f(\varepsilon) = \exp[-\{\alpha(\gamma)|\varepsilon\}^{1/\gamma}] / \{2\gamma\Gamma(\gamma)\alpha(\gamma)\}$$

for $\gamma > 0$, where $\alpha(\gamma) = \{\Gamma(\gamma)/\Gamma(3\gamma)\}^{1/2}$ (also used by Nelson [28] to model the innovation of the exponential GARCH model). We have $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$ as necessary, Assumption A(q) is satisfied for all $q > 0$, and $E|\varepsilon_0|^{2\rho} = \Gamma((2\rho + 1)\gamma)/\{\Gamma(\gamma)^{1-\rho}\Gamma(3\gamma)^\rho\}$. In case $\gamma = 0.5$, (33) is the normal density, for which $\hat{\theta}_T$ is asymptotically efficient. Here $E|\varepsilon_0|^{2\rho} = 2^\rho\Gamma(\rho + 0.5)/\sqrt{\pi}$, and numerical calculations for FIGARCH(0, d_0 , 0) cast doubt on (32). In case $\gamma = 1$, (33) is the Laplace density, with $E|\varepsilon_0|^{2\rho} = 2^{\rho-1}\Gamma(2\rho + 1)$. As γ increases, $E|\varepsilon_0|^{2\rho}$ can be made small for fixed $\rho < 1$, for example, with $\rho = 0.95$, it is 0.64 when $\gamma = 10$ and 0.42 when $\gamma = 20$.

4. Generalized exponential and hyperbolic models. FGARCH(n , d_0 , m) [and FIGARCH(n , d_0 , m)] processes require $d_0 \in (0, 1)$. For $d = 1$, (29) reduces to (23), and for $d > 1$, at least one coefficient in the expansion of (23) is negative, leading to the possibility of negative $\psi_j(\zeta)$. Because FGARCH $\psi_j(\zeta)$ decay like j^{-d-1} , a large mathematical gap is left relative to GARCH processes. Even if exponential decay is anticipated, there is a case for more direct modeling of the $\psi_j(\zeta)$ than provided by GARCH(n , m), since it is the $\psi_j(\zeta)$ and their derivatives that must be formed in point and interval estimation based on the PMLE.

Consider the choices

$$(34) \quad \psi_j(\zeta) = \sum_{i=1}^m \Gamma(f_i + 1)^{-1} e_i d^{f_i+1} j^{f_i} e^{-dj},$$

$$(35) \quad \psi_j(\zeta) = \sum_{i=1}^m \Gamma(f_i + 1)^{-1} e_i d \ln^{f_i} (j + 1) (j + 1)^{-d-1},$$

where $d = d(\zeta)$ and the $e_i = e_i(\zeta)$, $f_i = f_i(\zeta)$ are such that Υ satisfies

$$(36) \quad d \in (0, \infty),$$

$$(37) \quad e_i > 0, \quad i = 1, \dots, m,$$

$$(38) \quad 0 \leq f_1 \leq \dots \leq f_m < \infty,$$

with $2m + 1 \geq r$. Given (1)–(4) and (22), we call x_t generated by (34) a generalized exponential, GEXP(m), process, and x_t generated by (35) a generalized hyperbolic, GHYP(m), process. Condition (38) is sufficient but not necessary for $\psi_j(\zeta) > 0$, all $j \geq 1$. By choosing m large enough in (34) or (35), any finite $\psi(1; \zeta)$ can be arbitrarily well approximated, but (34) and (35) can also achieve parsimony. For real $x \geq 1$, $x^f e^{-dx}$ and $(\ln x)^f x^{-d-1}$ decay monotonically if $f = 0$, and for $f > 0$, have single maxima at f/d and $e^{f/(d+1)}$, respectively. Thus, with $m = 1$ and $f_1 = 0$, we have monotonic decay in (34) and (35); otherwise, both can exhibit lack of monotonicity, while eventually decaying exponentially or hyperbolically. The scale factors in (34) and (35) are so expressed because $x^f e^{-dx}$ and $(\ln x)^f x^{-d-1}$ integrate over $(0, \infty)$ to $\Gamma(f + 1)/d^{f+1}$ and $\Gamma(f + 1)/d$, respectively, so that $\psi(1; \zeta) \doteq \sum_{i=1}^m e_i$ in both cases, but the approximation may

not be very close and the “integrated” case is less easy to distinguish than in GARCH and FGARCH models (though it would be possible to alternatively scale the weights by infinite sums to achieve equality).

The following corollary covers (34) and (35) simultaneously, and implies the special case when the f_i are specified a priori, for example, to be nonnegative integers; strictly speaking, when the true value of f_1 is unknown, Assumption C prevents it from being zero.

COROLLARY 2. *Let $\psi(z; \zeta)$ be given by (22) and (34) or (35) with $m \geq 1$ and let d and the e_i, f_i be continuously differentiable. For some $\delta > 0$, let Assumptions A(2 + δ), B, C and E hold, with all $\zeta \in \Upsilon$ satisfying (36)–(38) and*

$$\text{rank} \left\{ \frac{\partial}{\partial \zeta} (e_1, f_1, \dots, e_m, f_m, d) \right\} = r.$$

Then (15) is true. Let also d and the e_i, f_i be thrice continuously differentiable and Assumption A(4) hold, and $d_0 = d(\zeta_0) > \frac{1}{2}$ in case of (35). Then (17) is true.

PROOF. Given (36)–(38) and the proofs of Corollaries 1 and 2, the verification of Assumptions D and F(l) is straightforward. We check Assumption G for (35) only, a very similar type of proof holding for (34). We have

$$\psi_j^{(1)} = \begin{bmatrix} E(u'_{1j}, \dots, u'_{mj})' \\ v_j \end{bmatrix} d(j+1)^{-d-1},$$

where

$$u_{ij} = (\ln \ln(j+1) - (\partial/\partial f_i) \ln \Gamma(f_{i+1}), 1)' \ln^{f_i}(j+1), \quad i = 1, \dots, r,$$

$$v_j = - \sum_{i=1}^m e_i \Gamma(f_{i+1})^{-1} \ln^{f_i+1}(j+1),$$

and E is the diagonal matrix whose $(2i - 1)$ st diagonal element is e_i , and whose even diagonal elements are all 1. Fixing ζ , we show first that the leading $(r - 1) \times (r - 1)$ submatrix of $\Psi_{(j_1, \dots, j_r)}(\zeta)$ has full rank, equivalently, that U_m has full rank, where, for $i = 1, \dots, m$, the $(2i) \times (2i)$ matrix U_i has (k, ℓ) th 2×1 sub-vector $u_{kj\ell}$, $k = 1, \dots, i$, $\ell = 1, \dots, 2i$. Suppose, for some $i = 1, \dots, m - 1$ and given j_1, \dots, j_{2i} , that U_i has full rank, and partition the rows and columns of U_{i+1} in the ratio $2i : 2$, calling its (k, ℓ) th submatrix $U_{k\ell}$ (so $U_{11} = U_i$). Take $j_{2i+2} = j_{2i+1}^2$. Because $\ln \ln x$ strictly increases in $x > 1$, it follows that U_{22} is nonsingular and $\|U_{22}^{-1}\| = O(\ln \ln j_{2i+1} \ln^{-f_{i+1}} j_{2i+1})$. Noting that $\|U_{12}\| = O(\ln \ln j_{2i+1} \ln^{f_i} j_{2i+1})$, while U_{11} and U_{21} depend only on j_1, \dots, j_{2i} , we can choose j_{2i+1} such that $U_{11} - U_{12}U_{22}^{-1}U_{21}$ differs negligibly from U_{11} . Thus, U_{i+1} has full rank. Since, for $f_1 \geq 0$, U_1 has full rank (e.g., when $j_1 = 1, j_2 = 2$), it follows by induction that U_m has full rank. Since v_j is dominated by a term of order $\ln^{f_m+1} j$, while $\|u_{ij}\| = O(\ln \ln j \ln^{f_i} j)$, a similar argument shows that j_r can then be chosen large enough, to complete verification of Assumption G. □

5. Technical lemmas. Define

$$\sigma_t^{*2}(\theta) = \omega + \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}^2, \quad \sigma_t^{*2} = \omega_U + \sum_{j=1}^{\infty} \sup_{\zeta \in \Upsilon} \psi_j(\zeta) x_{t-j}^2.$$

LEMMA 1. Under Assumptions **B** and **D**, for all $\theta \in \Theta$, $t \in \mathbb{Z}$,

$$K^{-1} \sigma_t^{*2}(\theta) \leq \sigma_t^2(\theta) \leq K \sigma_t^{*2}(\theta) \quad a.s.$$

PROOF. A simple extension of [21], Lemma 1. \square

LEMMA 2. Under Assumptions **A(2)**, **B**, **C**, **D** and **E**, for all $t \in \mathbb{Z}$,

$$(39) \quad E|x_t|^{2\rho} < E\sigma_t^{2\rho} \leq E \sup_{\theta \in \Theta} \sigma_t^{2\rho}(\theta) \leq K E\sigma_t^{*2\rho} \leq K E|x_t|^{2\rho} \leq K,$$

$$(40) \quad \inf_{\theta \in \Theta} \sigma_t^2(\theta) > 0, \quad \sup_{\theta \in \Theta} \sigma_t^2(\theta) < K \sigma_t^{*2} < \infty \quad a.s.,$$

$$(41) \quad E \sup_{\theta \in \Theta} |\ln \sigma_t^2(\theta)| \leq K.$$

PROOF. With respect to (39), the first inequality follows from Jensen's inequality, the second is obvious, the third follows from Lemma 1, the fourth follows from the c_r -inequality, (7) and (9), while the last one is due to (10). The proof of (40) uses Lemma 1, $\sigma_t^2(\theta) \geq \omega_L$, (10) and [23], page 121. To prove (41), $|\ln x| \leq x + x^{-1}$ for $x > 0$ and Lemma 2 give

$$E \sup_{\theta \in \Theta} |\ln \sigma_t^2(\theta)| \leq \rho^{-1} E \sup_{\theta \in \Theta} \sigma_t^{2\rho}(\theta) + E \left\{ \inf_{\theta \in \Theta} \sigma_t^2(\theta) \right\}^{-1} \leq K. \quad \square$$

LEMMA 3. Under Assumptions **D**, **E** and **F(l)**, for all $\theta \in \Theta$, $\sigma_t^2(\theta)$, $q_t(\theta)$ and their first l derivatives are strictly stationary and ergodic.

PROOF. Follows straightforwardly from the assumptions. \square

LEMMA 4. Under Assumption **A(2)**, for positive integer $k < (b+1)n/2$,

$$(42) \quad E \left(\sum_{t=1}^n \varepsilon_t^2 \right)^{-k} < \infty.$$

PROOF. Denote by $M_X(t) = E(e^{tX})$ the moment-generating function of a random variable X . By Cressie et al. [8], the left-hand side of (42) is proportional to

$$(43) \quad \int_0^\infty t^{k-1} M_{\sum \varepsilon_t^2}(-t) dt = \int_0^\infty t^{k-1} M_{\varepsilon_0^2}^n(-t) dt \\ \leq \int_0^1 t^{k-1} dt + \int_1^\infty t^{k-1} M_{\varepsilon_0^2}^n(-t) dt.$$

It suffices to show that the last integral is bounded. For all $\delta > 0$, there exists $\eta > 0$ such that $L(\varepsilon^{-1}) \leq \varepsilon^{-\delta}$, $\varepsilon \in (0, \eta)$, so

$$M_{\varepsilon_0^2}(-t) = \int_{-\infty}^{\infty} e^{-t\varepsilon^2} f(\varepsilon) d\varepsilon \leq K \int_0^\eta e^{-t\varepsilon^2} \varepsilon^{b-\delta} d\varepsilon + 2e^{-t\eta^2}.$$

The last integral is bounded by

$$Kt^{(\delta-b-1)/2} \int_0^\infty e^{-\varepsilon} \varepsilon^{(\delta-b-1)/2} d\varepsilon \leq Kt^{(\delta-b-1)/2}.$$

Thus, (43) is finite if $k + n(\delta - b - 1)/2 < 0$, that is, since δ is arbitrary, if $k < (b + 1)n/2$. \square

The previous version of the paper included a longer, independently obtained, proof of the following lemma which we have been able to shorten in one respect by using an idea of Berkes, Horváth and Kokoszka [3] in a corresponding lemma covering the GARCH(n, m) case.

LEMMA 5. Under Assumptions A(q), B, C and D, for $p < q/2$,

$$E \sup_{\theta \in \Theta} \left(\frac{\sigma_t^2}{\sigma_t^2(\theta)} \right)^p \leq K < \infty.$$

PROOF. We have

$$\sigma_t^2 = \omega_0 + \psi_{01}x_{t-1}^2 + \sum_{j=2}^{\infty} \psi_{0j}x_{t-j}^2 \leq \omega_0 + \psi_{01}\sigma_{t-1}^2\varepsilon_{t-1}^2 + K\sigma_{t-1}^2$$

from (8). Thus, $\sigma_t^2/\sigma_{t-1}^2 \leq K(1 + \varepsilon_{t-1}^2)$ and thence, for fixed $j \geq 1$, $\sigma_t^2/\sigma_{t-j}^2 \leq Kh_{tj}$, where $h_{tj} = \prod_{i=1}^j (1 + \varepsilon_{t-i}^2)$. For any $M < \infty$,

$$\begin{aligned} \frac{\sigma_t^2}{\sigma_t^2(\theta)} &\leq \frac{K\sigma_t^2}{\sigma_t^{*2}(\theta)} \leq K \left(\frac{\omega}{\sigma_t^2} + \sum_{j=1}^M \psi_j(\zeta)\varepsilon_{t-j}^2 \frac{\sigma_{t-j}^2}{\sigma_t^2} \right)^{-1} \\ &\leq \frac{Kh_{tM}/\{\inf_{\zeta \in \Upsilon} \inf_{j=1, \dots, M} \psi_j(\zeta)\}}{\sum_{j=1}^M \varepsilon_{t-j}^2}. \end{aligned}$$

The proof can now be completed much as in the proof of Lemma 5.1 of [3], using Hölder’s inequality as there but employing our Lemma 4 and taking $M > 2pq/[(b + 1)(q - 2p)]$. \square

LEMMA 6. Under Assumptions A(2), B, C, D, E and F(l), for all $p > 0$ and $k \leq l$,

$$(44) \quad E \sup_{\theta \in \Theta} \left| \frac{1}{\sigma_t^2(\theta)} \frac{\partial^k \sigma_t^2(\theta)}{\partial \theta_{i_1} \cdots \partial \theta_{i_k}} \right|^p < \infty,$$

$$(45) \quad E \sup_{\theta \in \Theta} \left| \frac{1}{\bar{\sigma}_t^2(\theta)} \frac{\partial^k \bar{\sigma}_t^2(\theta)}{\partial \theta_{i_1} \cdots \partial \theta_{i_k}} \right|^p < \infty.$$

PROOF. Take $i_1 \leq i_2 \leq \dots \leq i_k$. First assume $i_1 \geq 3$, whence, for given k and i_1, \dots, i_k ,

$$\frac{\partial^k \sigma_t^2(\theta)}{\partial \theta_{i_1} \dots \partial \theta_{i_k}} = \sum_{j=1}^{\infty} \xi_j(\zeta) x_{t-j}^2(\mu),$$

where $\xi_j(\zeta) = \partial^k \psi_j(\zeta) / \partial \zeta_{i_1-2} \dots \partial \zeta_{i_k-2}$. Now

$$\left| \sum_{j=1}^{\infty} \xi_j(\zeta) x_{t-j}^2(\mu) \right| \leq 2 \sum_{j=1}^{\infty} |\xi_j(\zeta)| (x_{t-j}^2 + \mu^2),$$

so using Lemma 1,

$$\left| \frac{1}{\sigma_t^2(\theta)} \frac{\partial^k \sigma_t^2(\theta)}{\partial \theta_{i_1} \dots \partial \theta_{i_k}} \right| \leq \frac{2 \sum_{j=1}^{\infty} |\xi_j(\zeta)| x_{t-j}^2}{\sigma_t^{*2}(\theta)} + K \sum_{j=1}^{\infty} |\xi_j(\zeta)|.$$

It suffices to take $p > 1$. By Hölder's inequality,

$$\sum_{j=1}^{\infty} |\xi_j(\zeta)| x_{t-j}^2 \leq \left\{ \sum_{j=1}^{\infty} |\xi_j(\zeta)|^{p/\rho} \psi_j(\zeta)^{1-p/\rho} x_{t-j}^2 \right\}^{\rho/p} \left\{ \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}^2 \right\}^{1-\rho/p},$$

so

$$\left\{ \frac{\sum_{j=1}^{\infty} |\xi_j(\zeta)| x_{t-j}^2}{\sigma_t^{*2}(\theta)} \right\}^p \leq K \sum_{j=1}^{\infty} |\xi_j(\zeta)|^p \psi_j(\zeta)^{\rho-p} |x_{t-j}|^{2\rho}.$$

By Assumption F(l), for all $\eta > 0$

$$\sup_{\zeta \in \Upsilon} |\xi_j(\zeta)|^p \psi_j(\zeta)^{\rho-p} \leq K \sup_{\zeta \in \Upsilon} \psi_j(\zeta)^{\rho-\eta p} \leq K j^{-(d+1)(\rho-\eta p)}.$$

Since $\rho(\underline{d} + 1) > 1$, we may choose η such that $(\underline{d} + 1)(\rho - p\eta) > 1$. Thus,

$$E \sup_{\theta \in \Theta} \left\{ \frac{\sum_{j=1}^{\infty} |\xi_j(\zeta)| x_{t-j}^2}{\sigma_t^{*2}(\theta)} \right\}^p < \infty.$$

The above proof implies that also

$$\sup_{\zeta \in \Upsilon} \left\{ \sum_{j=1}^{\infty} |\xi_j(\zeta)| \right\}^p < \infty,$$

whence, the proof of (44) with $i_1 \geq 3$ is concluded. Next take $i_1 = 2$. If $i_2 > 2$,

$$(46) \quad \frac{\partial^k \sigma_t^2(\theta)}{\partial \theta_{i_1} \dots \partial \theta_{i_k}} = -2 \sum_{j=1}^{\infty} \xi_j(\zeta) x_{t-j}(\mu),$$

where now $\xi_j(\zeta) = \partial^{k-1} \psi_j(\zeta) / \partial \zeta_{i_2-2} \cdots \partial \zeta_{i_k-2}$, while if $i_2 = 2, i_3 > 2$,

$$\frac{\partial^k \sigma_t^2(\theta)}{\partial \theta_{i_1} \cdots \partial \theta_{i_k}} = -2 \sum_{j=1}^{\infty} \xi_j(\zeta),$$

where now $\xi_j(\zeta) = \partial^{k-2} \psi_j(\zeta) / \partial \zeta_{i_3-2} \cdots \partial \zeta_{i_k-2}$. In the first of these cases the proof is seen to be very similar to that above after noting that, by the Cauchy inequality, (46) is bounded by

$$K \left\{ \sum_{j=1}^{\infty} |\xi_j(\zeta)| x_{t-j}^2 \sum_{j=1}^{\infty} |\xi_j(\zeta)| \right\}^{1/2} + K \sum_{j=1}^{\infty} |\xi_j(\zeta)|,$$

while in the second it is more immediate; we thus omit the details. We are left with the cases $i_1 = i_2 = i_3 = 2$ and $i_1 = 1$, both of which are trivial. The details for (45) are very similar (the truncations in numerator and denominator match), and are thus omitted. \square

Define

$$g_t(\theta) = u_t(\theta)u'_t(\theta), \quad G_T(\theta) = T^{-1} \sum_{t=1}^T g_t(\theta).$$

LEMMA 7. For some $\delta > 0$, under Assumptions **A**($2 + \delta$), **B**, **C**, **D**, **E** and **F**(1),

$$(47) \quad \sup_{\theta \in \Theta} |Q_T(\theta) - Q(\theta)| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty,$$

and $Q(\theta)$ is continuous in θ . If also Assumption **F**(2) holds,

$$(48) \quad \sup_{\theta \in \Theta} \|G_T(\theta) - G(\theta)\| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty,$$

and $G(\theta)$ is continuous in θ . If also Assumption **F**(3) holds,

$$(49) \quad \sup_{\theta \in \Theta} \|H_T(\theta) - H(\theta)\| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty,$$

and $H(\theta)$ is continuous in θ .

PROOF. To prove (47), note first that, by Lemmas 1, 2, 3 and 5,

$$\sup_{\Theta} E|q_0(\theta)| \leq \sup_{\Theta} E|\log \sigma_0^2(\theta)| + \sup_{\Theta} E\chi_0(\theta) < \infty.$$

Thus, by ergodicity

$$Q_T(\theta) \rightarrow Q(\theta) \quad \text{a.s.,}$$

for all $\theta \in \Theta$. Then uniform convergence follows on establishing the equicontinuity property

$$\sup_{\tilde{\theta}: \|\tilde{\theta} - \theta\| < \varepsilon} |Q_T(\tilde{\theta}) - Q_T(\theta)| \rightarrow 0 \quad \text{a.s.,}$$

as $\varepsilon \rightarrow 0$, and continuity of $Q(\theta)$. By the mean value theorem it suffices to show that

$$\sup_{\Theta} \left\| \frac{\partial Q_T(\theta)}{\partial \theta} \right\| + \sup_{\Theta} \left\| \frac{\partial Q(\theta)}{\partial \theta} \right\| < \infty \quad \text{a.s.},$$

which, by Loève ([23], page 121) and identity of distribution, is implied by $E \sup_{\Theta} \|u_0(\theta)\| < \infty$. By the definition of $u_t(\theta)$, and $x_t^2(\mu) \leq K(x_t^2 + 1)$, $\|v_t(\theta)\| \leq 2(|x_t| + 1)$, we have

$$\|u_t(\theta)\| \leq K \left[\|\tau_t(\theta)\| \left\{ 1 + \varepsilon_t^2 \frac{\sigma_t^2}{\sigma_t^2(\theta)} \right\} + |\varepsilon_t| \frac{\sigma_t}{\sigma_t(\theta)} + 1 \right].$$

Thus, $E \sup_{\Theta} \|u_0(\theta)\|$ is bounded by a constant times

$$\begin{aligned} E \sup_{\Theta} \|\tau_0(\theta)\| + \left[E \sup_{\Theta} \left\{ \frac{\sigma_0^2}{\sigma_0^2(\theta)} \right\}^p \right]^{1/p} \left[E \sup_{\Theta} \|\tau_0(\theta)\|^{p/(p-1)} \right]^{1-1/p} \\ + E \sup_{\Theta} \left\{ \frac{\sigma_0}{\sigma_0(\theta)} \right\} + 1 \end{aligned}$$

for all $p > 1$. On choosing $p < 1 + \delta/2$, this is finite by Lemmas 5 and 6. (Our use of Lemmas 5 and 6 is similar to Berkes, Horváth and Kokoszka's [3] use of their Lemmas 5.1 and 5.2 in the GARCH(n, m) case.) This completes the proof of (47). Then (48) and (49) follow by applying analogous arguments to those above, and so we omit the details; indeed, (48) and (49) are only used in the proof of consistency of $\bar{G}_T(\hat{\theta}_T)$, $\bar{H}_T(\hat{\theta}_T)$ for G_0, H_0 , where convergence over only a neighborhood of θ_0 would suffice. \square

LEMMA 8. Under Assumptions A($2 + \delta$), B, C, D, E and F(1),

$$(50) \quad \sup_{\theta \in \Theta} |Q_T(\theta) - \bar{Q}_T(\theta)| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty.$$

If also Assumption F(2) holds,

$$(51) \quad \sup_{\theta \in \Theta} \|G_T(\theta) - \bar{G}_T(\theta)\| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty.$$

If also Assumption F(3) holds,

$$(52) \quad \sup_{\theta \in \Theta} \|H_T(\theta) - \bar{H}_T(\theta)\| \rightarrow 0 \quad \text{a.s. as } T \rightarrow \infty.$$

PROOF. We have $\hat{Q}_T(\theta) - Q_T(\theta) = A_T(\theta) + B_T(\theta)$, where

$$A_T(\theta) = T^{-1} \sum_{t=1}^T \ln \left[\frac{\bar{\sigma}_t^2(\theta)}{\sigma_t^2(\theta)} \right], \quad B_T(\theta) = T^{-1} \sum_{t=1}^T x_t^2(\mu) \{ \bar{\sigma}_t^{-2}(\theta) - \sigma_t^{-2}(\theta) \}.$$

Because

$$\sigma_t^2(\theta) = \bar{\sigma}_t^2(\theta) + \sum_{j=0}^{\infty} \psi_{j+t}(\zeta) x_{-j}^2(\mu),$$

$\ln(1+x) \leq x$ for $x > 0$ and $\sigma_t^2(\theta) \geq \omega_L > 0$, it follows that

$$\begin{aligned} |A_T(\theta)| &\leq KT^{-1} \sum_{t=1}^T \{\sigma_t^2(\theta) - \bar{\sigma}_t^2(\theta)\} \\ &\leq KT^{-1} \sum_{t=1}^T \sum_{j=t}^{\infty} \psi_j(\zeta) x_{t-j}^2(\mu) \\ &\leq KT^{-1} \sum_{t=0}^{\infty} \left\{ \sum_{j=t+1}^{t+T} \psi_j(\zeta) \right\} x_{-t}^2(\mu). \end{aligned}$$

Now from (7),

$$\sup_{\zeta \in \Upsilon} \sum_{j=t+1}^{t+T} \psi_j(\zeta) \leq K \sum_{j=t+1}^{t+T} j^{-d-1} \leq K \min(t+1, T)(t+1)^{-d-1}.$$

Thus,

$$(53) \quad \sup_{\Theta} A_T(\theta) \leq KT^{-1} \sum_{t=0}^T (t+1)^{-d}(x_{-t}^2 + 1) + K \sum_{t=T}^{\infty} t^{-d-1}(x_{-t}^2 + 1).$$

From the c_r -inequality, (9) and (10), $\sum_{t=1}^{\infty} (t+1)^{-d-1} x_{-t}^2$ has finite ρ th moment, and thus, by Loève ([23], page 121), is a.s. finite. Thus, the second term of (53) tends to zero a.s. as $T \rightarrow \infty$, while the first does so for the same reasons combined with the Kronecker lemma. Next,

$$\begin{aligned} |B_T(\theta)| &\leq KT^{-1} \sum_{t=1}^T \chi_t(\theta) \sum_{j=t}^{\infty} \psi_j(\zeta) x_{t-j}^2(\mu) \\ (54) \quad &\leq KT^{-1} \sum_{t=1}^T \chi_t(\theta) \sum_{j=t}^{\infty} j^{-d-1}(x_{t-j}^2 + 1). \end{aligned}$$

From previous remarks, $\sum_{j=t}^{\infty} j^{-d-1}(x_{t-j}^2 + 1) \rightarrow 0$ a.s. Also, for each θ , a.s.

$$T^{-1} \sum_{t=1}^T \chi_t(\theta) \rightarrow E\chi_0(\theta) \leq K \left\{ E \left(\frac{\sigma_0^2}{\sigma_0^2(\theta)} \right) + 1 \right\} \leq K$$

by ergodicity and Lemma 5. Thus, (54) $\rightarrow 0$ a.s. by the Toeplitz lemma. The convergence is uniform in θ because, from the proof of Lemma 7, for all $\theta \in \Theta$,

$$\sup_{\tilde{\theta} : \|\tilde{\theta} - \theta\| < \varepsilon} \|\chi_0(\tilde{\theta}) - \chi_0(\theta)\| \rightarrow 0 \quad \text{a.s.,}$$

as $\varepsilon \rightarrow 0$. This completes the proof of (50). We omit the proofs of (51) and (52) as they involve the same kind of arguments. \square

LEMMA 9. For some $\delta > 0$, under Assumptions A(2 + δ), B, C, D, E, F(1) and G, $M(\theta)$ is finite and positive definite for all $\theta \in \Theta$.

PROOF. Fix $\theta \in \Theta$. Finiteness of $M(\theta)$ follows from Lemma 6. Positive definiteness follows (by an argument similar to that of Lumsdaine [24] in the GARCH(1, 1) case) if, for all nonnull $(r + 2) \times 1$ vectors λ , $\lambda' M(\theta) \lambda = E\{\lambda' \tau_0(\theta)\}^2 > 0$, that is, that

$$(55) \quad \lambda' \tau_0(\theta) \sigma_0^2(\theta) \neq 0 \quad \text{a.s.},$$

since $0 < \sigma_0^2(\theta) < \infty$ a.s. Define

$$\begin{aligned} \tau_{t\omega}(\theta) &= \frac{\partial}{\partial \omega} \ln \sigma_t^2(\theta) = \sigma_t^{-2}(\theta), \\ \tau_{t\mu}(\theta) &= \frac{\partial}{\partial \gamma} \ln \sigma_t^2(\theta) = -2\sigma_t^{-2}(\theta) \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}(\mu), \\ \tau_{t\zeta}(\theta) &= \frac{\partial}{\partial \zeta} \ln \sigma_t^2(\theta) = \sigma_t^{-2}(\theta) \sum_{j=1}^{\infty} \psi_j^{(1)}(\zeta) x_{t-j}^2(\mu), \end{aligned}$$

so that $\tau_t(\theta) = (\tau_{t\omega}(\theta), \tau_{t\mu}(\theta), \tau_{t\zeta}(\theta))'$. Write $\lambda = (\lambda_1, \lambda_2, \lambda_3)'$, where λ_1 and λ_2 are scalar and λ_3 is $r \times 1$. Consider first the case $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$. Suppose (55) does not hold. Then we must have

$$\sum_{j=1}^{\infty} \lambda_3' \psi_j^{(1)}(\zeta) x_{t-j}^2(\mu) = 0 \quad \text{a.s.}$$

If $\lambda_3' \psi_1^{(1)}(\zeta) \neq 0$, it follows that

$$(56) \quad (\sigma_{t-1} \varepsilon_{t-1} + \mu_0 - \mu)^2 = -\{\lambda_3' \psi_j^{(1)}(\zeta)\}^{-1} \sum_{j=2}^{\infty} \lambda_3' \psi_j^{(1)}(\zeta) x_{t-j}^2(\mu).$$

Since $\sigma_{t-1} > 0$ a.s., the left-hand side involves the nondegenerate random variable ε_{t-1} , which is independent of the right-hand side, so (56) cannot hold. Thus, $\lambda_3' \psi_j^{(1)}(\zeta) = 0$. Repeated application of this argument indicates that, for all ζ , $\lambda_3' \psi_j^{(1)}(\zeta) = 0$, $j = 1, \dots, j_r(\zeta)$. This is contradicted by Assumption G, so (56) cannot hold. Next consider the case $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$. If (56) does not hold, we must have

$$(57) \quad \sum_{j=1}^{\infty} \psi_j(\zeta) x_{t-j}(\mu) = 0 \quad \text{a.s.}$$

Let k be the smallest integer such that $\psi_k(\zeta) \neq 0$. Then (57) implies

$$\varepsilon_{t-k} = \sigma_{t-k}^{-1}(\theta) \left\{ \mu - \mu_0 - \psi_k^{-1}(\zeta) \sum_{j=k+1}^{\infty} \psi_j(\theta) x_{t-j}(\mu) \right\}.$$

But the left-hand side is nondegenerate and independent of the right-hand side, so (57) cannot hold. Next consider the case $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$. If (55) is not true, then, taking $\lambda_2 = 1$, we must have

$$(58) \quad \sum_{j=1}^{\infty} \{ \lambda'_3 \psi_j^{(1)}(\zeta) x_{t-j}(\mu) - 2\psi_j(\zeta) \} x_{t-j}(\mu) = 0 \quad \text{a.s.}$$

Let k be the smallest integer such that either $\lambda'_3 \psi_k^{(1)}(\zeta) \neq 0$ or $\psi_k(\zeta) \neq 0$; the preceding argument indicates that there exists such k . Then we have

$$\begin{aligned} & \{ 2\psi_k(\zeta) - \lambda'_3 \psi_k^{(1)}(\zeta) (\sigma_{t-k} \varepsilon_{t-k} + \mu_0 - \mu) \} \{ \sigma_{t-k} \varepsilon_{t-k} + \mu_0 - \mu \} \\ &= \sum_{j=k+1}^{\infty} \{ \lambda'_3 \psi_j^{(1)}(\zeta) x_{t-j}(\mu) - 2\psi_j(\zeta) \} x_{t-j}(\mu) \quad \text{a.s.} \end{aligned}$$

The left-hand side is a.s. nonzero and involves the nondegenerate random variable ε_{t-k} , which is independent of the right-hand side, so (58) cannot hold. We are left with the cases where $\lambda_1 \neq 0$. Taking $\lambda_1 = -1$ and noting that $\sigma_t^2(\theta) \tau_{t\omega}(\theta) \equiv 1$, the preceding arguments indicate that there exist no λ_2 and λ_3 such that

$$\lambda_2 \sigma_t^2(\theta) \tau_{t\mu}(\theta) + \lambda'_3 \sigma_t^2(\theta) \tau_{t\zeta}(\theta) = 1 \quad \text{a.s.} \quad \square$$

LEMMA 10. For some $\delta > 0$, under Assumptions A(2 + δ), B, C, D, E, F(1) and H,

$$\inf_{\substack{\theta \in \Theta \\ \theta \neq \theta_0}} Q(\theta) > Q(\theta_0).$$

PROOF. We have

$$Q(\theta) - Q(\theta_0) = E \left[\frac{\sigma_0^2}{\sigma^2(\theta)} - \ln \left\{ \frac{\sigma_0^2}{\sigma^2(\theta)} \right\} - 1 \right] + (\mu - \mu_0)^2 E \left[\frac{1}{\sigma_0^2(\theta)} \right].$$

The second term on the right-hand side is zero only when $\mu = \mu_0$ and is positive otherwise. Because $x - \ln x - 1 \geq 0$ for $x > 0$, with equality only when $x = 1$, it remains to show that

$$(59) \quad \ln \sigma_0^2(\theta) = \ln \sigma_0^2 \quad \text{a.s., some } \theta \neq \theta_0.$$

By the mean value theorem, (59) implies that $(\theta - \theta_0)' \tau_0(\bar{\theta}) = 0$ a.s., for $\theta \neq \theta_0$ and some $\bar{\theta}$ such that $\|\bar{\theta} - \theta_0\| \leq \|\theta - \theta_0\|$. But by Lemma 9 there is no such $\bar{\theta}$. □

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