Central limit theorems for long range dependent spatial linear processes†

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ABSTRACT

Central limit theorems are established for the sum, over a spatial region, of observations from a linear process on a $d$–dimensional lattice. This region need not be rectangular, but can be irregularly-shaped. Separate results are established for the cases of positive strong dependence, short range dependence, and negative dependence. We provide approximations to asymptotic variances that reveal differential rates of convergence under the three types of dependence. Further, in contrast to the one dimensional (i.e., the time series) case, it is shown that the form of the asymptotic variance in dimensions $d > 1$ critically depends on the geometry of the sampling region under positive strong dependence and under negative dependence and that there can be nontrivial edge-effects under negative dependence for $d > 1$. Precise conditions for the presence of edge effects are also given.

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1 Introduction

The presence of long range dependence in spatial data has been noted in various empirical studies but a suitable formulation and systematic study of such spatial processes is lacking. For example, the ‘law of environmental variation’ of Fairfield Smith (1938), based on ‘Agricultural Field Trials’ data, posits that the covariance function of the yield in the plane decays as the inverse of the Euclidean distance. Thus, the covariance functions of such spatial processes are not absolutely summable and may exhibit long range dependence. More recently, the effect of spatial long range dependence has been noted in Atmospheric sciences (cf. Kashyap and Lasa (1988), Gneiting (2000)), Economics (Leonenko and Taufer (2013)), Oceanography (cf. Percival et al. (2008)) and Solid State Physics (cf. Carlos-Davila et al. (1985)), among others. See also Lavancier (2006) for some specific examples and other applications of spatial long range dependence. The traditional approach of quantifying spatial dependence through various notions of mixing is inadequate for dealing with long range dependence. In this paper, we consider a class of stationary spatial linear processes that allow for long range dependence, as well as the properties of short range- and negative-dependence, and establish central limit theorems for the sum over the entire range of such dependence.

The dependence structure of a real-valued stationary process on a $d$-dimensional spatial lattice can be nonparametrically modeled by the linear process

$$Z(i) = \mu + \sum_{j \in \mathbb{Z}^d} \alpha(i - j) \epsilon(j), \quad i \in \mathbb{Z}^d, \quad (1.1)$$

where the collection of real numbers $\{\alpha(i), i \in \mathbb{Z}^d\}$ satisfies

$$\sum_{i \in \mathbb{Z}^d} \alpha(i)^2 < \infty, \quad (1.2)$$

and $\{\epsilon(i), \ i \in \mathbb{Z}^d\}$ is a collection of independent homoscedastic random variables with zero mean and finite variance. If the $\epsilon(i)$ are only uncorrelated, (1.1) and (1.2) represent the class of purely non-deterministic processes on $\mathbb{Z}^d$, for which $Z(i)$ has a more parsimonious "half-plane" representation (see e.g. Whittle (1954)), to generalize the one-sided Wold representation in the time series case $d = 1$. However, we impose independence in order to establish central limit theorems (CLTs) for

$$S_n = \sum_{i \in D_n} Z(i)$$

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as $n \to \infty$. Here it is supposed that we observe $Z(i)$ within a spatial region $R_n \subset \mathbb{R}^d$ (to be described in detail subsequently) whose volume is regarded as increasing with the integer $n \geq 1$, the data sites being given by

$$D_n = R_n \cap \mathbb{Z}^d.$$ 

Extending time series notions (cf. Robinson (1997)), we consider three different sub-classes of (1.1), (1.2), broadly described as

- **negatively dependent (ND):**
  $$\sum_{i \in \mathbb{Z}^d} |\alpha(i)| < \infty, \quad \sum_{i \in \mathbb{Z}^d} \alpha(i) = 0,$$

- **short-range dependent (SRD):**
  $$\sum_{i \in \mathbb{Z}^d} |\alpha(i)| < \infty, \quad \sum_{i \in \mathbb{Z}^d} \alpha(i) \neq 0,$$

- **positively strongly dependent (PSD):**
  $$\sum_{i \in \mathbb{Z}^d} |\alpha(i)| = \infty,$$

though our results rest on conditions that respectively imply these. Denoting by $f(\lambda)$ the spectral density of $Z(i)$, for ND processes $f(0) = 0$, for SRD processes $f(0) \in (0, \infty)$, and for PSD processes $f$ may diverge at frequency 0. The three sub-classes are also associated with different rates of increase of

$$\sigma_n^2 = \text{Var}(S_n),$$

where we assume

$$\sigma_n^2 \to \infty \quad \text{as} \quad n \to \infty, \quad (1.3)$$

which is a necessary condition for a CLT. Define $N_n = |D_n|$, where $|B|$ denotes the size (i.e. number of elements) of a finite set $B$, so $N_n$ denotes sample size. Under additional conditions, we have

$$N_n^{-1} \sigma_n^2 \to 0 \quad \text{as} \quad n \to \infty \quad \text{when} \quad Z(i) \text{ is ND},$$
$$N_n^{-1} \sigma_n^2 \to \sigma_0^2 \in (0, \infty) \quad \text{as} \quad n \to \infty \quad \text{when} \quad Z(i) \text{ is SRD},$$
$$N_n^{-1} \sigma_n^2 \to \infty \quad \text{as} \quad n \to \infty \quad \text{when} \quad Z(i) \text{ is PSD}.$$

The condition of PSD is also referred to as **long range dependence** in the literature. Under (1.1), (1.2) and (1.3), and for $d = 1$ with $D_n = (1, 2, \ldots n)$, Ibragimov and Linnik (1971, pp. 359-360) established that
\[ \frac{S_n - ES_n}{\sigma_n} \rightarrow_d N(0, 1) \quad \text{as} \quad n \rightarrow \infty. \] (1.4)

Their main achievement was to allow arbitrarly slowly increasing \( \sigma_n \), in particular to cover all ND \( Z(i) \), as well as SRD and PSD ones. Again for \( d = 1 \), Hannan (1979) relaxed independence of the \( \epsilon(i) \) to a martingale difference assumption, but only covered SRD and PSD \( Z(i) \). On the other hand, Rosenblatt (1961), Taqqu (1975) and others established non-central limit theorems when \( d = 1 \) and \( Z(i) \) does not satisfy (1.1) but is a non-linear function of a PSD Gaussian process, and more generally. Mention must also be made of the many CLTs for \( d = 1 \) where (1.1) is replaced by mixing conditions, following Rosenblatt (1956), implying \( Z(i) \) is SRD, and extended to \( d > 1 \) by a number of authors; see, e.g., Bolthausen (1982), Doukhan (1994), Guyon (1995), the latter two authors also discussing the mixing properties of linear processes. CLTs for SRD spatial processes over irregular sampling regions are given by Lahiri (2003) and El Machkouri, Volný, and Wu (2013), allowing more general processes than linear fields. But relatively less attention has been paid to PSD processes with \( d > 1 \), under either linear or other assumptions. For rectangular regions, CLTs and invariance principles for PSD spatial linear processes have been proved by Lavancier (2007) and for fractional Brownian sheets by Wang (2013). Dobrushin and Major (1979) and Surgailis (1982) proved central- and non-central limit theorems for functionals of PSD Gaussian processes and for functionals of PSD linear fields, respectively. There is also a small body of literature on statistical inference on the mean and covariance parameters of PSD spatial processes; see the papers by Boissy et al. (2005), Beran et al. (2009) and Wang and Cai (2010) and the monographs by Ivanov and Leonenko (1989) and Bertail et al. (2006), and the references therein. We know of no spatial work under ND.

A major limitation of the existing work on spatial PSD processes is that it deals exclusively with rectangular spatial sampling regions. In contrast to the temporal case, in most practical applications spatial sampling regions are non-rectangular, and possibly of a non-standard shape (cf. Cressie (1993), Lahiri et al (1999)). As a result, existing results are of limited use. The present paper attempts to fill the gap by introducing a general framework for studying linear spatial processes over sampling regions of non-standard shapes. The main results of the paper establish separate CLTs for sums of observations from, respectively, LRD, ND and SRD processes over possibly non-rectangular sampling regions, replacing \( \sigma_n \) in (1.4) by concise approximations which indicate the differing rates of convergence in different situations, and highlighting an intricate interplay between the spatial dependence structure and the geometry of the sampling regions.
It will be observed that the asymptotic variance of the centered sum \( S_n - ES_n \) shows a very complex pattern of interactions among (i) the (effective) rate of decay of the coefficients \( \alpha(\cdot) \), (ii) the sample size \( N_n \) or equivalently, the volume of the sampling region \( R_n \), and (iii) the shape of the sampling region \( R_n \). For simplicity of exposition, suppose for the time being that \( \alpha(i) \) decays as \( ||i||^{-\beta} \) as \( ||i|| \to \infty \) and that the volume of \( R_n \) grows at rate \( c_0 \lambda_n^{d/2} \) for some \( c_0 \in (0, \infty) \) and \( \lambda_n \to \infty \). Then, square-summability of the \( \alpha(\cdot) \)'s implies that \( \beta > d/2 \) and it can be shown that \( \beta \in (d/2, d) \) leads to the case of PSD. In this case, we show that

\[
\left( \lambda_n^{3d-2\beta} \right)^{-1/2} [S_n - ES_n] \to^d N(0, \sigma^2_{psd})
\]

where \( \sigma^2_{psd} \) depends on certain limiting characteristics of the co-efficients \( \alpha(\cdot) \) and the shape of the sampling region \( R_n \) (cf. Theorem 3.2 below). Note that in the PSD case, i.e., for \( \beta \in (d/2, d) \), the scaling sequence is given by \( \lambda_n^{(3d-2\beta)/2} \) which is of a larger order of magnitude than \( c_0^{1/2} \lambda_n^{d/2} \), the square root of the volume of the sampling region \( R_n \). Thus, in the PSD case, the variance of the sum grows at a rate faster than the usual rate \( N_n^{1/2} \) (since the sample size \( N_n \) here also grows at the rate \( c_0 \lambda_n^d \)). Further, the limiting variance of the sum does not depend on the values of \( \alpha(i) \) for \( i \) in any given bounded neighborhood of the origin.

In the ND case, the sum shows a very different limit behavior that critically depends on \( \beta \) as well as on the values of the coefficients \( \alpha(i) \), for both small as well as large values of \( ||i|| \). Indeed, for \( \beta \in (d, d + 1/2) \), a normal limit similar to (1.5) holds, albeit with a different asymptotic variance, which now depends on properties of both \( R_n \) and \( R_n^c \). On the other hand, for \( \beta \) beyond the critical level \( d + 1/2 \), the edge-effect of the sampling region \( R_n \) becomes asymptotically dominant in dimensions \( d \geq 2 \), which in turn determines the asymptotic distribution. The corresponding scaling sequence is now given by \( \lambda_n^{(d-1)/2} \) which, quite surprisingly, no longer depends on the values of \( \beta \in (d + 1/2, \infty) \), i.e., on the rate of decay of the coefficients \( \alpha(i) \). Thus, in dimensions \( d \geq 2 \), the slowest possible rate for the variance of the sum in the ND case is given by \( \lambda_n^{(d-1)/2} \) for all \( \beta > d + 1/2 \). This may be contrasted with the one dimensional case, where the edge-effect is asymptotically negligible and the growth rate of the variance of the sum can be very slow (e.g., \( O(\lambda_n^{3-2\beta}) \) with \( \beta \) close to \( d + 1/2 = 3/2 \)). See Section 4 for full details. For the sake of completeness, we also prove a CLT for the SRD case. Here the sum has the usual rate of \( N^{1/2} \) and the limiting variance depends on the \( \alpha(i) \) only through their sum \( A \), agreeing with a familiar result in the time series case \( d = 1 \) (cf. Section 5). The following table summarizes different limit behavior of the sum under PSD, ND and SRD for \( d \geq 2 \).

The rest of the paper is organized as follows. In Section 2, we consider the spatial linear process allowing non-identically distributed (but homoscedastic) errors and describe
Table 1: A summary of the limit behavior of the sum $S_n$ under PSD, ND and SRD when $\alpha(i) = c_1\|i\|^{-\beta}$ for $\|i\| > c_2$ and when $\text{vol}(R_n) \sim c_3\lambda_n^d$, for some constants $c_1, c_2, c_3 \in (0, \infty)$ and for some $\lambda_n \to \infty$, where $\text{vol}(R_n)$ denotes the volume of $R_n$. Note that here the sample size $N_n \sim \text{vol}(R_n)$ and $A = \sum_{i \in \mathbb{Z}^d} \alpha(i)$. The cases ND-EE and ND-NEE in the first column stand for the ND case with- and without- edge effects, respectively.

<table>
<thead>
<tr>
<th>Growth rate of $\text{Var}(S_n)$</th>
<th>Effects on Limit Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficients $\alpha(i)$</td>
<td>Irregular shape of $R_n$</td>
</tr>
<tr>
<td>PSD</td>
<td>$\lambda_n^{(3d-2\beta)}$, $\beta \in (\frac{d}{2}, d)$</td>
</tr>
<tr>
<td>ND-NEE</td>
<td>$\lambda_n^{(3d-2\beta)}$, $\beta \in (d, d + \frac{1}{2})$</td>
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<tr>
<td>ND-EE</td>
<td>$\lambda_n^{(d-1)}$, $\beta \in (d + \frac{1}{2}, \infty)$</td>
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<tr>
<td>SRD</td>
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an asymptotic framework that can accommodate a large class of sampling regions of non-standard shapes. In Section 2, we also state a regular variation condition on the co-efficients that, in particular, allows the co-efficients to have different rates of decay along different directions, and give some examples to illustrate the scope of the formulation. In Sections 3 and 4, we establish the limit distribution of the sum under PSD and ND, respectively. We prove the CLT in the SRD case in Section 5. Proofs of the main results are given in Section 6. Here we also present a very general version of the CLT for a spatial linear process observed on bounded regions that may be of independent interest.

2 The theoretical framework

In Subsection 2.1, we specify the spatial linear process and in Subsection 2.2, we give a formulation for the sampling regions $R_n$. In Subsection 2.3, we introduce the regularity conditions on the co-efficients $\alpha(i)$ and give some illustrative examples in Subsection 2.4. Under these conditions, it is possible to determine the exact order of the variance term $\sigma_n^2$ and derive explicit expressions for the asymptotic variance.
2.1 Spatial linear processes

We define a spatial linear process \( \{ Z(\cdot) \} \) as:

\[
Z(i) = \mu + \sum_{j \in \mathbb{Z}^d} \alpha(i - j) \epsilon(j), \quad i \in \mathbb{Z}^d, \tag{2.1}
\]

where \( \{ \epsilon(j) : j \in \mathbb{Z}^d \} \) is a collection of independent zero mean random variables with common variance 1 (w.l.o.g.), the \( \{ \epsilon(j)^2 : j \in \mathbb{Z}^d \} \) are uniformly integrable and \( \{ \alpha(j) : j \in \mathbb{Z}^d \} \) is a sequence of real numbers satisfying \( \sum_{i \in \mathbb{Z}^d} |\alpha(i)|^2 < \infty \).

2.2 Sampling regions

Next we specify the structure of the sampling region \( R_n \). Let \( R_0 \) be an open connected subset of \((-1/2, 1/2)^d\) containing the origin. We regard \( R_0 \) as a ‘prototype’ of the sampling region \( R_n \). Let \( \{ \lambda_n \} \) be a sequence of positive numbers such that \( \lambda_n \to \infty \) as \( n \to \infty \). We assume that the sampling region \( R_n \) is obtained by ‘inflating’ the set \( R_0 \) by the scaling factor \( \lambda_n \) (cf. Lahiri et al. (1999)), i.e.,

\[
R_n = \lambda_n R_0. \tag{2.2}
\]

Since the origin is assumed to lie in \( R_0 \), the shape of \( R_n \) remains the same for different values of \( n \). To avoid pathological cases, we assume that the boundary \( \partial R_0 \) of \( R_0 \) has \( d \)-dimensional Lebesgue measure zero. A stronger version of this condition will be needed for the ND case, which is stated as Condition (C.3) in Section 2.3 below. The (stronger) boundary condition holds for most regions \( R_n \) of practical interest, including common convex subsets of \( \mathbb{R}^d \), such as spheres, ellipsoids, polyhedrons, as well as for many nonconvex star-shaped sets in \( \mathbb{R}^d \). (Recall that a set \( A \subset \mathbb{R}^d \) is called star-shaped if for any \( x \in A \), the line segment joining \( x \) to the origin lies in \( A \)). The latter class of sets may have fairly irregular shapes (cf. Sherman and Carlstein (1994) and Lahiri (1999)). Some practical applications and studies involving sampling regions that satisfy the regularity conditions above are given by the wheat yield data of Mercer and Hall (1911) on agricultural field trials, the coal ash data of Gomez and Hazen (1970) from Mining, and the cancer mortality counts data of Riggan et al. (1987) from Epidemiology, among others.

2.3 Regularity conditions

For \( x = (x_1, \ldots, x_d)’ \in \mathbb{R}^d \), let \( \| x \| = (x_1^2 + \ldots + x_d^2)^{1/2} \) and let \( [x] = ([x_1], \ldots, [x_d])’ \) where \( [y] \) denotes the integer part of a real number \( y \). For \( \delta \in (0, \infty) \), \( x \in \mathbb{R}^d \) and \( A \subset \mathbb{R}^d \), let
$A^\delta = \{ y \in \mathbb{R}^d : \| z - y \| \leq \delta \text{ for some } z \in A \}$ and $d(x, A) \equiv d(A, x) = \min \{ \| z - x \| : z \in A \}$. Let $\gamma(t) = \max \{ |\alpha(tu)| : \| u \| = 1 \}, \ t > 0$. With this notation, we are now ready to state the regularity conditions.

(C.1) Suppose that

$$\gamma(t) = t^{-\beta}L(t), \ t > 0,$$

for some $\beta > d/2$ and some function $L : (0, \infty) \rightarrow [0, \infty)$ that is slowly varying at infinity in the sense that $L(\cdot)$ is bounded on any bounded subinterval of $(0, \infty)$ and

$$\lim_{t \rightarrow \infty} \sup \{ L(at)/L(t) : a \in [a_0, a_1] \} = 1 \text{ for any } 0 < a_0 < a_1 < \infty \text{ (cf. Taqqu (1975)).}$$

(C.2) Let $g_t(x) = \alpha(t|x|) / \gamma(t), \ x \in \mathbb{R}^d, \ t > 0$. Suppose that there exists a function $g_\infty : \mathbb{R}^d \rightarrow \mathbb{R}$ such that for every $\delta \in (0, \infty)$,

$$\int_{\{x \in \mathbb{R}^d : \| x \| \geq \delta \}} \left| g_t(x) - g_\infty(x) \right|^b dx \rightarrow 0 \text{ as } t \rightarrow \infty,$$

where $b = b(d, \beta) = 2$ if $\beta \leq d$ and $b = 1$ otherwise.

(C.3) For any measurable function $f : [0, \infty) \rightarrow [0, \infty)$, there exists $C_f \in (1, \infty)$ such that

$$\int_{[\partial R_0]} f(\| d(x, \partial R_0) \|) \nu(dx) \leq C_f \int_0^\epsilon f(t)dt \text{ for all } 0 < \epsilon < C_f^{-1},$$

where $\nu$ is the Lebesgue measure on $\mathbb{R}^d$.

Condition (C.1) requires that the radial maximum of the collection of co-efficients $\{ \alpha(i) : i \in \mathbb{Z}^d \}$ be regularly varying (at infinity). The requirement that $\beta > d/2$ in (C.1) is imposed to ensure that $\sum_{i \in \mathbb{Z}^d} \alpha(i)^2 < \infty$. Condition (C.2) is a weak form of spatial regular variation condition on the $\alpha(i)$. It is weaker than assuming directional separability of the co-efficients, and allows for differential rates of decay along different directions. See the Examples below. It is a variant of the standard form of regular variation that requires the function $g_t(\cdot)$ to satisfy (cf. Section 5.4, Resnick (1987))

$$\lim_{t \rightarrow \infty} g_t(x) = \| x \|^{-\beta} a(\| x \|) \text{ for all } x \in \mathbb{R}^d, x \neq 0,$$

for some function $a(\cdot)$ on the unit disc $\{ x \in \mathbb{R}^d : \| x \| = 1 \}$. In this case, the limit function $g_\infty$ is given by

$$g_\infty(x) = \| x \|^{-\beta} a(\| x \|) \mathbb{1}(x \neq 0), \ x \in \mathbb{R}^d,$$

where $\mathbb{1}(\cdot)$ denotes the indicator function. By comparison, Condition (C.2) requires convergence of $g_t$ to $g_\infty$ in $L^b$. Conditions (C.1) and (C.2) together quantify the behavior of
the function $\alpha(i)$ for large $\|i\|$ which plays an important role in determining the form of the asymptotic variance of the sum under PSD and ND.

Condition (C.3) is a regularity condition on the boundary of the prototype set $R_0$ which is equivalent to requiring that

$$\int_{[\partial R_0]^{\varepsilon}} f\left(d(x, \partial R_0)\right) \nu(dx) = O\left(\int_0^\varepsilon f(t) dt \right) \quad \text{as} \quad \varepsilon \downarrow 0$$

for each nonnegative measurable $f$. We need this condition to hold for $f \equiv 1$ and for $f(t) = t^{-b}L^2(t)$ for certain values of $b = b(\beta)$ (cf. the proof of Theorem 4.1 below). In particular, when $f \equiv 1$, this reduces to the condition

$$\nu\left((\partial R_0)^{\varepsilon}\right) = O(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0,$$

(2.4)

which is satisfied by most sampling regions of common interest (cf. Section 2.2). For $d = 2$, a sufficient condition is that the boundary of $R_0$ is delineated by a rectifiable curve of a finite length. We shall use (C.3) for proving the results only in the ND case where more precise information on the boundary is needed to determine the asymptotic variance.

Next we give a few examples to illustrate the range of spatial dependence covered by the regularity conditions above.

### 2.4 Examples

**Example 2.1**: *(Isotropic spatial linear processes)*. Let

$$\alpha(i) = a(\|i\|)(1 + \|i\|)^{-\beta}, \quad i \in \mathbb{Z}^d,$$

for some bounded function $a : [0, \infty) \rightarrow \mathbb{R}$, where $\beta \in (\frac{d}{2}, \infty)$. Also, suppose that $a(t) \rightarrow c_0 \neq 0$ as $t \rightarrow \infty$. Then, using the fact that

$$\sup \left\{ \|t u\| - t \|u\| : u \in \mathbb{R}^d \setminus \{0\} \right\} \leq \sqrt{d},$$

(2.5)

it is easy to see that Condition (C.1) holds with $\gamma(t) = t^{-\beta}L(t)$ where $L(t) \rightarrow |c_0|$ as $t \rightarrow \infty$. Further, using (2.5), one can show that for any $\eta > 0$,

$$g_t(x) \rightarrow \frac{c_0}{|c_0|}\|x\|^{-\beta} \equiv g_\infty(x) \quad \text{as} \quad t \rightarrow \infty \quad \text{for all} \quad \|x\| > \eta$$

and

$$\left| g_t(x) - g_\infty(x) \right| \leq C_1\|x\|^{-\beta} \quad \text{for all} \quad t > C_1,$$

for some constant $C_1 \equiv C_1(\eta, \beta) \in (0, \infty)$. Hence, Condition (C.2) holds.
This gives an example of an ‘isotropic’ spatial linear process where the coefficients \( \alpha(i) \) have an identical rate of decay in all directions. \( \square \)

**Example 2.2**: (A class of anisotropic spatial linear processes). Suppose that \( O \) is a \( d \times d \) orthonormal matrix with rows \( o'_i, \ i = 1, \ldots, d \). Let \( \phi_i(x) = |o'_i x|/\|x\|, \ x \in \mathbb{R}^d \setminus \{0\} \). Let \( \delta \in (0, \frac{1}{\sqrt{d}}) \). Suppose that

\[
\alpha(x) = \prod_{i=1}^{d} \left\{ |o'_i x|^{-a_i} \mathbb{1} \left( \phi_i(x) > \delta \right) \right\} \quad \text{for all} \quad x \in \mathbb{R}^d \setminus \Gamma
\]

for some \( a_1, \ldots, a_d \in [0, \infty) \) with \( a_1 + \ldots + a_d > d/2 \) and for some open neighborhood \( \Gamma \) of the origin. There is no restriction on the definition of the function \( \alpha(x) \) on \( \Gamma \).

It is easy to check that

\[
\gamma(t) = \sup \left\{ \left| \alpha(\lfloor ut \rfloor) \right| : \|u\| = 1 \right\}
\]

\[
= \sup \left\{ \left| \alpha(\lfloor O' ut \rfloor) \right| : \|u\| = 1 \right\}
\]

\[
= \sup \left\{ \prod_{i=1}^{d} \left| ut \right|^{-a_i} \mathbb{1} \left( \phi_i(\lfloor tu \rfloor) > \delta \right) : \|u\| = 1 \right\} \left(1 + o(1)\right)
\]

\[
= c_0 t^{-(a_1 + \ldots + a_d)} \left(1 + o(1)\right) \quad \text{as} \quad t \to \infty,
\]

for some \( c_0 \in (0, \infty) \).

Next let \( D = \{0\} \cup \{y : \phi_i(y) = \delta \text{ for } i = 1, \ldots, d\} \). Note that the \( d \)-dimensional Lebesgue measure of \( D \) is zero. And, for any \( x \not\in D \),

\[
g_t(x) = \frac{\alpha(\lfloor tx \rfloor)}{\gamma(t)}
\]

\[
= \gamma(t)^{-1} \prod_{i=1}^{d} \left| tu_i x \right|^{-a_i} \mathbb{1} \left( \phi_i(\lfloor tu \rfloor) > \delta \right)
\]

\[
= \left[ c_0^{-1} \prod_{i=1}^{d} \left\{ |o'_i x|^{-a_i} \mathbb{1} \left( \phi_i(x) > \delta \right) \right\} \right] \left(1 + o(1)\right) \quad \text{as} \quad t \to \infty,
\]

\[
\equiv g_\infty(x)(1 + o(1)) \quad \text{as} \quad t \to \infty.
\]

Thus, the point-wise limit of the functions \( g_t(\cdot) \) exists for all \( x \not\in D \). Now using the Dominated Convergence Theorem (DCT), it is easy to check that for any \( \eta > 0 \),

\[
\lim_{t \to \infty} \int_{\{\|x\| \geq \eta\}} \left| g_t(x) - g_\infty(x) \right|^b dx = 0.
\]

Thus, Conditions (C.1) and (C.2) are satisfied by the co-efficients generated by the function \( \alpha(\cdot) \).
Note that in this example, the coefficients $\alpha(i)$ are zero whenever $|\phi_i(i)| \leq \delta$ for some $i \in \{1, \ldots, d\}$ but they are nonzero for all $x$ such that $\phi_i(x) > \delta$ for all $i = 1, \ldots, d$. Since $\delta$ is small, the latter condition is satisfied in a conic region in each of the $2^d$ quadrants. Thus, this gives an example of an anisotropic spatial process. The maximal rate of decay of $\alpha(\cdot)$ over the set $\{i \in \mathbb{Z}^d : |\phi_i(i)| > \delta\}$ for all $i = 1, \ldots, d$ can vary with the choice of $a_1, \ldots, a_d$, allowing all possible types of long-range (as well as short-range) dependence. Also, note that here the ND case can be realized by a suitable choice of $\alpha(i)$ for $i \in \Gamma$. The rate of convergence of the sum in this example depends only on a single parameter, namely, the combined exponent $\beta = a_1 + \ldots + a_d$; individual $a_i$'s do not have an impact. □

**Example 2.3 :** (Spatial linear processes with nonuniform directional decay rates). Let $\mathcal{I}$ be a finite set and let $\{o_i : i \in \mathcal{I}\} \subset \{x \in \mathbb{R}^d : \|x\| = 1\}$. Here we suppose that the $o_i$'s are distinct but they are not necessarily orthogonal. Let $\phi_i(\cdot)$ be as in Example 2.2, i.e., $\phi_i(x) = |o'_i x|/\|x\|$, $x \in \mathbb{R}^d \setminus \{0\}$, $i \in \mathcal{I}$. Define $\psi_i(x) = \phi_i(x)\mathbb{I}(\phi_i(x) > \delta_i)$ for some $\delta_i \in (0, 1)$, $i \in \mathcal{I}$. Suppose that
\[
\alpha(x) = \sum_{i \in \mathcal{I}} \frac{\psi_i(x)}{1 + \|x\|^{a_i}} \quad \text{for all} \quad x \in \mathbb{R}^d \setminus \Gamma,
\]
where $\{a_i : i \in \mathcal{I}\} \subset (0, \infty)$ and where $\Gamma$ is an open neighborhood of the origin, as in Example 2.2. Let $a_0 = \min\{a_i : i \in \mathcal{I}\}$ and let $\mathcal{I}_0 = \{i \in \mathcal{I} : a_i = a_0\}$. Note that for any $i \in \mathcal{I}$ and any $x \in \mathbb{R}^d \setminus \{0\}$,
\[
\phi_i(\lfloor tx \rfloor) = \phi_i(x)(1 + o(1)) \quad \text{as} \quad t \to \infty, \quad \text{provided} \quad \phi_i(x) > 0,
\]
and
\[
\|\lfloor tx \rfloor\| = t\|x\|(1 + o(1)) \quad \text{as} \quad t \to \infty.
\]
Hence, it follows that
\[
\gamma(t) = c_1 t^{-a_0}(1 + o(1)) \quad \text{as} \quad t \to \infty,
\]
where $c_1 = \sup\{\sum_{i \in \mathcal{I}_0} \psi_i(u) : \|u\| = 1\}$.

Next we identify the limit function $g_\infty(\cdot)$. By arguments as above, it follows that for any $x \neq 0$ with $\psi_i(x) > 0$ for some $i \in \mathcal{I}_0$,
\[
ge_i(x) = \gamma(t)^{-1} \left[ \sum_{i \in \mathcal{I}} \frac{\psi_i(\lfloor tx \rfloor)}{1 + \|\lfloor tx \rfloor\|^{a_i}} \right] = \sum_{i \in \mathcal{I}_0} \frac{\psi_i(x)}{c_1 \|x\|^{a_0}} (1 + o(1)) \quad \text{as} \quad t \to \infty, \quad \text{a.e.}
\]
Since for any $i$, the set $\{x \in \mathbb{R}^d \setminus \{0\} : \phi_i(x) = \delta_i\}$ has $d$-dimensional Lebesgue measure zero, it follows that $g_t \to g_\infty$ as $t \to \infty$ (a.e.), where
\[ g_\infty(x) = \frac{\sum_{i \in I_0} \phi_i(x)}{c_1 \|x\|^{a_0}}, \quad x \in \mathbb{R}^d \setminus \{0\}. \]

Now using the DCT, one can show that for any $\eta > 0$,
\[ \lim_{t \to \infty} \int_{\{\|x\| \geq \eta\}} \left| g_t(x) - g_\infty(x) \right|^b dx = 0. \]

Thus, Conditions (C.1) and (C.2) are satisfied by the co-efficients generated by the function $\alpha(\cdot)$.

Note that for $\delta_i$ close to 1, the $i$th component $\psi_i(x)$ in $\alpha(x)$ takes non-zero values in a thin cone around $o_i$ and it may or may not intersect with the $j$th cone (for any given $j \neq i$) depending on the relative magnitudes of $\delta_i$ and $\delta_j$ and the angle between $o_i$ and $o_j$. As a result, with different choices of $o_i, \delta_i$ and $a_i$ for $i \in I$, the co-efficients $\alpha(i)$ here may have different rates of decay along the directions $o_i$ for $i \in I \setminus I_0$, allowing any combinations of short- and long-range dependent rates along different directions. However, the limit distribution of the sum depends only on $a_0$ which is the minimum of the exponents $\{a_i, i \in I\}$. \[ \square \]

In the next section, we describe the limit behavior of the sum $S_n$ depending on the rate of decay $\beta$ in (C.1)

### 3 Results under PSD

From the proofs given in Section 6, it follows that for $\beta \in (d/2, d)$, the variance of the sum $S_n$ grows at the rate $\lambda_n^{3d-2\beta} L(\lambda_n)^2$ and hence, the correct scaling factor sequence is given by $\lambda_n^{3d-2\beta} L(\lambda_n)$. Since $\beta \in (d/2, d)$, this scaling sequence grows at a rate faster than square-root of the sample size $|N_n|^{1/2} \sim [\lambda_n^d \text{vol}(R_0)]^{1/2}$, where $\text{vol}(B)$ denotes the volume (i.e., the Lebesgue measure) of a Borel set $B$ in $\mathbb{R}^d$. As a result, for $\beta \in (d/2, d)$, $N_n^{-1} \sigma_n^2 \to \infty$ and the spatial process $\{Z(\cdot)\}$ exhibits PSD.

To describe the limit distribution of the centered and scaled sum under PSD, define
\[ G_\infty(x) = \int_{R_0} g_\infty(y - x) dy, \quad x \in \mathbb{R}^d. \] (3.6)

The first result shows that the function $G_\infty$ is well defined on all of $\mathbb{R}^d$ for $\beta \in (0, d)$. 

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Proposition 3.1: Suppose that Conditions (C.1) and (C.2) hold for some \( \beta \in (0, d) \). Then the integral in (3.6) exists and is finite for all \( x \in \mathbb{R}^d \).

The function \( G_\infty \) determines the asymptotic variance of the sum \( S_n \) for \( \beta \in (d/2, d) \). We make this precise in the following result that gives the limit distribution of \( S_n \) under PSD.

Theorem 3.2: Let \( \{Z(\cdot)\} \) be the linear process given by (2.1) such that Conditions (C.1) and (C.2) hold for some \( \beta \in (d/2, d) \). Then \( G_\infty \in L^2(\mathbb{R}^d) \) and

\[
\frac{[S_n - ES_n]}{[\lambda_n^{3d-2\beta} L(\lambda_n)^2]^{1/2}} \rightarrow^d N \left(0, \int_{\mathbb{R}^d} G_\infty^2(x)dx\right) \text{ as } n \rightarrow \infty. \tag{3.7}
\]

Theorem 3.2 shows that for \( \beta \in (d/2, d) \), the growth rate of (the variance of) the sum \( S_n \) is \([\lambda_n^{3d-2\beta} L(\lambda_n)^2]^{1/2}\), which is of a larger order than the usual order \( N_n^{1/2} \). Since \( G_\infty \) is continuous, the asymptotic variance is non-zero if \( G_\infty(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R}^d \).

From (3.7), also note that the limiting variance of \( S_n \) depends on the prototype set \( R_0 \) as well as the function \( g_\infty \) of Condition (C.2). Thus, unlike the time series case, the geometry of the sampling region plays an important role in the spatial case under PSD.

4 Results under ND

When \( Z(i) \) is ND, it can be shown (cf. the proofs of Theorems 4.1-4.3) that \( N^{-1}_n \sigma_n^2 \rightarrow 0 \). The limit behavior of the sum \( S_n \) in the ND case critically depends on the behavior of the terms

\[
\theta_n(i) = \sum_{j \in [R_n-i] \cap \mathbb{Z}^d} \alpha(j), \quad i \in \mathbb{Z}^d,
\]

in a shrinking neighborhood of the set, \( \partial R_n \), the boundary of \( R_n \). In the parlance of spatial statistics, this represents an instance of edge effect (cf. Cressie (1993)) that may have a nontrivial effect on the limit behavior of the sum. Indeed, depending on the relative orders of contributions from the boundary terms and the non-boundary terms, we get different growth rates for the sum in the ND case. Further, the limiting variances are also different. For clarity of exposition, we present the two subcases of the ND case separately.
4.1 ND with asymptotically negligible edge effects

First we consider the relatively simple case where the contribution from the boundary $\theta_n(i)$'s is asymptotically negligible. Suppose that $\beta > d$, so that $\sum_{i \in \mathbb{Z}^d} |\alpha(i)| < \infty$, and that $A \equiv \sum_{i \in \mathbb{Z}^d} \alpha(i) = 0$. In this case, it will be shown that the asymptotic distribution of the sum depends on the $\alpha(i)$ only through (an analog of) the function $G_\infty$ of (3.6). However, Proposition 3.1 no longer holds, i.e., the function $G_\infty$ of (3.6) may not be well defined for all $x \in \mathbb{R}^d$ for $\beta > d$. To appreciate why, consider the special case where $g_\infty(x) = \|x\|^{-\beta} \mathbb{1}(x \neq 0)$ (cf. (2.3)). In this case, $\int_{\delta \leq \|y\| \leq 1} g_\infty(y) = O(\delta^{d-\beta}) = o(1)$ as $\delta \downarrow 0$ for all $\beta \in (d/2, d)$, but the integral blows up for $\beta > d$. As a result, the limit in (3.6) may not exist for all $x \in \mathbb{R}^d \setminus \{0\}$ in the case $\beta > d$. However, using the condition $A = 0$, we can define $G_\infty(x)$ (and a suitable variant of it) for a restricted set of $x$'s that would be adequate for our purpose.

To that end, note that the set $[R_0 \cup \partial R_0]^c$ is open and hence, for all $x \in [R_0 \cup \partial R_0]^c$, there exists a $\eta = \eta(x) > 0$ such that

$$B(x; \eta) \subset [R_0 \cup \partial R_0]^c,$$

where $B(x; \eta) \equiv \{y \in \mathbb{R}^d : \|y - x\| < \eta\}$ denotes the open ball of radius $\eta$ around $x$. As a consequence, for $x \in [R_0 \cup \partial R_0]^c$, $B(0; \eta) \cap [R_0 - x] = \emptyset$ and

$$\int_{R_0} |g_\infty(y - x)|dy = \int_{R_0 - x} |g_\infty(y)|dy \leq \left[ \int_{\|y\| \geq \eta} |g_\infty(y)|^2 dy \right]^{1/2} \left[ \text{vol}(R_0) \right]^{1/2} < \infty$$

whenever Condition (C.2) holds with $b = 2$. But, for $\beta > d$, $b = 1$ in Condition (C.2).

Nonetheless, the square integrability of $g_\infty(\cdot)$ on sets of the form $B_\eta \equiv \{y \in \mathbb{R}^d : \|y\| \geq \eta\}$, $\eta > 0$, follows from (C.2) and the fact that $|g_\infty(\cdot)| \leq C(\eta)$ a.e. (w.r.t. the Lebesgue measure on $\mathbb{R}^d$) on $B_\eta$, for some $C(\eta) \in (0, \infty)$ (See (6.6) below). Hence,

$$G_\infty(x) = \int_{R_0} g_\infty(y - x)dy \in \mathbb{R}$$

for all $x \in [R_0 \cup \partial R_0]^c$. By similar arguments, the integral $\int_{[R_0 \cup \partial R_0]^c} g_\infty(y - x)dy$ is well defined for all $x \in R_0$. Since $\partial R_0$ has $d$-dimensional Lebesgue measure zero, the value of the integrals remains unchanged (with any measurable extension of $g_\infty(\cdot)$ if $[R_0 \cup \partial R_0]^c$ is replaced by $R_0^c$ (cf. Billingsley (1995) or Athreya and Lahiri (2006; page 49)). With this convention, define the function $G_\infty^+(\cdot)$ as

$$G_\infty^+(x) = \begin{cases} \int_{R_0} g_\infty(y - x)dy & \text{if } x \in [R_0 \cup \partial R_0]^c \\ \int_{R_0} g_\infty(y - x)dy & \text{if } x \in R_0 \\ 0 & \text{if } x \in \partial R_0. \end{cases}$$

(4.8)
For \( \beta \in (d, d+1/2) \), the asymptotic distribution of the sum depends only on the function \( G^\dagger_{\infty}(\cdot) \), as shown by the following result.

**Theorem 4.1:** Let \( \{Z(\cdot)\} \) be the linear process given by (2.1) such that Conditions (C.1)-(C.3) hold with \( \beta \in (d, d+1/2) \) in (C.1). Also suppose that \( A = 0 \). Then, \( G^\dagger_{\infty} \in L^2(\mathbb{R}^d) \) and

\[
\frac{[S_n - ES_n]}{\lambda_n^{3d-2\beta} L^2(\lambda_n)} \rightarrow^d N \left( 0, \int_{\mathbb{R}^d} [G^\dagger_{\infty}(x)]^2 dx \right) \quad \text{as} \quad n \to \infty, \quad (4.9)
\]

where the function \( G^\dagger_{\infty} \) is as defined in (4.8).

Theorem 4.1 shows that the asymptotic variance of the centered and scaled sum depends on the coefficients \( \alpha(i) \) only through the integral of the function \( G^\dagger_{\infty}(\cdot) \) over \( \mathbb{R}^d \). Thus, the behavior of the \( \alpha(i) \) for large values of \( ||i|| \) determines the asymptotic variance. The exact values of the \( \alpha(i) \) for small values of \( ||i|| \) have no direct effect except for the condition \( A = 0 \). Further, the growth rate of the sum under the ND case is \( \lambda_n^{d-2\beta} L(\lambda_n)^2 = o(N_n^{1/2}) \), which is slower than the PSD rate and, also slower than the SRD rate, given by \( N_n^{1/2} \) (cf. Theorem 5.1 below). To compare the asymptotic variances under the ND case without edge effects and the PSD case, note that the integrals of \( G^\dagger_{\infty}(\cdot) \) and \( G_{\infty}(\cdot) \) over \( R_0^c \) are the same and hence, the difference in the asymptotic variances in the ND and the PSD cases comes from the integrals of the respective functions over \( R_0 \).

### 4.2 ND with asymptotically non-negligible edge effects

Next consider the case where \( \beta \geq d + 1/2 \). In this case, we may write the variance of the sum as the sum of two terms, one involving the sum of \( \theta_n(i)^2\sigma^2 \) for \( i \) near the boundary of the sampling region \( R_n \) and the other over the rest of the \( \theta_n(i)^2\sigma^2 \). It can be shown that the growth rate of the second term is of the order \( \lambda_n^{3d-2\beta} L(\lambda_n)^2 \). On the other hand, under Condition (C.3) (cf. (2.4)), for any sequence \( \{t_n\} \subset (0, \infty) \) with \( t_n^{-1} + \lambda_n^{-1} t_n = o(1) \) as \( n \to \infty \), the volume of the \( t_n \)-enlargement of the boundary of \( R_n \) is of the order of \( \lambda_n^{d-1} t_n \). It is easy to check that for \( \beta \geq d + 1/2 \), this boundary term can be of a larger order of magnitude than \( \lambda_n^{3d-2\beta} L(\lambda_n)^2 \). As a result, the contribution from \( \theta_n(i)^2 \) for \( i \) near the boundary of the sampling region \( R_n \) may become dominant and additional care must be taken to determine the exact growth rate of \( \sigma_n^2 \). The following example serves to illustrate such dominating
'edge effects' in the ND case:

**Example 4.1:** Suppose that $d = 2$, $R_0 = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ and let

$$\alpha(i, j) = \begin{cases} b(i)b(j), & \text{if } i, j \in \mathbb{Z} \setminus \{0\} \\ 0 & \text{if } ij = 0, (i, j) \neq (0, 0) \\ -4B^2 & \text{if } (i, j) = (0, 0) \end{cases}$$

where $\{b(i) : i \geq 1\} \subset (0, \infty)$, $b(-i) = b(i)$ for $i \geq 1$ and $B \equiv \sum_{i=1}^{\infty} b(i) \in (0, \infty)$. Further, suppose that $b(i) \sim c_0 i^{-\beta}$ as $i \to \infty$, for some $\beta > d + 1/2 = 2.5$. Then, $A = \sum_{i \in \mathbb{Z}^2} \alpha(i) = 0$ and $\alpha(i, j) = (0, 0)$.

Further, we may write $\sigma_n^2$ as

$$\sigma_n^2 = \sum_{k=1}^{3} \sum_{i \in I_k} \theta_n(i)^2$$

where $I_1 = [-\lambda_n/2 + c_n, \lambda_n/2 - c_n] \cap \mathbb{Z}^2$, $I_2 = \mathbb{Z}^2 \setminus [-\lambda_n/2 - c_n, \lambda_n/2 + c_n]^2$, and $I_3 = \mathbb{Z}^2 \setminus [I_1 \cup I_2]$ respectively denote the collections of integer vectors $i$ that lie in the interior, the exterior, and the boundary parts of $R_n$, where $c_n$ is a suitably chosen sequence satisfying $c_n^{-1} + \lambda_n^{-1}c_n = o(1)$. It can be shown (cf. Section 6.4 below) that for $\beta > d + 1/2 = 2.5$,

$$\sum_{k=1}^{2} \sum_{i \in I_k} \theta_n(i)^2 = o(\lambda_n) \quad \text{and} \quad \sum_{i \in I_3} \theta_n(i)^2 = \sigma_0^2 \lambda_n(1 + o(1)) \quad (4.10)$$

where

$$\sigma_0^2 = 16B^2 \left[ B^2 + \sum_{k=1}^{\infty} \left( \sum_{j=k}^{\infty} b(j) \right)^2 \right] .$$

Hence, in this case, the contribution of the boundary part dominates the other two terms, and the scaling is given by $\lambda_n \equiv \lambda_n^{d-1}$, which is the $d - 1$-dimensional Lebesgue measure of the boundary of the sampling region $R_n$. Note that the rate of convergence no longer depends on $\beta \in (d + 1/2, \infty)$. \hfill \Box

The main reason why the edge effect dominates in the ND case as highlighted by Example 4.1 can be explained by noting the form of the constant $\sigma_0^2$ in (4.10). Although the condition $A = 0$ makes the sum of the $\alpha(i)$ over large open neighborhoods of the origin small, sums of the $\alpha(i)$ over half-planes, as determined by the $\theta_n(i)$ near the boundary of $R_n$ are not small. As a result, the combined contribution of these terms near the edge of $R_n$ determines the asymptotic behavior of the sum $S_n$ for $\beta > d + 1/2$. 

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The next result proves the CLT in presence of nontrivial edge effects, for $\beta > d + 1/2$. The case $\beta = d + 1/2$ will be treated in Theorem 4.3 below.

**Theorem 4.2:** Let $\{Z(\cdot)\}$ be the linear process given by (2.1) such that Conditions (C.1) hold with $\beta \in (d + 1/2, \infty)$. Also suppose that $d \geq 2$, $A = 0$ and the following condition holds:

$$\lim \limsup_{\delta \downarrow 0} \lim_{n \to \infty} \left| \lambda_n^{-(d-1)} \sum_{i \in [\partial R_n]^{\beta \lambda_n \cap \mathbb{Z}^d}} |\theta_n(i)|^2 - \sigma_{EE}^2 \right| = 0 \quad (4.11)$$

for some $\sigma_{EE}^2 \in (0, \infty)$. Then

$$\lambda_n^{-(d-1)/2} [S_n - ES_n] \to^d N(0, \sigma_{EE}^2). \quad (4.12)$$

Thus, it follows that under the conditions of the theorem, only the $\theta(i)$ with indices $i$ close to the boundary of $R_n$ contribute to the asymptotic variance of the sum. The contribution of the $\theta(i)$ for the rest of $i$-values becomes asymptotically negligible for $\beta > d + 1/2$. It can be shown that in Example 4.1, the limiting variance $\sigma_{EE}^2$ is given by $\sigma_0^2$. Although we do not explicitly state it, note that the boundary condition (2.4) on $R_0$ is implicit in the formulation of (4.11). Also, note that this edge-effect phenomenon in the ND case appears ONLY in dimensions $d \geq 2$.

Next we consider the case where $\beta = d + 1/2$. In this case, the edge effect may or may not have a nontrivial effect on the limit distribution, depending on the growth rate of the slowly varying function $L(\cdot)$. More precisely, we have the following results:

**Theorem 4.3:** Let $\{Z(\cdot)\}$ be the linear process given by (2.1) such that Conditions (C.1) holds with $\beta = d + 1/2$ for some $d \geq 2$. Further suppose that $A = 0$ and that (4.11) holds.

(i) If $L(t) = o(1)$ as $t \to \infty$, then (4.12) holds.

(ii) Suppose that Condition (C.2) holds and $G_{\infty}^\dagger \in L^2(\mathbb{R}^d)$ where $G_{\infty}^\dagger$ is as in (4.8).

(a) If $L(t) = c_0(1 + o(1))$ as $t \to \infty$ for some $c_0 \in (0, \infty)$, then

$$\lambda_n^{-(d-1)/2} [S_n - ES_n] \to^d N(0, \sigma_{EE}^2 + c_0^2 \int_{\mathbb{R}^d} [G_{\infty}^\dagger(x)]^2 dx).$$

(b) If $L(t)^{-1} = o(1)$ as $t \to \infty$, then (4.9) holds.

Theorem 4.3 shows that the edge effect is non-trivial under ND whenever $\lambda_n^{3d-2\beta} L^2(\lambda_n) = O(\lambda_n^{(d-1)})$, i.e., whenever the contribution to $\sigma_n^2$ from the $\theta_n(i)^2$ near the boundary is at least
as large as that from the remaining $\theta_n(i)^2$. When the slowly varying function $L(\cdot)$ is bounded, both $\lambda_n^{d-2\beta}L^2(\lambda_n)$ and $\lambda_n^{(d-1)}$ are of the same order and the asymptotic variance depends on both $\sigma_{EE}^2$ and the function $G_\infty^+(\cdot)$ of (4.8). On the other hand, when the factors $\lambda_n^{d-2\beta}L^2(\lambda_n)$ and $\lambda_n^{(d-1)}$ are not asymptotically equivalent, the scaling sequence and the asymptotic variance of the centered sum are determined by the dominant factor.

**Remark 4.1:** Note that in dimensions $d \geq 2$, the slowest possible growth rate of the variance of the sum in the ND case is $\lambda_n^{(d-1)}$. This may be contrasted with the one dimensional ND case where the variance of the sum grows at rate $\lambda_n^{3-2\beta}L(\lambda_n)^2$ which, in turn, can grow very slowly for $\beta$ close to $d + 1/2 = 3/2$. The main reason for this unusual behavior of the sum in higher dimensions is the presence of the edge effect which is not rate adaptive, i.e., it does not become asymptotically smaller even when the co-efficients $\alpha(i)$ or the function $\gamma(t)$ have a faster rate of decay.

**Remark 4.2:** For the one dimensional ND case, the variance of the sum $\sigma_n^2$ does not necessarily go to infinity for $\beta \geq 3/2$ and hence, rate adaptivity of the variance for $d = 1$ is meaningful only when $\beta < 3/2$, which is covered by Theorem 4.1. It can be shown that when $\beta = d + 1/2$ and $d = 1$, part (ii) (b) of Theorem 4.3 holds. However, no analog of parts (i) and (ii) (a) holds for $d = 1$, as the edge effects are asymptotically negligible in the one dimensional case. Also, for $\beta \in (d + 1/2, \infty)$, CLTs for the sum are not available for $d = 1$ in the ND case (as $\sigma_n^2 \not\to \infty$), but they are available in dimensions $d \geq 2$ (cf. Theorem 4.2).

## 5 Result under SRD

For completeness, we also give the result in the SRD case. Suppose that $\beta \in [d, \infty)$ with $\int_0^\infty t^{d-1}\gamma(t)dt < \infty$. Then it follows that $A = \sum_{i \in \mathbb{Z}^d} \alpha(i) \in \mathbb{R}$. If $A \neq 0$, then the spatial process is SRD and we have the following result.

**Theorem 5.1:** Let $\{Z(\cdot)\}$ be the linear process given by (2.1) such that Condition (C.1) holds with $\beta \in [d, \infty)$ and that $\int_0^\infty t^{d-1}\gamma(t)dt < \infty$ and $A \neq 0$. Then, as $n \to \infty$,

$$N_n^{-1/2}[S_n - ES_n] \to^d N(0, A^2).$$

Thus, under the conditions of Theorem 5.1, the sum $S_n$ is asymptotically normal and the asymptotic variance grows at the standard rate, namely, the square root of the sample
size. Note that in this case, we only assume condition (C.1), but not (C.2) or (C.3). The asymptotic variance of the sum depends on the co-efficients $\alpha(\cdot)$ only through the sum $A$, but not on the relative behavior of the $\alpha(\cdot)$ and $\gamma(\cdot)$ at infinity.

**Remark 5.1:** The asymptotic variance in the SRD case is determined by the $\alpha(i)$ for $i$ in arbitrarily large compact neighborhoods of the origin, while in the PSD case, it is determined by the relative behavior of $\alpha(\cdot)$ and $\gamma_t(\cdot)$ near infinity - the values of $\alpha(i)$ for any fixed compact neighborhood of the origin has no effect on the asymptotic variance. In the ND case, the asymptotic variance depends on the $\alpha(i)$ for both - (i) for smaller $i$ through the condition $A = 0$ and (ii) for large $i$ through the relative behavior of $g_t(\cdot)$ and $\gamma(\cdot)$ near infinity - in absence of the edge-effect. For $d \geq 2$, in the ND case with non trivial edge effects, the asymptotic variance depends on the $\alpha(i)$ for $i$ in arbitrarily large compact neighborhoods of the origin, as in the SRD case, but not on the relative behavior of $\alpha(\cdot)$ and $\gamma(\cdot)$.

### 6 Proofs

#### 6.1 Notation

Let $C = [0,1)^d$ denote the unit cube in $\mathbb{R}^d$. Let $L^2$ denote the collection of all square integrable functions (with respect to the Lebesgue measure on $\mathbb{R}^d$) from $\mathbb{R}^d$ to $\mathbb{R}$, and let $\| \cdot \|$ denote the $L^2$ norm, i.e., $\| f \|^2 = \int f^2(x) dx$, $f \in L^2$. Let $\| \cdot \|_1$ and $\| \cdot \|_\infty$ respectively denote the $\ell^1$ and $\ell^\infty$-norms on $\mathbb{R}^d$, i.e., for $(x_1, \ldots, x_d)' \in \mathbb{R}^d$,

$$\|(x_1, \ldots, x_d)\|_1 = \sum_{i=1}^d |x_i| \quad \text{and} \quad \|(x_1, \ldots, x_d)\|_\infty = \max\{|x_i| : 1 \leq i \leq d\}.$$  

Recall that for any set $A \subset \mathbb{R}^d$ and $\delta \in (0, \infty)$, let $A^\delta = \{x \in \mathbb{R}^d : \|x - y\| \leq \delta \text{ for some } y \in A\}$ denote the $\delta$-enlargement of $A$. Similarly, define the set $A^{-\delta} = \{x \in A : B(x; \delta) \subset A\}$ where $B(x; \delta) = \{y \in \mathbb{R}^d : \|x - y\| < \delta\}$. Let $\partial A$ denote the boundary of $A$. Let $C, C(\cdot)$ denote generic constants that do not depend on $n$. Unless otherwise specified, all limits (including those in the order symbols) are taken by letting $n \to \infty$. Also, for notational simplicity, we set $\mu = 0$ for the rest of this section (except in cases where there is a chance of confusion).
6.2 Auxiliary results

Here we prove a general version of the CLT for spatial linear processes without structural conditions on the coefficients $\alpha(i)$ and the sampling regions; This result forms the basis for proving the results from Sections 3-5, and may be of independent interest.

**Theorem 6.1:** Let $\{Z(j) : j \in \mathbb{Z}^d\}$ be the spatial linear process in (2.1) with $\mu = 0$. Let $\Lambda_n$ be a finite subset of $\mathbb{Z}^d$ such that $|\Lambda_n| \rightarrow \infty$ as $n \rightarrow \infty$. Let $S_n = \sum_{j \in \Lambda_n} Z(j)$ and $\sigma_n^2 = \sum_{i \in \mathbb{Z}^d} \theta_n(i)^2$, where $\theta_n(i) = \sum_{j \in \Lambda_n} \alpha(j - i), i \in \mathbb{Z}^d$. Suppose that as $n \rightarrow \infty$, 

$$\frac{1}{\sigma_n} \max \{\|\theta_n(i)\| : i \in \mathbb{Z}^d\} \rightarrow 0. \quad (6.1)$$

Then $S_n/\sigma_n \xrightarrow{d} N(0, 1)$ as $n \rightarrow \infty$.

**Proof:** First we shall show that there exists a sequence of integers $m_n \rightarrow \infty$ such that

$$\sigma_n^{-2} \sum_{\|i\| > m_n} \theta_n(i)^2 = o(1). \quad (6.2)$$

To that end, note that $|\theta_n(i)| \leq \left[ \sum_{j \in (\Lambda_n - i)} \alpha(j) \right]^{1/2} = |\Lambda_n|^{1/2}$ for all $i \in \mathbb{Z}^d$. Since $\sum_{i \in \mathbb{Z}^d} \alpha(i)^2 < \infty$, there exists $m_{1n} \rightarrow \infty$ such that $\sigma_n^{-2} |\Lambda_n|^{2} \sum_{\|i\| > m_{1n}} \alpha(i)^2 = o(1)$. Define $m_n = \max\{\|i\| : j \in \Lambda_n\} + m_{1n}$. Then, it follows that

$$\sigma_n^{-2} \sum_{\|i\| > m_n} \theta_n(i)^2 \leq \sigma_n^{-2} |\Lambda_n| \sum_{\|i\| > m_n} \sum_{j \in \Lambda_n} \alpha(j - i)^2 \leq \sigma_n^{-2} |\Lambda_n|^2 \sum_{\|i\| > m_{1n}} \alpha(i)^2 = o(1),$$

proving (6.2).

Next define $U_n = \{i \in \mathbb{Z}^d : \|i\| \leq m_n\}$ and $\bar{U}_n = \{i \in \mathbb{Z}^d : \|i\| > m_n\}$. Define $\bar{S}_n = \sum_{i \in U_n} \theta_n(i)\epsilon(i)$ and $\bar{\sigma}_n^2 = \sum_{i \in U_n} \theta_n(i)^2$. Then, by (6.2), $\sigma_n^{-2}[\sigma_n^2 - \bar{\sigma}_n^2] = o(1)$ and hence,

$$\sigma_n^{-1} S_n - \bar{\sigma}_n^{-1} \bar{S}_n = \sigma_n^{-1} \left[ \sum_{i \in U_n} \theta_n(i)\epsilon(i) \right] + \left[ \frac{\bar{\sigma}_n - \sigma_n}{\sigma_n \bar{\sigma}_n} \right] \bar{S}_n = o_p(1), \quad (6.3)$$

provided $\bar{\sigma}_n^{-1} \bar{S}_n = O_p(1)$. Hence, it is enough to show that $\bar{\sigma}_n^{-1} \bar{S}_n \xrightarrow{d} N(0, 1)$ as $n \rightarrow \infty$.

By Lindeberg’s CLT, this would follow if for all $\delta > 0$,

$$\lim_{n \rightarrow \infty} \sum_{i \in U_n} EY_n(i)^2 \mathbb{I}(\|Y_n(i)\| > \delta) = 0, \quad (6.4)$$

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where \( Y_n(i) = \epsilon(i) \theta_n(i) / \sigma_n \), \( i \in U_n \). Now, by uniform integrability of \( \{ \epsilon(i)^2 : i \in \mathbb{Z}^d \} \) and (6.1), for any \( \delta > 0 \),

\[
\sum_{i \in U_n} EY_n(i)^2 \mathbb{I}(\left| Y_n(i) \right| > \delta) = \tilde{\sigma}_n^{-2} \sum_{i \in U_n} \theta_n(i)^2 E\epsilon(i)^2 \mathbb{I}(\left| \theta_n(i) \epsilon(i) \right| > \delta \tilde{\sigma}_n) \leq \max_{i \in \mathbb{Z}^d} E\epsilon(i)^2 \mathbb{I} \left( |\epsilon(i)| > \delta \frac{\tilde{\sigma}_n}{\max_{j \in \mathbb{Z}^d} \left| \theta_n(j) \right|} \right) = o(1).
\]

Hence, (6.4) holds and the result is proved. \( \square \)

**Corollary 6.2**: Let \( \Lambda_n \) be as in Theorem 6.1. Then, (6.1) holds if either of the following two conditions holds:

(i) \( \max\{ |\theta_n(i)| : i \in \mathbb{Z}^d \} = O(1) \) and \( \sigma_n^2 \to \infty \) as \( n \to \infty \).

(ii) \( \lim \inf_{n \to \infty} \frac{\sigma_n^2}{|\Lambda_n|} > 0 \).

**Proof**: Sufficiency of (i) for (6.1) is trivial. Consider (ii). Fix a sequence \( \{ p_n \} \subset (0, \infty) \) such that \( p_n^{-1} + |\Lambda_n|^{-1/d} p_n = o(1) \). Then, by the Cauchy-Schwarz inequality and (ii),

\[
\max_{j \in \mathbb{Z}^d} \frac{|\theta_n(j)|}{\sigma_n} \leq \max_{j \in \mathbb{Z}^d} \sigma_n^{-1} \left[ \sum_{|i-j| \leq p_n, i \in \Lambda_n} |\alpha(i-j)| + \sum_{|i-j| > p_n, i \in \Lambda_n} |\alpha(i-j)| \right] \leq \sigma_n^{-1} \left[ C(d) p_n^{d/2} \left( \sum_{i \in \mathbb{Z}^d} \alpha(i)^2 \right)^{1/2} + |\Lambda_n|^{1/2} \left( \sum_{|i| > p_n} \alpha(i)^2 \right)^{1/2} \right] = \frac{|\Lambda_n|^{1/2}}{\sigma_n} \left[ O \left( |\Lambda_n|^{-1/2} p_n^{-d/2} \right) + o(1) \right] = o(1).
\]

This completes the proof of the Corollary. \( \square \)

### 6.3 Proofs of the results from Section 3

**Proof of Proposition 3.1**: Note that by definition, \( |g_t(z)| \leq 1 \) for all \( \|z\| = 1 \) and \( t > 0 \). For any \( t > 0 \) and for any \( x \neq 0 \), writing \( x = rz \) with \( \|z\| = 1 \) and \( r = \|x\| \), we have

\[
|g_t(x)| = |g_{tr}(z)| \frac{\gamma(tr)}{\gamma(t)} \leq \frac{\gamma(tr)}{\gamma(t)} = \|x\|^{-\beta} \frac{L(t\|x\|)}{L(t)}.
\]

(6.5)
Hence, \( \eta \) there exists a sequence \( G \)

**Proof of Theorem 3.2:** First we shall show that \( \beta \in \mathbb{R} \)

Next, letting \( t \rightarrow \infty \), we have

\[
|g_\infty(x)| \leq \|x\|^{-\beta} \quad \text{for all } x \neq 0. \tag{6.6}
\]

Hence, \( \int_{R^d} |g_\infty(y - x)| dy \leq \int_{R^d} \|y - x\|^{-\beta} I(y \neq x) dy < \infty \) for all \( x \in \mathbb{R}^d \) and for all \( \beta \in (0, d) \). This completes the proof of the proposition. \( \square \)

**Proof of Theorem 3.2:** First we shall show that \( G_\infty \in L^2(\mathbb{R}^d) \). Note that \( R_0 \subset B(0, 2^{-1}\sqrt{d}) \) and hence, by (6.6), \( |G_\infty(x)| \leq \int_{\|y\| \leq 2\sqrt{d}} |g_\infty(y)| dy \leq C(d, \beta) \) for all \( \|x\|^2 \leq d \) while \( |G_\infty(x)| \leq \int_{\|y-x\| \leq \sqrt{d}/2} |g_\infty(y)| dx \leq C(d, \beta) \|x\|^{-\beta} \) for all \( \|x\|^2 > d \), implying that \( G_\infty \in L^2(\mathbb{R}^d) \).

Next we apply Corollary 6.2 to establish Theorem 3.2. Note that by Condition (C.2), there exists a sequence \( \eta_n \downarrow 0 \) such that

\[
\int_{\{2\|x\| > \eta_n \}} |g_\infty(x) - g_{\lambda_n}(x)|^2 dx = o(1) \quad \text{as } n \rightarrow \infty. \tag{6.7}
\]

W.l.o.g, suppose that \( \lambda_n \eta_n \gg \lambda_n^\delta \) for some \( \delta \in (0, 1/2) \). Next, for \( i \in \mathbb{Z}^d \), write

\[
\theta_n(i) = \sum_{j \in (R_n - 1) \cap \mathbb{Z}^d, \|j\| > \lambda_n \eta_n} \alpha(j) + \sum_{j \in (R_n - 1) \cap \mathbb{Z}^d, \|j\| \leq \lambda_n \eta_n} \alpha(j)
\]

\[
\equiv \theta_{2n}(i) + \theta_{1n}(i), \quad \text{say.}
\]

We shall first show that the contribution from the \( \theta_{2n}(i) \)-terms to \( \sigma_n^2 \) is negligible. To that end, note that by definition, \( \theta_{2n}(i) = 0 \) for all \( i \not\in [-2\lambda_n, 2\lambda_n]^d \). Hence,

\[
\sum_{i \in \mathbb{Z}^d} \theta_{2n}(i)^2 \leq \sum_{i \in [-2\lambda_n, 2\lambda_n]^d} \left( \sum_{\|j\| \leq \lambda_n \eta_n} |\alpha(j)| \right)^2
\]

\[
\leq (4\lambda_n)^d \left( \sum_{\|j\| \leq \lambda_n \eta_n} |\alpha(j)| \right)^2
\]

\[
\leq C(d) \lambda_n^d \left[ \lambda_n \eta_n \right]^{d-\beta} L(\lambda_n \eta_n)
\]

\[
= o\left( \lambda_n^{3d-2\beta} L^2(\lambda_n) \right). \tag{6.8}
\]

Next consider the \( \theta_{1n}(i) \)-terms. Note that by definition, for any \( t > 0 \) and \( k \in \mathbb{Z}^d \), \( g_t(x) = g_t(k) \) for all \( x \in t^{-1}(k + C) \). Hence,

\[
\theta_{1n}(i) = \sum_{j \in (R_n - 1) \cap \mathbb{Z}^d, \|j\| \leq \lambda_n \eta_n} \alpha(j)
\]
that \[ [R_0 - \gamma_n^{-1}i] \subset [R_0 - \lambda_n^{-1}i] \cap \{x \in \mathbb{R}^d : \|x\| > \eta_n - a_n\}.\] Write \[ g_t^{(n)}(x) = g_t(x) \mathbb{I}(\|x\| > \eta_n - a_n), \quad x \in \mathbb{R}^d. \] Then,

\[
\begin{align*}
\sum_{i \in \mathbb{Z}_d} \int_{\lambda_n^{-1}(i+C)} \left( \int \left| g_{\lambda_n}(x) - g_\infty(x) \right| \mathbb{I} \left( x \in [R_0 - \lambda_n^{-1}i] \right) dx \right)^2 dz & \\
\leq \sum_{i \in \mathbb{Z}_d} \int_{\lambda_n^{-1}(i+C)} \left( \int \left| g_{\lambda_n}(x) - g_\infty(x) \right| \mathbb{I} \left( x \in R_0^{2a_n} - z \right) \mathbb{I} \left( \|x\| > \eta_n - a_n \right) dx \right)^2 dz & \\
= \int \int \left( \int \left| g_{\lambda_n}^{(n)}(x - z) - g_\infty^{(n)}(x - z) \right| \left| g_{\lambda_n}^{(n)}(y - z) - g_\infty^{(n)}(y - z) \right| dx \right)^2 dz \times \mathbb{I} \left( x \in R_0^{2a_n} \right) \mathbb{I} \left( y \in R_0^{2a_n} \right) dxdydz & \\
\leq \int \int \left( \int \left| g_{\lambda_n}^{(n)}(x - z) - g_\infty^{(n)}(x - z) \right|^2 dz \right)^{1/2} \left( \int \left| g_{\lambda_n}^{(n)}(y - z) - g_\infty^{(n)}(y - z) \right|^2 dz \right)^{1/2} \times \mathbb{I} \left( x \in R_0^{2a_n} \right) \mathbb{I} \left( y \in R_0^{2a_n} \right) dxdy & \\
= \|g_{\lambda_n}^{(n)} - g_\infty^{(n)}\|_2^2 \left[ \int \mathbb{I} \left( x \in R_0^{2a_n} \right) dx \right]^2 & \\
= o(1). \quad (6.10)
\end{align*}
\]

Next note that the symmetric difference of the sets \([R_0 - \lambda_n^{-1}i] \subset [R_0 - \lambda_n^{-1}i] \cap \{x \in \mathbb{R}^d : \|x\| > \eta_n - a_n\}.\] Write \[ g_t^{(n)}(x) = g_t(x) \mathbb{I}(\|x\| > \eta_n - a_n), \quad x \in \mathbb{R}^d. \] Then,
\( C(d) \lambda_n \), while it is contained in \(((\partial R_0)^{2\alpha_n} - z) \cup \{x : \|x\| \leq \eta_n + a_n\}\) for all \( z \in \lambda_n^{-1}(i + \mathcal{C}) \), for \( \|l\| \leq C(d) \lambda_n \). Now, using the set inclusion relations given above (for the first inequality), arguments similar to (6.10) above (for the first term of the last inequality), and the bounded convergence theorem and the regularity conditions on the boundary of \( R_0 \) (that the \( d \)-dimensional Lebesgue measure of \( \partial R_0 \) is zero) (for the second and the third terms in the last inequality), we have

\[
\sum_{i \in \mathbb{Z}^d} \int_{\lambda_n^{-1}(i+c)} \left( \int g_\infty(x) \left[ \mathbb{I} \left( x \in [R_0 - \lambda_n^{-1}i_n] \right) - \mathbb{I} \left( x \in R_0 - z \right) \right] dx \right)^2 dz
\]

\[
\leq \sum_{\|l\| > C(d) \lambda_n} \int_{\lambda_n^{-1}(i+c)} \left\{ \int |g_\infty(x)| \mathbb{I} \left( x \in (\partial R_0)^{2\alpha_n} - z \right) dx \right\}^2 dz
\]

\[
+ \sum_{\|l\| \leq C(d) \lambda_n} \int_{\lambda_n^{-1}(i+c)} \left[ \int |g_\infty(x)| \left\{ \mathbb{I} \left( x \in (\partial R_0)^{2\alpha_n} - z \right) + \mathbb{I} \left( \|x\| \leq \eta_n + a_n \right) \right\} dx \right]^2 dz
\]

\[
\leq \left\| g_\infty(x) \mathbb{I} \left( \|x\| > C(d) \right) \right\|^2_2 \left\{ \int \mathbb{I} \left( x \in (\partial R_0)^{2\alpha_n} \right) dx \right\}^2
\]

\[
+ 2 \int_{\|z\| \leq C(d)} \left\{ \int |g_\infty(x)| \mathbb{I} \left( x \in \partial R_0^{2\alpha_n} - z \right) dx \right\}^2 dz
\]

\[
+ C(d) \left\{ \int |g_\infty(x)| \mathbb{I} \left( \|x\| \leq \eta_n + a_n \right) dx \right\}^2
\]

\[
= o(1), \quad (6.11)
\]

as \( \int |g_\infty(x)| \mathbb{I} \left( \|x\| \leq C \right) dx + \int g_\infty(x)^2 \mathbb{I} \left( \|x\| \geq C \right) dx \leq \infty \) for all \( C \in (0, \infty) \) and for all \( \beta \in (d/2, d) \). This completes the proof of (6.9). Note that (6.9) implies that \( \int \tilde{G}_\infty^2(x) dx \to \int G_\infty^2(x) dx \) as \( n \to \infty \). Hence, by Corollary 6.2(ii), Theorem 3.2 follows. \( \square \)

### 6.4 Proofs of the results from Section 4

**Proof of Theorem 4.1:** By Corollary 6.2(i), it is enough to show that \( G^\dagger_\infty \in L^2 \) and

\[
\sigma_n^2 = \left[ \lambda_n^{3d-2\beta} L(\lambda_n)^2 \int G^\dagger_\infty(x)^2 dx \right] (1 + o(1)). \quad (6.12)
\]

Note that by (6.6),

\[
|G^\dagger_\infty(x)| \leq C(d, \beta) \begin{cases} d(x, R_0)^{d-\beta} & \text{if } x \in [R_0 \cup \partial R_0]^c \text{ and } \|x\| \leq 1, \\
\|x\|^{-\beta} & \text{if } \|x\| > 1 \end{cases}
\]

and \( |G^\dagger_\infty(x)| \leq C(d, \beta) d(x, R_0^{\circ})^{d-\beta} \) for \( x \in R_0 \). Since \( \beta \in (d, d + 1/2) \), by the boundary condition (C.3), it follows that \( G^\dagger_\infty \in L^2 \).
Next consider (6.12). For \( \beta \in (d, d + 1/2) \), by Condition (C.2), (6.5) and (6.6), there exists a sequence \( \eta_n \downarrow 0 \) such that
\[
\int_{\{2 \|x\| > \eta_n\}} |g_\infty(x) - g_{\lambda_n}(x)|^p \, dx = o(1) \quad \text{as} \quad n \to \infty, \tag{6.13}
\]
for \( p = 1, 2 \). For \( p = 1 \), this follows directly from (C.2), as \( \beta > d \). As for \( p = 2 \), we use the trivial bound \( |f(x)|^2 \leq |f(x)| \max_{x \in B\{f(x)|} \} \) for a function \( f : B \to \mathbb{R} \) in conjunction with the \( p = 1 \) relation and the bounds (6.5) and (6.6), which may require replacing the \( \eta_n \) for the \( p = 1 \) case by a possibly coarser sequence that still decreases to zero. Hence, (6.13) holds for both \( p = 1, 2 \).

W.l.o.g, suppose that \( \eta_n \gg \lambda_n^{-\delta} \) (i.e., \( \lambda_n \eta_n \to \infty \)) for some \( \delta \in (0, 1) \). Let \( t_n = \lambda_n \eta_n \) and let \( u_n = \lambda_n^{[2d-2\beta+1]/2} \). Then \( u_n^{-1} + t_n^{-1} = o(1) \) and \( u_n = o\left(\lambda_n^{[2d-2\beta+1]} L_n(\lambda_n)^2\right) \).

Also, define
\[
V_{1n} = \{i \in \mathbb{Z}^d : B(i; t_n) \subset R_n\}
\]
\[
V_{2n} = \{i \in \mathbb{Z}^d : B(i; t_n) \subset R_n^c\}
\]
\[
V_{3n} = \mathbb{Z}^d \setminus [V_{1n} \cup V_{2n}]. \tag{6.14}
\]

First consider the sum of \( \theta_n(i)^2 \) for \( i \in V_{3n} \). Note that \( V_{3n} \subset \{i \in \mathbb{Z}^d : i \in (\partial R_n)^t\} \). By (C.3) (with \( f \equiv 1 \)), \( \nu((\partial R_n)^t) = O(\epsilon) \) as \( \epsilon \to 0 \). This implies \( \nu((\partial R_n)^{u_n}) = O(\lambda_n^{d-1} u_n) \) and hence,
\[
\sum_{i \in \partial R_n^c \cap \mathbb{Z}^d} |\theta_n(i)|^2 \leq C(d) \lambda_n^{d-1} u_n \left[ \sum_{i \in \mathbb{Z}^d} |\alpha(i)| \right]^2 = o\left(\lambda_n^{[3d-2\beta]} L(\lambda_n)^2\right). \tag{6.15}
\]

If \( u_n > t_n \), then this shows that \( \sum_{i \in V_{3n}} \theta_n(i)^2 = o(\lambda_n^{[3d-2\beta]} L(\lambda_n)^2) \). Hence, w.l.o.g., suppose that \( u_n \leq t_n \). Let \( v_n(i) = d(i, \partial R_n), i \in [(\partial R_n)^{u_n}]^c \). By the condition \( A = 0 \), uniformly in \( i \notin (\partial R_n)^{u_n} \), we have
\[
|\theta_n(i)| \leq \sum_{\|l\| > v_n(i)} |\alpha(l)| \leq \sum_{\|l\| > v_n(i)} \|l\|^{-\beta} L(\|l\|) \leq C(d, \beta) v_n(i)^{d-\beta} L(v_n(i)).
\]

Hence, by Condition (C.3), it follows that
\[
\sum_{i \in [\partial R_n^c \setminus \partial R_n^{u_n}] \cap \mathbb{Z}^d} \theta_n(i)^2 \\
\leq C(d, \beta) \sum_{i \in [\partial R_n^c \setminus \partial R_n^{u_n}] \cap \mathbb{Z}^d} \left\{ v_n(i)^{d-\beta} L(v_n(i)) \right\}^2 \\
\leq C(d, \beta) \max \left\{ L(i)^2 : u_n \leq \|i\| \leq t_n \right\} \cdot \lambda_n^{3d-2\beta} \int_{\partial R_n^{u_n} \setminus \lambda_n} d(x, \partial R_0)^{2d-2\beta} \, dx
\]

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Then, it follows that
\[ \sum_{i \in V_{3n}} \theta_n(i)^2 = o \left( \lambda_n^{3d-2\beta} L(\lambda_n)^2 \right). \]

Next, using arguments similar to those in the proof of Theorem 3.2, we have
\[ \sum_{i \in V_{1n}} \theta_n(i)^2 = \sum_{i \in V_{1n}} \left[ \sum_{j \in R_{n-1}} \alpha(i) \right]^2 \]
\[ = \sum_{i \in V_{1n}} \left[ \sum_{j \in R_{n-1}} \alpha(i) \right]^2 \quad \text{(as } A = 0) \]
\[ = \lambda_n^{2d} \gamma(\lambda_n)^2 \sum_{i \in V_{1n}} \left[ \int g_{\lambda_n}(x) \mathbb{I} \left( x \in [R_0 - \lambda_n^{-1}i]_n \right) dx \right]^2, \]

and similarly,
\[ \sum_{i \in V_{2n}} \theta_n(i)^2 = \lambda_n^{2d} \gamma(\lambda_n)^2 \sum_{i \in V_{2n}} \left[ \int g_{\lambda_n}(x) \mathbb{I} \left( x \in [R_0 - \lambda_n^{-1}i]_n \right) dx \right]^2, \]

where \([R_0 - \lambda_n^{-1}i]_n\) is as defined in the proof of Theorem 3.2. Define the function \(\tilde{G}_n(\cdot)\) by
\[ \tilde{G}_n(x) = \begin{cases} \int g_{\lambda_n}(y) \mathbb{I} \left( y \in [R_0 - \lambda_n^{-1}i]_n \right) dy & \text{if } x \in \lambda_n^{-1}(i + C), i \in V_{1n} \\ \int g_{\lambda_n}(y) \mathbb{I} \left( y \in [R_0 - \lambda_n^{-1}i]_n \right) dy & \text{if } x \in \lambda_n^{-1}(i + C), i \in V_{2n} \\ 0 & \text{otherwise} \end{cases} \]

Then, it follows that
\[ \sigma_n^2 = \lambda_n^{3d} \gamma(\lambda_n)^2 \int \tilde{G}_n(x)^2 dx + \sum_{i \in V_{1n}} |\theta_n(i)|^2 \]
\[ = \lambda_n^{3d} \gamma(\lambda_n)^2 \int \tilde{G}_n(x)^2 dx + o \left( \lambda_n^{3d} \gamma(\lambda_n)^2 \right). \]
that (from the proof of $G_0(1)$. Hence, it follows that

$$
\leq \sum_{i \in V_{2n}} \int_{\lambda_n^{-1}(i+c)} \left( \int \|g_{\lambda_n}^{(n)}(x) - g_\infty^{(n)}(x)\| \mathbb{I} \left( x \in R_0^{2a_n} - z \right) dx \right)^2 dz 
$$

$$
= \int_{\Gamma_n} \left( \int \|g_{\lambda_n}^{(n)}(x - z) - g_\infty^{(n)}(x - z)\| \mathbb{I} \left( x \in R_0^{2a_n} \right) dx \right)^2 dz 
$$

$$
= \int \int \left( \int \|g_{\lambda_n}^{(n)}(x - z) - g_\infty^{(n)}(x - z)\| \|g_{\lambda_n}^{(n)}(y - z) - g_\infty^{(n)}(y - z)\| \times \mathbb{I} \left( x \in R_0^{2a_n} \right) \mathbb{I} \left( y \in R_0^{2a_n} \right) dx dy dz \right.

$$

$$
\leq \int \int \left( \int \|g_{\lambda_n}^{(n)}(x - z) - g_\infty^{(n)}(x - z)\|^2 \times \mathbb{I} \left( x \in R_0^{2a_n} \right) \mathbb{I} \left( y \in R_0^{2a_n} \right) dx dy dz \right.

$$

$$
= \left\| \left( g_{\lambda_n}^{(n)} - g_\infty^{(n)} \right) \right\|_2^2 \left[ \int \mathbb{I} \left( x \in R_0^{2a_n} \right) dx \right]^2 
$$

$$
= o(1).
$$

By similar arguments,

$$
\sum_{i \in V_{1n}} \int_{\lambda_n^{-1}(i+c)} \left( \int [g_{\lambda_n}(x) - g_\infty(x)] \mathbb{I} \left( x \in [R_0^c - \lambda_n^{-1}i]_n \right) dx \right)^2 dz 
$$

$$
\leq C(d) \cdot \left[ \int_{\|y\| \geq \eta_n} \left| g_{\lambda_n}(y) - g_\infty(y) \right| dy \right]^2 = o(1).
$$

Next note that $\partial [R_0^c] = \partial R_0$. By repeating the arguments in (6.11), one can conclude that

$$
\sum_{i \in V_{1n}} \int_{\lambda_n^{-1}(i+c)} \left( \int g_\infty(x) \left[ \mathbb{I} \left( x \in [R_0^c - \lambda_n^{-1}i]_n \right) - \mathbb{I} \left( x \in R_0^c - z \right) \right] dx \right)^2 dz 
$$

$$
+ \sum_{i \in V_{2n}} \int_{\lambda_n^{-1}(i+c)} \left( \int g_\infty(x) \left[ \mathbb{I} \left( x \in [R_0^c - \lambda_n^{-1}i]_n \right) - \mathbb{I} \left( x \in R_0^c - z \right) \right] dx \right)^2 dz 
$$

$$
= o(1).
$$

Finally, using the boundary condition (C.3) and the bounds on $|G_\infty^+(x)|$ for $x \in (\partial R_0)^n$ (from the proof of $G_\infty^+ \in L^2(\mathbb{R}^d)$), one gets $\sum_{i \in V_{3n}} \int_{\lambda_n^{-1}(i+c)} cG_\infty^2(x) dx \leq C(d) \int_0^n t^{2(d-\beta)} dt = o(1)$. Hence, it follows that

$$
\sigma_n^2 = \left[ \gamma(\lambda_n) \right]^2 \lambda_n^3 \int [G_\infty^+(x)]^2 dx (1 + o(1)).
$$

This completes the proof of Theorem 4.1. □
Proofs of claims in Example 4.1  Let $c_n = \lceil \log \lambda_n \rceil$, $n \geq 1$. Also, let $||(x, y)||_\infty = \max\{|x|, |y|\}$, $x, y \in \mathbb{R}$. For a set $A \subset \mathbb{R}^2$ and $\delta > 0$, write $A_\delta = \{x \in \mathbb{R}^2 : \| x - y \|_\infty \leq \delta \}$ and $A^{-\delta} = \{x \in A : \| x - y \|_\infty < \delta \}$ implies $y \in A$ for the $\delta$-enlargement and the $\delta$-interior of a set $A$ in the $|| \cdot ||_\infty$-norm. As before, set $\theta_n(i) = \sum_{j+i \in R_n} \alpha(j)$. Let $\mathcal{I}_n(i_0) \equiv \{i \in \mathbb{Z} : -\frac{\lambda_n}{2} - i_0 < i < \frac{\lambda_n}{2} - i_0\}$, $i_0 \in \mathbb{Z}$. For $i = (i_0, j_0) \in R_n^{-\infty}$, it is easy to verify that both $\mathcal{I}_n(i_0)$ and $\mathcal{I}_n(j_0)$ contain the set $\{i \in \mathbb{Z} : |i| < c_n\}$. Hence, using the fact that $\sum_{i \in \mathbb{Z}^2} \alpha(i) = 0$, one gets

$$\sum_{i \in \mathcal{I}_n(i_0) \times \mathcal{I}_n(j_0)} \alpha(i, j)$$

Hence, it follows that

$$\sum_{i \in R_n^{-\infty}} \theta_n(i)^2 \leq \left[ \sum_{i_0 = -\lambda_n/2 + c_n}^{\lambda_n/2 - c_n} \left\{ \sum_{|i| > \frac{\lambda_n}{2} - |i_0|} b(i) \right\} \right]^2$$

$$\leq C(\beta, B) \left[ \sum_{j=0}^{\lambda_n/2 - c_n} \left| \frac{\lambda_n}{2} - j \right|^{2-2\beta} \right]^2$$

$$\leq C(\beta, B) \left( \frac{c}{n} \right)^{2-2\beta}$$

for some (generic) constant $C(\beta, B) \in (0, \infty)$. By similar arguments, it can be shown that

$$\sum_{i \notin R_n^{-\infty}} \theta_n(i)^2 = o(\lambda_n).$$

Hence, it remains to determine the contribution of the boundary terms to $\sigma_n^2$. For $|i_0| \leq \lambda_n/2 - c_n$ and $-c_n \leq k \equiv \lceil 2^{-1} \lambda_n - j_0 \rceil \leq c_n$ (this corresponds to a part of the upper boundary line of $R_n$), note that

$$\theta_n(i_0, j_0) = \sum_{|i+i_0|<\lambda_n/2} \sum_{j=-\lambda_n/2-j_0}^{\lambda_n/2-j_0} \alpha(i, j)$$

$$= \left[ \sum_{i \in \mathbb{Z}} \sum_{j=-\lambda_n/2-j_0}^{\lambda_n/2-j_0} \alpha(i, j) \right] (1 + o(1))$$
\[
= o(1) + \begin{cases} 
-4B^2 + (2B) \left[ \sum_{j=1}^k b(j) + B \right] & \text{if } k > 0 \\
-2B^2 & \text{if } k = 0 \\
(2B) \left[ \sum_{j=-\infty}^k b(j) \right] & \text{if } k < 0
\end{cases}
\]

uniformly in \((i_0, j_0)\). Note that by absolute summability of the \(\alpha(i)\),

\[
\sum_{|i_0 + \lambda_n/2| \leq c_n} \sum_{|j_0 - \lambda_n/2| \leq c_n} \theta_n(i_0, j_0)^2 = O(c_n^2).
\]

Hence, it follows that

\[
\sum_{|i_0| \leq \lambda_n/2 - c_n} \sum_{|j_0 - \lambda_n/2| \leq c_n} \theta_n(i_0, j_0)^2 = \left[ \lambda_n \sigma_0^2/4 \right] (1 + o(1)).
\]

Now using similar arguments for the other three boundary arms, one gets the results of Example 4.1. □

**Proof of Theorem 4.2:** Using (4.11), one can show that there exists \(\eta_n \to 0^+\) such that

\[
\limsup_{n \to \infty} \left| \frac{\lambda_n}{\lambda_n} (d-1) \sum_{i \in [\delta R_n] \cap \lambda_n \cap \mathbb{Z}^d} |\theta_n(i)|^2 - \sigma_{\text{EE}}^2 \right| = 0.
\]

By (4.11) and the monotonicity of the sum of \(\theta_n(i)^2\) over increasing index sets, we may always replace \(\eta_n\) by a coarser sequence going to zero. Hence, w.l.o.g. assume that \(\eta_n \geq [\log(\lambda_n)]^{-1}\).

Next set \(t_n = \lambda_n \eta_n\) and (re-)define the sets \(V_{k,n}\), \(k = 1, 2, 3\) in (6.14) with this choice of \(t_n\).

Further, write \(V_{21n} = V_{2n} \cap \{i \in \mathbb{Z}^d : \|i\| \leq \lambda_n \sqrt{d}\}\) and \(V_{22n} = V_{2n} \setminus V_{21n}\).

Next note that uniformly in \(i \notin R_{t_n}^n\),

\[
|\theta_n(i)| \leq \sum_{j \in R_{t_n}^n} |\alpha(j)| \\
\leq \sum_{j : \|j\| \geq d(\partial R_n, i)} \gamma(\|j\|) \\
\leq C(d, \beta) \min \left\{ d(\partial R_n, i)^{d-\beta} L(d(\partial R_n, i)), \lambda_n^d d(\partial R_n, i)^{-\beta} L_n^*(i) \right\}.
\]

where \(L_n^*(i) \equiv \max\{L(\|j\|) : j \in [R_n - i] \cap \mathbb{Z}^d\}\). And, using the fact that \(A = 0\), one can similarly show that uniformly in \(i \in R_{t_n}^n\),

\[
|\theta_n(i)| = \left| - \sum_{j \in R_{t_n}^n} \alpha(j) \right| \leq C(d, \beta) d(\partial R_n^c, i)^{d-\beta} L(d(\partial R_n^c, i)).
\]
Hence, it follows that
\[ \sum_{i \in V_{1n}} \theta_n(i)^2 + \sum_{i \in V_{21n}} \theta_n(i)^2 \leq C(d) \lambda_n^d \max\{ \theta_n(i)^2 : i \in V_{1n} \cup V_{21n} \} \]
\[ \leq C(d, \beta) \lambda_n^{d-2} \max\{ L^2(t) : t_n \leq t \leq d \lambda_n \} = o(\lambda_n^{d-1}). \] (6.22)

Next note that for all \( i \in V_{22n}, \)
\[ ||i||/2 \leq ||i|| - \lambda_n \sqrt{d} / 2 \leq d(R_n, i) \leq ||i|| \]
and \[ \max\{ ||j|| : j \in [R_n - i] \cap \mathbb{Z}^d \} \subset [||i||/2, 3||i||/2]. \]
Hence, by (6.21)
\[ \sum_{i \in V_{22n}} \theta_n(i)^2 \leq C(d) \lambda_n^{2d} \sum_{||i|| > \sqrt{d} \lambda_n} ||i||^{2-2\beta} L^2(||i||) \]
\[ \leq C(d, \beta) \lambda_n^{3d-2\beta} L^2(\lambda_n) = o(\lambda_n^{d-1}) \] (6.23)
for \( \beta > d + 1/2. \) Hence, from (4.11), (6.22) and (6.23), it follows that
\[ \sigma_n^2 = \sum_{k=1}^{3} \sum_{i \in V_{kn}} \theta_n(i)^2 = \lambda_n^{d-1} \sigma_{EE}^2(1 + o(1)). \] (6.24)

By Corollary 6.2 (i), Theorem 4.2 follows. \( \square \)

**Proof of Theorem 4.3:** Theorem 4.3 follows from Corollary 6.2 (i), by comparing the orders of the terms \( \sum_{i \in V_{kn}} \theta_n(i)^2, \) \( k = 1, 2, 3. \) Specifically, for part (i), we use the arguments in the proof of Theorem 4.2, and the bounds from (6.20), (6.22) and (6.23) to conclude that (6.24) holds. For part (ii), note that for \( \beta = d + 1/2, \) the conditions and the arguments in the proof of Theorem 4.1 no longer ensure that \( \int_{||x|| \leq \eta_n} G_\infty^t(x)^2 d\mathbf{x} = O(1), \) which is why we need to make the assumption that \( G_\infty^t \in L^2. \) However, under (C.1) and (C.2), the arguments in the proof of Theorem 4.1 leading to the convergence of \( \int_{||x|| \geq \eta_n} |\dot{G}_n(x) - G_\infty^t(x)| d\mathbf{x} \) to zero still holds. Both parts of (ii) now follow by deriving the limits of the terms \( \sum_{i \in V_{kn}} \theta_n(i)^2, \) using (6.20) for \( k = 3 \) and using the steps from the proof of Theorem 4.1 for \( k = 1, 2. \) We omit the routine details. \( \square \)

### 6.5 Proof of the results from Section 5

**Proof of Theorem 5.1:** In view of Corollary 6.2(ii), it is enough to show that
\[ \sigma_n^2 = N_n A^2(1 + o(1)). \] (6.25)
Let $c_n$ be a sequence of positive real numbers satisfying
\[ c_n^{-1} + \lambda_n^{-1} c_n = o(1). \]  
(6.26)

Also, let
\[
U_{1n} = \{ i \in \mathbb{Z}^d : \| \lambda_n^{-1} i - x \| \leq c_n / \lambda_n \ \text{for some} \ x \in \partial R_0 \},
\]
\[
U_{2n} = \{ i \in \mathbb{Z}^d : B(i; c_n) \subset R_n \},
\]
\[
U_{3n} = \{ i \in [-2d\lambda_n, 2d\lambda_n]^d : i \not\in [U_{1n} \cup U_{2n}] \}, \quad \text{and}
\]
\[
U_{4n} = \mathbb{Z}^d \setminus U_{3n}.
\]  
(6.27)

Then $\sigma_n^2$ can be written as
\[
\sigma_n^2 = \sum_{i=1}^{4} \sum_{i \in U_{in}} \theta_n(i)^2 \equiv I_{1n} + I_{2n} + I_{3n} + I_{4n}, \quad \text{say.} \]  
(6.28)

Note that by (6.26), the boundary condition on $R_0$ (that $\nu(\partial R_0) = 0$) and the absolute summability of $\alpha(i)$'s,
\[
I_{1n} \leq \left[ \sum_{j \in \mathbb{Z}^d} |\alpha(j)| \right]^2 \left| \left\{ i : \lambda_n^{-1} i \in [\partial R_0]_{c_n / \lambda_n} \right\} \right| 
= \lambda_n^d \cdot O\left( \text{vol.} \left( [\partial R_0]_{c_n / \lambda_n} \right) \right) 
= o(\lambda_n^d).
\]

Next consider $I_{2n}$. By definition of $U_{2n}$, $B(0; c_n) \subset R_n - i$ for all $i \in U_{2n}$. Hence, it follows that
\[
\sup\{ |\theta_n(i) - A| : i \in U_{2n} \} \leq \sum_{j \in \mathbb{Z}^d : |j| \geq c_n} |\alpha(j)| = o(1).
\]  
(6.29)

Next, note that for any $i \in U_{3n}$, $B(i; c_n / 2)$ is contained in the set $R_n^c$, and hence, $\emptyset = [R_n - i] \cap B(0; c_n / 2) = [R_n \cap B(i; c_n / 2)] - i$. Hence, it follows that
\[
I_{3n} \leq C(d)\lambda_n^d \sup\{ \theta_n(i)^2 : i \in U_{3n} \} \leq C(d)\lambda_n^d \sum_{j \in \mathbb{Z}^d : |j| \geq c_n} |\alpha(j)| = o(\lambda_n^d).
\]

Finally, by Condition (C.1) and the definition of $U_{4n}$,
\[
\sum_{i \in U_{4n}} \theta_n(i)^2 
\leq \sum_{i \in U_{4n}} \left[ \left( \sum_{j \in [R_n - i] \cap \mathbb{Z}^d} \alpha(j)^2 \right) \times N_n \right]
\]
\[ N_n \sum_{i \in U_n} \sum_{j \in \mathbb{Z}^d : \|j\|_1 \geq k} \alpha(j)^2 \leq C(d) N_n \sum_{i \in U_n} \sum_{k \geq \|i\|_1 - d\lambda_n} k^{d-1} \sup\{\gamma(t)^2 : t \in [k/\sqrt{d}, k]\} \]
\[ \leq C(d) N_n \int_{\|i\|_1 - d\lambda_n}^{\infty} t^{d-1-2\beta} L(t)^2 dt \]
\[ \leq C(d) N_n \lambda_n^{2d-2\beta} L(\lambda_n)^2 = o(N_n), \]

since \( \lambda_n^{d-\beta} L(\lambda_n) = o(1) \) for all \( \beta > d \) and also for \( \beta = d \) by the integrability condition \( \int_1^{\infty} \gamma(t) dt < \infty \). Since \( |U_{2n}| = |D_n| - O(|U_{1n}|) = N_n - o(\lambda_n^d) = N_n(1 + o(1)) \), (6.25) follows from (6.28), (6.29) and the bounds for \( I_{1n}, I_{3n} \) and \( I_{4n} \) above. This completes the proof of the theorem. \( \square \)

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**References**


