CHAPTER 7

Studentization in Edgeworth expansions for estimates of semiparametric index models

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1 Introduction

During the 1970s, Takeshi Amemiya considerably advanced the asymptotic theory of estimation of parametric econometric models for cross-sectional data. Previously, most work had concerned closed form estimates, such as generalized least squares and instrumental variable estimates of linear regressions, or two- or three-stage least squares estimates of linear (in equations and parameters) simultaneous equation systems. Prompted by Jennrich's (1969) work on strong consistency and asymptotic normality of nonlinear least squares, Amemiya developed asymptotic theory for implicitly defined extremum estimates of a variety of econometric models.

Let Y_i, X_i, $i = 1, 2, \ldots,$ be sequences of, respectively, scalar and $d \times 1$ vector observables, and define

$$Y_i = (\beta^\tau X_i + \epsilon_i)\, 1(\beta^\tau X_i + \epsilon_i > 0), \qquad i = 1, 2, \ldots, \tag{7.1.1}$$

where ϵ_i, $i = 1, 2, \ldots,$ is a sequence of unobservable zero-mean random variables, β is a $d \times 1$ unknown vector, τ denotes transposition, and $1(\cdot)$ is the indicator function. (7.1.1) is called a Tobit model. Least squares regression of Y_i on X_i, using either all observations or all observations such that $Y_i > 0$, inconsistently estimates β. Assuming the ϵ_i are independent and identically distributed (iid) normal variates, maximum likelihood (ML) estimates based on (7.1.1) can be consistent. These, however, are only implicitly defined. Amemiya (1973) established their strong consistency and asymptotic normality, later extending these results (Amemiya 1974a) to a multivariate version of (7.1.1).

Another model of econometric interest is

$$\frac{Y_i^\lambda - 1}{\lambda} 1(\lambda > 0) + (\log Y_i) 1(\lambda = 0) = \beta^\tau X_i + \epsilon_i, \tag{7.1.2}$$

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where the scalar λ is unknown. This is called a Box–Cox transformation model. If λ is specified incorrectly, least squares regression inconsistently estimates β. Thus, methods have been proposed for estimating β and λ simultaneously. One such purports to be ML, based on normal ϵ_i, but unless $\lambda = 0$ or $1/\lambda$ is odd, the left hand side of (7.1.2) cannot possibly be conditionally normal. Alternative, logically consistent distributions have been proposed, e.g., Amemiya and Powell (1981), but if the distribution is misspecified, inconsistent estimates again result. Amemiya and Powell (1981) also applied nonlinear two-stage least squares estimation, which applies to a general class of models including (7.1.2), and whose asymptotic theory was earlier developed by Amemiya (1974b). This estimate, which again is only implicitly defined, is consistent over a wide class of ϵ_i. Amemiya (1977) also developed asymptotic theory for nonlinear three-stage least squares and ML estimates of nonlinear simultaneous equations, to provide an extension to vector dependent variables.

Both models (7.1.1) and (7.1.2) are of the single linear index type

$$E(Y_i \mid X_i) = G(\beta^{\tau} X_i), \qquad i = 1, 2, \ldots, \tag{7.1.3}$$

almost surely (a.s.), for a function $G : R \to R$. Let F be the distribution function of ϵ_i. In (7.1.1),

$$G(u) = u \int_{-u}^{\infty} dF(v) - \int_{-\infty}^{-u} v \, dF(v).$$

If F is an unknown, nonparametric function, then so is G. Then β can be identified only up to scale. But if we can estimate β up to scale in (7.1.3), with unknown G, we have a form of robustness with respect to F. In (7.1.2),

$$G(u) = \int \{1 + \lambda(u + v)\}^{1/\lambda} \, dF(v) \, 1(\lambda > 0) + e^{u} \int e^{v} \, dF(v) \, 1(\lambda = 0),$$

so the same considerations arise. As already noted, we can robustly estimate β (and also λ) in (7.1.2) using nonlinear two-stage least squares. However, the general index form (7.1.3) indicates that we may be able to estimate β up to scale whether or not the transformation of Y_i is of Box–Cox type. Note that β can be identifiable on the basis of objective functions used in other semiparametric methods such as LAD (Powell 1984), symmetrically trimmed least squares (Powell 1986), semiparametric M-estimation (Horowitz 1988), and semiparametric least squares (Horowitz 1986, Lee 1992).

We can estimate β up to scale by the density-weighted averaged derivative statistic

$$U = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U_{ij},$$

where

$$U_{ij} = h^{-d-1} K'\left(\frac{X_i - X_j}{h}\right)(Y_i - Y_j),$$

such that $K'(u) = (\partial/\partial u)K(u)$, where $K : R^d \to R$ is a differentiable (kernel) function such that $\int_{R^d} K(u)\,du = 1$, and $h = h_n$ is a positive (bandwidth or smoothing) sequence which tends to zero slowly as $n \to \infty$. For an unknown scalar c, $n^{1/2}(U - c\beta)$ was shown to be asymptotically normal when the Y_i, X_i are iid (Powell, Stock, and Stoker 1989) and when they are weakly dependent (Robinson 1989), and to be possibly asymptotically nonnormal in case of an element of long-range dependence (Cheng and Robinson 1994).

Thus, in case of the Tobit model (7.1.1), for example, U achieves the same rate of convergence as that of the ML estimate established by Amemiya (1973) where the ϵ_i are normal, and by Robinson (1982) where the ϵ_i are normal but actually weakly dependent. (Robinson (1982) also established consistency when the ϵ_i are long-range dependent normal.) On the other hand, the smoothing entailed in U might be expected to produce inferior higher-order asymptotic properties, since these more closely approximate the finite-sample situation. We know of no explicit treatment of higher-order properties of the Tobit MLE (or of the Box–Cox estimates we have mentioned), but general results of Pfanzagl (1971, 1973), Bhattacharya and Ghosh (1978), and Linton (1996) suggest that, under suitable conditions, they are likely to have an $O(n^{-1/2})$ Berry–Esseen bound (uniform rate of convergence to normality) and valid Edgeworth expansion in powers of $n^{-1/2}$, and Robinson (1991) established a Berry–Esseen bound for an optimal version of Amemiya's (1977) nonlinear three-stage least squares estimate. Robinson (1995) showed that while in general U has a Berry–Esseen bound of order greater than $n^{-1/2}$, it can be implemented (using suitable h and K) to have an $O(n^{-1/2})$ bound. Correspondingly, Nishiyama and Robinson (2000) (hereafter NR) established that the leading Edgeworth expansion term is $O(n^{-1/2})$ or larger.

Theorems 1 and 2 of NR established valid theoretical and empirical Edgeworth expansions of $Z = n^{-1/2}\sigma_\nu^{-1}\nu^\tau(U - c\beta)$ for any $d \times 1$ vector ν, where $\sigma_\nu^2 = \nu^\tau \Sigma \nu$ and Σ is the asymptotic variance matrix of $n^{1/2}(U - c\beta)$. Of course Σ is unknown, so that these Edgeworth expansions fall short of being operational. For a consistent estimate, $\hat{\Sigma}$, of Σ, we are led to consideration of $\hat{Z} = n^{1/2}\hat{\sigma}_\nu^{-1}\nu^\tau(U - c\beta)$, where $\hat{\sigma}_\nu^2 = \nu^\tau \hat{\Sigma} \nu$. NR in fact proposed such a (jackknife) estimate $\hat{\Sigma}$, and reported valid theoretical and empirical Edgeworth expansions for $\hat{Z}$ in their Theorems 3 and 4. NR also derived a choice of h that is optimal in the sense of minimizing the maximal deviation of Edgeworth correction terms from the normal approximation, and proposed also a consistent estimate of the scale factor of this, leading to a feasible approximately optimal h. NR also reported a Monte Carlo examination of their Edgeworth expansions, and of their bandwidth choice proposal. However, NR did not include the proofs of their Theorems 3 and 4, which entail additional regularity conditions and a considerable and lengthy development beyond that of their Theorems 1 and 2. By marked contrast with the routine application of Slutsky's lemma, which is all that is needed to deduce asymptotic normality of $\hat{Z}$ from that of Z, the Edgeworth expansions for $\hat{Z}$ involve considerable extra work and actually differ from those for Z. The present chapter fills this gap, by providing the proofs of NR's Theorems 3 and 4, while taking for granted the proofs of their Theorems 1 and 2. Callaert and Veraverbeke (1981) and Helmers (1985, 1991) have established higher-order asymptotics for studentized versions of standard U-statistics. Though we follow their broad approach, our U is a U-statistic with an n-dependent "kernel" (through h) which significantly complicates matters, whence we must also make substantial use of lemmas established by Robinson (1995) and NR.

The following section presents regularity conditions and theorem statements. Section 3 contains the main details of the proofs, with some detailed technical material left to appendices.

2 Theoretical and empirical Edgeworth expansions

Our conditions below imply that X has a probability density, $f(x)$, and the existence of the conditional moments $g = E(Y \mid X), q = E(Y^2 \mid X), r = E(Y^3 \mid X)$, where, for a function $h: R^d \to R$, we write $h = h(X)$. For such a function, suitably smooth, we define $h' = (\partial/\partial X)h(X)$, $h'' = (\partial/\partial X^\tau)h'(X)$, and $h''' = (\partial/\partial X^\tau)\,\mathrm{vec}(h'')$. Write $e = fg$, $\mu = \mu(X, Y) = Yf' - e'$, $a = g'f - E(g'f)$, $\mu\cdot = E(\mu) = -E(g'f)$, and $\Sigma = 4\,\mathrm{Var}(\mu)$. We introduce the following assumptions.

(i) $E(Y^6) < \infty$.

(ii) Σ is finite and positive definite.

(iii) The underlying measure of (X^τ, Y) can be written as $\mu_X \times \mu_Y$, where μ_X and μ_Y are Lebesgue measure on R^d and R respectively, and (X_i^τ, Y_i) are iid observations on (X^τ, Y).

(iv) f is $L+1$ times differentiable, and f and its first $L+1$ derivatives are bounded, for $2L > d+2$.

(v) g is $L+1$ times differentiable, and e and its first $L+1$ derivatives are bounded, for $L \geq 1$.

(vi) q is twice differentiable, and $q', q'', g', g'', g''', E(|Y|^3 \mid X)f$, and qf' are bounded.

(vii) $f, gf, g'f$, and qf vanish on the boundaries of their convex (possibly infinite) supports.

(viii) $K(u)$ is even and differentiable,

$$\int_{R^d} \{(1+\|u\|^L)|K(u)| + \|K'(u)\|\}\, du + \sup_{u\in R^d} \|K'(u)\| < \infty,$$

and for the same L as in (iv),

$$\int_{R^d} u_1^{l_1}\cdots u_d^{l_d} K(u)\, du \begin{cases} =1 & \text{if } l_1+\cdots+l_d=0, \\ =0 & \text{if } 0<l_1+\cdots+l_d<L, \\ \neq 0 & \text{if } l_1+\cdots+l_d=L. \end{cases}$$

(ix) $(\log n)^9/nh^{d+2} + nh^{2L} \to 0$ as $n \to \infty$.

(x) $\sup_{\nu^\tau\nu=1} \limsup_{|t|\to\infty} |E\ \exp\{it\sigma_\nu^{-1}\nu^\tau(\mu-\mu.)\}| < 1$.

These assumptions are the same as those of Theorem 1 of NR except that (i) strengthens their third-moment assumption to sixth moments, our treatment of studentization requiring finite third moments of certain squared terms.

In our studentized statistic $\hat{Z}$ we take

$$\hat{\Sigma} = \frac{4}{(n-1)(n-2)^2}\sum_{i=1}^{n}\left\{\sum_{j\neq i}^{n}(U_{ij}-U)\right\}\left\{\sum_{k\neq i}^{n}(U_{ik}-U)^\tau\right\}, \tag{7.2.1}$$

a jackknife estimate of Σ. We are concerned with approximating

$$\hat{F}(z) = P(\hat{Z} \le z)$$

by the Edgeworth expansion

$$F^+(z) = \Phi(z) - \phi(z) \times \left[n^{1/2}h^L\kappa_1 - \frac{\kappa_2}{nh^{d+2}}z - \frac{4}{3n^{1/2}}\{(2z^2+1)\kappa_3 + 3(z^2+1)\kappa_4\}\right],$$

where $\Phi(z)$ and $\phi(z)$ are respectively the distribution function and density function of the standard normal, and, with

$$\Delta^{(l_1,\ldots,l_d)} = \frac{\partial^{l_1+\cdots+l_d}}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}},$$

$$\kappa_1 = \frac{2(-1)^L\sigma_\nu^{-1}}{L!}\sum_{\substack{0\le l_1,\ldots,l_d\le L \\ l_1+\cdots+l_d=L}}\cdots\sum\left\{\int\prod_{i=1}^{d}u_i^{l_i}K(u)\,du\right\}E\left[\left(\Delta^{(l_1,\ldots,l_d)}\nu^\tau f'\right)g\right],$$

$$\kappa_2 = 2\sigma_\nu^{-2}\int\{\nu^\tau K'(u)\}^2\,du\ E\{(q-g^2)f\},$$

$$\kappa_3 = \sigma_\nu^{-3}E[\{r-3(q-g^2)g-g^3\}(\nu^\tau f')^3 - 3(q-g^2)(\nu^\tau f')^2(\nu^\tau a) - (\nu^\tau a)^3],$$

$$\kappa_4 = -\sigma_\nu^{-3}E[f(q-g^2)(\nu^\tau f')(\nu^\tau a'\nu) - f(\nu^\tau f')\{\nu^\tau(q'-2gg')\}(\nu^\tau a) - f(q-g^2)(\nu^\tau a)(\nu^\tau f''\nu) + f(\nu^\tau g')(\nu^\tau a)^2].$$

Theorem A. *Under assumptions* (i)–(x), *as* $n \to \infty$,

$$\sup_{\nu:\nu^{\tau}\nu=1} \sup_{z} |\hat{F}(z) - F^{+}(z)| = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}).$$

The correction terms in $F^{+}(z)$ are of the same orders as those in the unstudentized case (see. Theorem 1 of NR), though their coefficients are mostly different.

The κ_i are unknown, but a feasible, empirical Edgeworth expansion is

$$\hat{F}^{+}(z) = \Phi(z) - \phi(z)\left[n^{1/2}h^{L}\tilde{\kappa}_1 - \frac{\tilde{\kappa}_2}{nh^{d+2}}z - \frac{4}{3n^{1/2}}\{(2z^2+1)\tilde{\kappa}_3 + 3(z^2+1)\tilde{\kappa}_4\}\right],$$

where

$$\tilde{\kappa}_1 = \frac{2(-1)^{L}\hat{\sigma}_{\nu}^{-1}}{L!} \sum_{\substack{0\le l_1,\ldots,l_d \le L \\ l_1+\cdots+l_d=L}} \cdots \sum \left\{ \int \prod_{i=1}^{d} u_i^{l_i} K(u)\, du \right\}$$
$$\times \frac{1}{n}\sum_{i=1}^{n} \{\Delta^{(l_1,\ldots,l_d)}\nu^{\tau}\bar{f}'(X_i)\} Y_i,$$

$$\tilde{\kappa}_2 = \hat{\sigma}_{\nu}^{-2}\binom{n}{2}^{-1} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} h^{d+2}\bar{W}_{ij}^{2}, \qquad \hat{\kappa}_3 = \frac{\hat{\sigma}_{\nu}^{-3}}{n}\sum_{i=1}^{n}\bar{V}_i^{3},$$

$$\hat{\kappa}_4 = \frac{\hat{\sigma}_{\nu}^{-3}}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}^{n} \nu^{\tau}U_{ij}\bar{V}_i\bar{V}_j,$$

where for positive b and a function $H : R^d \to R$

$$\bar{f}(X_i) = \frac{1}{(n-1)b^d}\sum_{j\neq i}^{n} H\left(\frac{X_i - X_j}{b}\right),$$

and

$$\bar{U}_i = \frac{1}{n-1}\sum_{j\neq i}^{n} U_{ij}, \quad \bar{V}_i = \nu^{\tau}(\bar{U}_i - U), \quad \bar{W}_{ij} = \nu^{\tau}(U_{ij} - \bar{U}_i - \bar{U}_j + U). \tag{7.2.2}$$

We impose the following additional assumptions, which are identical to those of Theorem 2 of NR:

(iv′) f is $L+2$ times differentiable, and f and its first $L+2$ derivatives are bounded, where $2L > d+2$.
(v′) g is $L+2$ times differentiable, and e and its first $L+2$ derivatives are bounded.

(ix′) $(\log n)^9/nh^{d+3} + nh^{2L} \to 0$ as $n \to \infty$.
(xi′) $H(u)$ is even and $L+1$ times differentiable, and

$$\int_{R^d} H(u)\,du = 1,$$

$$\int_{R^d} \left\|\Delta^{(l_1,\ldots,l_d)} H'(u)\right\| du + \sup_{u\epsilon R^d} \left\|\Delta^{(l_1,\ldots,l_d)} H'(u)\right\| < \infty$$

for any integers $l_1, \ldots, l_d$ satisfying $0 \le l_1 + \cdots + l_d \le L$ and $0 \le l_i \le L, i = 1, \ldots, d$.
(xii) $b \to 0$ and $(\log n)^2/nb^{d+2+2L} = O(1)$ as $n \to \infty$.

Theorem B. *Under assumptions* (i)–(iii), (iv′), (v′), (vi)–(viii), (ix′), *and* (x)–(xii),

$$\sup_{\nu:\nu^\tau\nu=1} \sup_z |\hat{F}(z) - \hat{F}^+(z)| = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) \qquad a.s.$$

3 Proof of Theorems A and B

Proof of Theorem A: In the sequel, C denotes a generic, finite, positive constant and the qualification "for sufficiently large n" may be omitted.

As is standard in U-statistic theory, we write

$$\begin{aligned} n^{1/2}\sigma_\nu^{-1}\nu^\tau(U - \mu.) &= \frac{2}{\sqrt{n}}\sum_{i=1}^{n} V_i + n^{1/2}\binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} W_{ij} + n^{1/2}\sigma_\nu^{-1}\nu^\tau(EU - \mu.) \\ &= \bar{V} + \bar{W} + \Delta. \end{aligned} \tag{7.3.1}$$

where $U_i = E(U_{ij} \mid i)$, $V_i = \sigma_\nu^{-1}\nu^\tau(U_i - EU)$, and $W_{ij} = \sigma_\nu^{-1}\nu^\tau(U_{ij} - EU) - V_i - V_j$, such that $E(\cdot \mid i_1, \ldots, i_r) = E(\cdot \mid (X_{i_j}, Y_{i_j}), j = 1, \ldots, r)$. Writing $S = 4\,\mathrm{Var}(U_i)$, $s^2 = \sigma_\nu^{-2}\nu^\tau S\nu$, Taylor's theorem gives

$$\begin{aligned} \sigma_\nu\hat{\sigma}_\nu^{-1} &= s^{-1} - \frac{s^{-3}}{2}\left(\hat{\sigma}_\nu^{-2}\hat{\sigma}_\nu^2 - s^2\right) + \frac{3}{8}\left\{s^2 + \theta\left(\hat{\sigma}_\nu^{-2}\hat{\sigma}_\nu^2 - s^2\right)\right\}^{-5/2}\left(\hat{\sigma}_\nu^{-2}\hat{\sigma}_\nu^2 - s^2\right)^2 \\ &= s^{-1} + \tilde{R} + \tilde{\tilde{R}} \end{aligned} \tag{7.3.2}$$

for some $\theta\epsilon[0, 1]$. Similarly to Callaert and Veraverbeke (1981), we expand $\tilde{R}$ as follows. With $\tilde{V}_i = E(V_j W_{ij} \mid i)$, $\tilde{W}_{jk} = E(W_{ij} W_{ik} \mid j, k)$, we have

$$\tilde{R} = T + Q + R, \qquad T = T_1 + T_2 + T_3, \qquad Q = Q_1 + Q_2,$$

$$R = R_1 + R_2 + R_3 + R_4 + R_5$$

where

$$T_1 = \frac{4\delta n}{(n-2)^2} E\left(W_{12}^2\right), \qquad T_2 = \frac{\delta}{n} \sum_{i=1}^{n} \left\{ \left(4V_i^2 - S^2\right) + 8\tilde{V}_i \right\},$$

$$T_3 = 4\delta \binom{n-1}{2}^{-1} \sum_{i<j}^{n} \tilde{W}_{ij},$$

$$Q_1 = 4\delta \binom{n}{2}^{-1} \sum_{i<j} \{(V_i + V_j) W_{ij} - \tilde{V}_i - \tilde{V}_j\},$$

$$Q_2 = -\frac{8\delta}{n} \binom{n-1}{2}^{-1} \sum_{i=1}^{n} \sum_{k<m}^{n}{}^{(i)} V_i W_{km},$$

$$R_1 = -4\delta \binom{n}{2}^{-1} \sum_{i<j} V_i V_j,$$

$$R_2 = \frac{4\delta}{n-2} \binom{n-1}{2}^{-1} \sum_{i=1}^{n} \sum_{k<m}^{n}{}^{(i)} (W_{ik} W_{im} - \tilde{W}_{km}),$$

$$R_3 = \frac{4\delta n}{(n-2)^2} \binom{n}{2}^{-1} \sum_{i<j} \left\{ W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E\left(W_{12}^2\right) \right\},$$

$$R_4 = \frac{8\delta}{(n-2)^2} \sum_{i=1}^{n} \left\{ \tilde{W}_{ii} - E\left(W_{12}^2\right) \right\},$$

$$R_5 = -\frac{4\delta n(n-1)}{(n-2)^2} \left\{ \binom{n-1}{2}^{-1} \sum_{i<j} W_{ij} \right\}^2,$$

where $\delta = -s^{-3}/2$, and

$$\sum_{k<m}^{n}{}^{(i)}$$

denotes summation with respect to k and m for $1 \leq k < m \leq n$ excluding $k = i$ and $m = i$. Because

$$\hat{Z} = (s^{-1} + \tilde{R} + \tilde{\tilde{R}})(\bar{V} + \bar{W} + \Delta),$$

by a standard inequality

$$\begin{aligned} \sup_z |\hat{F}(z) - F^+(z)| \leq \sup_z |P((s^{-1} + T + Q)(\bar{V} + \bar{W}) + s^{-1}\Delta \leq z) - F^+(z)| \\ + P(|(R + \tilde{\tilde{R}})(\bar{V} + \bar{W} + \Delta) + (T + Q)\Delta| \geq a_n) + O(a_n) \end{aligned} \tag{7.3.3}$$

for $a_n > 0$; here and subsequently we drop reference to $\sup_{\nu:\nu'\nu=1}$. Taking

$a_n = (1/\log n)\max(n^{-1/2}, n^{-1}h^{-d-2}, n^{1/2}h^L)$, we bound the second term on the right of (7.3.3) by

$$P\left(|(R+\tilde{\tilde{R}})(\bar{V}+\bar{W}+\Delta)| \geq \frac{a_n}{2}\right) + P\left(|(T+Q)\Delta| \geq \frac{n^{1/2}h^L}{2\log n}\right)$$

$$\leq P\left(|R+\tilde{\tilde{R}}| \geq \frac{a_n}{2\log n}\right) + P(|\bar{V}+\bar{W}+\Delta| \geq \log n)$$

$$+ P\left(|(T+Q)\Delta| \geq \frac{n^{1/2}h^L}{2\log n}\right). \tag{7.3.4}$$

The first term in (7.3.4) is, by an elementary inequality, bounded by

$$P\left(|R| \geq \frac{a_n}{4\log n}\right) + P\left(\frac{|\tilde{\tilde{R}}|}{\tilde{R}^2} \geq C_0\right) + P\left(\tilde{R}^2 \geq \frac{a_n}{4C_0\log n}\right) \tag{7.3.5}$$

for a constant C_0 determined later. The third term of (7.3.5) is bounded by

$$P\left(T_2^2\frac{a_n}{12C_0\log n}\right) + P\left(|T_1+T_3|^2 \geq \frac{a_n}{12C_0\log n}\right) + P\left(|Q+R|^2 \geq \frac{a_n}{12C_0\log n}\right)$$

$$= \quad \text{(a)} \quad + \quad \text{(b)} \quad + \quad \text{(c)}.$$

Lemmas 10–19 and Markov's inequality give, for $\zeta > 0$,

$$\text{(a)} \leq \frac{E|T_2|^{2(1+\zeta)}}{\left(\frac{a_n}{12C_0\log n}\right)^{1+\zeta}} \leq \frac{Cn^{-(1+\zeta)}(\log n)^{2(1+\zeta)}}{n^{-\frac{1}{2}(1+\zeta)}} = o(n^{-1/2}),$$

$$\text{(b)} \leq \frac{E|T_1+T_3|^{2(1+\zeta)}}{\left(\frac{a_n}{12C_0\log n}\right)^{1+\zeta}} \leq \frac{C(n^{-1}h^{-d-2})^{2(1+\zeta)}(\log n)^{2(1+\zeta)}}{(n^{-1}h^{-d-2})^{1+\zeta}}$$

$$= o(n^{-1}h^{-d-2}),$$

$$\text{(c)} \leq \frac{E|R+Q|^2}{\frac{a_n}{12C_0\log n}} \leq \frac{Cn^{-2}h^{-d-2}(\log n)^2}{n^{-1/2}} = o(n^{-1}h^{-d-2}),$$

where $\zeta = \frac{2}{7}$ suffices in (b), and ζ arbitrarily small suffices in (a).

The first term of (7.3.5) is, using Markov's inequality, (ix), and Lemmas 15–19, bounded by

$$\frac{16E(R^2)(\log n)^2}{a_n^2} \leq C(n^{-1}+n^{-2}h^{-2d-4})(\log n)^4$$

$$= o(n^{-1/2}+n^{-1}h^{-d-2}).$$

Now, in view of (7.3.2), $\tilde{\tilde{R}} = \frac{3}{2}s(1-2\theta s\tilde{R})^{-5/2}\tilde{R}^2$ so that because $\tilde{\tilde{R}} \geq 0$ and

$0 \le \theta \le 1$,

$$P\left(\frac{\tilde{\tilde{R}}}{\tilde{R}^2} \ge C_0\right) = P\left(\frac{3}{2}s(1 - 2\theta s\tilde{R})^{-5/2} \ge C_0\right)$$
$$\le P\left(|\tilde{R}| \ge \frac{1}{2s}\left\{1 - \left(\frac{3s}{2C_0}\right)^{2/5}\right\}\right). \qquad (7.3.6)$$

Taylor's expansion of s^r around $s^2 = 1$ and Lemma 2 of Robinson (1995) give, for integer r,

$$s^r = 1 + O\left(\sigma_\nu^{-2}\nu^\tau(S - \Sigma)\nu\right) = 1 + O(h^L), \qquad (7.3.7)$$

so that we can choose C_0 such that $C_0 > \frac{3}{2}s$ for sufficiently large n by (ii). Then by (7.3.7) and Markov's inequality, (7.3.6) is bounded by a constant times $E|T + Q + R|^3 = O(n^{-3/2} + n^{-3}h^{-3d-6})$ from Lemmas 10–19, so that the second term of (7.3.5) is $O(n^{-3/2} + n^{-3}h^{-3d-6})$. Therefore,

$$P\left(|R + \tilde{\tilde{R}}| \ge \frac{a_n}{\log n}\right) = O(n^{-1/2} + n^{-1}h^{-d-2}). \qquad (7.3.8)$$

Put $F(z) = P[n^{1/2}\sigma_\nu^{-1}\nu^\tau(U - \mu.) \le z]$. Then

$$P(|\bar{V} + \bar{W} + \Delta| \ge \log n) = 1 - F(\log n) + F(-\log n). \qquad (7.3.9)$$

NR proved in Theorem 1 that

$$\sup_z |F(z) - \tilde{F}(z)| = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L),$$

where

$$\tilde{F}(z) = \Phi(z) - \phi(z)\left\{n^{1/2}h^L\kappa_1 + \frac{\kappa_2}{nh^{d+2}}z + \frac{4(\kappa_3 + 3\kappa_4)}{3n^{1/2}}(z^2 - 1)\right\}, \qquad (7.3.10)$$

which implies that for any z

$$1 - F(z) + F(-z) = 1 - \tilde{F}(z) + \tilde{F}(-z) + o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L). \qquad (7.3.11)$$

Now by (7.3.10),

$$1 - \tilde{F}(z) + \tilde{F}(-z) = 1 - \Phi(z) + \Phi(-z) + \phi(z)\frac{2\kappa_2}{nh^{d+2}}z$$
$$= 2 - 2\Phi(z) + \phi(z)\frac{2\kappa_2}{nh^{d+2}}z. \qquad (7.3.12)$$

Substituting (7.3.12) into (7.3.11) and putting $z = \log n$, because $1 - \Phi(\log n) = o(n^{-1/2})$ and $\phi(\log n)\log n = o(n^{-1/2})$, we have

$$1 - F(\log n) + F(-\log n) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L). \qquad (7.3.13)$$

By (7.3.9) and (7.3.13),

$$P(|\bar{V} + \bar{W} + \Delta| \geq \log n) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}). \tag{7.3.14}$$

Finally, Markov's inequality, (ix), Lemma 1 of Robinson (1995), and Lemmas 10–14 bound the last term of (7.3.4) by

$$\begin{aligned}\frac{\Delta^2 E\,|T + Q|^2 (2\log n)^2}{nh^{2L}} &\leq C(n^{-1} + n^{-2}h^{-2d-4})(\log n)^2 \\ &= o(n^{-1/2} + n^{-1}h^{-d-2}). \end{aligned} \tag{7.3.15}$$

Substituting (7.3.8), (7.3.14), and (7.3.15) into (7.3.4),

$$\begin{aligned}&P(|(R + \tilde{\tilde{R}})(\bar{V} + \bar{W} + \Delta) + (T + Q)\Delta| \geq a_n) \\ &\quad = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^{L}). \end{aligned} \tag{7.3.16}$$

To deal with the first term on the right of (7.3.3), write $b_2 = s^{-1}\bar{V}$, $b_3 = s^{-1}\bar{W}$, $\tilde{b}_2 = (T + Q)\bar{V}$, $\tilde{b}_3 = (T + Q)\bar{W}$, $b_1 = b_2 + b_3$, $\tilde{b}_1 = \tilde{b}_2 + \tilde{b}_3$, $B = b_1 + \tilde{b}_1$, and define

$$\begin{aligned}\chi^+(t) &= \int e^{itz}\, dF^+(z) \\ &= e^{-t^2/2}\left[1 + \left\{n^{1/2}h^{L}\kappa_1 - \frac{4(\kappa_3 + 2\kappa_4)}{n^{1/2}}\right\}(it)\right. \\ &\quad \left. - \frac{\kappa_2}{nh^{d+2}}(it)^2 - \frac{4(2\kappa_3 + 3\kappa_4)}{3n^{1/2}}(it)^3\right].\end{aligned}$$

Essen's smoothing lemma gives, for $N_0 = \log n \min(\eta n^{1/2}, nh^{d+2})$, $\eta = (E|2s^{-1}V_1|^3)^{-1}$,

$$\begin{aligned}&\sup_z |P((s^{-1} + T + Q)(\bar{V} + \bar{W}) + s^{-1}\Delta \leq z) - F^+(z)| \\ &\quad \leq \int_{-N_0}^{N_0} \left|\frac{Ee^{it(B + s^{-1}\Delta)} - \chi^+(t)}{t}\right| dt + O(N_0^{-1}),\end{aligned}$$

which, for $p = \min(\log n, \varepsilon n^{1/2}, nh^{d+2})$, is bounded by

$$\begin{aligned}&\int_{-p}^{p} \left|\frac{Ee^{it(B + s^{-1}\Delta)} - \chi^+(t)}{t}\right| dt + \int_{p \leq |t| \leq N_0} \left|\frac{Ee^{it(B + s^{-1}\Delta)}}{t}\right| dt \\ &\quad + \int_{|t| \geq p} \left|\frac{\chi^+(t)}{t}\right| dt + o(n^{-1/2} + n^{-1}h^{-d-2}) \\ &\quad = \text{(I)} + \text{(II)} + \text{(III)} + o(n^{-1/2} + n^{-1}h^{-d-2}).\end{aligned}$$

Here we can set $\varepsilon \in (0, \eta]$ for sufficiently large n as discussed in the proof of the theorem of Robinson (1995), using (7.3.7) also, so that $p \leq N_0$. We first mention an inequality frequently used hereafter:

$$\left| e^{ix} - 1 - ix - \cdots - \frac{(ix)^{k-1}}{(k-1)!} \right| \leq \frac{|x|^k}{k!} \tag{7.3.17}$$

for integer k and real x.

Estimation of (I): Since $s^{-1}\Delta$ is nonstochastic,

$$E\{e^{it(B+s^{-1}\Delta)}\} = e^{its^{-1}\Delta} E(e^{itB}), \tag{7.3.18}$$

where (7.3.7) and (7.3.17) yield

$$e^{its^{-1}\Delta} = 1 + it\Delta + O(t^2\Delta^2 + |t|h^L\Delta). \tag{7.3.19}$$

Writing $\tilde{b}_2 = \tilde{b}_2' + \tilde{b}_2''$, where $\tilde{b}_2' = T\bar{V}$ and $\tilde{b}_2'' = Q\bar{V}$, and applying (7.3.17) repeatedly, we have

$$\begin{aligned} E(e^{itB}) &= E(e^{itb_1}) + E(e^{itB} - e^{itb_1}) \\ &= E(e^{itb_1}) + \{Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3)} - Ee^{it(b_1+\tilde{b}_2')}\} + \{Ee^{it(b_1+\tilde{b}_2')} - E(e^{itb_1})\} \\ &= E(e^{itb_1}) + O(|t|E|\tilde{b}_2'' + \tilde{b}_3|) + \{Ee^{it(b_1+\tilde{b}_2')} - E(e^{itb_1}) \\ &\quad - it\, E(\tilde{b}_2' e^{itb_1})\} + itE(\tilde{b}_2' e^{itb_1} - \tilde{b}_2' e^{itb_2}) + itE(\tilde{b}_2' e^{itb_2}) \\ &= E(e^{itb_1}) + itE(\tilde{b}_2' e^{itb_2}) + O(|t|E|\tilde{b}_2'' + \tilde{b}_3| + t^2(E|\tilde{b}_2'|^2 + E|\tilde{b}_2'\tilde{b}_3|)). \end{aligned} \tag{7.3.20}$$

Write

$$E(e^{itb_1}) = E\left[e^{itb_2}\left\{1 + itb_3 + \frac{(it)^2}{2}b_3^2\right\}\right] + O(|t|^3 E|b_3|^3), \tag{7.3.21}$$

and put $\gamma(t) = E(e^{it(2/\sqrt{ns})V_1})$. As in Appendix A of NR,

$$E(e^{itb_2}) = \{\gamma(t)\}^n, \tag{7.3.22}$$

$$\begin{aligned} E(b_3 e^{itb_2}) = \{\gamma(t)\}^{n-2}\Bigg[\frac{4(it)^2}{n^{1/2}} E(W_{12}V_1V_2) \\ + O\left(\frac{t^2h^L}{n^{1/2}} + \left(\frac{t^4}{n^{3/2}} + \frac{|t|^3}{n}\right)h^{-\frac{2}{3}d-1}\right)\Bigg], \end{aligned} \tag{7.3.23}$$

$$\begin{aligned} E\left(b_3^2 e^{itb_2}\right) = \frac{2}{n-1}\{\gamma(t)\}^{n-2}\left[E\left(W_{12}^2\right) + O\left(\frac{h^L}{nh^{d+2}}n + |t|n^{-1/2}h^{-\frac{4}{3}d-2}\right)\right] \\ + \{\gamma(t)\}^{n-3}O\left(|t|n^{-3/2}h^{-\frac{4}{3}d-2}\right) \\ + \{\gamma(t)\}^{n-4}O\left(t^4n^{-1} + t^8n^{-3}h^{-\frac{4}{3}d-2} + t^6n^{-2}h^{-\frac{4}{3}d-2}\right). \end{aligned} \tag{7.3.24}$$

Since, for $m = 0, 1, 2, 3,$

$$\{\gamma(t)\}^{n-m} = e^{-t^2/2}\left\{1 + \frac{E(2V_1)^3}{6n^{1/2}s^3}(it)^3\right\} + o\left(n^{-1/2}(|t|^3 + t^6)e^{-t^2/4}\right), \tag{7.3.25}$$

by Lemma 1 of Robinson (1995), (a) in Appendix B, and (7.3.18)–(7.3.25),

$$\begin{aligned} E\{e^{it(B+s^{-1}\Delta)}\} &= \{1 + it\Delta + O(t^2nh^{2L} + |t|n^{1/2}h^{2L})\} \\ &\times \left\{\left[e^{-t^2/2}\left\{1+\frac{4E(2V_1^3)}{3n^{1/2}s^3}(it)^3\right\} + o\left(n^{-1/2}(|t|^3+t^6)e^{-t^2/4}\right)\right]\right. \\ &\times \left[1 + \frac{4(it)^3}{n^{1/2}}E(W_{12}V_1V_2) + \frac{(it)^2}{n}E(W_{12}^3) - \frac{2(it)^2}{n}E(W_{12}^2)\right. \\ &\left. - \frac{(it)^3 + (it)}{n^{1/2}}\{4E(V_1^3) + 8E(W_{12}V_1V_2)\} + O(A_n)\right] \\ &\left. + (|t|E|\tilde{b}_2'' + \tilde{b}_3| + t^2(E|\tilde{b}_2'|^2 + E|\tilde{b}_2'b_3|))\right\}, \end{aligned} \tag{7.3.26}$$

where

$$\begin{aligned} A_n &= \frac{|t|^3h^L}{n^{1/2}} + \left(\frac{|t|^5}{n^{3/2}} + \frac{t^4}{n}\right)h^{-\frac{2}{3}d-1} + \frac{t^2}{n^2h^{d+2}} + \frac{t^2h^L}{n^2h^{d+2}} \\ &+ \frac{|t|^3}{n^{3/2}h^{\frac{4}{3}d+2}} + \frac{t^6}{n} + \frac{t^{10}}{n^3h^{\frac{4}{3}d+2}} + \frac{t^8}{n^2h^{\frac{4}{3}d+2}} + \frac{|t|^3}{(nh^{d+2})^{3/2}} \\ &+ \frac{t^2}{n^2h^{d+2}} + \frac{|t|^3}{n^{3/2}h^{d+2}} + \frac{t^2h^L}{nh^{d+2}} + \frac{t^2}{n} + \frac{|t|^3 + t^4}{n} + \frac{|t|^3}{n^{3/2}} \\ &+ \frac{|t|^7}{n^3h^{d+2}} + \frac{t^6}{n^{5/2}h^{d+2}} + \frac{|t|^5 + t^4}{n^2h^{d+2}} \\ &= o\left(\frac{t^2 + t^{10}}{nh^{d+2}} + \frac{t^2 + t^6}{n^{1/2}}\right). \end{aligned}$$

Expanding (7.3.26), we have

$$\begin{aligned} E\{e^{it(B+s^{-1}\Delta)}\} &= e^{-t^2/2}\left[1 + \left\{\Delta - \frac{4E(V_1^3) + 8E(W_{12}V_1V_2)}{n^{1/2}}\right\}(it)\right. \\ &\left. - \frac{E(W_{12}^2)}{n}(it)^2 - \frac{4\{2E(V_1^3) + 3E(W_{12}V_1V_2)\}}{3n^{1/2}}(it)^3\right] + D_n, \end{aligned} \tag{7.3.27}$$

where

$$
\begin{aligned}
D_n = O\Bigg(&\left\{e^{-t^2/2}\frac{|t|^3}{n^{1/2}} + o\left(n^{-1/2}(t^6+|t|^3)e^{-t^2/4}\right)\right\}\left\{\frac{t^2}{n^2h^{d+2}} + \frac{|t|^3+|t|}{n^{1/2}} + A_n\right\} \\
&+ e^{-t^2/2}(|t|n^{1/2}h^L + t^2h^{2L} + |t|n^{1/2}h^{2L})\left(\frac{t^2}{n^2h^{d+2}} + \frac{|t|^3+|t|}{n^{1/2}} + A_n\right) \\
&+ (|t|n^{1/2}h^L + t^2nh^{2L})\left\{e^{-t^2/2}\frac{|t|^3}{n^{1/2}} + o\left(n^{-1/2}(t^6+|t|^3)e^{-t^2/4}\right)\right\} \\
&+ (|t|n^{1/2}h^L + t^2nh^{2L})\left\{e^{-t^2/2}\frac{|t|^3}{n^{1/2}} + o\left(n^{-1/2}(t^6+|t|^3)e^{-t^2/4}\right)\right\} \\
&\times\left(\frac{t^2}{n^2h^{d+2}} + \frac{|t|^3+|t|}{n^{1/2}} + A_n\right) + (|t| + t^2n^{1/2}h^L + |t|^3nh^{2L}) \\
&\times E|\tilde{b}_2'' + \tilde{b}_3| + (t^2 + |t|^3n^{1/2}h^L + t^4nh^{2L})(E|\tilde{b}_2'|^2 + E|\tilde{b}_2'b_3|)\Bigg).
\end{aligned}
\tag{7.3.28}
$$

By Hölder's inequality, equation (14) of Robinson (1995), and Lemmas 9–14,

$$
E|\tilde{b}_2''| = E|Q\bar{V}| \le (E|Q|^2E|\bar{V}|^2)^{1/2} = O\left(n^{-1}h^{-(d+2)/2}\right), \tag{7.3.29}
$$

$$
\begin{aligned}
E|\tilde{b}_3| &= E|(T+Q)\bar{W}| \le (E|T+Q|^2E|\bar{W}|^2)^{1/2} \\
&= O((n^{-1/2} + n^{-1}h^{-d-2})(n^{-1}h^{-d-2})^{1/2}).
\end{aligned}
\tag{7.3.30}
$$

Writing $E|\tilde{b}_2'|^2 \le C(|T_1|^2E|\bar{V}|^2 + E|T_2\bar{V}|^2 + E|T_3\bar{V}|^2)$, Lemmas 9, 10, and 12 and Hölder's inequality give

$$
|T_1|^2E|\bar{V}|^2 + E|T_3\bar{V}|^2 \le |T_1|^2E|\bar{V}|^2 + (E|T_3|^4E|\bar{V}|^4)^{1/2} = O(n^{-2}h^{-2d-4}),
$$

and (7.3.7), (i), (iii), Lemma 1(d) of NR, and (7.A.1) give

$$
\begin{aligned}
E|T_2\bar{V}|^2 &\le \frac{C}{n^3}E\left|\sum_{i=1}^{n}\left(4V_i^2 - s^2 + 8\bar{V}_i\right)\sum_{j=1}^{n}V_j\right|^2 \\
&= \frac{C}{n^3}\{nE\left|\left(4V_1^2 - s^2 + 8\bar{V}_1\right)V_1\right|^2 \\
&\quad + n(n-1)E\left|\left(4V_1^2 - s^2 + 8\bar{V}_1\right)V_2\right|^2\} \\
&= O(n^{-1}).
\end{aligned}
$$

Thus

$$
E|\tilde{b}_2'|^2 = E|T\bar{V}|^2 = O(n^{-1} + n^{-2}h^{-2d-4}). \tag{7.3.31}
$$

Hölder's inequality, (7.3.31), and equation (14) of Robinson (1995) yield

$$E|\tilde{b}_2' b_3| = \left(E|\tilde{b}_2'|^2 E|b_3|^2\right)^{1/2} = O((n^{-1/2} + n^{-1}h^{-d-2})(n^{-1}h^{-d-2})^{1/2}). \tag{7.3.32}$$

It is straightforward due to (C.1) of NR that $E(V_1^3) = E(v_1^3) + o(1)$ and $E(W_{12}V_1V_2) = E(W_{12}v_1v_2) + o(1)$, where $v_i = \sigma_v^{-1}\nu^\tau\{\mu(X_i, Y_i) - \mu.\}$. Therefore, using (7.3.27)–(7.3.32) and Lemmas 11–13 of NR,

$$\text{(I)} \le \int_{-\log n}^{\log n} \left| \frac{Ee^{it(B+s^{-1}\Delta)} - \chi^+(t)}{t} \right| dt = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L).$$

Estimation of (II): Put $\tilde{b}_3' = T\bar{W}$, $\tilde{b}_3'' = Q\bar{W}$; then, noting that $\tilde{b}_3 = \tilde{b}_3' + \tilde{b}_3''$ and $B = b_1 + \tilde{b}_2 + (\tilde{b}_3' + \tilde{b}_3'')$, we have, using (7.3.17),

$$\begin{aligned} |Ee^{itB}| &\le \left|Ee^{itB} - Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3')} - it\, E\tilde{b}_3'' e^{it(b_1+\tilde{b}_2+\tilde{b}_3')}\right| \\ &\quad + \left|Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3')}\right| + |t|\,\left|E\tilde{b}_3'' e^{it(b_1+\tilde{b}_2+\tilde{b}_3')}\right| \\ &\le |t|^2 E|\tilde{b}_3''|^2 + \left|Ee^{it(b_1+\tilde{b}_2+\tilde{b}_3')}\right| + |t|\,\left|E\tilde{b}_3'' e^{it(b_1+\tilde{b}_2+\tilde{b}_3')}\right|. \end{aligned} \tag{7.3.33}$$

Writing $E|b_3''|^2 \le C(E|Q_1\bar{W}|^2 + E|Q_2\bar{W}|^2)$, we have from Hölder's inequality, equation (14) of Robinson (1995), and Lemma 14

$$E|Q_2\bar{W}|^2 \le \left(E|Q_2|^4\right)^{1/2}(E|\bar{W}|^4)^{1/2} = O((n^{-4}h^{-3d-4})^{1/2}n^{-1}h^{-d-2}) \tag{7.3.34}$$

and

$$\begin{aligned} E|Q_1\bar{W}|^2 &\le \frac{C}{n^7} E\left| \sum\sum_{i<j} \{(V_i + V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\} \sum\sum_{k<l} W_{kl} \right|^2 \\ &\le \frac{C}{n^7} E\left| \sum\sum\sum\sum_{i<j<k<l} \{(V_i + V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\} W_{kl} \right|^2 \\ &\quad + \frac{C}{n^7} E\left| \sum\sum\sum_{i<j<l} \{(V_i + V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\} W_{il} \right|^2 \\ &\quad + \frac{C}{n^7} E\left| \sum\sum_{i<j} \{(V_i + V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j\} W_{ij} \right|^2 \end{aligned}$$

$$\le \frac{C}{n^7}\sum\sum\sum\sum_{i<j<k<l} E|\{(V_i+V_j)W_{ij} - \tilde V_i - \tilde V_j\}W_{kl}|^2$$
$$+ \frac{C}{n^7}\sum\sum_{j>l\ge 2} n^2 E|\{(V_1+V_j)W_{1j} - \tilde V_1 - \tilde V_j\}W_{1l}|^2$$
$$+ \frac{C}{n^7} n^4 E|\{(V_1+V_2)W_{12} - \tilde V_1 - \tilde V_2\}W_{12}|^2$$
$$= O(n^{-3}h^{-3d-4}), \tag{7.3.35}$$

where the third inequality uses the Theorem of Dharmardhikari, Fabian, and Jogdeo (1968; abbreviated to DFJ hereafter), and the equality uses nested conditional expectation, Lemmas 1(d) and 4 of NR, Lemma 4 of Robinson (1995), and Lemma 2. Therefore by (7.3.34) and (7.3.35),

$$E|\tilde b_3''|^2 = E|Q\tilde W|^2 = O(n^{-3}h^{-3d-4}). \tag{7.3.36}$$

To investigate the second term of (7.3.33), let

$$d_i = \left(4V_i^2 - s^2\right) + 8\tilde V_i,\; e_{ij} = 4\left\{(V_i+V_j)W_{ij} - \tilde V_i - \tilde V_j + \frac{n}{n-2}\tilde W_{ij}\right\}; \tag{7.3.37}$$

then

$$\tilde b_2 = -\frac{s^{-3}}{2}\left\{\frac{4n}{(n-2)^2}E\left(W_{12}^2\right) + \frac{1}{n}\sum_{i=1}^{n} d_i + \binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} e_{ij}\right.$$
$$\left. - \frac{4}{n}\binom{n-1}{2}^{-1}\sum_{i=1}^{n}\sum_{k=1}^{n-1}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)} V_i W_{kl}\right\}\tilde V,$$
$$\tilde b_3' = -\frac{s^{-3}}{2}\left\{\frac{4n}{(n-2)^2}E\left(W_{12}^2\right) + \frac{1}{n}\sum_{i=1}^{n} d_i + \binom{n}{2}^{-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} 4\tilde W_{jk}\right\}\tilde W.$$

Define

$$b_{3m} = s^{-1}n^{1/2}\binom{n}{2}^{-1}\sum_{i=1}^{m}\sum_{j=i+1}^{n} W_{ij},$$

$$\tilde b_{2m} = -\frac{s^{-3}}{2}\left[\frac{8n^{1/2}}{(n-2)^2}E\left(W_{12}^2\right)\sum_{i=1}^{m} V_i + \frac{2}{n^{3/2}}\left(\sum_{i=1}^{n}\sum_{s=1}^{m} d_i V_s + \sum_{i=1}^{m}\sum_{s=m+1}^{n} d_i V_s\right)\right.$$
$$+ \frac{2}{n^{1/2}}\binom{n-1}{2}^{-1}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{m} e_{ij}V_s + \sum_{i=1}^{m}\sum_{j=i+1}^{n}\sum_{s=m+1}^{n} e_{ij}V_s\right)$$

$$
\begin{aligned}
&-\frac{8}{n^{3/2}}\binom{n-1}{2}^{-1}\left(\sum_{i=1}^{n}\sum_{k<l}^{(i)}\sum_{s=1}^{m}V_iW_{kl}V_s\right.\\
&+\sum_{i=1}^{n}\sum_{k=1}^{m}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n}V_iW_{kl}V_s\\
&\left.+\sum_{i=1}^{m}\sum_{k=m+1}^{n-1}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n}V_iW_{kl}V_s\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}'_{3m} = -\frac{s^{-3}}{2}\Bigg[&\frac{4n^{3/2}}{(n-2)^2}E(W_{12}^2)\binom{n}{2}^{-1}\sum_{l=1}^{m}\sum_{s=l+1}^{n}W_{ls}\\
&+\frac{1}{\sqrt{n}}\binom{n}{2}^{-2}\left(\sum_{i=1}^{m}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n}d_iW_{ls}+\sum_{i=m+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}d_iW_{ls}\right)\\
&+n^{1/2}\binom{n}{2}^{-2}\left(\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n}4\tilde{W}_{jk}W_{ls}\right.\\
&\left.+\sum_{j=m+1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}4\tilde{W}_{jk}W_{ls}\right)\Bigg]
\end{aligned}
$$

for $m = 1, \ldots, n-1$. Note that $b_1 - b_{3m}$, $\tilde{b}_2 - \tilde{b}_{2m}$, and $\tilde{b}'_3 - \tilde{b}'_{3m}$ are independent of $(X_1^\tau, Y_1), \ldots, (X_m^\tau, Y_m)$. Putting $\bar{B}_m = (b_1 - b_{3m}) + (\tilde{b}_2 - \tilde{b}_{2m}) + (\tilde{b}'_3 - \tilde{b}'_{3m})$, and using (7.3.17) repeatedly, we have

$$
\begin{aligned}
\left|Ee^{it(b_1+\tilde{b}_2+\tilde{b}'_3)}\right| &\le \frac{t^2}{2}E|\tilde{b}_{2m}+\tilde{b}'_{3m}|^2 + \left|Ee^{it(\bar{B}_m+b_{3m})}\right|\\
&\quad + |t|\left|Ee^{it(\bar{B}_m+b_{3m})}(\tilde{b}_{2m}+\tilde{b}'_{3m})\right|\\
&\le \frac{t^2}{2}E|\tilde{b}_{2m}+\tilde{b}'_{3m}|^2 + \left[\frac{|t|^3}{6}E|b_{3m}|^3\right.\\
&\quad \left.+\left|Ee^{it\bar{B}_m}\left\{1+itb_{3m}+\frac{(it)^2}{2}b_{3m}^2\right\}\right|\right]\\
&\quad + [t^2E|b_{3m}||\tilde{b}_{2m}+\tilde{b}'_{3m}| + |t||Ee^{it\bar{B}_m}(\tilde{b}_{2m}+\tilde{b}'_{3m})|]\\
&= \left[\frac{t^2}{2}E|\tilde{b}_{2m}+\tilde{b}'_{3m}|^2 + t^2E|b_{3m}||\tilde{b}_{2m}+\tilde{b}'_{3m}|\right]\\
&\quad + \left[\frac{|t|^3}{6}E|b_{3m}|^3 + \left|Ee^{it\bar{B}_m}\left\{1+itb_{3m}+\frac{(it)^2}{2}b_{3m}^2\right\}\right|\right]\\
&\quad + [|t||Ee^{it\bar{B}_m}(\tilde{b}_{2m}+\tilde{b}'_{3m})|]. \qquad (7.3.38)
\end{aligned}
$$

By elementary inequalities, (ix), items (d) and (e) in Appendix B, and equation (14) of Robinson (1995), the first bracketed term is bounded by

$$Ct^2\{E|\tilde{b}_{2m}|^2 + E|\tilde{b}'_{3m}|^2 + (E|b_{3m}|^2)^{1/2}(E|\tilde{b}_{2m}|^2 + E|\tilde{b}'_{3m}|^2)^{1/2}\}$$
$$\le Cmt^2\left\{\frac{1}{n^3h^{2d+4}} + \frac{1}{n^2} + \frac{1}{n^4h^{3d+6}} + \frac{1}{(n^2h^{d+2})^{1/2}}\left(\frac{1}{(n^3h^{2d+4})^{1/2}} + \frac{1}{n} + \frac{1}{(n^4h^{3d+6})^{1/2}}\right)\right\}$$
$$\le Cmt^2\left(\frac{1}{n^2h^{(d+2)/2}} + \frac{1}{n^{5/2}h^{(3d+6)/2}}\right). \tag{7.3.39}$$

The second bracketed term on the right of (7.3.38) is bounded by

$$C\left[|t|^3\left(\frac{m}{n^2h^{d+2}}\right)^{3/2} + \left\{1 + \frac{m|t|}{n^{1/2}h} + \frac{m|t|}{n^2h^{d+2}} + \frac{t^2m^2}{nh^2}\right\}|\gamma(t)|^{m-4}\right], \tag{7.3.40}$$

which is verified as in equations (13)–(19) of Robinson (1995), because s^{-1} is bounded due to (7.3.7) and $\bar{B}_m$ is the sum of $(2/\sqrt{ns})\sum_{i=1}^{n} V_i$ and $(b_3 - b_{3m}) + (\tilde{b}_2 - \tilde{b}_{2m}) + (\tilde{b}'_3 - \tilde{b}'_{3m})$, the latter being independent of $(X_1^\tau, Y_1), \ldots, (X_m^\tau, Y_m)$. Items (b) and (c) in Appendix B bound the last term in (7.3.38) by

$$\frac{Cm|t|}{n^{1/2}h^3}|\gamma(t)|^{m-4}. \tag{7.3.41}$$

Now we investigate the third term on the right of (7.3.33). Using elementary inequalities, (7.3.17), (7.3.36), equation (14) of Robinson (1995), and (d), (e), (f) of Appendix B, we have

$$\left|E\tilde{b}''_3e^{it(b_1+\tilde{b}_2+\tilde{b}'_3)}\right|$$
$$\le \left|E\tilde{b}''_3e^{it(b_1+\tilde{b}_2+\tilde{b}'_3)} - E\tilde{b}''_3e^{it(b_1-b_{3m}+\tilde{b}_2-\tilde{b}_{2m}+\tilde{b}'_3-\tilde{b}'_{3m})}\right| + |E\tilde{b}''_3e^{it\bar{B}_m}|$$
$$\le |t|E|\tilde{b}''_3||b_{3m} + \tilde{b}_{2m} + \tilde{b}'_{3m}| + |E\tilde{b}''_3e^{it\bar{B}_m}|$$
$$\le C|t|(E|\tilde{b}''_3|^2)^{1/2}[(E|b_{3m}|^2)^{1/2} + (E|\tilde{b}_{2m}|^2)^{1/2} + (E|\tilde{b}'_{3m}|^2)^{1/2}] + |E\tilde{b}''_3e^{it\bar{B}_m}|$$
$$\le \frac{C|t|h}{(nh^{d+2})^{3/2}}\left[\left(\frac{m}{n^2h^{d+2}}\right)^{1/2} + \left(\frac{m}{n^3h^{2d+4}}\right)^{1/2} + \left(\frac{m}{n^2}\right)^{1/2} + \left(\frac{m}{n^4h^{3d+6}}\right)^{1/2}\right] + \frac{Cn^{1/2}}{h^2}|\gamma(t)|^{m-5}$$
$$\le \frac{C|t|hm^{1/2}}{n^{1/2}(nh^{d+2})^2} + \frac{Cn^{1/2}}{h^2}|\gamma(t)|^{m-5}. \tag{7.3.42}$$

Therefore, by (7.3.33), (7.3.36), (7.3.38)–(7.3.42),

$$|Ee^{itB}| \le \frac{Ct^2}{n^3h^{3d+4}} + Cmt^2\left(\frac{1}{n^2h^{(d+2)/2}} + \frac{1}{n^{5/2}h^{(3d+6)/2}}\right)$$
$$+ C\left[|t|^3\left(\frac{m}{n^2h^{d+2}}\right)^{3/2} + \left\{1 + \frac{m|t|}{n^{1/2}h} + \frac{m|t|}{n^2h^{d+2}} + \frac{t^2m^2}{nh^2}\right\}|\gamma(t)|^{m-4}\right]$$
$$+ \frac{Cm|t|}{n^{1/2}h^3}|\gamma(t)|^{m-4} + \frac{Chm^{1/2}t^2}{n^{1/2}(nh^{d+2})^2} + \frac{Cn^{1/2}|t|}{h^2}|\gamma(t)|^{m-5}. \tag{7.3.43}$$

Now divide (7.3.43) by $|t|$ and integrate over $p \le |t| \le N_0$, where we partition the range of integration into two parts, $p \le |t| \le N_1$ and $N_1 \le |t| \le N_0$, for $N_1 = \min(\eta n^{1/2}, nh^{d+2})$:

(i) $p \le |t| \le N_1$. We can choose $m = [(9n \log n)/t^2]$ to satisfy $1 \le m \le n-1$ for large n. For this m, since $E(2V_1/s) = 0$ and $\mathrm{Var}(2V_1/s) = 1$,

$$|\gamma(t)|^{m-4} \le \exp\left(-\frac{m-4}{3n}t^2\right) \le C\exp(-3\log n) = \frac{C}{n^3}. \tag{7.3.44}$$

By (7.3.43), (7.3.44), and (ix), we obtain

$$\int_{p\le|t|\le N_1}\left|\frac{Ee^{itB}}{t}\right| dt$$
$$\le \frac{C}{n^3h^{3d+4}}\int_{p\le|t|\le nh^{d+2}}|t|\,dt$$
$$+ C\left(\frac{n\log n}{n^2h^{(d+2)/2}} + \frac{n\log n}{n^{5/2}h^{(3d+6)/2}}\right)\int_{p\le|t|\le\eta n^{1/2}}\frac{dt}{|t|}$$
$$+ C\left(\frac{n\log n}{n^2h^{d+2}}\right)^{3/2}\int_{p\le|t|\le\eta n^{1/2}}\frac{dt}{|t|}$$
$$+ \frac{C}{n^3}\int_{p\le|t|\le\eta n^{1/2}}\left\{\frac{1}{|t|} + \frac{n\log n}{n^{1/2}ht^2} + \frac{(n\log n)^2}{nh^2|t|^3}\right\}dt$$
$$+ \frac{Cn\log n}{n^{7/2}h^3}\int_{p\le|t|\le\eta n^{1/2}}\frac{dt}{t^2} + \frac{Ch(n\log n)^{1/2}}{n^{1/2}(nh^{d+2})^2}$$
$$\times \int_{p\le|t|\le nh^{d+2}}dt + \frac{C}{n^{5/2}h^2}\int_{p\le|t|\le\eta n^{1/2}}dt$$
$$= o(n^{-1/2} + n^{-1}h^{-d-2}). \tag{7.3.45}$$

(ii) $N_1 \le |t| \le N_0$. For sufficiently large n, there exists $\xi > 0$ such that $|\gamma(t)| < 1 - \xi$ by assumption (x). We may take $m = [-(3\log n)/\log(1-\xi)]$ to satisfy $1 \le m \le n-1$ for sufficiently large n. Since

$$|\gamma(t)|^{m-4} \le Cn^{-3},$$

$$\int_{N_1\le|t|\le N_0}\left|\frac{Ee^{itB}}{t}\right| dt$$

$$\le \frac{C}{n^3h^{3d+4}}\int_{N_1\le|t|\le nh^{d+2}\log n}|t|\,dt$$

$$+C\left(\frac{\log n}{n^2h^{(d+2)/2}}+\frac{\log n}{n^{5/2}h^{(3d+6)/2}}\right)\int_{N_1\le|t|\le n^{1/2}\log n}|t|\,dt$$

$$+C\left(\frac{\log n}{n^2h^{d+2}}\right)^{3/2}\int_{N_1\le|t|\le n^{1/2}\log n}|t|^2\,dt$$

$$+\frac{C}{n^3}\int_{N_1\le|t|\le n^{1/2}\log n}\left\{\frac{1}{|t|}+\frac{\log n}{n^{1/2}h}+\frac{(\log n)^2}{nh^2}|t|\right\}dt$$

$$+\frac{C\log n}{n^{7/2}h^3}\int_{N_1\le|t|\le n^{1/2}\log n}dt$$

$$+\frac{Ch(\log n)^{1/2}}{n^{1/2}(nh^{d+2})^2}\int_{N_1\le|t|\le nh^{d+2}\log n}|t|\,dt$$

$$+\frac{C}{n^{5/2}h^2}\int_{N_1\le|t|\le n^{1/2}\log n}dt$$

$$=o(n^{-1/2}+n^{-1}h^{-d-2}) \qquad (7.3.46)$$

by (ix). Therefore, by (7.3.45) and (7.3.46),

$$(\mathrm{II}) = o(n^{-1/2}+n^{-1}h^{-d-2}).$$

Estimation of (III).

$$(\mathrm{III}) \le C\left[\int_p^\infty \frac{1}{t}e^{-t^2/2}\,dt + n^{1/2}h^L\int_p^\infty e^{-t^2/2}\,dt + \frac{1}{nh^{d+2}}\int_p^\infty te^{-t^2/2}\,dt + \frac{1}{n^{1/2}}\int_p^\infty (t+t^2)e^{-t^2/2}\,dt\right]. \qquad (7.3.47)$$

The first integral in (7.3.47) is bounded by

$$\frac{1}{p^2}\int_p^\infty te^{-t^2/2}\,dt = \frac{1}{p^2}e^{-p^2/2} = o(n^{-1}),$$

because $p = \min(\log n, \epsilon n^{1/2})$. The remaining integrals are clearly $o(1)$ as $p \to \infty$. Therefore,

$$(\text{III}) = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L).$$

to complete the proof. □

Proof of Theorem B: In view of the proof of Theorem 2 of NR, $\tilde{\kappa}_i \to \kappa_i$, $i = 1, 2, 3, 4$, a.s. Combine this with Theorem A. □

Appendix A

Lemma 1. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|V_1W_{12}|^r = E|V_2W_{12}|^r = O\big(h^{-(r-1)d-r}\big) \qquad \text{for} \quad 1 \le r \le 3.$$

Proof: Using Lemmas 1(d) and 4 of NR,

$$\begin{aligned} E|V_1W_{12}|^r &\le E\{|V_1|^r E(|W_{12}|^r|1)\} \\ &\le CE\{(|Y_1|^r + 1)^2\}h^{-(r-1)d-r} \\ &\le Ch^{-(r-1)d-r} \quad \text{for } 1 \le r \le 3 \qquad \text{by (i).} \end{aligned}$$

$E|V_1W_{12}|^r = E|V_2W_{12}|^r$ is obvious by the symmetry of W_{12} and (iii). □

Lemma 2. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|\tilde{V}_1|^r = O(1) \qquad \text{for} \quad 1 \le r \le 6.$$

Proof: As in the proof of Lemma 3 of Robinson (1995),

$$|\tilde{V}_1|^r = |E(V_2W_{12}|1)|^r \le C(|Y_1|^r + 1) \qquad \text{a.s.,} \tag{7.A.1}$$

so (i) immediately produces the conclusion. □

Lemma 3. *Under assumptions* (i), (iii), (iv), (v), (vi), *and* (viii),

(a) $E|\tilde{W}_{11}|^r = O(h^{-r(d+2)})$ for $1 \le r \le 3$,
(b) $E|\tilde{W}_{12}|^r = O(h^{-r(r-1)d-2r})$ for $1 \le r \le 6$.

Proof: (a): $\tilde{W}_{11} = E(W_{12}^2|1) \le C(|Y_1|^2 + 1)h^{-d-2}$ a.s. by Lemma 4 of NR, so again application of (i) completes the proof.

(b): Apply Lemma 6 of NR. □

Lemma 4. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), and (viii), *for* d_1 *given in* (7.3.37),

$$\begin{aligned} &\text{(a)} \quad E|d_1 V_2|^r = O(1) \quad \text{for} \quad 1 \le r \le 3, \\ &\text{(b)} \quad E|d_1 V_1|^r = O(1) \quad \text{for} \quad 1 \le r \le 2. \end{aligned}$$

Proof: (a): By (iii), $E|d_1 V_2|^r = E|d_1|^r E|V_2|^r$, where the second factor is bounded due to Lemma 1(d) of NR. From Lemma 1(d) of NR and (7.A.1),

$$|d_1|^r \le C\left(|V_1^2|^r + |\tilde{V}_1|^r + 1\right) \le C(|Y_1|^{2r} + 1); \tag{7.A.2}$$

then apply (i).

(b): By an elementary inequality and (7.3.7), $E|d_1 V_1|^r \le C(E|V_1^3|^r + E|\tilde{V}_1 V_1|^r + E|V_1|^r)$. By Lemma 1(d) of NR and (7.A.1), $E|V_1^3|^r + E|V_1|^r = O(1)$ *for* $1 \le r \le 2$, and

$$E|\tilde{V}_1 V_1|^r \le CE(|Y_1|^r + 1)^2 = O(1) \tag{7.A.3}$$

for $1 \le r \le 3$ by (i). □

Lemma 5. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$\begin{aligned} &\text{(a)} \quad E|W_{12} V_1 V_3|^r = E|W_{12} V_2 V_3|^r = O(h^{-(r-1)d-r}) \quad \text{for} \quad 1 \le r \le 3, \\ &\text{(b)} \quad E|W_{12} V_1 V_1^2|^r = E|W_{12} V_2^2|^r = O(h^{-(r-1)d-r}) \quad \text{for} \quad 1 \le r \le 2. \end{aligned}$$

Proof: (a): Using (iii), Lemma 1(d) of NR, and Lemma 1, for $1 \le r \le 3$,

$$E|W_{12} V_1 V_3|^r = E|W_{12} V_1|^r E|V_3|^r = O\left(h^{-(r-1)d-r}\right).$$

$E|W_{12} V_1 V_3|^r = E|W_{12} V_2 V_3|^r$ is straightforward by (iii) and symmetry of W_{12}.

(b): By Lemmas 1(d) and 4 of NR, the left side is

$$E\{|V_1|^{2r} E(|W_{12}|^r|1)\} \le E\left\{C(|Y_1|^{3r} + 1)h^{-(r-1)d-r}\right\} = O\left(h^{-(r-1)d-r}\right).$$

$E|W_{12} V_1^2|^r = E|W_{12} V_2^2|^r$ is straightforward by (iii) and symmetry of W_{12}. □

Lemma 6. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii), *with* e_{12} *given in* (7.3.37),

$$\begin{aligned} &\text{(a)} \quad E|e_{12} V_3|^r = O(h^{-(r-1)d-2r)}) \quad \text{for} \quad 1 \le r \le 3, \\ &\text{(b)} \quad E|e_{12} V_1|^r = E|e_{12} V_2|^r = O(h^{-(r-1)d-2r)}) \quad \text{for} \quad 1 \le r \le 2. \end{aligned}$$

Proof: (a): By (iii) and Lemma 1(d) of NR, write

$$E|e_{12} V_3|^r = E|e_{12}|^r E|V_3|^r \le C(E|V_1 W_{12}|^r + E|\tilde{V}_1|^r + E|\tilde{W}_{12}|^r).$$

Then apply Lemmas 1, 2, and 3(b).

(b): An elementary inequality gives

$$E|e_{12}V_1|^r \le C\big(E|V_1V_2W_{12}|^r + E\big|V_1^2W_{12}\big|^r + E|\tilde{V}_1V_1|^r + E|\tilde{V}_1V_2|^r + E|\tilde{W}_{12}V_1|^r\big). \quad (7.A.4)$$

Writing $E|V_1V_2W_{12}|^r = \{E|V_1|^r E(|V_2W_{12}|^r|1)\}$, the proof of Lemma 4 of NR applies to yield $E(|V_2W_{12}|^r|1) \le C(|Y_1|^r + 1)h^{-(r-1)d-r}$ a.s. Thus, for $1 \le r \le 3$,

$$E|V_1V_2W_{12}|^r = O\big(h^{-(r-1)d-r}\big). \quad (7.A.5)$$

The second term in (7.A.4) has the same order bound as (7.A.5) by Lemma 5(b) for $1 \le r \le 2$. The third term in (7.A.4) is bounded due to (7.A.3), while the fourth term is bounded due to Lemma 1(d) of NR and Lemma 2. We handle the last term in (7.A.4) similarly to Lemma 6 of NR:

$$E|\tilde{W}_{12}V_1|^r = E|E(W_{13}W_{23} \mid 1, 2)V_1|^r = O\big(h^{-(r-1)d-2r}\big). \quad (7.A.6)$$

$E|e_{12}V_1|^r = E|e_{12}V_2|^r$ is straightforward by (iii) and symmetry of e_{12}. □

Lemma 7. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

(a) $E|d_1W_{23}|^r = O(h^{-(r-1)d-r})$ for $1 \le r \le 3$,
(b) $E|d_1W_{12}|^r = E|d_2W_{12}|^r = O(h^{-(r-1)d-r})$ for $1 \le r \le 2$.

Proof: (a): Using (7.A.2) and Lemma 4 of Robinson (1995),

$$E|d_1W_{23}|^r = E|d_1|^r E|W_{23}|^r = O\big(h^{-(r-1)d-r}\big)$$

for $1 \le r \le 3$.

(b): Using (7.A.2) and Lemma 4 of NR, the left side is

$$E\{|d_1|^r E(|W_{12}|^r|1)\} \le E\big\{(|Y_1|^{2r} + 1)C(|Y_1|^r + 1)h^{-(r-1)d-r}\big\} \le CE(|Y_1|^{3r} + 1)h^{-(r-1)d-r} = O\big(h^{-(r-1)d-r}\big)$$

for $1 \le r \le 2$ under (i). $E|d_1W_{12}|^r = E|d_2W_{12}|^r$ is straightforward by (iii) and symmetry of W_{12}. □

Lemma 8. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

(a) $E|\tilde{W}_{12}W_{12}|^r = O(h^{-(2r-1)d-3r})$ for $1 \le r \le 3$,
(b) $E|\tilde{W}_{12}W_{13}|^r = O(h^{-2(r-1)d-3r})$ for $1 \le r \le 3$,
(c) $E|\tilde{W}_{12}W_{23}|^r = O(h^{-2(r-1)d-3r})$ for $1 \le r \le 3$,
(d) $E|\tilde{W}_{12}W_{34}|^r = O(h^{-2(r-1)d-3r})$ for $1 \le r \le 6$.

Proof: (a): In view of the proof of Lemma 6 of NR,

$$\begin{aligned}E|\tilde{W}_{12}W_{12}|^r &= E|\tilde{W}_{12}|^r|W_{12}|^r\\ &\le h^{-r(d+2)}CE(1+|Y_1|^r+|Y_2|^r+|Y_1|^r|Y_2|^r)|W_{12}|^r\\ &\le Ch^{-r(d+2)}(E|W_{12}|^r+E|Y_1W_{12}|^r\\ &\quad +E|Y_2W_{12}|^r+E|Y_1Y_2W_{12}|^r).\end{aligned}$$

The first term in parentheses is $O(h^{-(r-1)d-r})$ by Lemma 4 of Robinson (1995). From inspecting their proofs, Lemma 1 and (7.A.5) still hold with V_1 and V_2 replaced by Y_1 and Y_2 so that the other terms are $O(h^{-(r-1)d-r})$ for $1 \le r \le 3$.

(b): Using Lemma 4 of NR, for $1 \le r \le 6$,

$$\begin{aligned}E|\tilde{W}_{12}W_{13}|^r &= E\{|\tilde{W}_{12}|^r E(|W_{13}|^r|1,2)\}\\ &\le E\left\{|\tilde{W}_{12}|^r C(|Y_1|^r+1)h^{-(r-1)d-r}\right\}\\ &= Ch^{-(r-1)d-r}(E|\tilde{W}_{12}Y_1|^r+E|\tilde{W}_{12}|^r).\end{aligned}$$

We may replace V_1 by Y_1 in (7.A.6), so that, using also Lemma 3(b),

$$E|\tilde{W}_{12}W_{13}|^r = O\left(h^{-2(r-1)d-3r}\right) \quad \text{for} \quad 1 \le r \le 3.$$

(c): The proof is as in (b).

(d): Writing $E|\tilde{W}_{12}W_{34}|^r = E|\tilde{W}_{12}|^r E|\tilde{W}_{12}|^r$ by (iii), the proof is straightforward by Lemma 4 of Robinson (1995) and Lemma 3(b). □

Lemma 9. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|\bar{V}|^r = O(1) \quad \text{for} \quad 2 \le r \le 6.$$

Proof: Since V_i, $i = 1, \ldots, n$, is an iid sequence, the result follows straightforwardly by DFJ and Lemma 1(d) of NR. □

Lemma 10. *Under assumptions* (i), (v), (vi), (vii), *and* (viii),

$$|T_1|^r = O\left(n^{-r}h^{-r(d+2)}\right) \quad \text{for} \quad r > 0.$$

Proof: Using Lemma 4 of Robinson (1995) and $|\delta| < C$ due to (7.3.7),

$$|T_1|^r \le \frac{C}{n^r}\left|E\left(W_{12}^2\right)\right|^r = O\left(n^{-r}h^{-r(d+2)}\right).$$ □

Lemma 11. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|T_2|^r = O(n^{-r/2}) \quad \text{for} \quad 2 \le r \le 3.$$

Proof: Using (7.3.7), write

$$E|T_2|^r \leq \frac{C}{n^r} E\left|\sum_{i=1}^{n}(4V_i^2 - s^2)\right|^r + \frac{C}{n^r} E\left|\sum_{i=1}^{n} 8\tilde{V}_i\right|^r.$$

Since $E(4V_i^2) = s^2$ and $E(\tilde{V}_i) = 0$, by (iii) both the $4V_i^2 - s^2$ and $\tilde{V}_i$ are martingale differences, and thus the Theorem of DFJ applies to yield

$$E\left|\sum_{i=1}^{n}(4V_i^2 - s^2)\right|^r \leq Cn^{r/2}E\left|4V_1^2 - s^2\right|^r = O(n^{r/2})$$

for $2 \leq r \leq 3$ by (7.3.7) and Lemma 1(d) of NR and

$$E\left|\sum_{i=1}^{n} 8\tilde{V}_i\right|^r \leq Cn^{r/2}E|\tilde{V}_1|^r = O(n^{r/2})$$

by Lemma 2. □

Lemma 12. *Under assumptions* (i), (iii), (iv), (v), (vi), *and* (viii),

$$E|T_3|^r = O\left(n^{-r}h^{-(r-1)d-2r}\right) \qquad \text{for} \quad 2 \leq r \leq 6.$$

Proof: Using (7.3.7), write

$$E|T_3|^r \leq Cn^{-2r}E\left|\sum_{k=1}^{n-1} Z_k\right|^r, \tag{7.A.7}$$

where $Z_k = \sum_{m=k+1}^{n} \tilde{W}_{km}$ for $k = 1, \ldots, n-1$. Since

$$E(\tilde{W}_{12}\,|\,2) - E(\tilde{W}_{12}\,|\,1) = E\{E(W_{13}W_{23}\,|\,1,2)\,|\,1\} = E(W_{13}W_{23}\,|\,1) = 0 \qquad \text{a.s.},$$

$Z_k, k = n-1, \ldots, 1$, is a martingale difference sequence. Thus we apply DFJ to bound (7.A.7) by $Cn^{-2r}(n-1)^{r/2-1}\sum_{k=1}^{n-1} E|Z_k|^r$. Since $E(\tilde{W}_{km}\,|\,m) = 0$ a.s. for $m = k+1, \ldots, n$, the $\tilde{W}_{km}$ are martingale differences. We use DFJ again and get, by Lemma 3(b),

$$E|Z_k|^r \leq C(n-k)^{r/2-1}\sum_{m=k+1}^{n} E|\tilde{W}_{km}|^r \leq C(n-k)^{r/2-1}(n-k)h^{-(r-1)d-2r}.$$

□

Lemma 13. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|Q_1|^r = O\left(n^{-r}h^{-(r-1)d-r}\right) \qquad \text{for} \quad 2 \leq r \leq 3.$$

Proof: Write $P_{ij} = (V_i + V_j)W_{ij} - \tilde{V}_i - \tilde{V}_j$. Then $\sum_{j=i+1}^{n} P_{ij}$ is a martingale difference sequence for $i = n-1, \ldots, 1$. We can proceed by replacing $\tilde{W}_{km}$ in Lemma 12 by P_{ij} due to the property $E(P_{ij} \mid j) = 0$ a.s. for $i \neq j$. Applying DFJ and (7.3.7),

$$E|Q_1|^r \le C\binom{n}{2}^{-r} E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P_{ij}\right|^r \le C\binom{n}{2}^{-r} n^{r/2-1}\sum_{i=1}^{n-1} E\left|\sum_{j=i+1}^{n} P_{ij}\right|^r.$$

Since P_{ij}, $j = n, \ldots, i+1$ is a martingale difference for fixed i, we can apply the theorem of DJF again and obtain $E|\sum_{j=i+1}^{n} P_{ij}|^r \le C(n-i)^{r/2-1} \sum_{j=i+1}^{n} E|P_{ij}|^r$. By Lemmas 1 and 2,

$$E|P_{ij}|^r \le C[E|\tilde{V}_i|^r + E|V_i W_{ij}|^r] = O\left(h^{-(r-1)d-r}\right) \quad \text{for} \quad 1 \le r \le 3. \quad \square$$

Lemma 14. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|Q_2|^r = O\left(n^{-r}h^{-(r-1)d-r}\right) \quad \text{for} \quad 2 \le r \le 6.$$

Proof: By an elementary inequality and (7.3.7),

$$E|Q_2|^r \le \frac{C}{n^r}\binom{n-1}{2}^{-r} E\left|\sum_{i=1}^{n}\sum_{k=1}^{n-1}{}^{(i)}\sum_{m=k+1}^{n}{}^{(i)} V_i W_{km}\right|^r$$
$$= \frac{C}{n^r}\binom{n-1}{2}^{-r} n^{r-1}\sum_{i=1}^{n} E\left|\sum_{k=1}^{n-1}{}^{(i)}\sum_{m=k+1}^{n}{}^{(i)} V_i W_{km}\right|^r.$$

$V_i W_{km}$, $m = k+1, \ldots, n$, is a martingale difference for fixed i, k, $k \neq i$ and $m \neq i$, and $\sum_{m=k+1}^{n(i)} V_i W_{km}$, $k = n-1, \ldots, 1$, is also a martingale difference for fixed i and $k \neq i$, so that we apply DFJ repeatedly as in the proof of the previous lemma and get

$$\sum_{i=1}^{n} E\left|\sum_{k=1}^{n-1}{}^{(i)}\sum_{m=k+1}^{n}{}^{(i)} V_i W_{km}\right|^r$$
$$\le C\sum_{i=1}^{n}(n-2)^{r/2-1}\sum_{k=1}^{n-1}{}^{(i)} E\left|\sum_{m=k+1}^{n}{}^{(i)} V_i W_{ij}\right|^r$$
$$\le C(n-1)^{r/2-1}\sum_{i=1}^{n}\sum_{k=1}^{n-1}{}^{(i)}(n-k)^{r/2-1}\sum_{m=k+1}^{n}{}^{(i)} E|V_i W_{km}|^r$$
$$\le Cn^{r+1}h^{-(r-1)d-r}$$

for $2 \le r \le 6$, by (iii), Lemma 1(d) of NR, and Lemma 4 of Robinson (1995).

$\square$

Lemma 15. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|R_1|^r = O(n^{-r}) \qquad \text{for} \quad 2 \le r \le 6.$$

Proof: Writing

$$E|R_1|^r \le C\binom{n}{2}^{-r} E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} V_i V_j\right|^r$$

due to (7.3.7), as in Lemma 12 or 13, $V_i V_j$, $i = 1, \ldots, j-1$ is a martingale difference sequence for fixed j as well as $\sum_{j=i+1}^{n} V_i V_j$, $i = n-1, \ldots, 1$. We use DFJ repeatedly again and (i), (iii), and Lemma 1(d) of NR to obtain

$$\begin{aligned} E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} V_i V_j\right|^r &\le C(n-1)^{r/2-1}\sum_{i=1}^{n-1} E\left|\sum_{j=i+1}^{n} V_i V_j\right|^r \\ &\le C(n-1)^{r/2-1}\sum_{i=1}^{n-1}(n-i)^{r/2-1}\sum_{j=i+1}^{n} E|V_i V_j|^r \\ &= O(n^r). \end{aligned}$$

□

Lemma 16. *Under assumptions* (i), (iii), (iv), (v), (vi), *and* (viii),

$$E|R_2|^r = O\left(n^{-3(r-1)}h^{-2(r-1)d-2r}\right) \qquad \text{for} \quad 2 \le r \le 3.$$

Proof: Using (7.3.7), write

$$E|R_2|^r = \frac{C}{n^r}\binom{n-1}{2}^{-r} E\left|\sum_{i=1}^{n}\sum_{k=1}^{n-1}{}^{(i)}\sum_{m=k+1}^{n}{}^{(i)}(W_{ik}W_{im} - \tilde{W}_{km})\right|^r.$$

Since R_2 has the same martingale structure as Q_2, the same method of proof as in Lemma 14 applies. The difference is in the moment bounds of the two summands, i.e.

$$E|V_i W_{km}|^r = O\left(h^{-(r-1)d-r}\right) \qquad \text{for} \quad 1 \le r \le 6$$

and

$$E|W_{ik}W_{im} - \tilde{W}_{km}|^r = O\left(h^{-2(r-1)d-2r}\right), \qquad i \ne k \ne m,$$

by Lemmas 1(d), 4 of NR and Lemma 3(b). □

Lemma 17. *Under assumptions* (i), (iii), (iv), (v), (vi), *and* (viii),

$$E|R_3|^r = O\left(n^{-2r}h^{-(2r-1)d-2r}\right) \qquad \text{for} \quad 2 \le r \le 3.$$

Proof: Write

$$E|R_3|^r \le \frac{C}{n^r}\binom{n}{2}^{-r} E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2)\}\right|^r$$

using (7.3.7). Since

$$E\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2) \mid j\} = E\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2) \mid i\} = 0$$

for $j > i$, R_3 has the same martingale structure as T_3. Therefore, we apply DFJ to obtain

$$\begin{aligned} & E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2)\}\right|^r \\ & \quad \le C(n-1)^{r/2-1}\sum_{i=1}^{n-1} E\left|\sum_{j=i+1}^{n}\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2)\}\right|^r \\ & \quad \le C(n-1)^{r/2-1}\sum_{i=1}^{n-1}(n-i)^{r/2-1} \\ & \qquad \times \sum_{j=i+1}^{n} E|\{W_{ij}^2 - \tilde{W}_{ii} - \tilde{W}_{jj} + E(W_{12}^2)\}|^r \\ & \quad = O(n^r h^{-(2r-1)d-2r}) \end{aligned}$$

by Lemma 4 of Robinson (1995) and Lemma 3(a). □

Lemma 18. *Under assumptions* (i), (iv), (v), (vi), *and* (viii),

$$E|R_4|^r = O(n^{-\frac{3}{2}r} h^{-r(d+2)}) \qquad \textit{for} \quad 1 \le r \le 3.$$

Proof: Write $E|R_4|^r \le (C/n^{2r})E|\sum_{i=1}^{n}\{\tilde{W}_{ii} - E(W_{12}^2)\}|^r$, using (7.3.7). Since $\tilde{W}_{ii} - E(W_{12}^2)$ is a martingale difference, by (iii), DFJ, and Lemma 3(a),

$$E\left|\sum_{i=1}^{n}\{\tilde{W}_{ii} - E(W_{12}^2)\}\right|^r \le Cn^{r/2-1}\sum_{i=1}^{n} E|\tilde{W}_{ii} - E(W_{12}^2)|^r = O(n^{r/2}h^{-r(d+2)}).$$

Lemma 19. *Under assumptions* (i), (iii), (iv), (v), (vi), (vii), *and* (viii),

$$E|R_5|^r = O(n^{-2r} h^{-r(d+2)}) \qquad \text{for} \quad 1 \le r \le 3.$$

Proof: Using (7.3.7), DFJ, and Lemma 6 of Robinson (1995),

$$E|R_5|^r \le \frac{C}{n^{4r}} E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} W_{ij}\right|^{2r} \le \frac{C}{n^{4r}}(n-1)^{r-1}\sum_{i=1}^{n-1} E\left|\sum_{j=i+1}^{n} W_{ij}\right|^{2r}$$
$$= O\left(n^{-2r}h^{-r(d+2)}\right). \qquad \square$$

Appendix B

Here, we present some of the derivations used in the proof of Theorem A, namely:

(a) $E(\tilde{b}_2' e^{itb_2})$

$$= -\{\gamma(t)\}^{n-1}\left\{it\frac{2}{n}E\left(W_{12}^2\right) + O\left(\frac{|t|}{n^2h^{d+2}} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{|t|h^L}{nh^{d+2}}\right)\right\}$$
$$-\{\gamma(t)\}^{n-1}\left\{\frac{4E\left(V_1^3\right) + 8E(W_{12}V_1V_2)}{n^{1/2}} + O\left(\frac{|t|}{n}\right)\right\}$$
$$-\{\gamma(t)\}^{n-2}\left[\frac{(it)^2}{n^{1/2}}\left\{4E\left(V_1^3\right) + 8E(W_{12}V_1V_2)\right\}\right.$$
$$\left. + O\left(\frac{t^2+|t|^3}{n} + \frac{t^4}{n^{3/2}}\right)\right] + \{\gamma(t)\}^{n-3}$$
$$\times \left[O\left(\frac{|t|^3+|t|}{n} + \frac{t^2}{n^{3/2}h^{d+2}} + \frac{t^6}{n^3h^{d+2}} + \frac{|t|^5}{n^{5/2}h^{d+2}} + \frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right],$$

(b) $|E(\tilde{b}_{2m}e^{it\tilde{B}_m})| \le \dfrac{Cm}{n^{1/2}h^2}|\gamma(t)|^{m-4}$,

(c) $|E(\tilde{b}_{3m}' e^{it\tilde{B}_m})| \le \dfrac{Cm}{n^{1/2}h^3}|\gamma(t)|^{m-4}$,

(d) $E|\tilde{b}_{2m}|^2 \le Cm\left(\dfrac{1}{n^3h^{2d+4}} + \dfrac{1}{n^2}\right)$,

(e) $E|\tilde{b}_{3m}'|^2 \le \dfrac{Cm}{n^4h^{3d+6}}$,

(f) $|E\tilde{b}_3'' e^{it\tilde{B}_m}| \le \dfrac{Cn^{1/2}}{h^2}|\gamma(t)|^{m-5}$.

for $1 \le m \le n-1$.

Proof: (a): Write

$$
\begin{aligned}
E(\bar{b}_2' e^{itb_2}) &= E(T\bar{V}e^{itb_2})\\
&= E\left(T_1\frac{2}{\sqrt{n}}\sum_{j=1}^{n}V_je^{itb_2}\right)+E\left(T_2\frac{2}{\sqrt{n}}\sum_{j=1}^{n}V_je^{itb_2}\right)\\
&\quad+E\left(T_3\frac{2}{\sqrt{n}}\sum_{j=1}^{n}V_je^{itb_2}\right)\\
&= \text{(A)}+\text{(B)}+\text{(C)}. \qquad (7.B.1)
\end{aligned}
$$

Thus

$$
\text{(A)} = -\frac{4n^{1/2}}{(n-2)^2s^3}E\left(W_{12}^2\right)\sum_{j=1}^{n}\left(V_je^{itb_2}\right). \qquad (7.B.2)
$$

Due to (iii), (7.3.17), $\gamma(t)=E(e^{it(2/\sqrt{n}s)V_1})$, $4E(V_1^2)=s^2$, (7.3.7), and Lemma 1(d) of NR,

$$
\begin{aligned}
E(V_je^{itb_2}) &= E\left(V_je^{it(2/\sqrt{n}s)V_j}\right)E\left(e^{it(2/\sqrt{n}s)\sum_{k\neq j}^{n}V_k}\right)\\
&= \left[E\left\{V_j\left(e^{it(2/\sqrt{n}s)V_j}-1-it\frac{2V_j}{n^{1/2}s}\right)\right\}\right.\\
&\quad\left.+(it)E\left(\frac{2V_j^2}{n^{1/2}s}\right)\right]\{\gamma(t)\}^{n-1}\\
&= \{\gamma(t)\}^{n-1}\left\{\frac{its}{2n^{1/2}}+O\left(\frac{t^2}{n}\right)\right\}\\
&= \{\gamma(t)\}^{n-1}\left\{\frac{it}{2n^{1/2}}+O\left(\frac{t^2}{n}+\frac{|t|h^L}{n^{1/2}}\right)\right\}. \qquad (7.B.3)
\end{aligned}
$$

Substituting (7.B.3) into (7.B.2),

$$
\begin{aligned}
\text{(A)} &= -\{\gamma(t)\}^{n-1}\frac{4n^{3/2}}{(n-2)^2s^3}E\left(W_{12}^2\right)\left\{\frac{it}{2n^{1/2}}+O\left(\frac{t^2}{n}+\frac{|t|h^L}{n^{1/2}}\right)\right\}\\
&= -\frac{\{\gamma(t)\}^{n-1}}{s^3}\left\{\frac{2it}{n}E\left(W_{12}^2\right)+O\left(\frac{|t|}{n^2h^{d+2}}+\frac{t^2}{n^{3/2}h^{d+2}}+\frac{|t|h^L}{nh^{d+2}}\right)\right\}.
\end{aligned}
$$

(7.B.4)

Now write (B) = (B′) + (B″), where

$$
\text{(B}'\text{)} = -\frac{1}{n^{3/2}s^3}\sum_{j=1}^{n}E\left(4V_j^2-s^2+8\tilde{V}_j\right)V_je^{itb_2}, \qquad (7.B.5)
$$

$$
\text{(B}''\text{)} = -\frac{1}{n^{3/2}s^3}\sum_{j=1}^{n}\sum_{k\neq j}^{n}E(4V_j-s^2+8\tilde{V}_j)V_ke^{itb_2}. \qquad (7.B.6)
$$

The summand of (B′) is, using (7.3.17) and Lemma 1(d) of NR,

$$E\{(4V_1^2 - s^2 + 8\tilde{V}_1)V_i e^{it\,2V_1/\sqrt{ns}}\}E\left(e^{it(2/\sqrt{ns})\sum_{l\neq 1}V_l}\right)$$
$$= \{\gamma(t)\}^{n-1}E\{(4V_1^2 - s^2 + 8\tilde{V}_1)V_1 e^{it\,2V_1/\sqrt{ns}}\}$$
$$= \{\gamma(t)\}^{n-1}\left\{4E\{(V_1^3) + 8E(W_{12}V_1V_2) + O\left(\frac{|t|}{n^{1/2}}\right)\right\}. \quad (7.B.7)$$

Substituting (7.B.7) into (7.B.5),

$$(B') = -\frac{1}{n^{1/2}s^3}\{\gamma(t)\}^{n-1}\left\{4E(V_1^3) + 8E(W_{12}V_1V_2) + O\left(\frac{|t|}{n^{1/2}}\right)\right\}. \quad (7.B.8)$$

For $j \neq k$, the summand of (B″) is, due to (iii),

$$E\{(4V_1^2 - s^2 + 8\tilde{V}_1)V_2 e^{it\,2(V_1+V_2)/\sqrt{ns}}\}E\left(e^{it(2/\sqrt{ns})\sum_{l\neq 1,2}V_l}\right)$$
$$= \{\gamma(t)\}^{n-2}E\{(4V_1^2 - s^2) + 8\tilde{V}_1)V_2 e^{it\,2(V_1+V_2)/\sqrt{ns}}\}$$
$$= \{\gamma(t)\}^{n-2}E\{(4V_1^2 - s^2) + 8\tilde{V}_1)e^{it\,2V_1/\sqrt{ns}}\}E(V_2 e^{it\,2V_2/\sqrt{ns}})$$
$$= \{\gamma(t)\}^{n-2}\left[E\left\{(4V_1^2 - s^2 + 8\tilde{V}_1)\left(e^{it\,2V_1/\sqrt{ns}} - 1 - it\frac{V_1}{\sqrt{ns}}\right)\right\}\right.$$
$$\left. + itE\frac{V_1}{\sqrt{ns}}(4V_1^2 - s^2 + 8\tilde{V}_1)\right]$$
$$\times\left[E\left\{V_2\left(e^{it\,2V_2/\sqrt{ns}} - 1 - it\frac{V_2}{\sqrt{ns}}\right)\right\} + it\,E\frac{V_2^2}{\sqrt{ns}}\right]$$
$$= \{\gamma(t)\}^{n-2}\left[itE\frac{V_1}{\sqrt{ns}}(4V_1^2 - s^2 + 8\tilde{V}_1) + O\left(\frac{|t|^2}{n}\right)\right]$$
$$\times\left[it\frac{E(V_2^2)}{\sqrt{ns}} + O\left(\frac{|t|^2}{n}\right)\right]$$
$$= \{\gamma(t)\}^{n-2}\left[\frac{it}{\sqrt{ns}}\{E(4V_1^3) + 8E(W_{12}V_1V_2)\}\right.$$
$$\left. + O\left(\frac{|t|^2}{n}\right)\right]\left[\frac{its}{\sqrt{n}} + O\left(\frac{|t|^2}{n}\right)\right]$$
$$= \{\gamma(t)\}^{n-2}\left[\frac{(it)^2}{n}\{E(4V_1^3) + 8E(W_{12}V_1V_2)\}\right.$$
$$\left. + O\left(\frac{|t|^3}{n^{3/2}} + \frac{|t|^4}{n^2}\right)\right] \quad (7.B.9)$$

by (7.3.17). Therefore, substituting (7.B.9) into (7.B.6) yields

$$(\mathrm{B}'') = -\frac{n(n-1)}{n^{3/2}s^3}\left[\frac{(it)^2}{n}\{4E(V_1^3) + 8E(W_{12}V_1V_2)\} + O\left(\frac{|t|^3}{n^{3/2}} + \frac{|t|^4}{n^2}\right)\right]\{\gamma(t)\}^{n-2}. \tag{7.B.10}$$

By (7.B.8) and (7.B.10),

$$\begin{aligned}(\mathrm{B}) = &-\frac{\{\gamma(t)\}^{n-1}}{s^3}\left\{\frac{4E(V_1^3) + 8E(W_{12}V_1V_2)}{n^{1/2}} + o\left(\frac{|t|}{n}\right)\right\} \\ &- \frac{\{\gamma(t)\}^{n-2}}{s^3}\left[\frac{(it)^2}{n^{1/2}}\{4E(V_1^3) + 8E(W_{12}V_1V_2)\} \right. \\ &\left. + O\left(\frac{t^2 + |t|^3}{n} + \frac{t^4}{n^{3/2}}\right)\right].\end{aligned} \tag{7.B.11}$$

Now, write

$$\begin{aligned}(\mathrm{C}) &= -\frac{4}{n^{1/2}s^3}E\left\{\binom{n-1}{2}^{-1}\sum_{1\le j<k\le n}\tilde{W}_{jk}\right\}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}V_l e^{itb_2} \\ &= -\frac{4}{n^{1/2}s^3}\binom{n-1}{2}^{-1}\sum_{l=1}^{n}\sum_{j<k}^{(l)}E(\tilde{W}_{jk}V_l e^{itb_2}) \\ &\quad - \frac{4}{n^{1/2}s^3}\binom{n-1}{2}^{-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}E(\tilde{W}_{jk}V_j e^{itb_2}) \\ &\quad - \frac{4}{n^{1/2}s^3}\binom{n-1}{2}^{-1}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}E(\tilde{W}_{jk}V_k e^{itb_2}) \\ &= (\mathrm{C}') + (\mathrm{C}'') + (\mathrm{C}''').\end{aligned}$$

Using (iii), Lemma 1(d) of NR, $E(V_j) = 0$, and $E(\tilde{W}_{km}) = E(\tilde{W}_{km} \mid k) = E(\tilde{W}_{km} \mid m) = 0$, the summand of (C$'$) is

$$\begin{aligned}&E\left\{\tilde{W}_{12}V_3 e^{it(2/\sqrt{n}s)(V_1+V_2+V_3)}\right\}E\left(e^{it(2\sqrt{n}s)\sum_{l\neq 1,2,3}^{n}V_l}\right) \\ &\quad = E\left\{\tilde{W}_{12}V_3 e^{it(2/\sqrt{n}s)(V_1+V_2+V_3)}\right\}\{\gamma(t)\}^{n-3} \\ &\quad = \left\{E\tilde{W}_{12}\left(e^{it\,2V_1/\sqrt{n}s} - 1 - it\frac{2V_1}{\sqrt{n}s}\right)\left(e^{it\,2V_2/\sqrt{n}s} - 1 - it\frac{2V_2}{\sqrt{n}s}\right)\right. \\ &\qquad + (it)^2\frac{4}{ns^2}E(\tilde{W}_{12}V_1V_2)\end{aligned}$$

$$+ (it)\frac{2}{\sqrt{ns}} E\left\{ \tilde{W}_{12} V_1 \left(e^{it\, 2V_2/\sqrt{ns}} - 1 - it\frac{2V_2}{\sqrt{ns}} \right) \right\}$$
$$+ (it)\frac{2}{\sqrt{ns}} E\left\{ \tilde{W}_{12} V_2 \left(e^{it\, 2V_1/\sqrt{ns}} - 1 - it\frac{2V_1}{\sqrt{ns}} \right) \right\}\Bigg\}$$
$$\times \left\{ E\left\{ V_3 \left(e^{it\, 2V_3/\sqrt{ns}} - 1 - it\frac{2V_3}{\sqrt{ns}} \right) \right\} + (it)\frac{2}{\sqrt{ns}} E\left(V_3^2\right) \right\} \{\gamma(t)\}^{n-3}$$
$$= \left\{ (it)^2 \frac{4}{ns^2} E(\tilde{W}_{12} V_1 V_2) + O\left(\frac{t^4}{n^2 h^{d+2}} + \frac{|t|^3}{n^{3/2} h^{d+2}} \right) \right\}$$
$$\times \left\{ (it)\frac{2}{\sqrt{ns}} E\left(V_3^2\right) + O\left(\frac{t^2}{n}\right) \right\} \{\gamma(t)\}^{n-3}.$$

The last equality uses (7.3.17) and

$$E\left|\tilde{W}_{12} V_1^2 V_2^2\right| \le \{E|\tilde{W}_{12}|^3\}^{1/2} (E|V_1 V_2|^3)^{2/3} \le Ch^{-\frac{2}{3}d-2} = O(h^{-d-2})$$

due to Hölder's inequality, Lemma 3(b), (i), (iii), and Lemma 1(d) of NR. Next,

$$(\mathrm{C}') = -\frac{\{\gamma(t)\}^{n-3}}{s^3} \frac{8n(n-1)(n-2)}{n^{5/2}}$$
$$\times \left\{ \frac{8(it)^3}{n^{3/2} s^3} E(\tilde{W}_{12} V_1 V_2) s^2 + O\left(\frac{t^6}{n^3 h^{d+2}} + \frac{|t|^5}{n^{5/2} h^{d+2}} + \frac{t^4}{n^2 h^{d+2}} \right) \right\}$$
$$= \frac{\{\gamma(t)\}^{n-3}}{s^3} O\left(\frac{|t|^3}{n} + \frac{t^6}{n^3 h^{d+2}} + \frac{|t|^5}{n^{5/2} h^{d+2}} + \frac{t^4 + |t|^3}{n^2 h^{d+2}} \right). \tag{7.B.12}$$

Here we use, due to (iii) and Lemma 2,

$$\begin{aligned} E(\tilde{W}_{12} V_1 V_2) &= E(W_{13} W_{23} V_1 V_2) = E[E(W_{13} V_1 \mid 3) E(W_{23} V_2 \mid 3)] \\ &= E\left(\tilde{V}_3^2\right) = O(1). \end{aligned} \tag{7.B.13}$$

The summand of (C″) can be expressed as follows using (iii), $E(\tilde{W}_{12} V_1 e^{it(2/\sqrt{ns})V_1}) = 0$, Lemma 3(b), and (7.3.17):

$$\begin{aligned} E\left(\tilde{W}_{jk} V_j e^{itb_2}\right) &= \{\gamma(t)\}^{n-2} E\left(\tilde{W}_{12} V_1 e^{it(2/\sqrt{ns})(V_1+V_2)}\right) \\ &= \{\gamma(t)\}^{n-2} E\Bigg\{ \tilde{W}_{12} V_1 e^{it\, 2V_1/\sqrt{ns}} \\ &\quad \times \left(e^{it\, 2V_2/\sqrt{ns}} - 1 - it\frac{2V_2}{\sqrt{ns}} \right) + \frac{2it}{\sqrt{ns}} \tilde{W}_{12} V_1 V_2 \Bigg\} \\ &= \{\gamma(t)\}^{n-2} \left\{ \frac{2it}{\sqrt{ns}} E(\tilde{W}_{12} V_1 V_2) + O\left(\frac{|t|^2}{nh^{d+2}} \right) \right\}. \end{aligned}$$

Thus, using (7.3.7) and (7.B.13),

$$(\mathrm{C}'') = \frac{\{\gamma(t)\}^{n-2}}{s^3}\left[-\frac{8n(n-1)}{n^{5/2}}\left\{\frac{2it}{\sqrt{n}s}E(\tilde{W}_{12}V_1V_2)+O\left(\frac{t^2}{nh^{d+2}}\right)\right\}\right]$$
$$= \frac{\{\gamma(t)\}^{n-2}}{s^3}O\left(\frac{|t|}{n}+\frac{t^2}{n^{3/2}h^{d+2}}\right). \tag{7.B.14}$$

Similarly,

$$(\mathrm{C}''') = \frac{\{\gamma(t)\}^{n-2}}{s^3}O\left(\frac{|t|}{n}+\frac{t^2}{n^{3/2}h^{d+2}}\right). \tag{7.B.15}$$

By (7.B.12), (7.B.14), and (7.B.15),

$$(\mathrm{C}) = (\mathrm{C}') + (\mathrm{C}'') + (\mathrm{C}''')$$
$$= \frac{\{\gamma(t)\}^{n-3}}{s^3}\left[O\left(\frac{|t|^3+|t|}{n}+\frac{t^2}{n^{3/2}h^{d+2}}+\frac{t^6}{n^3h^{d+2}}\right.\right.$$
$$\left.\left.+\frac{|t|^5}{n^{5/2}h^{d+2}}+\frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right]. \tag{7.B.16}$$

Therefore, by (7.3.7), (7.B.1), (7.B.4), (7.B.11), and (7.B.16),

$$E(\tilde{b}_2'E^{itb_2}) = (\mathrm{A}) + (\mathrm{B}) + (\mathrm{C})$$
$$= -\{\gamma(t)\}^{n-1}\left\{it\frac{2}{n}E(W_{12}^2)+O\left(\frac{|t|}{n^2h^{d+2}}+\frac{t^2}{n^{3/2}h^{d+2}}+\frac{|t|h^L}{nh^{d+2}}\right)\right\}$$
$$-\{\gamma(t)\}^{n-1}\left\{\frac{4E(V_1^3)+8E(W_{12}V_1V_2)}{n^{1/2}}+O\left(\frac{|t|}{n}\right)\right\}$$
$$-\{\gamma(t)\}^{n-2}\left[\frac{(it)^2}{n^{1/2}}\{4E(V_1^3)+8E(W_{12}V_1V_2)\}\right.$$
$$\left.+O\left(\frac{t^2+|t|^3}{n}+\frac{t^4}{n^{3/2}}\right)\right]+\{\gamma(t)\}^{n-3}\left[O\left(\frac{|t|^3+|t|}{n}\right.\right.$$
$$\left.\left.+\frac{t^2}{n^{3/2}h^{d+2}}+\frac{t^6}{n^3h^{d+2}}+\frac{|t|^5}{n^{5/2}h^{d+2}}+\frac{t^4+|t|^3}{n^2h^{d+2}}\right)\right].$$

(b): Writing, using (7.3.7) and Lemma 4 of Robinson (1995),

$$|E(\tilde{b}_{2m}e^{it\tilde{B}_m})| \le \frac{C}{n^{3/2}h^{d+2}}\sum_{j=1}^{m}|E(V_je^{it\tilde{B}_m})|$$
$$+\frac{C}{n^{3/2}}\left\{\sum_{j=1}^{n}\sum_{k=1}^{m}|E(d_jV_ke^{it\tilde{B}_m})|+\sum_{j=1}^{m}\sum_{k=m+1}^{n}|E(d_jV_ke^{it\tilde{B}_m})|\right\}$$

$$+\frac{C}{n^{5/2}}\left\{\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{s=1}^{m}|E(e_{jk}V_s e^{it\bar{B}_m})|\right.$$

$$\left.+\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{s=m+1}^{n}|E(e_{jk}V_s e^{it\bar{B}_m})|\right\}$$

$$+\frac{C}{n^{7/2}}\left[\sum_{j=1}^{n}\sum_{k<l}^{n\ (j)}\sum_{s=1}^{m}|E(V_j W_{kl} V_s e^{it\bar{B}_m})|\right.$$

$$+\sum_{j=1}^{n}\sum_{k=m+1}^{m_{(j)}}\sum_{l=k+1}^{n\ (j)}\sum_{s=m+1}^{n}|E(V_j W_{kl} V_s e^{it\bar{B}_m})|$$

$$\left.+\sum_{j=1}^{m}\sum_{k=m+1}^{n-1\ (j)}\sum_{l=k+1}^{n\ (j)}\sum_{s=m+1}^{n}|E(V_j W_{kl} V_s e^{it\bar{B}_m})|\right], \qquad (7.B.17)$$

$$|E(V_j e^{it\bar{B}_m})| = \left|E\left(V_j e^{it(2/\sqrt{n}s)V_j}\right)E\left\{e^{it((2/\sqrt{n}s)\sum_{k\neq j}^{n}V_k+b_3-b_{3m}+\bar{b}_2-\bar{b}_{2m}+\bar{b}_3'-\bar{b}_{3m}')}\right\}\right|$$
$$\le E|V_j||\gamma(t)|^{m-1} \qquad (7.B.18)$$

for $j = 1, \ldots, m$, since $b_3 - b_{3m} + \bar{b}_2' - b_{2m}' + \bar{b}_3' - \bar{b}_{3m}'$ is independent of $V_1, \ldots, V_m$.

For $j \le m$, $k \le m$, and $j \neq k$,

$$|E(d_j V_k e^{it\bar{B}_m})|$$
$$= \left|E\left[d_j V_k e^{it\left\{(2/\sqrt{n}s)\left(V_j+V_k+\sum_{l=m+1}^{n}V_l\right)+b_3-b_{3m}+\bar{b}_2-\bar{b}_{2m}+\bar{b}_3'-\bar{b}_{3m}'\right\}}\right]\right|$$
$$\times\left|E\left\{e^{it(2/\sqrt{n}s)\sum_{l\neq j,k}^{m}V_l}\right\}\right|$$
$$\le E|d_j V_k||\gamma(t)|^{m-2}. \qquad (7.B.19)$$

For $j = k \le m$,

$$|E(d_j V_j e^{it\bar{B}_m})|$$
$$= \left|E\left[d_j V_j e^{it\left\{(2/\sqrt{n}s)\left(V_j+\sum_{k=m+1}^{n}V_k\right)+b_3-b_{3m}+\bar{b}_2-\bar{b}_{2m}+\bar{b}_3'-\bar{b}_{3m}'\right\}}\right]\right|$$
$$\times\left|E\left\{e^{it(2/\sqrt{n}s)\sum_{k\neq j}^{m}V_k}\right\}\right|$$
$$\le E|d_j V_j||\gamma(t)|^{m-1} \le E|d_j V_j||\gamma(t)|^{m-2}. \qquad (7.B.20)$$

For $j \leq m$ and $k \geq m+1$,

$$
\begin{aligned}
&|E(d_j V_k e^{it\bar{B}_m})| \\
&\quad = \left| E\left[d_j V_k e^{it\left\{(2/\sqrt{n}s)\left(V_j + \sum_{l=m+1}^{n} V_l\right) + b_3 - b_{3m} + \bar{b}_2 - \bar{b}_{2m} + \bar{b}_3' - \bar{b}_{3m}'\right\}} \right] \right| \\
&\qquad \times \left| E\left\{ e^{it\,(2/\sqrt{n}s)\sum_{l\neq j}^{m} V_l} \right\} \right| \\
&\quad \leq E|d_j V_k||\gamma(t)|^{m-1} \\
&\quad \leq E|d_j V_k||\gamma(t)|^{m-2}. \qquad (7.B.21)
\end{aligned}
$$

For $j \geq m+1$ and $k \leq m$, similarly to (7.B.21),

$$
|E(d_j V_k e^{it\bar{B}_m})| \leq E|d_j V_k||\gamma(t)|^{m-2}. \qquad (7.B.22)
$$

Therefore, by (7.B.19)–(7.B.22) and Lemma 4, for all j, k,

$$
|E(d_j V_k e^{it\bar{B}_m})| \leq E|d_j V_k||\gamma(t)|^{m-2}. \qquad (7.B.23)
$$

Similarly to the derivation of (7.B.23), for any j, k, l, s,

$$
|E(e_{jk} V_s e^{it\bar{B}_m})| \leq E|e_{jk} V_s||\gamma(t)|^{m-3}, \qquad (7.B.24)
$$

$$
|E(V_j W_{kl} V_s e^{it\bar{B}_m})| \leq E|V_j W_{kl} V_s||\gamma(t)|^{m-4}. \qquad (7.B.25)
$$

Substituting (7.B.18), (7.B.23)–(7.B.25) into (7.B.17), using $|\gamma(t)| \leq 1$,

$$
\begin{aligned}
|E(\bar{b}_{2m} e^{it\bar{B}_m})| \leq C|\gamma(t)|^{m-4} \Bigg[& \frac{1}{n^{3/2}h^{d+2}} \sum_{j=1}^{m} E|V_j| \\
& + \frac{1}{n^{3/2}} \left(\sum_{j=1}^{n} \sum_{k=1}^{m} E|d_j V_k| + \sum_{j=1}^{m} \sum_{k=m+1}^{n} E|d_j V_k| \right) \\
& + \frac{1}{n^{5/2}} \left(\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \sum_{s=1}^{m} E|e_{jk} V_s| + \sum_{j=1}^{m} \sum_{k=j+1}^{n} \sum_{s=m+1}^{n} E|e_{jk} V_s| \right) \\
& + \frac{1}{n^{7/2}} \Bigg(\sum_{j=1}^{n} \sum_{k<l}^{n\;(j)} \sum_{s=1}^{m} E|V_j W_{kl} V_s| \\
& + \sum_{j=1}^{n} \sum_{k=l}^{m\;(j)} \sum_{l=k+1}^{n\;(j)} \sum_{s=m+1}^{n} E|V_j W_{kl} V_s| \\
& + \sum_{j=1}^{m} \sum_{k=m+1}^{n-1\;(j)} \sum_{l=k+1}^{n\;(j)} \sum_{s=m+1}^{n} E|V_j W_{kl} V_s| \Bigg) \Bigg]. \qquad (7.B.26)
\end{aligned}
$$

The summations in the square brackets have the following bounds.

$$\sum_{j=1}^{m} E|V_j| \le C_m \qquad \text{by Lemma 1(d) of NR;} \tag{7.B.27}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{m} E|d_j V_k| = \sum_{j=1}^{m} E|d_j V_j| + \sum_{j=1}^{n}\sum_{k=l}^{m}{}^{(j)} E|d_j V_k|$$
$$\le C(m+mn) \qquad \text{by Lemma 4;} \tag{7.B.28}$$

$$\sum_{j=1}^{m}\sum_{s=m+1}^{n} E|d_j V_s| \le Cmn \qquad \text{by Lemma 4(a);} \tag{7.B.29}$$

$$\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{s=1}^{m} E|e_{jk} V_s| = \sum_{j=1}^{n=1}\sum_{k=j+1}^{n}\sum_{s=1}^{m}{}^{(j,k)} E|e_{jk} V_s|$$
$$+ \sum_{j=1}^{m}\sum_{k=j+1}^{n} E|e_{jk} V_j| + \sum_{j=1}^{m-1}\sum_{k=j+1}^{m} E|e_{jk} V_k|$$
$$\le C(mn^2 + mn + m^2)h^{-2} \qquad \text{by Lemma 6,} \tag{7.B.30}$$

$\sum_s^{(i_1,i_2,\ldots,i_r)}$ denoting summation excluding $s = i_1,\ i_2, \ldots, i_r$;

$$\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{s=m+1}^{n} E|e_{jk} V_s|$$
$$= \sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{s=m+1}^{n}{}^{(j)} E|e_{jk} V_s| + \sum_{j=1}^{m}\sum_{k=j+1}^{n} E|e_{jk} V_k|$$
$$\le C(mn^2 + mn)h^{-2} \qquad \text{by Lemma 6;} \tag{7.B.31}$$

$$\sum_{j=1}^{n}\sum_{k<l}^{n}{}^{(j)}\sum_{s=1}^{m} E|V_j W_{kl} V_s|$$
$$= \sum_{j=1}^{n}\sum_{k<l}^{n}{}^{(j)}\sum_{s=1}^{m}{}^{(j,k,l)} E|V_j|E|W_{kl}|E|V_s| + \sum_{j=1}^{m}\sum_{k<l}^{m}{}^{(j)} E\left|V_j^2 W_{kl}\right|$$
$$+ \sum_{j=1}^{n}\sum_{k<l}^{m}{}^{(j)} E|V_j W_{kl} V_k| + \sum_{j=1}^{n}\sum_{k<l}^{m}{}^{(j)} E|V_j W_{kl} V_l|$$
$$\le C(mn^3 + mn^2 + m^2 n)h^{-1} \tag{7.B.32}$$

by (iii), Lemma 1(d) of NR, Lemma 4 of Robinson (1995), and Lemma 5;

$$\sum_{j=1}^{n}\sum_{k=l}^{m}{}^{(j)}\sum_{l=k+1}^{n}{}^{(j)}\sum_{s=m+1}^{n} E|V_j W_{kl} V_s|$$
$$= \sum_{j=1}^{n}\sum_{k=l}^{m}{}^{(j)}\sum_{l=k+1}^{n}{}^{(j)}\sum_{s=m+1}^{n}{}^{(j,l)} E|V_j|E|W_{kl}|E|V_s|$$
$$+ \sum_{j=m+1}^{n}\sum_{k=l}^{m}{}^{(j)}\sum_{l=k+1}^{n}{}^{(j)} E\left|V_j^2 W_{kl}\right| + \sum_{j=1}^{n}\sum_{k=l}^{m}{}^{(j)}\sum_{l=k+1}^{n}{}^{(j)} E|V_j W_{kl} V_l|$$
$$\le C(mn^3 + mn^2)h^{-1} \tag{7.B.33}$$

by (iii), Lemma 1(d) of NR, Lemma 4 of Robinson (1995), and Lemma 5; and

$$\sum_{j=1}^{m}\sum_{m<k<l}^{n}{}^{(j)}\sum_{s=m+1}^{n} E|V_j W_{kl} V_s|$$
$$= \sum_{j=1}^{m}\sum_{m<k<l}^{n}{}^{(j)}\sum_{s=m+1}^{n}{}^{(k,l)} E|V_j|E|W_{kl}|E|V_s|$$
$$+ \sum_{j=1}^{m}\sum_{m<k<l}^{n}{}^{(j)}(E|V_j W_{kl} V_k| + E|V_j W_{kl} V_l|)$$
$$\le C(mn^3 + mn^2)h^{-1} \tag{7.B.34}$$

by (iii), Lemma 1(d) of NR, Lemma 4 of Robinson (1995), and Lemma 5. Therefore, substituting (7.B.27)–(7.B.34) into (7.B.26), using $1 \le m \le n-1$,

$$|E(\tilde{b}'_{2m}e^{it\bar{B}_m})| \le C|\gamma(t)|^{m-4}\left(\frac{m}{n^{3/2}h^{d+2}} + \frac{mn}{n^{3/2}} + \frac{mn^2}{n^{5/2}h^2} + \frac{mn^3}{n^{7/2}h}\right)$$
$$\le \frac{Cm}{n^{1/2}h^2}|\gamma(t)|^{m-4}, \tag{7.B.35}$$

the third term in parentheses dominating for sufficiently large n by assumption (ix).

(c): Using (7.3.7) and Lemma 4 of Robinson (1995), we write

$$|E(\tilde{b}'_{3m}e^{it\bar{B}_m})|$$
$$\le C\left[\frac{1}{n^{5/2}h^{d+2}}\sum_{l=1}^{m}\sum_{s=l+1}^{n}|E(W_{ls}e^{it\bar{B}_m})|\right.$$
$$+ \frac{1}{n^{5/2}}\left(\sum_{j=1}^{m}\sum_{l<s}^{n}|E(d_j W_{ls}e^{it\bar{B}_m})| + \sum_{j=m+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}|E(d_j W_{ls}e^{it\bar{B}_m})|\right)$$

$$+\frac{1}{n^{7/2}}\left(\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{l<s}^{n}|E(\tilde{W}_{jk}W_{ls}e^{it\bar{B}_m})|\right.$$

$$\left.\left.+\sum_{m<j<k}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}|E(\tilde{W}_{jk}W_{ls}e^{it\bar{B}_m})|\right)\right]. \tag{7.B.36}$$

Similarly to (7.B.23)–(7.B.25), for all j, k, l, s,

$$|E(W_{ls}e^{it\bar{B}_m})| \le E|W_{ls}||\gamma(t)|^{m-2}, \tag{7.B.37}$$

$$|E(d_j W_{ls}e^{it\bar{B}_m})| \le E|d_j W_{ls}||\gamma(t)|^{m-3}, \tag{7.B.38}$$

$$|E(\tilde{W}_{jk}W_{ls}e^{it\bar{B}_m})| \le E|\tilde{W}_{jk}W_{ls}||\gamma(t)|^{m-4}. \tag{7.B.39}$$

Substituting (7.B.37)–(7.B.39) into (7.B.36), we have, due to $|\gamma(t)| \le 1$,

$$|E(\tilde{b}'_{3m}e^{it\bar{B}_m})| \le C|\gamma(t)|^{m-4}\left[\frac{1}{n^{5/2}h^{d+2}}\sum_{l=1}^{m}\sum_{s=l+1}^{n}E|W_{ls}|\right.$$

$$+\frac{1}{n^{5/2}}\left(\sum_{j=1}^{m}\sum_{l<s}^{n}E|d_j W_{ls}| + \sum_{j=m+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}E|d_j W_{ls}|\right)$$

$$+\frac{1}{n^{7/2}}\left(\sum_{j=1}^{m}\sum_{k=j+1}^{n}\sum_{l<s}^{n}E|\tilde{W}_{jk}W_{ls}|\right.$$

$$\left.\left.+\sum_{j=m+1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=1}^{m}\sum_{s=l+1}^{n}E|\tilde{W}_{jk}W_{ls}|\right)\right].$$

Applying Lemma 4 of Robinson (1995), Lemmas 7 and 8, and (ix),

$$\begin{aligned}|E(\tilde{b}'_{3m}e^{it\bar{B}_m})| &\le C|\gamma(t)|^{m-4}\left\{\frac{mn}{n^{5/2}h^{d+3}}+\frac{mn^2}{n^{5/2}h}+\frac{1}{n^{7/2}}\left(\frac{mn}{h^{d+3}}+\frac{mn^3}{h^3}\right)\right\}\\ &= Cm|\gamma(t)|^{m-4}\left(\frac{1}{n^{3/2}h^{d+3}}+\frac{1}{n^{1/2}h}+\frac{1}{n^{5/2}h^{d+3}}+\frac{1}{n^{1/2}h^3}\right)\\ &\le \frac{Cm}{n^{1/2}h^3}|\gamma(t)|^{m-4}.\end{aligned}$$

(d): Write, using (7.3.7) and Lemma 4 of Robinson (1995),

$$E|\tilde{b}_{2m}|^2 \le C\left[\frac{1}{n^3h^{2d+4}}E\left|\sum_{i=1}^{m}V_i\right|^2 + \frac{1}{n^3}\left(E\left|\sum_{i=1}^{n}\sum_{s=1}^{m}d_iV_s\right|^2 + E\left|\sum_{i=1}^{m}\sum_{s=m+1}^{n}d_iV_s\right|^2\right)\right.$$

$$+\frac{1}{n^5}\left(E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{m}e_{ij}V_s\right|^2 + E\left|\sum_{i=1}^{m}\sum_{j=i+1}^{n}\sum_{s=m+1}^{n}e_{ij}V_s\right|^2\right)$$

$$+\frac{1}{n^7}\left(E\left|\sum_{i=1}^{n}\sum_{k<l}^{(i)}\sum_{s=1}^{m} V_i W_{kl} V_s\right|^2 + E\left|\sum_{i=1}^{n}\sum_{k=1}^{m}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n} V_i W_{kl} V_s\right|^2\right.$$

$$\left.\left.+E\left|\sum_{i=1}^{m}\sum_{k=m+1}^{n-1}{}^{(i)}\sum_{l=k+1}^{n}{}^{(i)}\sum_{s=m+1}^{n} V_i W_{kl} V_s\right|^2\right)\right]. \tag{7.B.40}$$

We show bounds only of some typical terms. Since V_i is an iid sequence with zero mean, by Lemma 1(d) of NR we have $E|\sum_{i=1}^{m} V_i|^2 m E|V_1|^2 \le Cm$. Writing

$$E\left|\sum_{i=1}^{n}\sum_{s=1}^{m} d_i V_s\right|^2 \le C\left(E\left|\sum_{i=1}^{m} d_i V_i\right|^2 + E\left|\sum_{i=1}^{m-1}\sum_{s=i+1}^{m} d_i V_s\right|^2\right.$$

$$\left.+E\left|\sum_{s=1}^{m}\sum_{i=s+1}^{n} d_i V_s\right|^2\right), \tag{7.B.41}$$

the first term in parentheses is bounded by

$$mE|d_1 V_1|^2 + m(m-1)E|d_1 V_1|E|d_2 V_2| \le Cm^2 \tag{7.B.42}$$

due to (iii) and Lemma 4(b). Since d_i and V_s are iid with zero mean,

$$E\left|\sum_{i=1}^{m-1}\sum_{s=i+1}^{m} d_i V_s\right|^2 = \sum_{i=1}^{m-1} E\left(d_i^2\right)\sum_{s=i+1}^{m} E\left(V_s^2\right) \le Cm^2 \tag{7.B.43}$$

by Lemma 1(d) of NR and (7.A.2) under (i). Similarly, using Lemma 4(a),

$$E\left|\sum_{s=1}^{m}\sum_{i=s+1}^{n} d_i V_s\right|^2 \le \sum_{s=1}^{m}\sum_{i=s+1}^{n} E\left(d_i^2\right)E\left(V_s^2\right) \le Cmn. \tag{7.B.44}$$

From (7.B.41)–(7.B.44),

$$E\left|\sum_{i=1}^{n}\sum_{s=1}^{m} d_i V_s\right|^2 \le C(m^2 + mn).$$

Similarly,

$$E\left|\sum_{i=1}^{m}\sum_{j=m+1}^{n} d_i V_s\right|^2 = \sum_{i=1}^{m} E\left(d_i^2\right)\sum_{s=m+1}^{n} E\left(V_s^2\right) \le Cmn.$$

We next consider

$$E\left|\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{s=1}^{m}e_{ij}V_s\right|^2 \le C\left\{E\left|\sum_{s=1}^{m-1}\sum_{i=s+1}^{n-1}\sum_{j=i+1}^{n}e_{ij}V_s\right|^2 + E\left|\sum_{i=1}^{m}\sum_{j=i+1}^{n}e_{ij}V_i\right|^2\right.$$

$$+E\left|\sum_{i=1}^{m-1}\sum_{s=i+1}^{m}\sum_{j=s+1}^{n}e_{ij}V_s\right|^2 + E\left|\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}e_{ij}V_j\right|^2$$

$$\left.+E\left|\sum_{i=1}^{m-2}\sum_{j=i+1}^{m-1}\sum_{s=j+1}^{m}e_{ij}V_s\right|^2\right\}. \tag{7.B.45}$$

Due to (iii), $E(e_{ij}\mid i)=E(e_{ij}\mid j)=0$, $E(V_s)=0$, and Lemma 6, the triple summation on the right of (7.B.45) is $O((m^3+m^2n+mn^2)h^{-d-4})$. Using Lemma 6 and Hölder's inequality, the second term in (7.B.45) is

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{j=i+1}^{n}E(e_{ij}V_i)^2+2\sum_{i=1}^{m-1}\sum_{k=i+1}^{m}\sum_{j=k+1}^{n}E(e_{ij}V_ie_{kj}V_k)\\&\quad\le C[mnE(e_{12}V_1)^2+m^2n\{E(e_{13}V_1)^2E(e_{23}V_2)^2\}^{1/2}]\\&\quad\le Cm^2nh^{-d-4}.\end{aligned} \tag{7.B.46}$$

Similarly, the fourth term of (7.B.45) is $O(m^3h^{-d-4})$. Using Lemma 5, as above, the terms involving $V_iW_{kl}V_s$ in (7.B.40) are $O((m^4+m^3n+m^2n^2+mn^3)h^{-d-2})$, so by (ix)

$$\begin{aligned}E|\tilde{b}_{2m}|^2 &\le \frac{C}{nh^{d+2}}\left(\frac{m}{n^2h^{d+2}}\right)+\frac{C}{n^3}(m^2+mn)+\frac{C}{n^5}(m^3+m^2n+mn^2)h^{-d-4}\\&\quad+\frac{C}{n^7}(m^4+m^3n+m^2n^2+mn^3)h^{-d-2}\\&\le Cm\left(\frac{1}{n^3h^{2d+4}}+\frac{1}{n^2}\right).\end{aligned}$$

(e): The derivation is similar, using Lemma 4 of Robinson (1995) and Lemmas 7 and 8. As in (d), we can show

$$\begin{aligned}E|\tilde{b}'_{3m}|^2 &\le \frac{C}{n^5h^{2d+4}}mnh^{-d-2}+\frac{C}{n^5}(m^3+m^2n+mn^2)h^{-d-2}\\&\quad+\frac{C}{n^7}(m^4+m^3n+m^2n^2+mn^3)h^{-3d-6}\\&\le\frac{Cm}{n^4h^{3d+6}}.\end{aligned}$$

(f): Write

$$|E\tilde{b}_3''e^{it\bar{B}_m}| = |EQ\tilde{W}e^{it\bar{B}_m}| \le |EQ_1\tilde{W}e^{it\bar{B}_m}| + |EQ_2\tilde{W}e^{it\bar{B}_m}|. \tag{7.B.47}$$

By (7.3.7),

$$\begin{aligned}
&|E(Q_1\tilde{W}e^{it\bar{B}_m})| \\
&\le \frac{C}{n^{7/2}}\left|\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n} E\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{ls}e^{it\bar{B}_m}\right| \\
&\le \frac{6C}{n^{7/2}}\sum_{j=1}^{n-3}\sum_{k=j+1}^{n-2}\sum_{l=k+1}^{n-1}\sum_{s=l+1}^{n} E|\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{ls}||\gamma(t)|^{m-4} \\
&\quad + \frac{6C}{n^{7/2}}\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{s=k+1}^{n} E|\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{ks}||\gamma(t)|^{m-3} \\
&\quad + \frac{C}{n^{7/2}}\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} E|\{(V_j+V_k)W_{jk}-\tilde{V}_j-\tilde{V}_k\}W_{jk}||\gamma(t)|^{m-2} \\
&\le Cn^{1/2}E|\{(V_1+V_2)W_{12}-\tilde{V}_1-\tilde{V}_2\}W_{34}||\gamma(t)|^{m-4} \\
&\quad + \frac{C}{n^{1/2}}E|\{(V_1+V_2)W_{12}-\tilde{V}_1-\tilde{V}_2\}W_{13}||\gamma(t)|^{m-3} \\
&\quad + \frac{C}{n^{3/2}}E|\{(V_1+V_2)W_{12}-\tilde{V}_1-\tilde{V}_2\}W_{12}||\gamma(t)|^{m-3}.
\end{aligned} \tag{7.B.48}$$

Using (i), (iii), Lemmas 1(d) and 4 of NR, (7.A.1), and Lemma 4 of Robinson (1995), the first expectation of (7.B.48) is bounded by

$$CE\{(|Y_1|+|Y_2|+1)|W_{12}|\}E|W_{34}| \le Ch^{-2}.$$

Using (i), Lemmas 1(d) and 4 of NR, (7.A.1), and Lemma 4 of Robinson (1995), the second expectation of (7.B.48) is bounded by

$$\begin{aligned}
&CE\{(|Y_1|+|Y_2|+1)|W_{12}||W_{13}|\} \\
&\quad \le CE\{(|Y_1|+|Y_2|+1)|W_{12}|E(|W_{13}||1)\} \\
&\quad \le Ch^{-1}E\{(|Y_1|+|Y_2|+1)(|Y_1|+1)|W_{12}|\}.
\end{aligned}$$

Similarly to Lemma 1 and (7.A.5), $E|Y_1W_{12}| + E|Y_1^2W_{12}| + E|Y_1Y_2W_{12}| = O(h^{-1})$, so that the above quantity is $O(h^{-2})$. The third expectation of (7.B.48) is bounded by

$$CE\left|V_1W_{12}^2\right| + E|\tilde{V}_1W_{12}| \le C(h^{-d-2}+h^{-1}) = O(h^{-d-2})$$

due to Lemmas 1(d) and 4 of NR, Lemma 4 of Robinson (1995), and Lemma 2. Therefore,

$$|E(Q_1\tilde{W}e^{it\bar{B}_m})| \le C\left(\frac{n^{1/2}}{h^2}+\frac{1}{n^{3/2}h^{d+2}}\right)|\gamma(t)|^{m-4} \le \frac{Cn^{1/2}}{h^2}|\gamma(t)|^{m-4}.$$

The second term of (7.B.47) is bounded, using (7.3.7), by

$$
\begin{aligned}
&\frac{C}{n^{9/2}}\left|\sum_{r=1}^{n}\sum_{j=1}^{n-1}{}^{(r)}\sum_{k=j+1}^{n}{}^{(r)}\sum_{l=1}^{n-1}\sum_{s=l+1}^{n} E(V_r W_{jk} W_{ls} e^{it\bar{B}_m})\right| \\
&\le \frac{C}{n^{9/2}}\sum_{r=1}^{n-4}\sum_{j=r+1}^{n-3}\sum_{k=j+1}^{n-2}\sum_{l=k+1}^{n-1}\sum_{s=l+1}^{n} E|V_r W_{jk} W_{ls}||\gamma(t)|^{m-5} \\
&+ \frac{C}{n^{9/2}}\sum_{r=1}^{n-3}\sum_{j=r+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{s=k+1}^{n} E|V_r W_{jk} W_{ks}||\gamma(t)|^{m-4} \\
&+ \frac{C}{n^{9/2}}\sum_{r=1}^{n-3}\sum_{j=r+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{s=k+1}^{n} E|V_r W_{jk} W_{rs}||\gamma(t)|^{m-4} \\
&+ \frac{C}{n^{9/2}}\sum_{r=1}^{n-2}\sum_{j=r+1}^{n-1}\sum_{k=j+1}^{n} E|V_r W_{jk} W_{jk}||\gamma(t)|^{m-3} \\
&+ \frac{C}{n^{9/2}}\sum_{r=1}^{n-2}\sum_{j=r+1}^{n-1}\sum_{k=j+1}^{n} E|V_r W_{jk} W_{rk}||\gamma(t)|^{m-3} \\
&\le Cn^{1/2}E|V_1|E|W_{23}|E|W_{45}||\gamma(t)|^{m-5} \\
&+ \frac{C}{n^{1/2}}(E|V_1|E|W_{23}W_{24}| + E|V_1 W_{14}|E|W_{23}|)|\gamma(t)|^{m-4} \\
&+ \frac{C}{n^{3/2}}(E|V_1|E|W_{23}|^2 + E|V_1 W_{23} W_{13}|)\gamma(t)|^{m-3} \\
&\le C\left(\frac{n^{1/2}}{h^2} + \frac{1}{n^{1/2}h^2} + \frac{1}{n^{3/2}h^{d+2}}\right)|\gamma(t)|^{m-5}
\end{aligned}
$$

by (i), (iii), Lemmas 1(d) and 4 of NR, Lemma 4 of Robinson (1995), and Lemma 1. Then apply (ix).

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