Research Article
Noncoherence of a Causal Wiener Algebra Used in Control Theory

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Received 18 March 2008; Accepted 13 June 2008

Recommended by Ülle Kotta

Let $C_{\geq 0} := \{ s \in \mathbb{C} | \text{Re}(s) \geq 0 \}$, and let $\mathcal{W}$ denote the ring of all functions $f : C_{\geq 0} \rightarrow \mathbb{C}$ such that $f(s) = f_\alpha(s) + \sum_{k=0}^\infty f_k e^{-st} (s \in C_{\geq 0})$, where $f_\alpha \in L^1(0, \infty)$, $(f_k)_{k=0}^\infty \in \ell^1$, and $0 = t_0 < t_1 < t_2 < \cdots$ equipped with pointwise operations. (Here $\hat{\cdot}$ denotes the Laplace transform.) It is shown that the ring $\mathcal{W}$ is not coherent, answering a question of Alban Quadrat. In fact, we present two principal ideals in the domain $\mathcal{W}$ whose intersection is not finitely generated.

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1. Introduction

The aim of this paper is to show that the ring $\mathcal{W}$ (defined below) is not coherent.

We first recall the notion of a coherent ring.

Definition 1.1. Let $R$ be a commutative ring with identity element 1, and let $R^m = R \times \cdots \times R$ ($m$ times). Suppose that $f = (f_1, \ldots, f_m) \in R^m$.

(1) An element $(g_1, \ldots, g_m) \in R^m$ is called a relation on $f$ if

$$g_1 f_1 + \cdots + g_m f_m = 0. \quad (1.1)$$

(2) Let $f^\perp$ denote the set of all relations on $f \in R^m$. (Then $f^\perp$ is an $R$-submodule of the $R$-module $R^m$.)

(3) The ring $R$ is called coherent if for all $m \in \mathbb{N}$ and all $f \in R^m$, $f^\perp$ is finitely generated, that is, there exists a $d \in \mathbb{N}$ and there exist $g_j \in f^\perp$, $j \in \{1, \ldots, d\}$, such that for all $g \in f^\perp$, there exist $r_j \in R$, $j \in \{1, \ldots, d\}$ such that $g = r_1 g_1 + \cdots + r_d g_d$. 
An integral domain is coherent if and only if the intersection of any two finitely generated ideals of the ring is again finitely generated; see [1, Theorem 2.3.2, page 45].

The coherence of some rings of analytic functions has been investigated in earlier works. For example, McVoy and Rubel [2] showed that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disc algebra $A(\mathbb{D})$ is not. Mortini and von Renteln proved that the Wiener algebra $W^*(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc) is not coherent [3]. In this article, we will show that the ring $\mathcal{K}^+$ (defined below, and which is useful in control theory) is not coherent.

**Notation 1.** Throughout the article, we will use the following notation:

\[ \mathbb{C}_{\geq 0} := \{ s \in \mathbb{C} \mid \text{Re}(s) \geq 0 \}. \]  

**Definition 1.2.** Let $\mathcal{K}^+$ denote the Banach algebra

\[ \mathcal{K}^+ = \left\{ f : \mathbb{C}_{\geq 0} \rightarrow \mathbb{C} \mid \begin{array}{c} f(s) = \tilde{f}_a(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \ (s \in \mathbb{C}_{\geq 0}), \\ f_a : (0, \infty) \rightarrow \mathbb{C}, \ f_a \in L^1(0, \infty), \\ \forall k \geq 0, \ f_k \in \mathbb{C}, \ (f_k)_{k \geq 0} \in \ell^1, \\ \forall k \geq 0, \ t_k \in \mathbb{R}, \ 0 = t_0 < t_1 < t_2 < \cdots \end{array} \right\} \]  

equipped with pointwise operations and the norm

\[ \| f \|_{\mathcal{K}^+} := \| f_a \|_{L^1} + \| (f_k)_{k \geq 0} \|_{\ell^1}. \]  

Here $\tilde{f}_a$ denotes the Laplace transform of $f_a$, given by

\[ \tilde{f}_a(s) = \int_0^{\infty} e^{-st} f_a(t) dt, \quad s \in \mathbb{C}_{\geq 0}. \]  

The above algebra arises as a natural class of transfer functions of stable distributed parameter systems in control theory; see [4, 5].

Our main result is the following.

**Theorem 1.3.** The ring $\mathcal{K}^+$ is not coherent.

The relevance of the coherence property in control theory can be found in [6, 7]. We will prove Theorem 1.3 following the same method as in the proof of the noncoherence of $W^*(\mathbb{D})$ given by Mortini and von Renteln in [3].

In Section 3, we will give the proof of Theorem 1.3. But before doing that, in Section 2, we first prove a few technical results needed in the sequel.

**2. Preliminaries**

We first recall the definition of the Hardy algebra $H^\infty$ of the open right half plane.
Definition 2.1. Let $H^\infty$ denote the Hardy space of all bounded analytic functions in the open right half plane equipped with the norm

$$
\|\varphi\|_\infty := \sup_{\text{Re}(s)>0} |\varphi(s)|, \quad \varphi \in H^\infty.
$$

(2.1)

In order to prove our main result (Theorem 1.3), we will use the relation between the convergence in $H^\infty$ versus that in $\mathcal{K}^+$.

Lemma 2.2. If $f \in \mathcal{K}^+$, then $f \in H^\infty$ and $\|f\|_\infty \leq \|f\|_{\mathcal{K}^+}$.

Proof. Let

$$
f(s) = \hat{f}_a(s) + \sum_{k=0}^\infty f_k e^{-stk} \quad (s \in \mathbb{C}_{\geq 0}).
$$

(2.2)

For $s \in \mathbb{C}_{\geq 0}$, we have

$$
|\hat{f}_a(s)| = \left| \int_0^\infty e^{-st} f_a(t) dt \right| \leq \int_0^\infty e^{-\text{Re}(s)t} |f_a(t)| dt \leq \int_0^\infty 1 \cdot |f_a(t)| dt = \|f_a\|_{L^1},
$$

(2.3)

and moreover,

$$
\left| \sum_{k=0}^\infty f_k e^{-stk} \right| \leq \sum_{k=0}^\infty |f_k| e^{-\text{Re}(s)tk} \leq \sum_{k=0}^\infty |f_k| \cdot 1 = \|f_k\|_{L^1}.
$$

(2.4)

So the result follows. \qed

The maximal ideal $m_0$ (defined below) of $\mathcal{K}^+$ will play an important role in the remainder of this article.

Notation 2. Let $m_0$ denote the kernel of the complex algebra homomorphism $f \mapsto f(0) : \mathcal{K}^+ \to \mathbb{C}$, that is,

$$
m_0 := \{ f \in \mathcal{K}^+ \mid f(0) = 0 \}.
$$

Then $m_0$ is a maximal ideal of $\mathcal{K}^+$, and this maximal ideal plays an important role in the proof of our main result in the next section. We will prove a few technical results about $m_0$ in this section, which will be used in the sequel. The following result is analogous to [3, Lemma 1].

Lemma 2.3. Let $L \neq (0)$ be an ideal in $\mathcal{K}^+$ contained in the maximal ideal $m_0$. If $L = Lm_0$, that is, if every function $f \in L$ can be factorized in a product $f = hg$ of two functions $h \in L$ and $g \in m_0$, then $L$ cannot be finitely generated.

Proof. Suppose that

$$
L = (f_1, \ldots, f_N) \neq (0)
$$

(2.5)
is a finitely generated ideal in $\mathcal{I}^+$ contained in the maximal ideal $m_0$. By our assumption, there are functions $h_n \in L$, $g_n \in m_0$ with

$$f_n = h_n g_n \quad (n = 1, \ldots, N). \tag{2.6}$$

Since $h_n \in L$, there exist functions $q_k^{(n)} \in \mathcal{I}^+$ with

$$h_n = \sum_{k=1}^{N} q_k^{(n)} f_k \quad (n = 1, \ldots, N; \ k = 1, \ldots, N). \tag{2.7}$$

From this it follows that

$$\sum_{n=1}^{N} |h_n| \leq NC \sum_{n=1}^{N} |f_n| = NC \sum_{n=1}^{N} |h_n g_n| \quad \text{in } \mathbb{C}_{\geq 0}, \tag{2.8}$$

where $C$ is a constant chosen so that

$$\|q_k^{(n)}\|_{\infty} \leq C, \quad \forall k \text{ and } n. \tag{2.9}$$

(Here $\|\|_{\infty}$ denotes the supnorm over $\mathbb{C}_{\geq 0}$.) This implies together with the Cauchy-Schwarz inequality that

$$\sum_{n=1}^{N} |h_n|^2 \leq \left( \sum_{n=1}^{N} |h_n| \right)^2 \leq N^2 C^2 \left( \sum_{n=1}^{N} |h_n g_n| \right)^2 \leq N^2 C^2 \left( \sum_{n=1}^{N} |h_n|^2 \right) \left( \sum_{n=1}^{N} |g_n|^2 \right). \tag{2.10}$$

This inequality holds for all $s \in \mathbb{C}_{\geq 0}$. With $\delta := 1/(N^2 C^2)$, we obtain the inequality

$$\delta \leq \sum_{n=1}^{N} |g_n(s)|^2 \tag{2.11}$$

for all points $s \in E$, where

$$E := \left\{ s \in \mathbb{C}_{\geq 0} \mid \sum_{n=1}^{N} |h_n(s)|^2 > 0 \right\}. \tag{2.12}$$

Since $L \neq (0)$, $E$ is a dense subset of $\mathbb{C}_{\geq 0}$ (for otherwise, if $s_0 \in \mathbb{C}_{\geq 0}$ is such that it has a neighbourhood $V$ in $\mathbb{C}_{\geq 0}$ where there is no point of $E$, then each $h_n$ is identically zero in $V$, and by the identity theorem for holomorphic functions, each $h_n$ is zero; consequently each $f_n$ is zero, and so $L = (0)$, a contradiction). So by continuity, inequality (2.11) holds in $\mathbb{C}_{\geq 0}$. But this contradicts the fact that each $g_n$ vanishes at 0.

**Remark 2.4.** Lemma 2.3 can be proved purely algebraically using Nakayama’s lemma. Indeed, it holds in the following more general algebraic situation: if $I$ is a nonzero ideal of a commutative domain $D$ contained in a maximal ideal $M$ and $I = IM$, then $I$ cannot be finitely generated. However, we have given an analytic proof in our special case above.
Since every maximal ideal is closed, \( m_0 \) is a commutative Banach subalgebra of \( \mathcal{K}^+ \), but obviously without identity element. But there is a substitute, namely the notion of the approximate identity, which turns out to be useful.

**Definition 2.5.** Let \( R \) be a commutative Banach algebra (without identity element). One says that \( R \) has an **approximate identity** if there exists a bounded sequence \( (e_n) \) of elements \( e_n \) in \( R \) such that for any \( f \in R \),

\[
\lim_{n \to \infty} \|e_n f - f\| = 0.
\]

We will now prove the following result, which shows that the maximal ideal \( m_0 \) in \( \mathcal{K}^+ \) has an approximate identity.

**Theorem 2.6.** Let

\[
e_n := \frac{s}{s + 1/n}, \quad n \in \mathbb{N}.
\]

Then \( (e_n)_{n \in \mathbb{N}} \) is an approximate identity for \( m_0 \).

The existence of an approximate identity for the maximal ideal \( m_0 \) in \( \mathcal{K}^+ \) is not obvious. In order to prove Theorem 2.6, we will need the following lemma.

**Lemma 2.7.** Suppose \( \tilde{f} \in m_0 \). Then, for all \( \epsilon > 0 \), there exists an \( \tilde{p} \in m_0 \) such that \( \tilde{p} \) has compact support in \( [0, \infty) \), and \( \|\tilde{f} - \tilde{p}\|_{\mathcal{K}^+} < \epsilon \).

**Proof.** Let \( \epsilon > 0 \) be given. Suppose that

\[
f = f_a + \sum_{k=0}^{\infty} f_k \delta(-t_k),
\]

where \( f_a \in L^1[0, \infty) \), \( (f_k)_{k \geq 0} \in \ell^1 \), and \( 0 = t_0 < t_1 < t_2 < \cdots \). Since \( \int_0^{\infty} |f_a(t)| \, dt < \infty \), we can choose an \( M > 0 \) large enough such that

\[
\int_M^{\infty} |f_a(t)| \, dt < \frac{\epsilon}{4}.
\]

With \( p_a(t) := f_a(t) \) if \( t \in [0, M] \), and 0 otherwise, we have that \( p_a \in L^1[0, \infty) \) is compactly supported and

\[
\|p_a - f_a\|_{L^1} < \frac{\epsilon}{4}.
\]

Furthermore, select \( N \in \mathbb{N} \) such that

\[
\sum_{k > N} |f_k| < \frac{\epsilon}{4}.
\]

Now let \( T \in (0, \infty) \) be any number satisfying \( t_N < T < t_{N+1} \), and define

\[
f_T := -\left( \int_0^{T} p_a(t) \, dt + \sum_{0 \leq k \leq N} f_k \right).
\]
Set
\[ p := p_a + \sum_{0 \leq k \leq N} f_k \delta(\cdot - t_k) + f_T \delta(\cdot - T). \]  
(2.20)

Then \( \hat{p} \in \mathcal{K}^* \) and
\[
\hat{p}(0) = \int_0^\infty p(t) dt = \int_0^\infty p_a(t) dt + \sum_{0 \leq k \leq N} f_k + f_T = 0.
\]  
(2.21)

So \( \hat{p} \in \mathfrak{m}_0 \). Clearly \( p \) has compact support contained in \([0, \infty)\). We have
\[
|f_T| = \left| \int_0^\infty p_a(t) dt + \sum_{0 \leq k \leq N} f_k \right| \\
= \left| \int_0^\infty f_a(t) dt + \sum_{k=0}^\infty f_k + \int_0^\infty (p_a(t) - f_a(t)) dt - \sum_{k>N} f_k \right| \\
\leq \left| \int_0^\infty f_a(t) dt \right| + \|p_a - f_a\|_{L^1} + \sum_{k>N} |f_k| \\
= |\hat{f}(0)| + \|p_a - f_a\|_{L^1} + \sum_{k>N} |f_k| \\
< 0 + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\]  
(2.22)

Thus
\[
\|\hat{f} - \hat{p}\|_{\mathcal{K}^*} = \|f_a - p_a\|_{L^1} + \sum_{k>N} |f_k| + |f_T| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\]  
(2.23)

This completes the proof. \( \square \)

We are now ready to prove the existence of an approximate identity for the maximal ideal \( \mathfrak{m}_0 \) in \( \mathcal{K}^* \).

**Proof of Theorem 2.6.** We have
\[
e_n = \frac{s}{s + 1/n} = \frac{s + 1/n - 1/n}{s + 1/n} = 1 - \frac{1}{n} + \frac{1}{s + 1/n} = 1 + \frac{1}{n} e^{-t/n}.
\]  
(2.24)

Thus for an \( n \in \mathbb{N} \),
\[
\|e_n\|_{\mathcal{K}^*} = \left\| \frac{1}{n} e^{-t/n} \right\|_{L^1} + |1| = 1 + 1 = 2.
\]  
(2.25)

Given \( \hat{f} \in \mathcal{K}^* \), and \( \epsilon > 0 \) arbitrarily small, in view of Lemma 2.7, we can find a \( \hat{p} \in \mathfrak{m}_0 \) such that \( p \) has compact support and \( \|\hat{f} - \hat{p}\|_{\mathcal{K}^*} < \epsilon \). Then
\[
\|e_n\hat{f} - \hat{p}\|_{\mathcal{K}^*} \leq \|e_n\hat{p} - \hat{p}\|_{\mathcal{K}^*} + \|e_n\|_{\mathcal{K}^*} \|\hat{f} - \hat{p}\|_{\mathcal{K}^*} + \|\hat{f} - \hat{p}\|_{\mathcal{K}^*}.
\]  
(2.26)
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So it is enough to prove that
\[
\lim_{n \to \infty} \|e_n \tilde{p} - \hat{p}\|_{\mathcal{K}_0} = 0
\]  \hspace{1cm} (2.27)

for all \( \tilde{p} \in \mathcal{M}_0 \) such that \( p \) has compact support in \([0, \infty)\). We do this below.

We have
\[ e_n \tilde{p} - \hat{p} = \frac{s + 1/n - 1/n}{s + 1/n} \tilde{p} - \hat{p} = -\frac{1}{n} \left( \frac{1}{s + 1/n} \tilde{p} - \hat{p} \right) = \frac{1}{n} (e^{-1/n} * p). \]  \hspace{1cm} (2.28)

Let \( C \) denote the convolution \( e^{-1/n} * p \):
\[ C(t) := \int_0^t e^{-(t-\tau)/n} p(\tau) d\tau. \]  \hspace{1cm} (2.29)

We note that \( C \in L^1(0, \infty) \), since \( L^1(0, \infty) \) is an ideal in \( \mathcal{K}_0^\ast \). Let \( T > 0 \) be such that \( \text{supp}(p) \subset [0, T] \).

We have
\[
\|e_n \tilde{p} - \hat{p}\|_{\mathcal{K}_0} = \frac{1}{n} \|C\|_{L^1} = \frac{1}{n} \int_0^\infty |C(t)| dt = \frac{1}{n} \int_0^T |C(t)| dt + \frac{1}{n} \int_T^\infty |C(t)| dt. \]  \hspace{1cm} (2.31)

We estimate \((I)\) as follows:
\[
(I) = \frac{1}{n} \int_0^T |C(t)| dt = \frac{1}{n} \int_0^T \left| \int_0^t e^{-(t-\tau)/n} p(\tau) d\tau \right| dt \leq \frac{1}{n} \int_0^T \int_0^t e^{-(t-\tau)/n} |p(\tau)| d\tau dt \leq \frac{1}{n} \left( \int_0^T \int_0^t 1 \cdot |p(\tau)| d\tau dt \right). \]  \hspace{1cm} (2.32)

Since the integral \((III)\) does not depend on \( n \), we obtain that
\[ \lim_{n \to \infty} \frac{1}{n} \int_0^T |C(t)| dt = 0. \]  \hspace{1cm} (2.33)

Furthermore,
\[
(II) = \frac{1}{n} \int_T^\infty |C(t)| dt \]
\[
= \frac{1}{n} \int_T^\infty e^{-t/n} \left| \int_0^t e^{\tau/n} p(\tau) d\tau \right| dt \]
\[
= \frac{1}{n} \int_T^\infty e^{-t/n} \left| \int_0^\infty e^{\tau/n} p(\tau) d\tau \right| dt \quad \text{(since \( \text{supp}(p) \subset [0, T] \))}
\]
\[
= \frac{1}{n} \int_T^\infty e^{-t/n} \left| \hat{p} \left( -\frac{1}{n} \right) \right| dt.
\]  \hspace{1cm} (2.34)
Since $p$ has compact support in $[0,T]$, $\hat{p}$ is an entire function by the Payley-Wiener theorem (see, e.g., [8, Theorem 7.2.3, page 122]). Consequently,

$$
(II) = \frac{1}{n} \int_{-\infty}^{\infty} e^{-t/n} \left| \hat{p}\left( -\frac{1}{n} \right) \right| \left| dt = e^{-T/n} \left| \hat{p}\left( -\frac{1}{n} \right) \right| \right| \rightarrow n \rightarrow \infty 1 \cdot |\hat{p}(0)| = 1 \cdot 0 = 0. \quad (2.35)
$$

This completes the proof. \(\square\)

We will also need the following lemma, which is basically a repetition of key steps from Browder’s proof of Cohen’s factorization theorem; see [9, Theorem 1.6.5, page 74]. We will need this version since in our application in the proof of Theorem 1.3, we are not able to use Cohen’s factorization theorem directly.

**Lemma 2.8.** Let $f_1, f_2 \in m_0$, and $\delta > 0$. Let $U(\mathcal{K}^*)$ denote the set of all invertible elements in $\mathcal{K}^*$. Then there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{K}^*$ such that

1. for all $n \in \mathbb{N}$, $g_n \in U(\mathcal{K}^*)$;
2. $(g_n)_{n \in \mathbb{N}}$ is convergent in $\mathcal{K}^*$ to a limit $g \in m_0$;
3. for all $n \in \mathbb{N}$, $\|g_n^{-1} f_i - g_n^{-1} f_j\|_{\mathcal{K}^*} \leq \delta/2^n$, $i = 1, 2$.

**Proof.** We will first prove two general results in steps (A) and (B), which we will use in the rest of the proof.

(A) Let $e \in m_0$ and $\|e\|_{\mathcal{K}^*} \leq K$, where $K > 1$. Then $1 - c + ce \in U(\mathcal{K}^*)$, where $c$ is a number chosen such that

$$
0 < c < \frac{1}{4K} < \frac{1}{4}. \quad (2.36)
$$

Indeed,

$$
\left\| \frac{c}{c-1} e \right\|_{\mathcal{K}^*} < \frac{1}{3/4 \cdot 3/4} \cdot K = \frac{1}{3} < 1, \quad (2.37)
$$

and so

$$(1 - c + ce)^{-1} = \frac{1}{1 - c} \sum_{k=0}^{\infty} \left( \frac{c}{c-1} \right)^k e^k. \quad (2.38)$$

(B) Furthermore, under the same assumptions and notation as in (A) above, we now show that if $\|eF - F\|_{\mathcal{K}^*}$ is small for some $F$, then so is $\|EF - F\|_{\mathcal{K}^*}$, where $E := (1 - c + ce)^{-1}$. Since

$$
1 = \frac{1}{1 - c} \sum_{k=0}^{\infty} \left( \frac{c}{c-1} \right)^k, \quad (2.39)
$$

we have

$$
\|EF - F\|_{\mathcal{K}^*} = \left\| \frac{1}{1 - c} \sum_{k=0}^{\infty} \left( \frac{c}{c-1} \right)^k (e^k F - F) \right\|_{\mathcal{K}^*} \leq \frac{1}{1 - c} \sum_{k=0}^{\infty} \left( \frac{c}{1 - c} \right)^k \|e^k F - F\|_{\mathcal{K}^*}. \quad (2.40)
$$

But

$$
\|e^k F - F\|_{\mathcal{K}^*} = \|e^{k+1} F - e^k F\|_{\mathcal{K}^*} \leq \sum_{j=0}^{k-1} \|e^j\|_{\mathcal{K}^*} \|eF - F\|_{\mathcal{K}^*} \leq \|e|F - F\|_{\mathcal{K}^*} \sum_{j=0}^{k-1} \|e^j\|_{\mathcal{K}^*} < \|eF - F\|_{\mathcal{K}^*} \frac{K^k}{K-1}. \quad (2.41)
$$
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Hence

\[ \|EF - F\|_{K^0} < \|eF - F\|_{K^0} \leq \frac{1}{1 - c} \sum_{k=0}^{\infty} \frac{1}{K - 1} \left( \frac{1}{4(1 - c)} \right)^k < \frac{2}{K - 1} \|eF - F\|_{K^0}. \] (2.42)

This estimate will be used in constructing the sequence of \( g_n \)'s.

Let \((e_n)_{n \in \mathbb{N}}\) denote the approximate identity for \( m_0 \) from Theorem 2.6. Let \( K > 1 \) be such that \( \|e_n\|_{K^0} \leq K \) for all \( n \in \mathbb{N} \). Choose \( c \) such that

\[ 0 < c < \frac{1}{4K} < \frac{1}{4}. \] (2.43)

We will inductively define a sequence \((e_{m_k})_{k \in \mathbb{N}}\) with terms from the approximate identity for \( m_0 \) such that if

\[ g_n := c \sum_{k=1}^{n} (1 - c)^{k-1} e_{m_k} + (1 - c)^n, \] (2.44)

then we have \( \|f_i - g_1^{-1} f_i\|_{K^0} < \delta/2, \ i = 1, 2, \) and

(P1) for all \( n \in \mathbb{N}, g_n \in \mathcal{U}(K^0), \)

(P2) for all \( n \in \mathbb{N}, \|g_n^{-1} f_i - g_{n+1}^{-1} f_i\|_{K^0} < \delta/2^n, \ i = 1, 2. \)

Since \((e_n)_{n \in \mathbb{N}}\) is an approximate identity for \( m_0 \), we can choose \( m_1 \) such that

\[ \|e_{m_i} f_i - f_i\|_{K^0} \leq \frac{\delta}{4} (K - 1), \ i = 1, 2. \] (2.45)

Define \( g_1 = ce_{m_1} + 1 - c \). So by (A), \( g_1 \in \mathcal{U}(K^0) \) and using the calculation in (B), we see that

\[ \|f_i - g_1^{-1} f_i\|_{K^0} < \frac{\delta}{2}, \ i = 1, 2. \] (2.46)

Suppose that \( e_{m_1}, \ldots, e_{m_n} \) have been constructed, so that \( g_n \) defined by (2.44) satisfies (P1) and (P2). We assert that if we choose \( e_{m_{n+1}} \) such that

\[ \|e_{m_{n+1}} f_i - f_i\|_{K^0} \quad (i = 1, 2), \quad \|e_{m_{n+1}} e_{m_k} - e_{m_k}\|_{K^0}, \quad (1 \leq k \leq n) \] (2.47)

are sufficiently small, then \( g_{n+1} \) defined by (2.44) satisfies (P1) and (P2), completing the induction step.

Indeed, if \( E := (1 - c + ce_{m_{n+1}})^{-1} \), we have

\[ g_n = E^{-1} c \sum_{k=1}^{n} (1 - c)^{k-1} E e_{m_k} + (1 - c)^n, \]

\[ g_{n+1} = E^{-1} c \sum_{k=1}^{n} (1 - c)^{k-1} E e_{m_k} + (1 - c)^n. \] (2.48)
Let $G_n$ be defined by

$$G_n = c \sum_{k=1}^{n} (1 - c)^{k-1} E e_{m_k} + (1 - c)^n. \quad (2.49)$$

Then we have

$$\|G_n - g_n\|_{\mathcal{K}} < \epsilon \sum_{k=1}^{n} (1 - c)^{k-1} \|E e_{m_k} - e_{m_k}\|_{\mathcal{K}} < \max_{1 \leq k \leq n} \|E e_{m_k} - e_{m_k}\|_{\mathcal{K}} < \frac{2}{K - 1} \max_{1 \leq k \leq n} \|e_{m_{i_k}} e_{m_k} - e_{m_k}\|_{\mathcal{K}}. \quad (2.50)$$

Hence $G_n \in U(\mathcal{K})$ and moreover $\|G_n^{-1} - g_n^{-1}\|_{\mathcal{K}}$ is small, provided only that $\|e_{m_{i_k}} e_{m_k} - e_{m_k}\|_{\mathcal{K}}$ is small for $k = 1, \ldots, n$. (Indeed, this is because $U(\mathcal{K})$ is an open set in $\mathcal{K}$.)

Since $g_{n+1} = E^{-1}G_n$, we have then $g_{n+1} \in U(\mathcal{K})$, $g_{n+1}^{-1} G_n = G_n^2 E$, and so for $i = 1, 2$,

$$\|g_{n+1}^{-1} f_i - g_n^{-1} f_i\|_{\mathcal{K}} = \|G_n^{-1} E f_i - g_n^{-1} f_i\|_{\mathcal{K}} \leq \|G_n^{-1} E f_i - g_n^{-1} E f_i\|_{\mathcal{K}} + \|g_n^{-1} E f_i - g_n^{-1} f_i\|_{\mathcal{K}} \leq \|G_n^{-1} - g_n^{-1}\|_{\mathcal{K}} \|E f_i\|_{\mathcal{K}} + \|g_n^{-1}\|_{\mathcal{K}} \|E f_i - f_i\|_{\mathcal{K}}. \quad (2.51)$$

Moreover, recall that by (B), we know that

$$\|E f_i - f_i\|_{\mathcal{K}} \leq \frac{2}{K - 1} \|e_{m_{i_k}} f_i - f_i\|_{\mathcal{K}}, \quad i = 1, 2. \quad (2.52)$$

Thus if $\|e_{m_{i_k}} f_i - f_i\|_{\mathcal{K}}$ (i = 1, 2) and $\|e_{m_{i_k}} e_{m_k} - e_{m_k}\|_{\mathcal{K}}$ (1 \leq k \leq n) are sufficiently small, we will have $\|g_{n+1}^{-1} f_i - g_n^{-1} f_i\|_{\mathcal{K}}$ as small as we please. This completes the induction step.

Since $\|e_{m_k}\|_{\mathcal{K}} \leq K, 0 < 1 - c < 1$, and $\mathcal{K}$ is a Banach algebra, it follows that

$$g_n \rightarrow \epsilon \sum_{k=1}^{\infty} (1 - c)^{k-1} e_{m_k} =: g \in m_0, \quad (2.53)$$

and the proof is completed. \qed

### 3. Noncoherence of $\mathcal{K}$

**Proof of Theorem 1.3.** We will use the characterization that an integral domain is coherent if and only if the intersection of any two finitely generated ideals of the ring is again finitely generated; see [1, Theorem 2.3.2, page 45]. In fact, we present two finitely generated ideals $I$ and $J$ such that $I \cap J$ is not finitely generated.
Let \( p, S \) be given by
\[
p = (1 - e^{-s})^3, \quad S = e^{-((1 + e^{-s})/(1 - e^{-s})).}
\] (3.1)

Clearly we have \( p \in m_0 \).

It is known (see, e.g., [3, Remark after Theorem 1, page 224]) that
\[
(1 - z)^3 e^{-(1+z)/(1-z)} \in W^+(D) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \mid (z \in D) \mid \sum_{n=0}^{\infty} |a_n| < \infty \right\}. \tag{3.2}
\]

Here \( \overline{D} := \{ z \in \mathbb{C} \mid |z| \leq 1 \} \). So if \( a_n \)'s are defined via
\[
(1 - z)^3 e^{-(1+z)/(1-z)} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in D,
\] (3.3)
then we have
\[
\sum_{k=0}^{\infty} |a_k| < \infty. \tag{3.4}
\]

If \( \text{Re}(s) > 0 \), then \( e^{-s} \in D \), and so from (3.3), we have
\[
pS = a_0 + a_1 e^{-s} + a_2 e^{-2s} + a_3 e^{-3s} + \cdots, \quad \text{Re}(s) > 0. \tag{3.5}
\]

Since \( \sum_{k=0}^{\infty} |a_k| < \infty \), the right-hand side in (3.5) belongs to \( \mathcal{K}^+ \). So \( pS \in \mathcal{K}^+ \).

We define the ideals \( I = (p) \) and \( f = (pS) \) of \( \mathcal{K}^+ \).

Let
\[
K := \{ pSf \mid f \in \mathcal{K}^+ \text{ and } Sf \in \mathcal{K}^+ \}. \tag{3.6}
\]

We claim that \( K = I \cap f \). Trivially \( K \subset I \cap f \). To prove the reverse inclusion, let \( g \in I \cap f \). Then there exist two functions \( f \) and \( h \) in \( \mathcal{K}^+ \) such that \( g = ph = pSf \). Since \( p \neq 0 \) and \( \mathcal{K}^+ \) is an integral domain, we obtain that \( Sf = h \in \mathcal{K}^+ \). So \( g \in K \).

Let \( L \) denote the ideal
\[
L := \{ f \in \mathcal{K}^+ \mid Sf \in \mathcal{K}^+ \}. \tag{3.7}
\]

Then \( K := pSL \). Since \( S \) has a singularity at \( s = 0 \), it follows that \( L \subset m_0 \). We will show that \( L = Lm_0 \). Let \( f \in L \). We would like to factor \( f = hg \) with \( h \in L \) and \( g \in m_0 \). Applying Lemma 2.8 with \( f_1 := f \in m_0 \) and \( f_2 := Sf \in m_0 \), for any \( \delta > 0 \), there exists a sequence \( (g_n)_{n \in \mathbb{N}} \) in \( \mathcal{K}^+ \) such that

(1) for all \( n \in \mathbb{N} \), \( g_n \in U(\mathcal{K}^+) \);
(2) \( (g_n)_{n \in \mathbb{N}} \) is convergent in \( \mathcal{K}^+ \) to a limit \( g \in m_0 \);
(3) for all \( n \in \mathbb{N} \),
\[
\| g_n^{-1} f - g_{n+1}^{-1} f \|_{\mathcal{K}^+} \leq \frac{\delta}{2^n}, \quad \| g_n^{-1} Sf - g_{n+1}^{-1} Sf \|_{\mathcal{K}^+} \leq \frac{\delta}{2^n}. \tag{3.8}
\]

Put
\[
h_n := g_n^{-1} f, \quad H_n := g_n^{-1} Sf. \tag{3.9}
\]
Then \( h_n \in m_0 \). Also \( H_n \in m_0 \), since \( |S| \) is bounded by 1 on \( \text{Re}(s) > 0 \) and \( f(0) = 0 \). The estimates above imply that \((h_n)_{n \in \mathbb{N}}\) and \((H_n)_{n \in \mathbb{N}}\) are Cauchy sequences in \( \mathcal{K}^* \). Since \( m_0 \) is closed, they converge to elements \( h \) and \( H \), respectively, in \( m_0 \), that is, \( h_n = g_n^{-1}f \to h \) and \( H_n = g_n^{-1}Sf = Sh_n \to H \). Since convergence in \( \mathcal{K}^* \) implies convergence in \( H^\infty \) (Lemma 2.2), it follows that

\[
\begin{align*}
    h_n &\to H^\infty h \quad \text{(since } h_n \to \mathcal{K}^* h) , \\
    Sh_n &\to H^\infty Sh \quad \text{(since } h_n \to H^\infty h, S \in H^\infty) , \\
    Sh_n &\to H^\infty H \quad \text{(since } H_n \to \mathcal{K}^* H) 
\end{align*}
\]

and so by the uniqueness of the limit of the sequence \((Sh_n)_{n \in \mathbb{N}}\) in \( H^\infty \), we have \( Sh = H \). Also, in \( \mathcal{K}^* \)-norm we have

\[
f = \lim_{n \to \infty} h_n g_n = h g \tag{3.11}
\]

since multiplication is continuous in the Banach algebra \( \mathcal{K}^* \). Since \( h \) and \( Sh = H \) belong to \( m_0 \subset \mathcal{K}^* \), we see that \( h \in L \). Moreover, as \( g \in m_0 \), we have got the desired factorization and \( L = L_{m_0} \).

But \( L \neq (0) \), since \( p \in L \). By Lemma 2.3, it follows that \( L \) cannot be finitely generated. Therefore, \( pSL = I \cap J \) is not finitely generated.

**Remark 3.1.** The ideal \( L \) in the above proof can be interpreted as an ideal of denominators; see [10, page 396]. Using the fact that \( pS \in \mathcal{K}^* \), we have \( S \in \mathcal{Q}(\mathcal{K}^*) \), where \( \mathcal{Q}(\mathcal{K}^*) \) denotes the field of fractions of \( \mathcal{K}^* \). We can then consider the fractional ideal \( M := \mathcal{K}^* + \mathcal{K}^* S \) of \( \mathcal{K}^* \) (see [11, page 19]) and the ideal of denominators \( L \) of \( S \), namely \( L = \mathcal{K}^* : M = \{ d \in \mathcal{K}^* \mid dS \in \mathcal{K}^* \} \).

Based on the results in [12, Theorem 3, Example 3], it follows that \( S \in \mathcal{Q}(\mathcal{K}^*) \) does not admit a weak coprime factorization, since \( L \) is not a principal ideal of \( \mathcal{K}^* \). In particular, \( S \) does not admit a coprime factorization, that is, there do not exist \( d, x, y, n \in \mathcal{K}^* \) such that \( d \neq 0, S = n/d, \) and \( dx - ny = 1 \). Moreover, \( S \) is not internally stabilizable, since otherwise \( L \) would be generated by two elements. Finally, the fact that \( L \) is not finitely generated implies that \( \mathcal{K}^* \) is not a greatest common divisor domain: indeed, were it the case that \( \mathcal{K}^* \) is a greatest common divisor domain, then by [12, Corollary 3], every element in \( \mathcal{Q}(\mathcal{K}^*) \) would admit a weak coprime factorization.

**Acknowledgment**

The author thanks all the referees for their careful review, and in particular, two of the referees for the Remarks 2.4 and 3.1.

**References**


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