Compactness and nuclearity of the Hankel operator and internal stability of infinite-dimensional state linear systems

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We examine the relationships between the exponential (or strong) stability of certain classes of regular linear systems and the compactness and nuclearity properties of the Hankel operator. New sufficient conditions for nuclearity are given for exponentially stable regular linear systems with an analytic semigroup.

1. Introduction

In the theory of realization and approximation of infinite-dimensional systems, the nuclearity and compactness properties of the Hankel operator play an essential role. While the relationships between the transfer function and these properties of the corresponding Hankel operator are well known, in this paper, we investigate the relationship between properties of the realization \((A, B, C)\) and the nuclearity and compactness properties of the corresponding Hankel operator.

In this paper we consider the Hankel operator of stable matrix-valued functions \(G \in H_\infty(C^{p \times m})\), where \(H_\infty(C^{p \times m})\) denotes the set of complex \(p \times m\) matrix-valued functions defined on the open right half-plane \(C_+\), which are bounded and analytic in \(C_+\). Let \(H_2(C_+)\) denote the set of all analytic functions \(f : C_+ \to C_+\) such that

\[
\|f\|_2 := \sup_{\zeta > 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\zeta + i\omega)\|^2 d\omega \right)^{1/2} < \infty
\]

The Hankel operator with symbol \(G\) is the operator \(H_G : H_2(C_+) \to H_2(C')\) are given by

\[
H_Gf = II(A_Gf_\bot) \quad \text{for } f \in H_2(C_+),
\]

where \(A_G\) is the multiplication map of \(L_2(i\mathbb{R}, C^{m})\) induced by \(G\) (see Curtain and Zwart 1995, Theorem A.6.26, p. 647), \(II\) is the orthogonal projection operator from \(L_2(i\mathbb{R}, C^{m})\) onto \(H_2(C')\) and \(f_\bot(s) := f(-s)\).

Practical control design is typically based on a reduced-order model of the original system. Many design methodologies utilize a rational approximation of a stable transfer in the \(H_\infty\)-norm (for example, see Curtain and Zwart 1995, Chapter 9, pp. 457–563). For this to be possible the Hankel operator with symbol \(G \in H_\infty(C^{p \times m})\) should be compact. We quote the following criterion (see for instance Partington 1988, Corollary 4.10, p. 46).

**Theorem 1** (Hartman’s theorem): \(G \in L_\infty(i\mathbb{R}, C^{p \times m})\) determines a compact Hankel operator iff \(G_\infty(\cdot) \in H_\infty(C^{p \times m}) + C^0(C^{p \times m})\), where \(C^0(C^{p \times m})\) denotes the space of continuous \(p \times m\) complex matrix-valued functions defined on \(i\mathbb{R}\), with a (unique) limit at \(\pm \infty\).

Since most models of infinite-dimensional systems are obtained, not as transfer functions, but as realizations, we are interested in deducing the properties of the Hankel operator from properties of the realization. To do this we introduce the time-domain Hankel operator, which is defined in terms of \(h\), the inverse-Laplace transform of \(G\). If \(h \in L_1([0, \infty), C^{p \times m})\) or \(L_2([0, \infty), C^{p \times m})\), we define the time-domain Hankel operator \(\Gamma_h : L_2([0, \infty), C^{m}) \to L_2([0, \infty), C'\)) by

\[
(\Gamma_hu)(t) = \int_0^\infty h(t + s)u(s) \, ds \quad \text{for all } u \in L_2([0, \infty), C^{m})
\]

In the case that \(h \in L_1([0, \infty), C^{p \times m})\), it is well-known that \(\Gamma_h\) is compact (see for example Curtain and Zwart 1995, Lemma 8.2.4, p. 399) and so \(\Gamma_h\) has countably many singular values (square roots of the eigenvalues of \(\Gamma_h^*\Gamma_h\)) \(\sigma_1 \geq \sigma_2 \geq \cdots \geq 0\) and these are also called the Hankel singular values of \(G\). If \(h \in L_2([0, \infty), C^{p \times m})\), then (2) may not be well-defined. If, however, we also assume that \(G \in L_\infty(i\mathbb{R}, C^{p \times m})\), then (2) always defines \(\Gamma_h\) as a bounded operator from \(L_2([0, \infty), C^{m})\) to \(L_2([0, \infty), C'\)) (see van Keulen 1990, Proposition 8, p. 224). In either of these cases, \(\Gamma_h\) is isomorphic to \(H_G\) under the Laplace (or Fourier) transform (see van Keulen 1990 and Curtain and Zwart 1995, Lemma 8.2.3, p. 397).

Let us consider the following two classes of systems; the first was the subject of the book by Curtain and Swart (1995) and the second was the main topic of the recent thesis by Oostveen (1999).

**Class 1**: The state linear system given by the triple \((A, B, C)\) where \(A\) is the infinitesimal generator of an exponentially stable strongly continuous semigroup
Theorem 2 (Coifman and Rochberg 1980): If $G \in H_\infty(C^{p \times m})$, then $H_G$, is nuclear if
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|G''(x+iy)\| \, dx \, dy < \infty \quad (3)
\]

Theorem 3 (Coifman and Rochberg 1980): The Hankel operator given by (2) is nuclear if $G$ possesses on $C_0^+ := \{ s \in C | \text{Re}(s) > 0 \}$ and expansion of the form
\[
G(s) \sum_{n=1}^{\infty} \frac{b_n}{s-a_n}
\]
where
\[
\sum_{n=1}^{\infty} \left| \frac{b_n}{s-a_n} \right| < \infty \quad (5)
\]
and $\text{Re}(a_n) < 0$.

These theorems made it possible to obtain sufficient conditions for the nuclearity of transfer functions of delay systems (see Zwart et al. 1998, Glover et al. 1990). See also the recent results on fractional transfer functions in Bonnet and Partington (2000). In this paper we are interested in obtaining sufficient conditions for nuclearity in terms of the state-space realizations. Our new results on nuclearity in §3 are:

1. Class 1 has a nuclear Hankel operator (Theorem 4).
2. A class of regular linear systems with an exponentially stable analytic semigroup and unbounded $B$ and $C$ has a nuclear Henkel operator (Theorem 6).

Analytic semigroups are generated by parabolic partial differential operators and hyperbolic partial differential operators with structural damping (see Pazy 1983). Consequently, the above results have important consequences for model reduction for distributed systems with an analytic semigroup with unbounded sensing and control. We illustrate the result in Theorem 6 with a less well-known analytic system which has the fractional transfer function $1/(1+s)^m$, $0 < m < 1$ with an appropriate choice of a branch.

2. Compactness of Hankel operators

In §1 we recalled that linear systems with bounded, finite-rank inputs and outputs and an exponentially stable semigroup always have a compact Hankel operator. One might hope that this would also be the case if we only have strong stability. However, by means of two examples, we show that the property of strong stability does not imply the compactness of the Hankel operator. Let us consider Class 2, since we know that $A - BB^*$ will generate at most a strongly stable semigroup $\{T_{B}(t)\}_{t \geq 0}$.
This class arises by stabilizing the open-loop system \( \Sigma(A, B, B^*) \) with transfer function \( G_0(s) = B'(sI - A)^{-1}B \), via the static output feedback \( u = -y \), which results in the closed-loop system \( \Sigma(A - BB^*, B, B^*) \) with transfer function \( \tilde{G}(s) = G_0(s)(I + G_0(s))^{-1} - B'(sI - A + BB^*)^{-1}B \). This closed-loop system has several nice properties (see Curtain and Zwart 1996).

\[ \text{C1: } A - BB^* \text{ generates a contraction semigroup } \{T_B(t)\}_{t \geq 0}. \]

\[ \text{C2: } \int_0^\infty \|Bt_B(t)x\|^2 \, dt \leq \frac{1}{2} \|x\|^2. \]

\[ \text{C3: } \int_0^\infty \|B't_B(t)x\|^2 \, dt \leq \frac{1}{2} \|x\|^2. \]

\[ \text{C4: } G(s) = B'(sI - A + BB^*)^{-1}B \in H_\infty(\mathcal{L}(U, Y)). \]

C2 and C3 show that the system has an impulse response \( h \in L_2([0, \infty), \mathcal{L}(U, Y)) \) and bounded observability and controllability maps \( C \) and \( B \) defined as follows:

1. \( B: L_2([0, \infty), U) \to X \) is defined by

\[ B_u = \int_0^\infty T_B(t)Bu(t) \, dt, \text{ for all } u \in L_2([0, \infty), U). \]

(6)

2. \( C: X \to L_2([0, \infty), Y) \) is defined by

\[ (Cx)(t) = B'T_B(t)x \text{ for all } t \geq 0 \text{ and for all } x \in X. \]

(7)

\[ \{T_B(t)\}_{t \geq 0} \text{ is not necessarily strongly stable.} \]

Sufficient conditions for \( \{T_B(t)\}_{t \geq 0} \) and \( \{T_B(t)^*\}_{t \geq 0} \) to be strongly stable can be found in Arendt and Batty (1988).

\[ \text{N1: } \text{The intersection of the spectrum of } A \text{ with the imaginary axis is at most countable.} \]

OR

\[ \text{N2: } \{x \in X | B'T(t)x = 0, \|T(t)x\|^2 = \|x\|^2 \text{ for all } t \geq 0\} = \{0\}. \]

Note that in N2, \( B'T(t)x = 0 \) can be replaced by \( B'T(t)^*x = 0 \), and if \( \Sigma(A, B, B^*) \) is approximately controllable or observable, then N2 holds. The Hankel operator \( \Gamma_h \) is equal to \( CB \). The controllability gramian \( L_B = BB^* \) and the observability gramian \( L_C = C'C \) always satisfy their respective Lyapunov equations.

N3: \( (A - BB^*)L_Bx + L_B(A - BB^*)^*x = -BB^*x, \) for all \( x \in D(A). \)

N4: \( (A - BB^*)^*L_Cx + L_C(A - BB^*)x = -BB^*x, \) for all \( x \in D(A^*). \)

If \( \{T_B(t)\}_{t \geq 0} \) is strongly stable, N3 has the unique solution \( L_B \) and if \( \{T_B(t)^*\}_{t \geq 0} \) is strongly stable, N4 has the unique solution \( L_C \). Suppose now that \( A \) is skew-symmetric. Then N3 and N4 have the unique solution \( \frac{1}{2}I \) and we now prove that \( \Gamma_h \) will not be compact. We have

\[ \Gamma_h^* \Gamma_h = B'C'CB = B'L_CB = B'B = \frac{1}{2}BB^* \]

If \( \Gamma_h \) is compact, then \( \Gamma_h^* \Gamma_h \) is also compact and it follows from the above that \( B \) must be compact (the compactness of \( B \) follows, for example, Weidmann 1980, Theorem 6.4.(c), p. 131). Consequently we obtain \( BB^* = L_B = \frac{1}{2}I \) must be compact, a contradiction.

So contrary to our expectations, if \( A \) is skew-symmetric and \( \{T_B(t)\}_{t \geq 0} \) and \( \{T_B(t)^*\}_{t \geq 0} \) are strongly stable, \( \Gamma_h \) can never be compact.

Many examples of partial differential equation systems of the Class 2 structure with a skew-symmetric operator \( A \) can be found in Oostveen (1999). Here we give a simple example which can be readily analysed and which captures the salient features of the partial differential equation examples.

Example 1: Let \( A: D(A)(\subset \ell_2(\mathbb{N})) \to \ell_2(\mathbb{N}) \) be the operator given by

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \]

with

\[ D(A) = \left\{ x \in s_2(\mathbb{N}) \left| \sum_{k=1}^\infty (|k(x, e_{2k-1})|^2 + |k(x, e_{2k})|^2) < \infty \right. \right\} \]

where \( \{e_n\}_{n \geq 1} \) denotes the standard orthonormal basis for \( \ell_2(\mathbb{N}) \). We first show that \( A \) is closed and densely defined. Clearly, all elements \( x \) in \( \ell_2(\mathbb{N}) \) with \( \langle x, e_n \rangle = 0 \) for all sufficiently large \( n \) lie in \( D(A) \) and form a dense set in \( \ell_2(\mathbb{N}) \). So \( A \) is densely defined.
Let \( \{x_m\}_{m \geq 1} \) be a sequence in \( D(A) \) and let \( x_m \to x_0 \) and \( Ax_m \to y_0 \) as \( m \to \infty \). Since the sequence \( \{Ax_m\}_{m \geq 1} \) is bounded, there exists a \( M > 0 \) such that
\[
\sum_{k=1}^{\infty} (|k(x_m, e_{2k-1})|^2 + |-k(x_m, e_{2k})|^2) < M \quad \text{for all } m \geq 1
\]
Consequently, for any \( N \in \mathbb{N} \)
\[
\sum_{k=1}^{N} (|k(x_m, e_{2k-1})|^2 + |-k(x_m, e_{2k})|^2) < M \quad \text{for all } m \geq 1
\]
Owing to the continuity of the inner product and the fact that \( x_m \to x_0 \), we obtain
\[
\sum_{k=1}^{\infty} (|k(x_0, e_{2k-1})|^2 + |-k(x_0, e_{2k})|^2) \leq M
\]
Since the choice of \( N \) was arbitrary, it follows that
\[
\sum_{k=1}^{\infty} (|k(x_0, e_{2k-1})|^2 + |-k(x_0, e_{2k})|^2) \leq M
\]
Consequently, \( x_0 \in D(A) \) with \( Ax_0 = y_0 \) and so \( A \) is closed.

In fact it can be easily checked that \( A \) is a Riesz spectral operator with the (totally disconnected) set of simple unstable eigenvalues \( \{ \pm n \}_{n \in \mathbb{N}} \) (see figure 1) and the corresponding (orthogonal) Riesz basis of eigenvectors \( \{(1/\sqrt{2})(e_n \pm i e_{n+1})\}_{n \in \mathbb{N}} \).

\( Ax + A^* x = 0 \) for all \( x \in D(A) = D(A^*) \) and so \( A \) is the infinitesimal generator of a contraction strongly continuous semigroup on the Hilbert space \( \ell_2(\mathbb{N}) \), \( 0 \in \rho(A) \) and
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]
\( \in \mathcal{L}(\ell_2(\mathbb{N})) \)

It can be easily seen that \( \mathcal{R}_n \to A^{-1} \) as \( n \to \infty \) in the uniform operator topology, where
\[
\begin{bmatrix}
0 & -1/n \\
1/n & 0
\end{bmatrix}
\]
\( \in \mathcal{L}(\ell_2(\mathbb{N})) \)

Since each \( \mathcal{R}_n \) has finite rank \( 2n \), \( A^{-1} \) is the uniform limit of a sequence of compact operators and so \( A^{-1} \) is compact (see for example Curtain and Zwart 1999, Theorem A.3.22.3, p. 587). From Kato (1966, Theorem 6.29, p. 187), it follows that \( A \) has compact resolvent.

Let \( B \in \mathcal{L}(\mathbb{C}, \ell_2(\mathbb{N})) \) be defined by
\[
B = \begin{bmatrix}
1 \\
0 \\
1/2 \\
0 \\
\vdots
\end{bmatrix}
\]
It follows from Curtain and Zwart (1995, Theorem 4.2.3, p. 164) that $\Sigma(A, B, \rightarrow)$ is approximately controllable. Dually, $\Sigma(A^*, \leftarrow, B^*)$ is approximately observable. Consequently, from Oostveen (1999, Lemma 2.2.6, p. 23), it follows that $A_B := A - BB^*$ and $A_B^* = A^* - B^*B$ generate strongly stable semigroups $\{T_B(t)\}_{t \geq 0}$ and $\{T_B(t)^*\}_{t \geq 0}$, respectively, on $\ell^2_{2}(\mathbb{N})$.

Thus $A - BB^*$ generates a strongly stable semigroup, but the state linear system $\Sigma(A - BB^*, B, B^*)$ has a time-domain Hankel operator $F_h$ that is bounded, but not complete.

The following example shows that a system with the Class 2 structure can have a compact Hankel operator although $\{T_B(t)\}_{t \geq 0}$ is not strongly stable.

**Example 2:** We start by showing that the transfer function

$$
G(s) = \frac{1}{1 + \sqrt{s^2 + 1}}
$$

(with an appropriate choice of an analytic branch of the complex square root function) has a realization of the form $\Sigma(A - BB^*, B, B^*)$. We do this by interpreting it as the closed-loop system formed by applying the static output feedback to the system with transfer function

$$
G_0(s) = \frac{1}{\sqrt{s^2 + 1}}
$$

Following Baras and Brockett (1975), we show that $G_0(s)$ has a realization $\Sigma(A, B, B^*)$, where $A$ generates a contraction semigroup. Let $A \in \mathcal{L}(\ell^2_{2}(\mathbb{Z}))$ be given by

$$
A = \frac{1}{2}
\begin{bmatrix}
\ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & -1 & 0 \\
1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
1 & 0 & \ddots & \ddots \\
\end{bmatrix}
$$

It is easy to see that $A$ is bounded and that $A + A^* = 0$. It can be shown that the spectrum of $A$ (see figure 2) is the set

$$
\{-i, i\} = \{s \in \mathbb{C} | -1 \leq \text{Im}(s) \leq 1\}
$$


Let $B \in \mathcal{L}(\mathbb{C}, \ell^2_{2}(\mathbb{Z}))$ be defined as

$$
B = \begin{bmatrix}
\vdots \\
0 \\
0 \\
\vdots \\
\end{bmatrix}
$$

**Claim:** $\Sigma(A, B, B^*)$ realizes the transfer function

$$
1/\sqrt{s^2 + 1}.
$$

We know that $s \mapsto B^*(sI - A)^{-1}B$ is analytic in $\mathbb{C}\\setminus\{-i, i\}$. Moreover for $\omega > 1$, it can be easily checked that $B^*(i\omega - A)^{-1}B = -i/\sqrt{\omega^2 - 1}$. Consider the map $s \mapsto 1/\sqrt{s^2 + 1}$. We shall take two copies of the $s$-plane, with cuts along the segment $[-i, i]$. The formation of an analytic branch of the map $s \mapsto 1/\sqrt{s^2 + 1}$ takes place in each of these planes. Now we shall join the edges of our cuts cross-wise with the help of two segments $[i, -i]$, the inner points of which will be considered to be different, although geometrically they coincide. This is the Riemann surface of the function $s \mapsto 1/\sqrt{s^2 + 1}$ (see figure 3). It is double-
3. Nuclearity

In this section we study the nuclearity property of the Hankel operator of a system in terms of a given realization \((A, B, C)\). First we show that an exponentially stable state linear system with bounded inputs and outputs and finite-dimensional input and output spaces is nuclear.

**Theorem 4:** Let \(A\) be the infinitesimal generator of an exponentially stable strongly continuous semigroup \(\{T(t)\}_{t>0}\) on the separate Hilbert space \(X\), with \(B \in \mathcal{L}(\mathbb{C}^n, X)\), \(C \in \mathcal{L}(X, \mathbb{C}^p)\). Then

1. The observability operator \(C: X \to L_2([0, \infty), \mathbb{C}^p)\) defined by \((C_x)(\cdot) = CT(\cdot)x\) is Hilbert–Schmidt.

2. The controllability operator \(B: L_2([0, \infty), \mathbb{C}^n) \to X\) defined by \(Bu = \int_0^\infty T(t)Bu(t)\,dt\) is Hilbert–Schmidt.

3. \(L^*_C = C^*C\), \(L_B = BB^*\) and \(\Gamma_h = CB\) are all nuclear.

**Proof:**

1. Define \(C_i: X \to L_2(0, \infty), i \in \{1, \ldots, p\}\) by

\[
(C_i x)(t) = \langle CT(t)x, e_i \rangle = \langle x, T(t)^* C^* e_i \rangle
\]

where \(\{e_1, \ldots, e_p\}\) is the standard basis for \(\mathbb{C}^p\).

We have

\[
|\langle C_i x \rangle(t)| = |\langle x, T(t)^* C^* e_i \rangle| \leq \|x\| \|T(t)^* C^* e_i\|
\]

\[
\leq \|x\| \|T(t)^*\| \|C^*\| \|e_i\|
\]

\[
\leq \|x\| \cdot M \cdot e^{-\alpha t} \cdot \|C^*\|
\]

\[
\text{and } \int_0^\infty |Me^{-\alpha t}| \cdot \|C^*\|^2 \,dt < \infty.
\]

We will now use the following result which is an adaptation of Weidmann (1980, Theorem 6.12, p. 140).

**Theorem 5:** Let \(K\) be a bounded linear operator from a Hilbert space \(H\) into \(L_2(0, \infty)\). If there exists a function \(K\in L_2(0, \infty)\) such that

\[
|\langle Kv(t)\rangle| \leq \kappa(t)\|v\|
\]

for almost all \(t \in (0, \infty)\), and all \(v \in H\), then \(K\) is Hilbert–Schmidt.

Applying this result we obtain that \(C_i\) is Hilbert–Schmidt. Consequently, for an arbitrary orthonormal basis \(\{x_i\}\) of \(X\),

\[
\sum_{j=1}^\infty \|C_i x_j\|_{L_2(0, \infty)}^2 < \infty \quad \text{for all } i \in \{1, \ldots, p\},
\]

and

\[
\sum_{i=1}^\infty \sum_{j=1}^\infty \|C_i x_j\|_{L_2(0, \infty)}^2 < \infty
\]

Thus

\[
\sum_{i=1}^\infty \|C_i x_i\|_{L_2(0, \infty)}^2 < \infty
\]

and so \(C\) is Hilbert–Schmidt.

(2) From Weidmann (1980, Theorem 6.9), it follows that \(B\) is Hilbert–Schmidt iff \(BB^*\) is Hilbert–Schmidt. But \(BB^*\) is Hilbert–Schmidt by applying the first part of the lemma to the dual system \(\Sigma(A^*, C^*, B^*)\).

(3) Using Weidmann (1980, Theorem 6.10(b)), we obtain that \(L_e = C^*C\), \(L_B = BB^*\) and \(\Gamma_h = CB\) are all nuclear.

We remark that the Hilbert–Schmidt property \(C\) was already shown in Dumortier (1998, Proposition 1.0.2, p. 24). We now show that a class of regular linear systems with an exponentially stable analytic semigroup and unbounded \(B\) and \(C\) has a nuclear Hankel operator. First we give a few preliminaries and introduce the notation used.

Let \(-A\) be the infinitesimal generator of an exponentially stable analytic semigroup \(\{T(t)\}_{t \geq 0}\) on \(X\). For each \(\theta \in \mathbb{R}\), let \(X_\theta = A^{\theta}X\) be \(D(A^\theta)\) with the norm \(\|x\|_{X_\theta} = \|A^\theta x\|_X\) and inner product \(\langle x_1, x_2 \rangle_{X_\theta} = \langle A^\theta x_1, A^\theta x_2 \rangle_X\). The restrictions of \(-A\) to \(X\) for \(\theta > 0\) and the extensions of \(-A\) to \(X_\theta\) for \(\theta < 0\) (which we still denote by \(-A\)) generate analytic semigroups on \(X_\theta\) for \(\theta \in \mathbb{R}\). These semigroups are all similar to each other, and they commute with \(A^\beta\) for all \(\beta \in \mathbb{R}\). We therefore denote all of them by the same symbol \(\{T(t)\}_{t \geq 0}\). The generator of the semigroup \(\{T(t)\}_{t \geq 0}\) on \(X_\theta\) is then \(-A \in \mathcal{L}(X_{\theta+1}, X_\theta)\). Moreover, for each \(t > 0\) and \(\theta \in \mathbb{R}\), \(T(t)\) maps \(X_\theta\) into \(\cap_{\beta \in \mathbb{R}} X_\beta\). For each \(\theta \geq 0\), there exists \(K_1 > 0\) and \(\epsilon > 0\) such that

\[
\|A^\theta T(t)\|_{X_\theta} \leq K_1 e^{-\epsilon t}, \quad t > 0
\]

where the norm represents the operator norm in any one of the spaces \(X_\theta\). If \(\theta < 0\), then there exist \(K_2 > 0\) and \(\epsilon > 0\) such that

\[
\|A^\theta T(t)\|_{X_\theta} \leq K_2 e^{-\epsilon t}, \quad t > 0
\]
(this follows from Pazy 1983, Lemma 6.3, p. 71). The same conclusion as above can be repeated with $A$ replaced by $A^*$ to give another chain of Hilbert spaces $X_0 = (A^*)^{-1} X$ with similar properties. We identify $X_0$ with the dual of $X_0$ by using $X$ as the pivot space. We note that $X_0 = X_0^* = X$.

**Theorem 6:** Let $-A$ generate an exponentially stable, analytic, strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $X$, $B \in \mathcal{L}(X_\alpha, X_\alpha)$, and $C \in \mathcal{L}(X_\alpha, \mathbb{C}^m)$, where $\alpha_B \leq \alpha_C < \alpha_B + 1$. Fix any $\gamma$ satisfying $\alpha_C = \frac{1}{\gamma} < \gamma < \alpha_B + \frac{1}{2}$.

1. $-A, B, C$ generate a regular linear system with state space $X_\gamma$, input space $\mathbb{C}^m$ and output space $\mathbb{C}^\gamma$. The transfer function is given by

$$G(s) = C(sI + A)^{-1} B$$

and it satisfies

$$\lim_{\|B\| \to \infty} \|G(s)\| = 0$$

(2) The controllability map $B \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), X_\gamma)$ and map $C \in \mathcal{L}(X_\gamma, L_2([0, \infty), \mathbb{C}^\gamma))$ are Hilbert–Schmidt operators.

(3) $h(\cdot) := CT(\cdot)B \in L_1([0, \infty), \mathbb{C}^{p \times m})$ and the Hankel operator $H_x$ satisfies $H_x = CB$, where $B \in \mathcal{L}(X_\gamma, X_\gamma)$ and $C \in \mathcal{L}(X_\gamma, \mathbb{C}^\gamma)$. Furthermore, $H_x$ is nuclear.

**Proof:**

1. This part follows from Staffans (2002, p. 251) or Staffans (1998).

2. Just as in Theorem 4, it is sufficient to prove this for the case $p = 1$. From part (1) above, we know that $C$ is an admissible observation operator for $\{T(t)\}_{t \geq 0}$ with state space $X_\gamma$, and so $C \in \mathcal{L}(X_\gamma, L_2([0, \infty), \mathbb{C}^\gamma))$. Since $C \in \mathcal{L}(X_\gamma, \mathbb{C}^\gamma)$, for every $x \in X_\gamma$, we have

$$\|Cx(t)\| = \|CT(t)x\|$$

$$\leq \|C\|_{\mathcal{L}(X_\gamma, \mathbb{C}^\gamma)} \|T(t)x\|_{X_\gamma}$$

$$= \|C\|_{\mathcal{L}(X_\gamma, \mathbb{C}^\gamma)} \|A^{\alpha_C-\gamma} T(t)x\|_{X_\gamma}$$

$$\leq \|C\|_{\mathcal{L}(X_\gamma, \mathbb{C}^\gamma)} \|A^{\alpha_C-\gamma} T(t)\|_{\mathcal{L}(X_\gamma)} \|x\|_{X_\gamma}$$

**Case 1:** If $\alpha_C - \gamma \geq 0$, then we have

$$\|Cx(t)\| = \|C\|_{\mathcal{L}(X_\gamma, \mathbb{C}^\gamma)} \|A^{\alpha_C-\gamma} T(t)\|_{\mathcal{L}(X_\gamma)} \|x\|_{X_\gamma}$$

and $e^{\alpha_C \tau}/e^{\alpha_C \gamma} \in L_2([0, \infty), \mathbb{C})$, for $\gamma > \alpha_C - \frac{1}{2}$.

So by Theorem 5, it follows that that $C$ is Hilbert–Schmidt for all $\gamma$ satisfying $\alpha_C \geq \gamma > \alpha_C - \frac{1}{2}$.

**Case 2:** If $\alpha_C - \gamma < 0$, then we have

$$\|Cx(t)\| = \|C\|_{\mathcal{L}(X_\gamma, \mathbb{C}^\gamma)} \|A^{\alpha_C-\gamma} T(t)\|_{\mathcal{L}(X_\gamma)} \|x\|_{X_\gamma}$$

$$\leq \|C\|_{\mathcal{L}(X_\gamma, \mathbb{C}^\gamma)} \|A^{\alpha_C-\gamma} T(t)\|_{\mathcal{L}(X_\gamma)} \|x\|_{X_\gamma}$$

and $e^{\alpha_C \tau}/e^{\alpha_C \gamma} \in L_2([0, \infty), \mathbb{C})$. So by Weidmann (1980, Theorem 6.12, p. 140) it follows that that $C$ is Hilbert–Schmidt for all $\gamma$ satisfying $\alpha_C < \gamma$.

From the above cases, it follows that $C$ is Hilbert–Schmidt for all $\gamma > \alpha_C - \frac{1}{2}$.

Since $B$ is an admissible control operator for $\{T(t)\}_{t \geq 0}$ with state space $X_\gamma$, $B$ is an element in $\mathcal{L}(L_2([0, \infty), \mathbb{C}^m), X_\gamma)$ and for $u \in L_2([0, \infty), \mathbb{C}^m)$

$$Bu = \int_0^\infty T(t)Bu(t) \, dt$$

Thus the dual operator $B^* \in \mathcal{L}(X_\gamma, L_2([0, \infty), \mathbb{C}^m))$ and for $x \in X_\gamma$

$$(B^*x)(\cdot) = B^* T(\cdot)^* x$$

Proceeding as above, it can be shown that $B^* \in \mathcal{L}(X_\gamma, L_2([0, \infty), \mathbb{C}^m))$ is Hilbert–Schmidt. Thus, using Proposition 2, Aubin (1979, Proposition 2, p. 261) it follows that $B$ is also Hilbert–Schmidt.

3. If $u \in \mathbb{C}^m$, then $Bu \in X_\alpha$ and $T(t)Bu \in \cap_{\beta \in \mathbb{R}} X_\beta$. Consequently $T(t)Bu \in X_\alpha$ and $CT(t)Bu \in \mathbb{C}^\gamma$. Moreover.

$$\|CT(t)Bu\|_{\mathbb{C}^\gamma}$$

$$\leq \|C\|_{\mathcal{L}(X_\alpha, \mathbb{C}^\gamma)} \|A^{\alpha_C-\gamma} T(t)\|_{\mathcal{L}(X_\gamma)} \|B\|_{\mathcal{L}(\mathbb{C}^m, \mathbb{C}^\gamma)} \|u\|_{\mathbb{C}^m}$$

But since $\alpha_C - \alpha_B \geq 0$, it follows that

$$\|CT(t)Bu\|_{\mathbb{C}^\gamma} \leq \|C\|_{\mathcal{L}(X_\alpha, \mathbb{C}^\gamma)} \|K_1\|_{\mathcal{L}(\mathbb{C}^m, \mathbb{C}^\gamma)} \|B\|_{\mathcal{L}(\mathbb{C}^m, \mathbb{C}^\gamma)} \|u\|_{\mathbb{C}^m}$$

Finally, since $\alpha_C - \alpha_B < 1$, we obtain that $h(\cdot) = CT(\cdot)B \in L_1([0, \infty), \mathbb{C}^{p \times m})$.

We know that $B \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), X_\gamma)$ and $C \in \mathcal{L}(X_\gamma, L_2([0, \infty), \mathbb{C}^\gamma))$. Consequently, if $u \in L_2([0, \infty), \mathbb{C}^m)$, $Bu \in X_\gamma$, and

$$(C(Bu))(\cdot) = CT(t)Bu = CT(t) \int_0^\infty Y(\tau)Bu(\tau) \, d\tau$$

But $CT(t) \in \mathcal{L}(X_\gamma, \mathbb{C}^\gamma)$ and so we have (see for example Curtain and Zwart 1995, A.5.23, p. 628)
(C(Bu))(t) = \int_0^\infty CT(t) T(\tau) Bu(\tau) \, d\tau \\
= \int_0^\infty CT(t + \tau) Bu(\tau) \, d\tau \\
= \int_0^\infty h(t + \tau) u(\tau) \, d\tau = (\Gamma_h u)(t)

Furthermore, B and C are Hilbert–Schmidt operators, and so it follows from Weidmann (1980, Theorem 7.10(b), p. 175) that $\Gamma_h = CB$ is nuclear.

As explained in the introduction, analytic semigroups are generated by parabolic and some hyperbolic partial differential equations and so the above theorem is relevant to many distributed parameter systems with unbounded sensing and control. For a change, we give a different example of an analytic system from the class of fractional transfer functions.

**Example 3** (The fractional transfer function $1/(1 + s)^m$, where $0 < m < 1$): Let $A : D(A) \subset L_2(0, \infty) \to L_2(0, \infty)$ be defined as

$$D(A) = \left\{ h \in L_2([0, \infty), \mathbb{C}) \mid \int_0^\infty (1 + x)^2 |h(x)|^2 \, dx < \infty \right\}$$

and $(Ah)(x) = (1 + x)h(x)$

Then $A$ is a non-negative self-adjoint operator and so using Example 1.25 of Kato (1966, p. 493) it follows that $-A$ is the infinitesimal generator of the exponentially stable analytic semigroup $\{T(t) \}_{t \geq 0}$, where $T(t) \in \mathcal{L}(L_2([0, \infty), \mathbb{C}))$ is given by

$$(T(t)h)(x) = e^{(1+x)t}h(x)$$

We note that

$$(-A)^\theta h(x) = (1 + x)^\theta h(x), \quad h \in D((-A)^\theta)$$

and for each $\theta \in \mathbb{R}$, we define $X_\theta$ as described earlier. Define $B \in \mathcal{L}(C, X_\alpha)$ for

$$\alpha_B < \frac{m - 1}{2} < 0$$

by

$$(Bu)(x) = u \cdot x^{-m/2}$$

We define $C = (\sin(m\pi)/\pi)B^* \in \mathcal{L}(X_\alpha, \mathbb{C})$ and for $h \in X_{-\alpha}$,

$$Ch = \frac{(\sin(m\pi)}{\pi} \int_0^\infty x^{-m/2}h(x) \, dx$$

So we have $\alpha_C = -\alpha_B$. We first choose $-\frac{1}{2} < \alpha_B < (m - 1)/2$; such a choice of $\alpha_B$ is possible, since $m \in (0, 1)$, and next we choose a $\gamma$ satisfying

$$-\alpha_B - \frac{1}{2} < \gamma < \alpha_B + \frac{1}{2}.$$

(Such a choice of $\gamma$ is possible, since we have $\alpha_B$ satisfying $-\frac{1}{2} < \alpha_B$). Thus all the conditions in Theorem 6 are satisfied and so $(\rightarrow A, B, C)$ defines a regular linear system on $X_\gamma$, and its transfer function is given by

$$G(s) = C(sI + A)^{-1}B$$

$$= \frac{\sin(m\pi)}{\pi} \int_0^\infty x^{-m/2}(s + 1 + x)^{-1} x^{-m/2} \, dx$$

$$= \frac{\sin(m\pi)}{\pi} \int_0^\infty x^{-m} \, dx$$

$$= \frac{1}{(1 + s)^m}$$

where we choose the following branch: for $s \in \mathbb{C}\setminus(-\infty, -1]$

$$\frac{1}{(1 + s)^m} = \exp[-m(\log |1 + s| + i \arg(1 + s))],$$

$$-\pi < \arg(1 + s) < \pi$$

(see for instance Lang 1999, pp. 187–188). By Theorem 6, $G$ has a nuclear Hankel operator.

The analyticity assumption in Theorem 6 is crucial. In general, if $A$ generates an exponentially stable semigroup and if $B$ or $C$ is unbounded, the Hankel operator will not be nuclear, as the following example illustrates.

**Example 4**; Let $G(s) = e^{\tau s}/(s + 1)$, where $\tau > 0$. We will first give a realization of $G$. Let $X = \mathbb{C} \times L_2(-\tau, 0)$ (with the obvious inner product). Consider the Sobolev space

$$W^{1,2}(-\tau, 0) = \left\{ f \in L_2(-\tau, 0) \mid \begin{array}{c} \text{the derivative of } f \text{ (in the} \\
\text{sense of distributions) is} \\
\text{a regular distribution } T_g, \end{array} \right\}$$

equipped with the inner product

$$\langle f, g \rangle_{W^{1,2}(-\tau, 0)} = \int_{-\tau}^0 [f(x)g(x) + f’(x)g’(x)] \, dx$$

(With this inner product, $W^{1,2}(-\tau, 0)$ is a Hilbert space; see for example, Yosida (1978, Proposition 5, p. 55). Define

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2(0) \\ x_2’ \end{bmatrix}$$

with

$$D(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in X \mid x_2 \in W^{1,2}(-\tau, 0), x_1 = x_2(0) \right\}$$
Then it can be shown that $A$ is the infinitesimal generator of an exponentially stable strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $X$ (see for instance van Keulen 1993, Example 2.8). The space $W := D(A)$ with the inner product

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \right\rangle_W = \langle x_2, \tilde{x}_2 \rangle_{W^{1/2}(\mathbb{R},0)}$$

is a Hilbert space and $\{T(t)\}_{t \geq 0}$ restricts to a strongly continuous semigroup on $W$. Let $B \in \mathcal{L}(\mathbb{C}, X)$ be defined as

$$Bu = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

Finally, if $C: D(A) \to \mathbb{C}$ is given by

$$C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2(-\tau)$$

then $C \in \mathcal{L}(W, \mathbb{C})$. We now choose $V = X$. In Pritchard and Salamon (1987) it is explained that $B$ and $C$ are Pritchard–Salamon admissible control and observation operators, respectively (for the relevant definitions and the theory of Pritchard–Salamon systems, we refer the reader to Curtain et al. (1997)). Finally we note that $D(A^k) \to W$ is trivially satisfied since $W = D(A)$, and so in fact we have a smooth Pritchard–Salamon system, with transfer function $e^{-\tau s}/(s + 1)$.

The Hankel singular values $\sigma_k = 1/\sqrt{\mu_k^2 + 1}$, where the $\mu_k$'s are the roots of the transcendental equation

$$\tan(\mu \tau) = \frac{-\mu(3 - \mu^2)}{1 - 3\mu^2}$$

(see for example, Curtain and Zwart 1995, Theorem 8.2.10, pp. 402–403.) For the sake of simplicity, we assume $\tau = 1$. Because of the periodicity of $\tan(\cdot)$, its monotonicity in each periodic interval of the type

$$\left( (2k - 1) \frac{\pi}{2}, (sk + 1) \frac{\pi}{2} \right), \quad k \in \mathbb{Z},$$

and the monotonicity of the function $f(x) = -x(3 - x^2)/(1 - 3x^2)$, for $|x| > \sqrt{3}$, the positive roots of the transcendental equation above satisfy $1 < \mu_k < k\pi$ for $k > 2$ (see figure 4).

Thus we obtain

$$\sigma_k = \frac{1}{\sqrt{\mu_k^2 + 1}} > \frac{1}{\sqrt{\mu_k^2 + \mu_k^2}} = \frac{1}{\sqrt{2\mu_k}} > \frac{1}{\sqrt{2\pi k}}$$

Hence $\sum \sigma_k$ diverges, and so $\Gamma_k$ is not nuclear. However, it is Hilbert–Schmidt. This follows from Curtain and Zwart (1995, Exercise 8.9).

Finally, we make the point that exponential stability is not a necessary condition for nuclearity: In the following example, the Hankel operator is nuclear, although the semigroup is not exponentially stable.

**Example 5:** We construct the example below following Ober (1987). Let $\{\lambda_n\}_{n \geq 1}$ be a decreasing bounded sequence of distinct positive numbers converging to 0. Consider the $\Sigma(A, B, B^*)$, with

$$A = \begin{bmatrix} -\lambda_k \lambda_k \\ jk(\lambda_k + \lambda_k) \end{bmatrix} \quad 1 \leq j, k < \infty$$

$$B = \begin{bmatrix} \lambda_1/1 \\ \lambda_2/2 \\ \lambda_3/3 \\ \vdots \end{bmatrix}$$

Then

(1) $A \in \mathcal{L}(\ell^2(\mathbb{N}))$ is Hilbert–Schmidt. This follows from the proof of Ober (1987, Proposition 3(i), p. 304), and Weidmann (1980, Theorem 6.22). Thus $0 \in \sigma(A)$ and this rules out exponential stability. However, from the proof of Ober (1987, Proposition 3(ii), p. 304), it follows that the semigroup $\{e^{tj}\}_{t \geq 0}$ is strongly stable.

(2) The Lyapunov equation

$$AA + AA = -BB^*$$

has a solution

$$A_0 := \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This is proved in Ober (1987, Proposition 3, p. 304) and so it follows from Hansen and Weiss (1997, Theorem 3.1, p. 10) that $B$ is an infinite-
time admissible control operator. Since $A = A^*$ generates a strongly stable semigroup, it follows from Hansen and Weiss (1997, p. 10) that (10) has the unique solution

$$BB^* = A_0 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is clearly compact. Hence it follows from Weidmann (1980, Theorem 6.4(c), p. 131) that $B$ is compact.

(3) The time domain Hankel operator $\Gamma_h \in \mathcal{L}(L_2([0, \infty) \mathbb{C}))$ is compact, with $\sigma_n(\Gamma_h) = \lambda_n$ for all $n \in \mathbb{N}$.

Finally, upon choosing the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} \lambda_n < \infty$, we can obtain nuclearity: for example, we can take $\lambda_n = 1/n^2$. We remark that this is an example of a balanced realization.

4. Conclusions

There already exist necessary and sufficient conditions for the compactness and nuclearity of the Hankel operator in terms of the transfer function. However, in most applications, the model is given in terms of a triple $(A, B, C)$ of operators. We have given new sufficient conditions for nuclearity for two classes of realizations: an exponentially stable realization with bounded $B$ and $C$ (Theorem 4) and an exponentially stable analytic realization (Theorem 6). On the other hand, by means of examples, we have shown that the state space properties of exponential and strong stability of the semigroup have little to do with the compactness and nuclearity of the Hankel operator. Figure 5 gives a concise overview of the results in this paper.

References


Figure 5. Overview of the results.


