Some algebraic properties of the Wiener–Laplace algebra

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Abstract. We denote by $W^+(\mathbb{C}_+)$ the set of all complex-valued functions defined in the closed right half plane $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\}$ that differ from the Laplace transform of functions from $L^1(0, \infty)$ by a constant. Equipped with pointwise operations, $W^+(\mathbb{C}_+)$ forms a ring. It is known that $W^+(\mathbb{C}_+)$ is a pre-Bézout ring. The following properties are shown for $W^+(\mathbb{C}_+)$:

1. $W^+(\mathbb{C}_+)$ is not a GCD domain, that is, there exist functions $F_1, F_2$ in $W^+(\mathbb{C}_+)$ that do not possess a greatest common divisor in $W^+(\mathbb{C}_+)$. 

2. $W^+(\mathbb{C}_+)$ is not coherent, and in fact, we give an example of two principal ideals whose intersection is not finitely generated.

We will also observe that $W^+(\mathbb{C}_+)$ is a Hermite ring, by showing that the maximal ideal space of $W^+(\mathbb{C}_+)$, equipped with the Gelfand topology, is contractible.

Keywords. GCD domain, Wiener–Laplace algebra, control theory.

2010 Mathematics Subject Classification. Primary 46J15; Secondary 13G05, 93D15.

1 Introduction

The aim of this paper is to study some algebraic properties of the ring $W^+(\mathbb{C}_+)$ (defined below).

We first recall the notion of a GCD domain, a coherent ring and a Hermite ring below.

Definition 1.1. Let $R$ be an integral domain (that is a commutative unital ring having no divisors of zero).

1. An element $d \in R$ is called a greatest common divisor (often abbreviated by gcd) of $a, b \in R$ if it is a divisor of $a$ and $b$, and moreover, if $k$ is another divisor, then $k$ divides $d$.

$R$ is said to be a GCD domain if for all $a, b \in R$, there exists a greatest common divisor $d$ of $a, b$.

2. $R$ is said to be pre-Bézout if for every $a, b \in R$ for which there exists a greatest common divisor $d$, there exist $x, y \in R$ such that $d = xa + yb$.
3. $R$ is called coherent if for any pair $(I, J)$ of finitely generated ideals in $R$ their intersection $I \cap J$ is finitely generated again.

4. A matrix $f \in R^{n \times k}$ is called left invertible if there exists a $g \in R^{k \times n}$ such that $gf = I_k$.

5. $R$ is called a Hermite ring if for all $k, n \in \mathbb{N}$ with $k < n$ and all left invertible matrices $f \in R^{n \times k}$, there exist $F, G \in R^{n \times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$. We shall also say that in that case $R$ has the matricial extension property.

Whether some rings of analytic functions have the above algebraic properties has been investigated in earlier works. For example, von Renteln showed that the Hardy algebra $H^\infty(\mathbb{D})$ (of all bounded and analytic functions in the open unit disc, with pointwise operations) is a GCD domain [17, p. 519], while the disc algebra $A(\mathbb{D})$ (the ring of continuous functions on the closed unit disc $\overline{\mathbb{D}}$, which are analytic in the open unit disc $\mathbb{D}$, with the usual pointwise operations) is not [16, p. 52]. In [10], the first author and von Renteln noted that the Wiener algebra $W^+(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc) is not a GCD domain. In this article, we will show that the ring $W^+(\mathbb{C}_+)$ (defined below) is not a GCD domain.

W.S. McVoy and L.A. Rubel [8] showed that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disc algebra $A(\mathbb{D})$ is not. In [10], it was also shown that the Wiener algebra $W^+(\mathbb{D})$ (of all absolutely convergent Taylor series in the open unit disc $\mathbb{D}$) is not coherent. In Section 4, we will show that the ring $W^+(\mathbb{C}_+)$ is not coherent, in the same manner as the noncoherence of $W^+(\mathbb{D})$ was shown in [10, Theorem 3, p. 226].

The Hermiteness of $H^\infty(\mathbb{D})$ was first shown by V. Tolokonnikov (see for example [21], [12, §10, p. 293]). A. Quadrat has proved in [15, Corollary 3.30] that $H^\infty(\mathbb{D})$ is a projective free ring, which implies in particular that it is Hermite. The Hermiteness of $A(\mathbb{D})$ and $W^+(\mathbb{D})$ follows from [6] and the fact that their maximal ideal space is homeomorphic to the closed unit disc $\overline{\mathbb{D}}$. Indeed, Lin’s theorem (Proposition 2.10 below) says that a Banach algebra whose maximal ideal space is contractible is a Hermite ring. We will show that the maximal ideal space of $W^+(\mathbb{C}_+)$ is homeomorphic to the closed unit disc, and hence, by [6], the ring $W^+(\mathbb{C}_+)$ is Hermite as well.

Throughout the article, we will use the following notation:

$$\mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\}.$$ 

**Definition 1.2.** We denote by $W^+(\mathbb{C}_+)$ the set of all functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that $F(s) = \widehat{f_a}(s) + f_0$ ($s \in \mathbb{C}_+$), where $f_a \in L^1(0, \infty)$, $f_0 \in \mathbb{C}$, and $\widehat{f_a}$ denotes
the Laplace transform of $f_a$:

$$\hat{f}_a(s) = \int_0^\infty e^{-st} f_a(t) \, dt, \quad s \in \mathbb{C}_+.$$  

Equipped with pointwise operations and the norm $\|F\|_{W^+} = \|f_a\|_{L^1} + |f_0|$, $W^+(\mathbb{C}_+)$ is a Banach algebra.

**Remark 1.3.**

1. From the application point of view, the above algebra arises as natural classes of transfer functions of stable distributed parameter systems in control theory; see [13].

   Following [11], we use the notation $W^+(\mathbb{C}_+)$ in order to highlight the similarity with $W^+(\mathbb{D})$ and call $W^+(\mathbb{C}_+)$ the *Wiener–Laplace algebra*.

2. The algebraic properties of $W^+(\mathbb{C}_+)$ investigated in this article have important consequences in control theory:

   The relevance in control theory of the question whether or not $W^+(\mathbb{C}_+)$ is a GCD domain can be found in [13]: Let $R$ denote a ring of stable transfer functions, and $Q(R)$ the field of fractions of $R$. Then every transfer function $p \in Q(R)$ admits a weak coprime factorisation iff $R$ is a GCD domain.

   The importance of the coherence property in control theory can be found in [15, Theorem 3.24, p. 286]. In fact, our Theorem 1.4 answers a question raised in [14, p. 30].

   The motivation for proving that $W^+(\mathbb{C}_+)$ is a Hermite ring is that if a transfer function $G$ (with entries from the field of fractions of $W^+(\mathbb{C}_+)$) has a right (or left) coprime factorisation, then $G$ has a doubly coprime factorisation, and then the standard Youla–Kučera parameterisation yields all stabilising controllers for $G$. For further details, see [24, Theorem 66, p. 347].

Our main results are the following:

**Theorem 1.4.** The ring $W^+(\mathbb{C}_+)$ is not a GCD domain.

**Theorem 1.5.** The ring $W^+(\mathbb{C}_+)$ is not coherent.

**Theorem 1.6.** The ring $W^+(\mathbb{C}_+)$ is Hermite.

The paper is structured as follows. In Section 2, we will first collect a few auxiliary results needed to prove our main theorems. Subsequently in Sections 3, 4 and 5 we will prove Theorem 1.4, 1.5, and 1.6, respectively.
2 Preliminaries

We begin by recalling different (but equivalent) versions of the corona theorem for $W^+(\mathbb{C}_+)$; see [3, p. 112] and [11, Proposition 4.4]. Let $L^1(0, \infty)$ denote the set of Laplace transforms of functions in $L^1(0, \infty)$. Note also that by the non-discrete version of the Riemann–Lebesgue Lemma

$$\lim_{s \to \infty \atop \Re(s) \geq 0} F(s) = 0$$

for any $F \in L^1(0, \infty)$. Hence, every function in $W^+(\mathbb{C}_+)$ admits a continuous extension at infinity. In particular, $W^+(\mathbb{C}_+) \subseteq H^\infty(\mathbb{C}_+)$, the set of bounded analytic functions on the open right half plane.

**Proposition 2.1.** The following assertions hold:

1. The set of maximal ideals in $W^+(\mathbb{C}_+)$ is given by

$$\mathcal{M}_a = \{ f \in W^+(\mathbb{C}_+) \mid f(a) = 0 \}, \quad a \in \mathbb{C}_+,$$

and

$$\mathcal{M}_\infty = \overline{L^1(0, \infty)} = \{ f \in W^+(\mathbb{C}_+) \mid \lim_{s \to \infty \atop \Re(s) \geq 0} f(s) = 0 \}.$$ 

2. The set $\{ \phi_a \mid \Re(a) > 0 \}$ of point evaluations $\phi_a(f) = f(a)$, where $f \in W^+(\mathbb{C}_+)$, is dense in the maximal ideal space of $W^+(\mathbb{C}_+)$.

3. For every $n$-tuple $(f_1, \ldots, f_n)$ of functions in $W^+(\mathbb{C}_+)$ satisfying

$$\delta := \inf_{\Re(s) > 0} \sum_{j=1}^n |f_j(s)| > 0,$$

there exists a solution $(g_1, \ldots, g_n) \in W^+(\mathbb{C}_+)^n$ of the Bézout equation

$$\sum_{j=1}^n g_j f_j = 1.$$

Using a general procedure that allows to pass from $n$-tuples to matrices (see [24, p. 340]), we obtain the following matricial version of the corona theorem:

**Proposition 2.2.** Let $R = W^+(\mathbb{C}_+)$. Let $F \in \mathbb{R}^{n \times k}$. Then the following are equivalent:

1. There exists a $G \in \mathbb{R}^{k \times n}$ such that $G(s)F(s) = I_k$, $s \in \mathbb{C}_+$.

2. There exists a $\delta > 0$ such that $F(s)^*F(s) \succeq \delta^2 I_k$, $s \in \mathbb{C}_+$. 
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Here $I_k$ is the identity matrix in $\mathbb{C}^k$ and $F^*$ is the conjugate transpose (or adjoint) of $F$. We note that conditions 1 or 2 imply that, automatically, $n \geq k$; in particular the rank of every $k \times k$-submatrix of $F(s)$ is $k$. Condition 2, also, is equivalent to

3. $\|F(s)[x]\|_2 \geq \delta \|x\|_2$ for every $x \in \mathbb{C}^k$, $s \in \mathbb{C}_+$.

In the following, the maximal ideal $\mathfrak{M}_0$, being the kernel of the complex homomorphism $F \mapsto F(0)$, will play an important role. Since every maximal ideal is closed, the set $\mathfrak{M}_0$ is a commutative Banach subalgebra of $W^+(\mathbb{C}_+)$. Obviously the subalgebra $\mathfrak{M}_0$ has no identity element. But there is a substitute, namely the notion of the bounded approximate identity, which will be useful in the sequel.

**Definition 2.3.** Let $R$ be a commutative Banach algebra (without identity element). Then $R$ has a bounded approximate identity if there exists a bounded sequence $(e_n)_n$ of elements $e_n$ in $R$ such that for any $f \in R$, $\lim_{n \to \infty} \|e_n f - f\| = 0$.

It is known that the maximal ideal $\mathfrak{M}_0$ has a bounded approximate identity; see [11, Theorem 4.2.(a), p. 6].

**Proposition 2.4.** Let $e_n(s) := \frac{s}{s+\frac{1}{n}}$ ($s \in \mathbb{C}_+$), $n \in \mathbb{N}$. Then $(e_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathfrak{M}_0$.

We will also need the following amazing factorisation theorem:

**Theorem 2.5** (Varopoulos, [23]). Let $R$ be a commutative Banach algebra with a bounded approximate identity. Then for every sequence $(a_n)_{n \geq 1}$ in $R$ there exists a sequence $(b_n)_{n \geq 1}$ in $R$ as well as an element $c \in R$ such that for all $n \geq 1$, $a_n = cb_n$.

The following was noted in [10, Remark after Theorem 1, p. 224] without proof. A proof is given below. Our proof of the Theorem 1.4 will follow the same method.

**Proposition 2.6** (Mortini–von Renteln, [10]). Let $f_1, f_2 \in W^+(\mathbb{D})$ be defined as follows:

$$f_1(z) := (1-z)^3 \quad \text{and} \quad f_2(z) := (1-z)^3 e^{-\frac{1+z}{1-z}} (z \in \mathbb{D}).$$

Then $f_1, f_2$ do not have a greatest common divisor in $W^+(\mathbb{D})$.

**Proof.** Suppose that $d$ is a gcd and let $f_1 = dq_1$, $f_2 = dq_2$. Then $q_1$ is not invertible in $W^+(\mathbb{D})$, otherwise $f_1$ would divide $f_2$, which is not the case. Since the only zero of $f_1$ is at $z = 1$, it follows that $q_1(1) = 0$. Similarly, $q_2(1) = 0$. So
$q_1$ and $q_2$ belong to the maximal ideal $m_1 := \{ f \in W^+(\mathbb{D}) \mid f(1) = 0 \}$, which has a bounded approximate identity (see [2] and [11]). Hence by Theorem 2.5 (applied to $(q_1, q_2, 0, 0, 0, \ldots)$), there is a common factor $c \in m_1$ of $q_1$ and $q_2$. Thus $k := dc$ divides $f_1$ and $f_2$. But $d$ is a gcd of $f_1$, $f_2$, and so $k$ must divide $d$, say $d'ch = d$ for some $h \in W^+(\mathbb{D})$. Since $f_1$ is never zero on $\mathbb{D}$, neither is $d'$. So we obtain that on $\mathbb{D}$, $ch = 1$. But $c \in m_1$ and $h$ is bounded and continuous on the closed unit disc. So by passing the limit as $z \to 1$ in $ch = 1$, we obtain the contradiction that $0 = 1$. \hfill \square

Instead of $f_1$ and $f_2$ above, we will use the following in the case of $W^+(\mathbb{C}_+)$:

**Lemma 2.7.** For $\text{Re}(s) > 0$, let

$$F_1(s) := \left(1 - \frac{1}{s + 1}\right)^3 \quad \text{and} \quad F_2(s) := \left(1 - \frac{1}{s + 1}\right)^3 e^{-\frac{s+2}{s}}.$$

Then $F_1, F_2 \in W^+(\mathbb{C}_+)$. Moreover, $F_1, F_2 \in \mathfrak{M}_0$.

**Proof.** It is easy to see that $F_1 \in W^+(\mathbb{C}_+)$. It was also noted in [10] that $f_2$ given by $f_2(z) := (1 - z)^3 e^{-\frac{1+z}{1-s}} (z \in \mathbb{D})$ belongs to $W^+(\mathbb{D})$. So if the complex numbers $a_n$ ($n \geq 0$) are defined via

$$(1 - z)^3 e^{-\frac{1+z}{1-s}} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}, \quad (1)$$

then $\sum_{k=0}^{\infty} |a_k| < \infty$. But if $\text{Re}(s) > 0$, then $\frac{1}{s+1} \in \mathbb{D}$, and so

$$F_2(s) = \left(1 - \frac{1}{s + 1}\right)^3 e^{-\frac{s+2}{s}} = \left(1 - \frac{1}{s + 1}\right)^3 e^{-\frac{1+z}{1-s}}$$

$$= a_0 + a_1 \frac{1}{s + 1} + a_2 \left(\frac{1}{s + 1}\right)^2 + a_3 \left(\frac{1}{s + 1}\right)^3 + \cdots. \quad (2)$$

We have $\|\frac{1}{s+1}\|_{W^+} = 1$ and so $\|(\frac{1}{s+1})^n\|_{W^+} \leq \|\frac{1}{s+1}\|_{W^+}^n = 1^n = 1$. Since $\sum_{k=0}^{\infty} |a_k| < \infty$, it follows that the series

$$a_0 + a_1 \frac{1}{s + 1} + a_2 \left(\frac{1}{s + 1}\right)^2 + a_3 \left(\frac{1}{s + 1}\right)^3 + \cdots \quad (3)$$

converges in norm to an element in $W^+(\mathbb{C}_+)$. For $f_a \in L^1(0, \infty)$, we have $|\hat{f}_a(s)| \leq \|f_a\|_{L^1} (s \in \mathbb{C}_+)$, and so for every $F \in W^+(\mathbb{C}_+)$, $|F(s)| \leq \|F\|_{W^+} (s \in \mathbb{C}_+)$. Thus norm-convergence implies pointwise convergence. But by (2), the pointwise limit for $\text{Re}(s) > 0$ is in fact $F_2$. Thus $F_2 \in W^+(\mathbb{C}_+)$. 

That $F_1 \in \mathfrak{M}_0$ is trivial. To see that $F_2 \in \mathfrak{M}_0$, we observe that for $s = x + iy \in \mathbb{C}_+, s \neq 0$, $F_2(s) = \left(\frac{s}{s+1}\right)^3 e^{-1} e^{-\frac{2x}{s^2 + y^2}}$ and 
\[
|s^3 e^{-2/s}| = s^3 e^{-\frac{2x}{s^2 + y^2}} \to 0 \quad \text{as } s \to 0. \quad \Box
\]

We will need the result below (a Nakayama type lemma) in order to prove that $W^+(\mathbb{C}_+)$ is not coherent. An analytic proof can be given along the lines to the analogous result for $W^+(\mathbb{D})$ [10, Lemma 1]:

**Lemma 2.8.** Let $\mathcal{L} \neq (0)$ be an ideal in $W^+(\mathbb{C}_+)$ contained in the maximal ideal $\mathfrak{M}_0$. If $\mathcal{L} = \mathcal{L}\mathfrak{M}_0$, that is, if every function $F \in \mathcal{L}$ can be factorised in a product $F = HG$ of two functions $H \in \mathcal{L}$ and $G \in \mathfrak{M}_0$, then $\mathcal{L}$ cannot be finitely generated.

We will also need the following technical result, the proof of which is basically a repetition of key steps from Browder’s proof of Cohen’s factorisation theorem; see [1, Theorem 1.6.5, p. 74]. It uses the fact that $\mathfrak{M}_0$ has a bounded approximate identity. For a detailed exposition, see also [18, Lemma 2.8].

**Lemma 2.9.** Let $R_1, R_2 \in \mathfrak{M}_0$ and $\delta > 0$. Let $U(W^+(\mathbb{C}_+))$ denote the set of all invertible elements in $W^+(\mathbb{C}_+)$. Then there exists a sequence $(G_n)_{n \in \mathbb{N}}$ in $W^+(\mathbb{C}_+)$ such that

1. for all $n \in \mathbb{N}$, $G_n \in U(W^+(\mathbb{C}_+))$;
2. $(G_n)_{n \in \mathbb{N}}$ is convergent in $W^+(\mathbb{C}_+)$ to a limit $G \in \mathfrak{M}_0$;
3. for all $n \in \mathbb{N}$, $\|G_n^{-1}R_i - G^{-1}_{n+1}R_i\|_{W^+} \leq \delta/2^n$, $i = 1, 2$.

We now state Lin’s result, which will be used to show that $W^+(\mathbb{C}_+)$ is Hermite; [6, Theorem 3, p. 127]. Recall that a topological space, $X$, is said to be contractible if there exists a continuous map $\kappa : X \times [0, 1] \to X$ and $x_0 \in X$ such that $\kappa(x, 0) = x$ for all $x$ and $\kappa(x, 1) = x_0$.

**Proposition 2.10.** Let $R$ be a commutative Banach algebra with identity. If the maximal ideal space $X(R)$ of the Banach algebra is contractible, then $R$ is a Hermite ring.

### 3 $W^+(\mathbb{C}_+)$ is not a GCD domain

**Proof of Theorem 1.4.** We claim that $F_1, F_2$ (defined as in Lemma 2.7) have no gcd. Suppose, on the contrary, that $D$ is a gcd, and let $F_1 = DQ_1$ and $F_2 = DQ_2$ with $Q_1, Q_2 \in W^+(\mathbb{C}_+)$. 

Step 1. $Q_1$ is not invertible in $W^+(\mathbb{C}_+)$. If not, then $D = F_1 Q_1^{-1}$ and so $F_2 = D Q_2 = F_1 (Q_1^{-1} Q_2)$. In particular, for $s \neq 0$, we have $e^{-\frac{s+2}{s}} = Q_1^{-1} Q_2 \in W^+(\mathbb{C}_+)$, which is not the case, since $\lim_{\mathbb{R} \ni \omega \to 0} e^{-i\omega + 2} \frac{s}{\omega}$ does not exist, while $\lim_{\mathbb{R} \ni \omega \to 0} Q_1^{-1}(i \omega)Q_2(i \omega)$ does (because $Q_1^{-1} Q_2 \in W^+(\mathbb{C}_+)$), a contradiction.

Since we have that $\lim_{\mathbb{C} \ni \omega \to 0} F_1(s) = 1 \neq 0$, it follows from $F_1 = D Q_1$ and the fact that $W^+(\mathbb{C}_+)$-functions have a continuous extension at infinity, that $\lim_{\mathbb{C} \ni \omega \to 0} Q_1(s) \neq 0$. So by the Corona Theorem for $W^+(\mathbb{C}_+)$ (Proposition 2.1), we conclude that $Q_1$ has a zero in $\mathbb{C}_+$. But the only zero of $F_1$ is 0, and since $F_1 = D Q_1$, we have that this zero of $Q_1$ must be 0. Consequently, $Q_1(0) = 0$, that is, $Q_1 \in \mathfrak{M}_0$.

Step 2. Also, $Q_2$ is not invertible in $W^+(\mathbb{C}_+)$. Otherwise $D = F_2 Q_2^{-1}$ and $F_1 = F_2(Q_2^{-1} Q_1)$, so that for $s \neq 0$, we have $1 = e^{-\frac{s+2}{s}} (Q_2^{-1} Q_1)$, and so $e^{-\frac{s+2}{s}} = Q_2^{-1} Q_1$. But $\lim_{\mathbb{R} \ni \omega \to 0} e^{-i\omega + 2} \frac{s}{\omega}$ does not exist, while $\lim_{\mathbb{R} \ni \omega \to 0} Q_2^{-1}(i \omega)Q_1(i \omega)$ does (because $Q_2^{-1} Q_1 \in W^+(\mathbb{C}_+)$), a contradiction.

Since we have that $\lim_{\mathbb{C} \ni \omega \to 0} F_2(s) = e^{-1} \neq 0$, it follows from $F_2 = D Q_2$ that $\lim_{\mathbb{C} \ni \omega \to 0} Q_2(s) \neq 0$. Again by the Corona Theorem for $W^+(\mathbb{C}_+)$, we conclude that $Q_2$ has a zero in $\mathbb{C}_+$. But the only zero of $F_2$ is 0, and since $F_2 = D Q_2$, we have $Q_2 \in \mathfrak{M}_0$ as well.

Step 3. So $Q_1$ and $Q_2$ belong to the maximal ideal $\mathfrak{M}_0$ which has a bounded approximate identity, by Proposition 2.4. From Theorem 2.5 (applied to the sequence $(Q_1, Q_2, 0, 0, 0, \ldots)$), it follows that there is a common factor $G \in \mathfrak{M}_0$ of $Q_1$ and $Q_2$. Thus $K := DG$ divides $F_1$ and $F_2$. But $D$ is a gcd of $F_1$, $F_2$, and so $K$ must divide $D$, say $DGH = D$ for some $H \in W^+(\mathbb{C}_+)$. Since $F_1$ is never zero for $s \neq 0$, neither is $D$. So we obtain that for $s \neq 0$, $GH = 1$. But $G \in \mathfrak{M}_0$, and $H$ is bounded and continuous in $\mathbb{C}_+$. Hence by passing the limit as $s \to 0$ in $GH = 1$, we obtain the contradiction that $0 = 1$.

\textbf{Remark 3.1.} (1) In a similar manner, one can also show that $\mathcal{A}$ is not a GCD domain, where $\mathcal{A}$ denotes the set of all functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that

$$F(s) = \hat{f}_a(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_+),$$

where $f_a \in L^1(0, \infty)$, $(f_k)_{k \geq 0} \in \ell^1$, $t_0 = 0$ and $t_k > 0$ for $k = 1, 2, \ldots$.

$\mathcal{A}$ is a Banach algebra when it is equipped with pointwise operations and the norm: $\|F\|_{\mathcal{A}} := \|f_a\|_{L^1} + \|(f_k)_{k \geq 0}\|_{\ell^1}$.
(2) Recall that an integral domain \( R \) is said to be pre-Bézout if for every \( a, b \in R \) for which there exists a greatest common divisor \( d \), there exist \( x, y \in R \) such that \( d = xa + yb \).

However, as opposed to \( W^+(\mathbb{C}_+) \), which is a pre-Bézout domain [11, Theorem 1.5], it turns out that \( \mathcal{A} \) is not a pre-Bézout domain. Indeed, consider the elements \( U_1, U_2 \in \mathcal{A} \) given by

\[
U_1(s) := \frac{1}{s + 1} \quad \text{and} \quad U_2(s) := e^{-s}.
\]

Note that \( U_1 \) is an outer function in \( H^\infty \) and \( U_2 \) an inner function. Then 1 is a greatest common divisor of \( U_1 \) and \( U_2 \), but if there exist \( G_1, G_2 \in \mathcal{A} \) such that

\[
1 = G_1 U_1 + G_2 U_2,
\]

then by passing to the limit \( s \to +\infty, s \in \mathbb{R} \), and using

\[
\lim_{(\mathbb{R} \ni s \to \infty)} U_1(s) = 0 = \lim_{(\mathbb{R} \ni s \to \infty)} U_2(s),
\]

we obtain the contradiction that \( 1 = 0 \).

4 \( W^+(\mathbb{C}_+) \) is not coherent

We use the same approach as the one used to show the noncoherence of \( W^+(\mathbb{D}) \) in [10, Theorem 3, p. 226]. Nevertheless, we record the details here for the sake of convenience of the reader.

Proof of Theorem 1.5. We present two principal ideals \( \mathcal{I} \) and \( \mathcal{J} \) such that \( \mathcal{I} \cap \mathcal{J} \) is not finitely generated.

Let

\[
P(s) = \left(1 - \frac{1}{1+s}\right)^3 \quad \text{and} \quad S(s) = e^{-\frac{s+2}{s}}
\]

for \( \text{Re}(s) > 0 \). Note that \( P = F_1 \) and \( PS = F_2 \), the functions in Lemma 2.7. Also, \( S \in H^\infty \), since \( |S| \leq e^{-1} \).

By Lemma 2.7, \( F_1, F_2 \in \mathcal{M}_0 \). We define the ideals \( \mathcal{I} = (P) \) and \( \mathcal{J} = (PS) \).

Let

\[
\mathcal{R} := \{PSF \mid F \in W^+(\mathbb{C}_+) \text{ and } SF \in W^+(\mathbb{C}_+)\}.
\]

We claim that \( \mathcal{R} = \mathcal{I} \cap \mathcal{J} \). Trivially \( \mathcal{R} \subset \mathcal{I} \cap \mathcal{J} \). To prove the reverse inclusion, let \( G \in \mathcal{I} \cap \mathcal{J} \). Then there exist two functions \( F \) and \( H \) in \( W^+(\mathbb{C}_+) \) such that

\[
G = PH = PSF. \quad \text{Hence } SF = H \in W^+(\mathbb{C}_+). \quad \text{So } G \in \mathcal{R}.
\]
Let $\mathcal{L}$ denote the ideal

$$
\mathcal{L} := \{ F \in W^+(\mathbb{C}_+) \mid SF \in W^+(\mathbb{C}_+) \}.
$$

Then $\mathcal{R} := PSL$. 

We first show that $L \subset M_0$. Let $F \in L$. We have

$$
\lim_{\omega \searrow 0} F(i\omega)S(i\omega) = \lim_{r \searrow 0} F(r)S(r) = 0,
$$

since $F$ is bounded in $\mathbb{C}_+$ and $\lim_{r \searrow 0} S(r) = 0$. Since for $\omega \in \mathbb{R} \setminus \{0\}$ we have

$$
S(i\omega) = e^{-1} e^{-\frac{2}{i\omega}},
$$

it follows that $S(i\omega)$ is invertible and $|[S(i\omega)]^{-1}| = e^{-1}$. Thus

$$
\lim_{\omega \searrow 0} F(i\omega) = \lim_{\omega \searrow 0} F(i\omega)S(i\omega)[S(i\omega)]^{-1} = 0,
$$

and so $F \in M_0$. Consequently, $L \subset M_0$.

We will show that $L = LM_0$. Let $F \in L$. Then $F \in M_0$. Also, since $|S|$ is bounded by 1 on $\text{Re}(s) > 0$ and $F(0) = 0$, it follows that $SF \in M_0$. We would like to factor $F = HG$ with $H \in L$ and $G \in M_0$. Applying Lemma 2.9 with $R_1 := F \in M_0$ and $R_2 := SF \in M_0$, for any $\delta > 0$, there exists a sequence $(G_n)_{n \in \mathbb{N}}$ in $W^+(\mathbb{C}_+)$ such that

1. $G_n \in U(W^+(\mathbb{C}_+))$ ($n \in \mathbb{N}$).
2. $(G_n)_{n \in \mathbb{N}}$ is convergent in $W^+(\mathbb{C}_+)$ to a limit $G \in M_0$.
3. $\|G_n^{-1}F - G_{n+1}^{-1}F\|_{W^+} \leq \frac{\delta}{2\pi}, \|G_n^{-1}SF - G_{n+1}^{-1}SF\|_{W^+} \leq \frac{\delta}{2\pi}$ ($n \in \mathbb{N}$).

Put

$$
H_n := G_n^{-1}F \quad \text{and} \quad K_n := G_n^{-1}SF.
$$

Then $H_n \in M_0$. Also $K_n \in M_0$. The estimates above imply that $(H_n)_{n \in \mathbb{N}}$ and $(K_n)_{n \in \mathbb{N}}$ are Cauchy sequences in $W^+(\mathbb{C}_+)$. Since $M_0$ is closed, they converge to elements $H$ and $K$, respectively, in $M_0$: $H_n \rightarrow H$ and $K_n \rightarrow K$.

Let $H^\infty(\mathbb{C}_+)$ denote the Hardy space of all bounded analytic functions in the open right half plane equipped with the norm $\|\varphi\|_{\infty} := \sup_{\text{Re}(s) > 0} |\varphi(s)|$, $\varphi \in H^\infty(\mathbb{C}_+)$. If $f_a \in L^1(0, \infty)$ and $f_0 \in \mathbb{C}$, then we have

$$
|\hat{f}_a(s) + f_0| \leq |\hat{f}_a(s)| + |f_0| \leq \|f_a\|_{L^1} + |f_0| \quad (s \in \mathbb{C}_+),
$$
and so it follows that \( \| F \|_\infty \leq \| F \|_{W^+} \) for all \( F \in W^+(\mathbb{C}_+) \). Hence convergence in \( W^+(\mathbb{C}_+) \) implies convergence in \( H^\infty(\mathbb{C}_+) \), and so

\[
H_n \xrightarrow{H^\infty(\mathbb{C}_+)} H \quad \text{(since } H_n \xrightarrow{W^+(\mathbb{C}_+)} \text{)}
\]

\[
SH_n \xrightarrow{H^\infty(\mathbb{C}_+)} SH \quad \text{(since } H_n \xrightarrow{W^+(\mathbb{C}_+)} H \text{ and } S \in H^\infty(\mathbb{C}_+))
\]

\[
SH_n \xrightarrow{H^\infty(\mathbb{C}_+)} K \quad \text{(since } K_n \xrightarrow{W^+(\mathbb{C}_+)} K)\]

and so \( SH = K \). Also, in \( W^+(\mathbb{C}_+) \) norm we have

\[
F = \lim_{n \to \infty} H_n G_n = HG.
\]

Since \( H \) and \( SH = K \) belong to \( \mathfrak{M}_0 \subset W^+(\mathbb{C}_+) \), we see that \( H \in \mathcal{L} \). Moreover, as \( G \in \mathfrak{M}_0 \), we have got the desired factorisation and \( \mathcal{L} = \mathfrak{M} \mathfrak{M}_0 \).

But \( \mathcal{L} \neq (0) \), since \( P \in \mathcal{L} \). By Lemma 2.8, it follows that \( \mathcal{L} \) cannot be finitely generated. Therefore, \( PS\mathcal{L} = \mathfrak{I} \cap \mathfrak{J} \) is not finitely generated. \( \square \)

**Remark 4.1.** 1. The ideal \( \mathcal{L} \) in the above proof can be interpreted as an *ideal of denominators*; see [4, p. 396]. Indeed, using the fact that \( PS \in W^+(\mathbb{C}_+) \), we have \( S \in Q(W^+(\mathbb{C}_+)) \), where \( Q(W^+(\mathbb{C}_+)) \) denotes the field of fractions of \( W^+(\mathbb{C}_+) \). The *ideal of denominators* \( \mathcal{L} \) of \( S \), namely

\[
\mathcal{L} = \{ d \in W^+(\mathbb{C}_+) \mid dS \in W^+(\mathbb{C}_+) \}
\]

is the ideal of \( W^+(\mathbb{C}_+) \) consisting of all possible denominators of \( S \), together with 0, when written as a fraction of elements from \( W^+(\mathbb{C}_+) \); see the book by Matusumura [7].

2. Following the proof in [10], the second author had proved that \( \mathcal{A} \) is also not coherent [18]. The following functions were used there:

\[
P(s) = \frac{(1 - e^{-s})^3}{s + 1} \quad \text{and} \quad S(s) = e^{-\frac{1+e^{-s}}{1-e^{-s}}}.
\]

Another way to see that \( W^+(\mathbb{C}_+) \) is not a GCD domain, is to use the following observation together with Theorem 1.5 and the fact that \( W^+(\mathbb{C}_+) \) is a pre-Bézout domain (see [11]):

**Observation 4.2.** Let \( R \) be a pre-Bézout domain. Let \( f, g \in R \). Suppose that the intersection of the associated principal ideals \( (f) \) and \( (g) \) is not finitely generated. Then \( f \) and \( g \) have no greatest common divisor.
Proof. Assuming the contrary, let \( d \) be a greatest common divisor of \( f \) and \( g \) and write \( f = dF, g = dG \). Then \( \gcd(F, G) = 1 \). We claim that \( (f) \cap (g) = (dFG) \). In fact, one trivially has that \( (dFG) \subseteq (f) \cap (g) \). Now let \( h = xdF = ydG \). Then \( d(xF - yG) = 0 \). Since there are now divisors of zero, and \( d \neq 0 \), \( xF = yG \). Now \( R \) has the pre-Bézout property; hence \( 1 = aF + bG \) for some \( a, b \in R \). Thus \( y = yaF + b(yG) = yaF + b(xF) = F(ya + bx) \). So \( h = y(dG) = F(ya + bx)dG \in (dFG) \). So \( (f) \cap (g) \subseteq (dFG) \).

5 \( W^+(\mathbb{C}_+) \) is a Hermite ring

Proof of Theorem 1.6. Consider the algebra

\[
A^+ := \left\{ F \left( \frac{1+z}{1-z} \right) \mid z \in \mathbb{D}, F \in W^+(\mathbb{C}_+) \right\},
\]

with pointwise operations and the same norm as in \( W^+(\mathbb{C}_+) \), that is,

\[
\| z \mapsto F \left( \frac{1+z}{1-z} \right) \|_{A^+} := \| F \|_{W^+(\mathbb{C}_+)}, \quad F \in W^+(\mathbb{C}_+).
\]

Then this is a Banach algebra that is isometrically isomorphic to \( W^+(\mathbb{C}_+) \). Since \( \lim_{s \to \infty, s \in \mathbb{C}_+} F(s) \) exists for each element \( F \in W^+(\mathbb{C}_+) \), we see that every \( f \in A^+ \) admits a continuous extension to \( z = 1 \) and hence \( A^+ \subseteq A(\mathbb{D}) \). Now we will show that the maximal ideal space of \( A^+ \) is homeomorphic to the closed unit disc \( \overline{\mathbb{D}} \) in \( \mathbb{C} \).

Using the formula

\[
1 = \frac{f - f(a)}{f(a)} + \frac{f}{f(a)}
\]

whenever \( f \in A^+ \) and \( f(a) \neq 0 \), \( a \in \overline{\mathbb{D}} \), we see that

\[
M_a := \{ f \in A^+ \mid f(a) = 0 \}
\]

is a maximal ideal in \( A^+ \). But these are all the maximal ideals of \( A^+ \). In fact, let \( M \) be any maximal ideal in \( A^+ \) and suppose that \( M \) is not contained in \( M_a \) for any \( a \in \overline{\mathbb{D}} \). Then by a compactness argument, there exists finitely many functions \( f_1, \ldots, f_n \in A^+ \) so that \( \sum_{j=1}^n |f_j| \geq \delta > 0 \) on \( \mathbb{D} \). Moving back to the algebra \( W^+(\mathbb{C}_+) \), we see from the corona theorem 2.1 that the ideal generated by the functions \( w \mapsto f(\frac{w}{w+1}), \Re(w) > 0 \) is \( W^+(\mathbb{C}_+) \). Hence \( (f_1, \ldots, f_n) \) is not proper either. This contradiction shows that \( M \subseteq M_a \) for some \( a \in \overline{\mathbb{D}} \). The
maximality of $M_a$ now implies that $M = M_a$. Finally, since $\overline{D}$ is compact, it is easy to see that the map $\Psi : \overline{D} \to X(A^+), a \mapsto \Phi_a$ is a homeomorphism; here $X(A^+)$ is the space of nonzero multiplicative linear functionals on $A^+$ endowed with the Gelfand topology and $\Phi_a$ is the evaluation functional at $a$.

To sum up, we have shown that the maximal ideal space of $A$ can be identified with $D$. Since $D$ is contractible, it follows that the maximal ideal space of $W^+(\mathbb{C}_+)$ is contractible. The claim that $W^+(\mathbb{C}_+)$ (and also $A^+$) are Hermite rings now follows via Lin’s result given in Proposition 2.10.

Remark 5.1. 1. The second author proved that $A$ is also Hermite [19].

2. It can be shown that a commutative ring $R$ with identity, and having Bass stable rank equal to 1 is a Hermite ring; see [22, p. 3155]. It is known that the Bass stable rank of $W^+(\mathbb{C}_+)$ is 1; see [9, Theorem 1.2]. So this gives another proof of Theorem 1.6.

We conclude this note by showing that $A^+$ is not contained in $W^+(\mathbb{D})$.

Proposition 5.2. The following assertions hold:

1. If $f \in W^+(\mathbb{D})$, then $f(\frac{1-z}{2}) \in A^+$.

2. For $\alpha > 0$,

$$f_\alpha(z) = (1-z)^\alpha e^{-\frac{1+z}{2}} \in W^+(\mathbb{D}) \iff \alpha > 1/2$$

but $f_\alpha \in A^+$ for all $\alpha > 0$;

3. $f_\alpha(\frac{s-1}{s+1}) = \frac{2^\alpha}{(1+s)^\alpha} e^{-s} \in L^1(0, \infty)$;

4. $\|z\|_{A^+} = 3$.

Proof. To show 1 and 4, we have to observe that $z = F(\frac{s+1}{s-1})$, where $F(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1} = 1 - 2e^{-t}$. Now let $f \in W^+(\mathbb{D})$, say $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n| < \infty$. As in the proof of Lemma 2.7, we see that the function $F$ given by $F(s) = f(\frac{1+1}{1+s}) \in W^+(\mathbb{C}_+)$. Replacing $s$ by $\frac{1+z}{2}$, $z \in \mathbb{D}$, we get that $f(\frac{1-z}{2}) = F(\frac{1+z}{1-s}) \in A^+$. But it is obvious that $h(z) := \sum a_n(\frac{1-z}{2})^n \in W^+(\mathbb{D})$ since $\frac{1-z}{2} \| W^+(\mathbb{D}) = 1$, and so the series for $h$ is a Cauchy-sequence in $W^+(\mathbb{D})$.

For 3, we observe that trivially, $f_\alpha(\frac{s-1}{s+1}) = \frac{2^\alpha}{(1+s)^\alpha} e^{-s} =: G(s)$. But $G(s)$ is the Laplace transform of the $L^1(0, \infty)$-function

$$g(t) = 2^\alpha \frac{e^{-(t-1)u(t-1)}}{\Gamma(1-\alpha)(t-1)^{1-\alpha}},$$
where \( u(t) \) is the Heaviside function given by

\[
u(t) = \begin{cases} 
0 & \text{if } t < 0, \\
1 & \text{if } t > 0.
\end{cases}
\]

Thus \( f_\alpha \in A^+ \) for each \( \alpha > 0 \).

The following proof that \( f_\alpha \in W^+(\mathbb{D}) \) is from Udo Klein [5]. Let

\[
g_\beta(z) := (1-z)^{-\beta-1} \exp \frac{\mu z}{z-1}, \quad \beta := -\alpha - 1, \quad \mu = 2.
\]

Then \( f_\alpha = e^{-1}g_\beta \). Hence it will suffice to show that \( g_\beta \in W^+(\mathbb{D}) \) if and only \( \beta < -\frac{3}{2} \). The Taylor coefficients of \( g \) though are (more or less by definition) generalised Laguerre polynomials. Indeed, we have the following expansion (see [20, p. 100]):

\[
g_\beta(z) = \sum_{n=0}^{\infty} L_n^{(\beta)}(\mu) z^n.
\]

The result now follows since the asymptotic behaviour of the terms \( L_n^{(\beta)}(\mu) \) is given by Fejér’s formula (see [20, p. 198])

\[
L_n^{(\beta)}(\mu) = \frac{1}{\sqrt{\pi}} e^{\mu} \mu^{-\frac{1}{2}} n^{\frac{\beta}{2} - \frac{1}{4}} \cos \left( \frac{\sqrt{4n\mu} - \frac{\beta\pi}{2} - \frac{\pi}{4}}{4} \right) + O(n^{\frac{\beta}{2} - \frac{3}{4}}).
\]

\( \square \)

Acknowledgments. The authors thank the anonymous referee for the careful review and for comments that improved the presentation of the article.

Bibliography


*Note 2*

Please update [19], if possible.


Received December 9, 2008; revised June 1, 2009.

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