NONCOHERENCE OF SOME RINGS OF FUNCTIONS

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Abstract. Let \( \mathbb{D}, \mathbb{T} \) denote the unit disc and unit circle, respectively, in \( \mathbb{C} \), with center 0. If \( S \subset \mathbb{T} \), then let \( A_S \) denote the set of complex-valued functions defined on \( \mathbb{D} \cup S \) that are analytic in \( \mathbb{D} \), and continuous and bounded on \( \mathbb{D} \cup S \). Then \( A_S \) is a ring with pointwise addition and multiplication. We prove that if the intersection of \( S \) with the set of limit points of \( S \) is not empty, then the ring \( A_S \) is not coherent.

1. Introduction

In this paper, we investigate the coherence of some rings of analytic functions. We first recall the notion of coherence.

Definition 1.1. Let \( R \) be a commutative ring with identity element \( e \), and let \( R^n = R \times \cdots \times R \) (\( n \) times). Let \( f = (f_1, \ldots, f_n) \in R^n \). An element \( (g_1, \ldots, g_n) \in R^n \) is called a relation on \( f \) if \( g_1 f_1 + \cdots + g_n f_n = 0 \). The set of all relations on \( f \in R^n \), denoted by \( f^\perp \), is a \( R \)-submodule of the \( R \)-module \( R^n \). The ring \( R \) is called coherent if for each \( f \in R^n \), \( f^\perp \) is finitely generated, that is, there exists a \( d \in \mathbb{N} \) and there exist \( g_j \in f^\perp, j \in \{1, \ldots, d\} \), such that for all \( g \in f^\perp \), there exist \( r_j \in R, j \in \{1, \ldots, d\} \) such that \( g = r_1 g_1 + \cdots + r_d g_d \).

In [5], McVoy and Rubel showed that while \( H^\infty \) is coherent, the disk algebra \( A \) is not coherent. In this paper, we prove the noncoherence of some rings that lie between \( A \) and \( H^\infty \). The rings that we consider are introduced below:

Definition 1.2. Let the open unit disk \( \{ z \in \mathbb{C} \mid |z| < 1 \} \) be denoted by \( \mathbb{D} \), and the unit circle \( \{ z \in \mathbb{C} \mid |z| = 1 \} \) be denoted by \( \mathbb{T} \). If \( S \) be a subset of \( \mathbb{T} \), then \( A_S = \{ f : \mathbb{D} \cup S \to \mathbb{C} \mid f \text{ is analytic in } \mathbb{D} \text{ and } f \text{ is continuous and bounded on } \mathbb{D} \cup S \} \), with pointwise addition and multiplication: if \( f, g \in A_S \), then

\[(f + g)(z) = f(z) + g(z) \quad \text{and} \quad (f \cdot g)(z) = f(z)g(z), \quad \text{for all } z \in \mathbb{D} \cup S.\]

Equipped with the supremum norm,

\[\|f\|_\infty = \sup_{z \in \mathbb{D} \cup S}|f(z)| \quad \text{for } f \in A_S,\]

the ring \( A_S \) forms a Banach algebra. We note that if \( S = \mathbb{T} \), then \( A_T \) is the usual disk algebra \( A \), while if \( S = \emptyset \), then one obtains the Hardy space \( H^\infty \).

The spaces \( A_S \) considered here have been studied earlier (see for instance [1], where among other things, it was shown that the corona theorem holds for \( A_S \)). These classes are also of interest in control theory, where they form natural families.
of irrational transfer functions (see [10], where it was shown that \( A_S \) is a Hermite domain, although not a Bézout domain, and the stable rank and topological stable rank of \( A_S \) are 1 and 2, respectively, and the consequences of these properties in control theory were elaborated).

In this note, we study the coherence of the rings \( A_S \). The relevance of the coherence property in control theory can be found in [7], [8]. For an expository background on coherent rings, we refer the reader to [2]. Noncoherence has been discussed in the context of other rings, for instance, see [6], [4].

Following the ideas of McVoy and Rubel from [5], used for proving the noncoherence of \( A_S \), we show that if \( S \) is a subset of the unit circle such that the intersection of \( S \) with the set of limit points of \( S \) is not empty (that is, \( S \cap S' \neq \emptyset \), where \( S' \) denotes the set of limit points of \( S \)), then \( A_S \) is not coherent. If \( S = \emptyset \), then \( S \cap S' = \emptyset \), and \( A_S = H^\infty \); in [5], it is shown that \( H^\infty \) is coherent. In the case when \( S \) is not empty, and \( S \cap S' = \emptyset \), I do not know if \( A_S \) is coherent or not, and this is an open problem.

2. Noncoherence of \( A_S \) for \( S \) intersecting \( S' \)

In this section we prove the following.

**Theorem 2.1.** If \( S \) is a subset of \( \mathbb{T} \) such that the intersection of \( S \) with the set of limit points of \( S \) is not empty, then the ring \( A_S \) is not coherent.

**Proof.** Let \( z_0 \) be a limit point of \( S \) that belongs to \( S \), and let \((z_n)_{n \in \mathbb{N}}\) be any sequence in \( S \setminus \{z_0\} \) with distinct terms, and with limit \( z_0 \):

\[
\forall n \in \mathbb{N}, \; z_n \in S \setminus \{z_0\}; \quad \forall m, n \in \mathbb{N}, \; z_m \neq z_n; \quad \lim_{n \to \infty} z_n = z_0 \in S.
\]

Let \( B_1 \) be the Blaschke product

\[
B_1(z) = \prod_{n=1}^{\infty} \frac{|\alpha_n| \alpha_n - z}{\alpha_n 1 - \alpha_n^* z}, \quad z \in \mathbb{D},
\]

where

\[
\alpha_n = \left(1 - \frac{1}{(n+1)^2}\right) z_0, \quad n \in \mathbb{N}.
\]

Let \( B_2 \) be the Blaschke product

\[
B_2(z) = \prod_{n=1}^{\infty} \frac{|\beta_n| \beta_n - z}{\beta_n 1 - \beta_n^* z}, \quad z \in \mathbb{D},
\]

where

\[
\beta_n = \left(1 - \frac{1}{\sqrt{2(n+1)^2}}\right) z_0, \quad n \in \mathbb{N}.
\]

If \( m, n \in \mathbb{N} \), then \( \alpha_m \neq \beta_n \) by construction. We also observe that \((\alpha_n)_{n \in \mathbb{N}} \) and \((\beta_n)_{n \in \mathbb{N}} \) are convergent with the same limit \( z_0 \). The functions \( B_1, B_2 \) are analytic in an open set \( \Omega \) containing \( \mathbb{D} \setminus \{z_0\} \), and the infinite products converge uniformly on compact subsets contained in \( \Omega \) (see for instance Exercise 12 on page 317 of [9]).

Define \( f_i : \mathbb{D} \to \mathbb{C}, \; i \in \{1, 2\} \), as follows:

\[
f_i(z) = \begin{cases} 
(z - z_0)B_i(z) & \text{if } z \in \overline{\mathbb{D}} \setminus \{z_0\}, \\
0 & \text{if } z = z_0.
\end{cases}
\]

Then \( f_1, f_2 \in A \).
Lemma 2.2. Let \( S \) be a subset of \( \mathbb{T} \), and let \( z_0 \in S \). Suppose that \( f_i, i \in \{1, 2\} \) are defined by (2.4). If \( g_1, g_2 \in A_S \) are such that for all \( z \in \mathbb{D} \cup S \), \( g_1(z)f_1(z) + g_2(z)f_2(z) = 0 \), then \( g_1(z_0) = 0 = g_2(z_0) \).

Proof. For all \( z \in \mathbb{D} \), we have \( |g_1(z)||z - z_0||B_1(z)| = |g_2(z)||z - z_0||B_2(z)| \), and in particular, \( |g_1(\beta_n)||\beta_n - z_0||B_1(\beta_n)| = 0 \). As \( B_1(\beta_n) \neq 0 \) and \( |\beta_n - z_0| \neq 0 \), it follows that \( g_1(\beta_n) = 0 \). Since \( g_1 \in A_S \), we obtain
\[
g_1(z_0) = g_1 \left( \lim_{n \to \infty} \beta_n \right) = \lim_{n \to \infty} g_1(\beta_n) = 0.
\]

Similarly, \( g_2(z_0) = 0 \).

We shall prove that the module of relations on \( (f_1, f_2) \) is not finitely generated. Assume, on the contrary, that the module of relations on \( (f_1, f_2) \) is generated by \( (g_{j,1}, g_{j,2}) \in A^2_S \), \( j \in \{1, \ldots, d\} \). From Lemma 2.2, it follows that for all \( i \in \{1, 2\} \) and all \( j \in \{1, \ldots, d\} \),
\[
\lim_{n \to \infty} g_{j,i}(z_n) = g_{j,i} \left( \lim_{n \to \infty} z_n \right) = g_{j,i}(z_0) = 0.
\]

Let \( K \) denote the compact set defined as follows: \( K = \{z_n \mid n \in \mathbb{N}\} \cup \{z_0\} \). Define the function \( \varphi : \mathbb{D} \cup S \to \mathbb{C} \) by
\[
\varphi(z) = \left( \max\{|z - z_0|, |g_{1,1}(z)|, \ldots, |g_{d,1}(z)|, |g_{1,2}(z)|, \ldots, |g_{d,2}(z)|\} \right)^{\frac{1}{2}}.
\]

Then it can be seen that the restriction of \( \varphi \) to \( K \) is a continuous function on \( K \). Also, we note that
\[
\forall n \in \mathbb{N}, \quad \varphi(z_n)^2 \geq |z_n - z_0| > 0,
\]
and moreover, using (2.5), it follows that
\[
\lim_{n \to \infty} \varphi(z_n) = 0.
\]

\( K \) is a closed set of Lebesgue measure 0 on the unit circle, and \( \varphi : K \to \mathbb{C} \) is a continuous function on \( K \). We now recall the following result (see for instance the theorem on page 81 of [3])

**Theorem 2.3 (Rudin).** Let \( K \) be a closed set of Lebesgue measure zero on the unit circle, and let \( F \) be any continuous complex-valued function on \( K \). Then there exists a function in the disk algebra \( A \) whose restriction to \( K \) is \( F \).

An application of this theorem yields the existence of a function \( \Phi \in A \) whose restriction to \( K \) is \( \varphi \). Observe that \( \Phi(z_0) = \varphi(z_0) = 0 \). Define \( \Gamma_i : \mathbb{D} \cup S \to \mathbb{C} \), \( i \in \{1, 2\} \) as follows:
\[
\Gamma_i(z) = \begin{cases} 
\Phi(z)B_i(z) & \text{if } z \neq z_0, \\
0 & \text{if } z = z_0.
\end{cases}
\]

Then we claim that \( \Gamma_1, \Gamma_2 \in A_S \), and this can be seen as follows:

Let \( i \in \{1, 2\} \). As \( \Phi \) and \( B_i \) are analytic in \( \mathbb{D} \), it follows that \( \Gamma_i \) is also analytic in \( \mathbb{D} \).

If \( z \in \mathbb{D} \cup S \), then \( |\Phi(z)| \leq \|\Phi\|_{\infty} < +\infty \). If \( z \in \mathbb{D} \), then \( |B_i(z)| \leq 1 \). If \( z \in S \setminus \{z_0\} \), then each factor in the partial product of (2.2) or (2.3) has modulus 1, and using the (uniform) convergence of \( B_i \) on compact subsets contained in \( \Omega \), it follows that \( |B_i(z)| = 1 \) for all \( z \in S \setminus \{z_0\} \). Consequently \( \Gamma_i \) is bounded on \( \mathbb{D} \cup S \).

Finally, continuity of \( \Gamma_i \) on \( \mathbb{D} \cup S \setminus \{z_0\} \) follows from the continuity of \( B_i \) and that of \( \Phi \) on \( \mathbb{D} \cup S \setminus \{z_0\} \). Continuity of \( \Gamma_i \) at \( z_0 \) follows by observing that it is
the product of a bounded function and the continuous function $\Phi$ which converges to 0. So we conclude that $\Gamma_1, \Gamma_2 \in A_S$.

Let $g_1 := \Gamma_2$ and $g_2 := -\Gamma_1$. Then $(g_1, g_2)$ is a relation on $(f_1, f_2)$. Indeed, if $z \in (D \cup S) \setminus \{z_0\}$, then

$$g_1(z)f_1(z) + g_2(z)f_2(z) = \Phi(z)B_2(z)(z - z_0)B_1(z) - \Phi(z)B_1(z)(z - z_0)B_2(z) = 0.$$ 

If $z = z_0$, then $g_1(z)f_1(z) + g_2(z)f_2(z) = 0 \cdot 0 + 0 \cdot 0 = 0$.

However, we now show that there cannot exist functions $h_1, \ldots, h_d$ in $A_S$ such that

$$(2.8) \quad h_1(g_1,1, g_1,2) + \cdots + h_d(g_1,d, g_2,d) = (g_1, g_2).$$ 

If (2.8) holds for some $h_1, \ldots, h_d$ in $A_S$, then for all $n \in \mathbb{N}$,

$$h_1(z_n)g_1,1(z_n) + \cdots + h_d(z_n)g_d,1(z_n) = g_1(z_n) = \Gamma_2(z_n) = \Phi(z_n)B_2(z_n) = \varphi(z_n)B_2(z_n).$$

Hence for all $n \in \mathbb{N}$, we have

$$|\varphi(z_n)| = |\varphi(z_n)B_2(z_n)| \leq |h_1(z_n)||g_1,1(z_n)| + \cdots + |h_d(z_n)||g_d,1(z_n)|$$

$$\leq d \cdot \max\{|h_1||z_n - z_0|, \ldots, |h_d||z_n - z_0|\} \cdot \max\{|g_1,1(z_n)|, \ldots, |g_d,1(z_n)|\}$$

$$\leq M \cdot \max\{|z_n - z_0|, \ldots, |g_d,1(z_n)|\}.$$ 

where $M := d \cdot \max\{|h_1|, \ldots, |h_d|\}$. Consequently, using (2.6), we have

$$(2.9) \quad \forall n \in \mathbb{N}, \quad 1 \leq M|\varphi(z_n)|.$$ 

But from (2.7), the right hand side of (2.9) tends to 0 as $n \to \infty$, and so we arrive at the contradiction that $1 \leq 0$. This completes the proof. \hfill \Box

References


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