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Differential Equations

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Introduction

0.1 What a differential equation is

In any subject, it is natural and logical to begin with an explanation of what the subject matter is. Often it's rather difficult, too. Our subject matter is differential equations, and the first order of business is to define a differential equation. The easiest way out, and maybe the clearest, is to list a few examples, and hope that although we do not know how to define one, we certainly know one when we see it. But that is shirking the job. Here is one definition: an *ordinary differential equation*[1] is an equation involving known and unknown functions of a single variable and their derivatives. (Ordinarily only one of the functions is unknown.) Some examples:

1. $\frac{d^2x}{dt^2} - 7t\frac{dx}{dt} + 8x \sin t = (\cos t)^4$.
2. $\left(\frac{dx}{dt}\right)^3 + \frac{d^2x}{dt^2} = 1$.
3. $\frac{dx}{dt}\frac{d^2x}{dt^2} + \sin x = t$.
4. $\frac{dx}{dt} + 2x = 3$.

Incidentally, the word "ordinary" is meant to indicate not that the equations are run-of-the-mill, but simply to distinguish them from partial differential equations (which involve functions of several variables and partial derivatives). We shall also deal with systems of ordinary differential equations, in which several unknown functions and their derivatives are linked by a system of equations. An example:

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1x_2 + x_2 \\ \frac{dx_2}{dt} &= x_1 - t^2x_2. \end{aligned}$$

A *solution* to a differential equation is, naturally enough, a function which satisfies the equation. It's possible that a differential equation has no solutions. For instance,

$$\left(\frac{dx}{dt}\right)^2 + x^2 + t^2 = -1$$

has none. But in general, differential equations have lots of solutions. For example, the equation

$$\frac{dx}{dt} + 2x = 3$$

[1] commonly abbreviated as 'ODE'

is satisfied by

$$x = \frac{3}{2}, \quad x = \frac{3}{2} + e^{-2t}, \quad x = \frac{3}{2} + 17e^{-2t},$$

and more generally by,

$$x(t) = \frac{3}{2} + ce^{-2t},$$

where c is any real number. However, in applications where these differential equations model certain phenomena, the equations often come equipped with *initial conditions*. Thus one may demand a solution of the above equation satisfying $x = 4$ when $t = 0$. This condition lets one solve for the constant c.

Why study differential equations? The answer is that they arise naturally in applications.

Let us start by giving an example from physics since historically that's where differential equations started. Consider a weight on a spring bouncing up and down. A physicist wants to know where the weight is at different times. To find that out, one needs to know where the weight is at some time and what its velocity is thereafter. Call the position x; then the velocity is $\frac{dx}{dt}$. Now the change in velocity $\frac{d^2x}{dt^2}$ is proportional to the force on the weight, which is proportional to the amount the spring is stretched. Thus $\frac{d^2x}{dt^2}$ is proportional to x. And so we get a differential equation.

Things are pretty much the same in other fields where differential equations are used, such as biology, economics, chemistry, and so on. Consider economics for instance. Economic models can be divided into two main classes: static ones and dynamic ones. In static models, everything is presumed to stay the same; in dynamic ones, various important quantities change with time. And the rate of change can sometimes be expressed as a function of the other quantities involved. Which means that the dynamic models are described by differential equations.

How to get the equations is the subject matter of economics (or physics or biology or whatever). What to do with them is the subject matter of these notes.

0.2 What these notes are about

Given a differential equation (or a system of differential equations), the obvious thing to do with it is to solve it. Nonetheless, most of these notes will be taken up with other matters. The purpose of this section is to try to convince the student that all those other matters are really worth discussing.

To begin with, let's consider a question which probably seems silly: what does it mean to solve a differential equation? The answer seems obvious: it means that one find all functions satisfying the equation. But it's worth looking more closely at this answer. A function is, roughly speaking, a rule which associates to each t a value $x(t)$, and the solution will presumable specify this rule. That is, solving a differential equation like

$$x + t^2x = e^t, \quad x(0) = 1$$

should mean that if I choose a value of t, say $\frac{11}{\pi}$, I end up with a procedure for determining x there.

There are two problems with this notion. The first is that it doesn't really conform to what one wants as a solution to a differential equation. Most of the times a function means (intuitively, at least) a formula into which you plug t to get x. Well, for the average differential equation, this formula doesn't exist–at least, we have no way of finding it. And even when it does exist, it is often unsatisfactory. For example, an equation like

$$x'' + tx' + t^2x = 0, \quad x(0) = 1, \quad x(1) = 0$$

has a solution which can be written as a power series:

$$x(t) = 1 - \frac{1}{12}t^4 + \frac{1}{90}t^6 + \frac{1}{3360}t^8 + \dots.$$

And this, at least at first, doesn't seem too helpful. Why not? That leads to the second problem: the notion of a function given above doesn't really tell us what we want to know. Consider for instance, a typical use of a differential equation in physics, like determining the motion of a vibrating spring. One makes various plausible assumptions, uses them to derive a differential equation, and (with luck) solves it. Suppose that the procedure works brilliantly and that the solutions to the equation describe the motion of the spring. Then we can use the solutions to answer questions like

"If the mass of the weight at the end of the spring is 7 grams, if the spring constant is 3 (in appropriate units), and if I start the spring off at some given position with some given velocity where will the mass be 3 seconds later?"

But there are also more qualitative question we can ask. For instance,

"Is it true that the spring will oscillate forever? Will the oscillations get bigger and bigger, will they die out, or will they stay roughly constant in size?"

If we only know the solution as a power series, it may not be easy to answer these questions. (Try telling, for instance, whether or not the function above gets large as $t \to \infty$.) But questions like this are obviously interesting and important if one wants to know what the physical system will do as time goes on.

For applications, another matter arises. Perhaps the most spectacular way of putting it is that every differential equation used in applications is wrong! First of all, the problem being considered is usually a simplification of real life, and that introduces errors. Next, there are errors in measuring the parameters used in the problem. Result: the equation is all wrong. Of course, these errors are slight (one hopes, anyway), and presumably the solutions to the equation bear some resemblance to what happens in the world. So the qualitative behaviour of solutions is very useful.

Another question is whether solutions exist and how many do. Since there is in general no formula for solving a differential equation, we have no guarantee that there are solutions, and it would be frustrating to spend a long time searching for a solution that doesn't exist. It is also very important, in many cases, to know that exactly one solution exists.

What all of this means is that these notes will be discussing these sorts of matters about differential equations. First how to solve the simplest ones. Second, how to get qualitative information about the solutions. And third, theorems about existence and uniqueness of solutions and the like. In all three, there will be theoretical material, but we will also see examples.

Contents

Chapter 1

Linear equations

1.1 Objects of study

Many problems in economics, biology, physics and engineering involve rate of change dependent on the interaction of the basic elements–assets, population, charges, forces, etc.–on each other. This interaction is frequently expressed as a system of ordinary differential equations, a system of the form

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t), \dots, x_n(t)), & (1.1)\\ x_2'(t) &= f_2(t, x_1(t), x_2(t), \dots, x_n(t)), & (1.2)\\ &\vdots \\ x_n'(t) &= f_n(t, x_1(t), x_2(t), \dots, x_n(t)). & (1.3) \end{aligned}$$

Here the (known) functions $(\tau, \xi_1, \dots, \xi_n) \mapsto f_i(\tau, \xi_1, \dots, \xi_n)$ take values in $\mathbb{R}$ (the real numbers) and are defined on a set in $\mathbb{R}^{n+1}$ ($\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, $n+1$ times).

We seek a set of n unknown functions $x_1, \dots, x_n$ defined on a real interval I such that when the values of these functions are inserted into the equations above, the equality holds for every $t \in I$.

Definition. A function $x : [t_0, t_1] \to \mathbb{R}^n$ is said to be a *solution* of (1.1)- (1.3) if x is differentiable on $[t_0, t_1]$ and it satisfies (1.1)- (1.3) for each $t \in [t_0, t_1]$.

In addition, an initial condition may also need to be satisfied: $x(0) = x_0 \in \mathbb{R}^n$, and a corresponding solution is said to satisfy the *initial value problem*

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

Introducing the vector notation

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad x' := \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix},$$

the system of differential equations can be abbreviated simply as

$$x'(t) = f(t, x(t)).$$

In this course we will mainly consider the case when the functions $f_1, \dots, f_n$ do not depend on t (that is, they take the same value for all t).

In most of this course, we will consider autonomous systems, which are defined as follows.

Definition. If f does not depend on t, that is, it is simply a function defined on some subset of $\mathbb{R}^n$, taking values in $\mathbb{R}^n$, the differential

$$x'(t) = f(x(t)),$$

is called *autonomous*.

But we begin our study with an even simpler case, namely when these functions are *linear*, that is,

$$f(\xi) = A\xi,$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Then we obtain the 'vector' differential equations

$$x'(t) = Ax(t),$$

which is really the system of scalar differential equations given by

$$x_1' = a_{11}x_1 + \cdots + a_{1n}x_n, \tag{1.4}$$

$$\vdots$$

$$x_n' = a_{n1}x_1 + \cdots + a_{nn}x_n. \tag{1.5}$$

In many applications, the equations occur naturally in this form, or it may be an approximation to a nonlinear system.

Exercises.

1. Classify the following differential equations as autonomous/nonautonomous. In each autonomous case, also identify if the system is linear or nonlinear.
   (a) $x'(t) = e^t$.
   (b) $x'(t) = e^{x(t)}$.
   (c) $x'(t) = e^t y(t)$, $y'(t) = x(t) + y(t)$.
   (d) $x'(t) = y(t)$, $y'(t) = x(t)y(t)$.
   (e) $x'(t) = y(t)$, $y'(t) = x(t) + y(t)$.
2. Verify that the differential equation has the given function or functions as solutions.
   (a) $x'(t) = e^{\sin x(t)} + \cos(x(t))$; $x(t) \equiv \pi$.
   (b) $x'(t) = ax(t)$, $x(0) = x_0$; $x(t) = e^{ta}x_0$.
   (c) $x_1'(t) = 2x_2(t)$, $x_2'(t) = -2x_1(t)$; $x_1(t) = \sin(2t)$, $x_2(t) = \cos(2t)$.
   (d) $x'(t) = 2t(x(t))^2$; $x_1(t) = \frac{1}{1-t^2}$ for $t \in (-1, 1)$, $x_2(t) \equiv 0$.
3. Find value(s) of m such that $x(t) = t^m$ is a solution to $2tx'(t) = x(t)$ for $t \geq 1$.
4. Show that every solution of $x'(t) = (x(t))^2 + 1$ is an increasing function.

1.2 Using Maple to investigate differential equations

1.2.1 Getting started

To start Maple, follow the sequence:

$$\texttt{Start} \longrightarrow \texttt{Programs} \longrightarrow \texttt{Mathematics} \longrightarrow \texttt{Maple10}.$$

Background material about Maple can be found at:

1. A pamphlet "Getting started with Maple" can be viewed online at `ittraining.lse.ac.uk/documentation/Files/Maple-95-Get-Started.pdf` .
2. Maple's own "New User's Tour", which can be found under `Help` in Maple.
3. MA100 Maple tutorials, which can be found at `www.maths.lse.ac.uk/Courses/MA100/maths_tutorial.mws` and `www.maths.lse.ac.uk/Courses/Tut2prog.mws` .

1.2.2 Differential equations in Maple

Here we describe some main Maple commands related to differential equations.

1. *Defining differential equations.* For instance, to define the differential equation $x' = x + t$, we give the following command.

   $$> \texttt{ode1} := \texttt{diff}(\texttt{x}(\texttt{t}), \texttt{t}) = \texttt{t} + \texttt{x}(\texttt{t});$$

   Here, `ode1` is the label or name given to the equation, $\texttt{diff}(x(t), t)$ means that the function $t \mapsto x(t)$ is differentiated with respect to t, and the last semicolon indicates that we want Maple to display the answer upon execution of the command. Indeed, on hitting the enter-key, we obtain the following output.

   $$ode1 \ := \frac{d}{dt}x(t) = t + x(t)$$

   The differentiation of x can also be expressed in another equivalent manner as shown below.

   $$> \texttt{ode1} := \texttt{D}(\texttt{x})(\texttt{t}) = \texttt{t} + \texttt{x}(\texttt{t});$$

   A second order differential equation, for instance $x'' = x' + x + \sin t$ can be specified by

   $$> \texttt{ode2} := \texttt{diff}(\texttt{x}(\texttt{t}), \texttt{t}, \texttt{t}) = \texttt{diff}(\texttt{x}(\texttt{t}), \texttt{t}) + \texttt{x}(\texttt{t}) + \texttt{sin}(\texttt{t});$$

   or equivalently by the following command.

   $$> \texttt{ode2} := \texttt{D}(\texttt{D}(\texttt{x}))(\texttt{t}) = \texttt{D}(\texttt{x})(\texttt{t}) + \texttt{x}(\texttt{t}) + \texttt{sin}(\texttt{t});$$

   A system of ODEs can be specified in a similar manner. For example, if we have the system

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1, \end{aligned}$$

   then we can specify this as follows:

   $$> \texttt{ode3a} := \texttt{diff}(\texttt{x1}(\texttt{t}), \texttt{t}) = \texttt{x2}(\texttt{t}); \texttt{ode3b} := \texttt{diff}(\texttt{x2}(\texttt{t}), \texttt{t}) = -\texttt{x1}(\texttt{t});$$

2. *Solving differential equations.* To solve say the equation `ode1` from above, we give the command

$$> \texttt{dsolve(ode1);}$$

which gives the following output:

$$x(t) = -t - 1 + e^t\_C1$$

The strange "$\_C1$" is Maple's indication that the constant is generated by Maple (and has not been introduced by the user).

To solve the equation with a given initial value, say with $x(0) = 1$, we use the command:

$$> \texttt{dsolve}(\{\texttt{ode1}, \texttt{x}(0) = 1\});$$

If our initial condition is itself a parameter α, then we can write

$$> \texttt{dsolve}(\{\texttt{ode1}, \texttt{x}(0) = \texttt{alpha}\});$$

which gives:

$$x(t) = -t - 1 + e^t(1 + \alpha)$$

We can also give a name to the equation specifying the initial condition as follows

$$> \texttt{ic1} := \texttt{x}(0) = 2;$$

and then solve the initial value problem by writing:

$$> \texttt{dsolve}(\{\texttt{ode1}, \texttt{ic1}\});$$

Systems of differential equations can be handled similarly. For example, the ODE system `ode3a, ode3b` can be solved by

$$> \texttt{dsolve}(\{\texttt{ode3a}, \texttt{ode3b}\});$$

and if we have the initial conditions $x_1(0) = 1$, $x_2(0) = 1$, then we give the following command:

$$> \texttt{dsolve}(\{\texttt{ode3a}, \texttt{ode3b}, \texttt{x1}(0) = 1, \texttt{x2}(0) = 1\});$$

3. *Plotting solutions of differential equations.* The tool one has to use is called `DEplot`, and so one has to activate this at the outset using the command:

$$> \texttt{with(DEtools)}:$$

Once this is done, one can use for instance the command `DEplot`, which can be used to plot solutions. This command is quite complicated with many options, but one can get help from Maple by using:

$$>?\texttt{DEplot};$$

For the equation `ode1` above, the command

$$> \texttt{DEplot}(\texttt{ode1}, \texttt{x}(\texttt{t}), \texttt{t} = -2..2, [[\texttt{x}(0) = 0]]);$$

will give a nice picture of a solution to the associated initial value problem, but it contains some other information as well. The various elements in the above command are: `ode1` is the label specifying which differential equation we are solving, `x(t)` indicated the dependent variable, `t=-2..2` indicates the independent variable and its range, and `[[x(0)=0]]` gives the initial value.

One can also give more than one initial value, for instance:

$$> \texttt{DEplot(ode1, x(t), t} = -2..2, [[\texttt{x(0)} = -1], [\texttt{x(0)} = 0], [\texttt{x(0)} = 1]]);$$

The colour of the plot can also be changed:

$$> \texttt{DEplot(ode1, x(t), t} = -2..2, [[\texttt{x(0)} = -1], [\texttt{x(0)} = 0], [\texttt{x(0)} = 1]], \texttt{linecolour} = \texttt{blue});$$

The arrows one sees in the picture show the *direction field*, a concept we will discuss in Chapter 2. One can hide these arrows:

$$> \texttt{DEplot(ode1, x(t), t} = -2..2, [[\texttt{x(0)} = -1], [\texttt{x(0)} = 0], [\texttt{x(0)} = 1]], \texttt{arrows} = \texttt{NONE});$$

To make plots for higher order ODEs, one must give the right number of initial values. We consider an example for `ode2` below:

$$> \texttt{DEplot(ode2, x(t), t} = -2..2, [[\texttt{x(0)} = 0, \texttt{D(x)(0)} = 0], [\texttt{x(0)} = 0, \texttt{D(x)(0)} = 2]]);$$

One can also handle systems of ODEs using `DEplot`, and we give an example below.

```
>  DEplot({ode3a, ode3b}, {x1(t), x2(t)}, t = 0..10, [[x1(0) = 1, x2(0) = 0]],
   scene = [t, x1(t)]);
```

This picture has sharp corners, since Maple only computes approximate solutions in making plots. One can specify the accuracy level using `stepsize`, which specifies the discretization level used by Maple to construct the solution to the ODE system. The finer one chooses *stepsize*, the better the accuracy, but this is at the expense of the time taken to make calculations. Compare the above plot with the one obtained with:

```
>  DEplot({ode3a, ode3b}, {x1(t), x2(t)}, t = 0..10, [[x1(0) = 1, x2(0) = 0]],
   scene = [t, x1(t)], stepsize = 0.1);
```

If one wants x_1 and x_2 to be displayed in the same plot, then we can use the command `display` as demonstrated in the following example.

```
>  with(plots) :
>  plot1 := DEplot({ode3a, ode3b}, {x1(t), x2(t)}, t = 0..10, [[x1(0) = 1, x2(0) = 0]],
   scene = [t, x1(t)], stepsize = 0.1) :
>  plot2 := DEplot({ode3a, ode3b}, {x1(t), x2(t)}, t = 0..10, [[x1(0) = 1, x2(0) = 0]],
   scene = [t, x2(t)], stepsize = 0.1, linecolour = red) :
>  display(plot1, plot2);
```

In Chapter 2, we will learn about 'phase portraits' which are plots in which we plot one solution against the other (with time giving this parametric representation) when one has a 2D system. We will revisit this subsection in order to learn how we can make phase portraits using Maple.

Exercises.

1. In each of the following initial-value problems, find a solution using Maple. Verify that the solution exists for some $t \in I$, where I is an interval containing 0.

   (a) $x' = x + x^3$ with $x(0) = 1$.

   (b) $x'' + x = \frac{1}{2}\cos t$ with $x(0) = 1$ and $x'(0) = 1$.

   (c) $\left.\begin{array}{rcl} x_1' & = & -x_1 + x_2 \\ x_2' & = & x_1 + x_2 + t \end{array}\right\}$ with $x_1(0) = 0$ and $x_2(0) = 0$.

2. In forestry, there is interest in the evolution of the population x of a pest called 'spruce budworm', which is modelled by the following equation:

$$x' = x\left(2 - \frac{1}{5}x - \frac{5x}{2+x^2}\right). \tag{1.6}$$

The solutions of this differential equation show radically different behaviour depending on what initial condition $x(0) = x_0$ one has in the range $0 \le x_0 \le 10$.

(a) Use Maple to plot solutions for several initial values in the range $[0, 10]$.

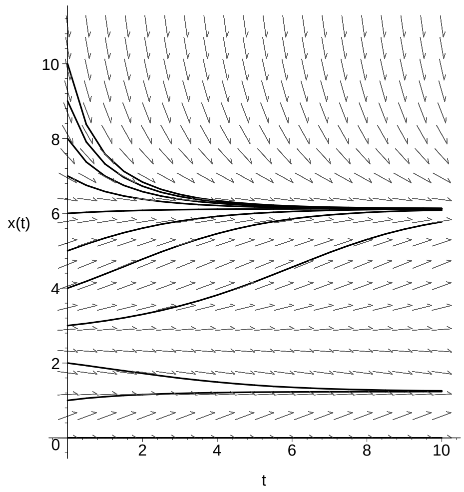

Figure 1.1: Population evolution of the budworm for various initial conditions.

(b) Use the plots to describe the different types of behaviour, and also give an interval for the initial value in which the behaviour occurs. (For instance: For $x_0 \in [0, 8)$, the solutions $x(t)$ go to 0 as t increases. For $x_0 \in [8, 10]$ the solutions $x(t)$ go to infinity as t increases.)

(c) Use Maple to plot the function

$$f(x) = x\left(2 - \frac{1}{5}x - \frac{5x}{2+x^2}\right),$$

in the range $x \in [0, 10]$. Can the differential equation plots be explained theoretically?

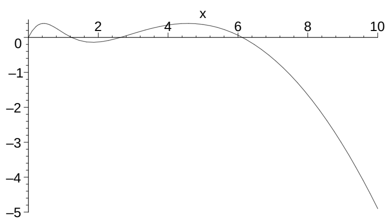

Figure 1.2: Graph of the function f.

Hint: See Figure 1.2.

1.3 High order ODE to a first order ODE. State vector.

Note that the system of equations (1.1)-(1.3) are *first order*, in that the derivatives occurring are of order at most 1. However, in applications, one may end with a model described by a set of high order equations. So why restrict our study only to first order systems? In this section we learn that such high order equations can be expressed as a system of first order equations, by introducing a 'state vector'. So throughout the sequel we will consider only a system of first order equations.

Let us consider the second order differential equation

$$y''(t) + a(t)y(t) + b(t)y(t) = u(t). \tag{1.7}$$

If we introduce the new functions x_1, x_2 defined by

$$x_1 = y \text{ and } x_2 = y',$$

then we observe that

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t), \\ x_2'(t) &= y''(t) = -a(t)y'(t) - b(t)y(t) + u(t) = -a(t)x_2(t) - b(t)x_1(t) + u(t), \end{aligned}$$

and so we obtain the system of first order equations

$$x_1'(t) = x_2(t), \tag{1.8}$$
$$x_2'(t) = -a(t)x_2(t) - b(t)x_1(t) + u(t), \tag{1.9}$$

which is of the form (1.1)-(1.3).

Solving (1.7) is equivalent to solving the system (1.8)-(1.9). To see the equivalence, suppose that (x_1, x_2) satisfies the system (1.8)-(1.9). Then x_1 is a solution to (1.7), since

$$(x_1'(t))' = x_2'(t) = -b(t)x_1(t) - a(t)x_1'(t) + u(t),$$

which is (1.7). On the other hand, if y is a solution to (1.7), then define $x_1 = y$ and $x_2 = y'$, and proceeding as in the preceding paragraph, this yields a solution of (1.8)-(1.9).

More generally, if we have an nth order scalar equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y'(t) + a_0(t)y(t) = u(t),$$

then by introducing the vector of functions

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}, \tag{1.10}$$

we arrive at the equivalent first order system of equations

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= x_3(t), \\ &\vdots \\ x_{n-1}'(t) &= x_n(t), \\ x_n'(t) &= -a_1(t)x_1(t) - \cdots - a_{n-1}(t)x_{n-1}(t) + u(t). \end{aligned}$$

The auxiliary vector in (1.10) comprising successive derivatives of the unknown function in the high order differential equation, is called a *state*, and the resulting system of first order differential equations is called a *state equation*.

Exercises. By introducing appropriate state variables, write a state equation for the following (systems of) differential equations:

1. $x'' + \omega^2 x = 0$.
2. $x'' + x = 0$, $y'' + y' + y = 0$.
3. $x'' + t \sin x = 0$.

1.4 The simplest example

The differential equation

$$x'(t) = ax(t) \tag{1.11}$$

is the simplest differential equation. It is also one of the most important. First, what does it mean? Here $x : \mathbb{R} \to \mathbb{R}$ is an unknown real-valued function (of a real variable t), and $x'(t)$ is its derivative at t. The equation (1.11) holds for every value of t, and a denotes a constant.

The solutions to (1.11) are obtained from calculus: if C is any constant, then the function f given by $f(t) = Ce^{ta}$ is a solution, since

$$f'(t) = Cae^{ta} = a(Ce^{ta}) = af(t).$$

Moreover, there are no other solutions. To see this, let u be any solution and compute the derivative of v given by $v(t) = e^{-ta}u(t)$:

$$\begin{aligned} v'(t) &= ae^{-ta}u(t) + e^{-ta}u'(t) \\ &= ae^{-ta}u(t) + e^{-ta}au(t) \quad (\text{since } u'(t) = au(t)) \\ &= 0. \end{aligned}$$

Therefore by the fundamental theorem of calculus,

$$v(t) - v(0) = \int_0^t v'(t)dt = \int_0^t 0dt = 0,$$

and so $v(t) = v(0)$ for all t, that is, $e^{-ta}u(t) = u(0)$. Consequently $u(t) = e^{ta}u(0)$ for all t.

So we see that the *initial value problem*

$$x'(t) = ax(t), \quad x(0) = x_0$$

has the unique solution

$$x(t) = e^{ta}x_0, \quad t \in \mathbb{R}.$$

As the constant a changes, the nature of the solutions changes. Can we describe qualitatively the way the solutions change? We have the following cases:

<u>1</u>° $a < 0$. In this case,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{ta}x(0) = 0x(0) = 0.$$

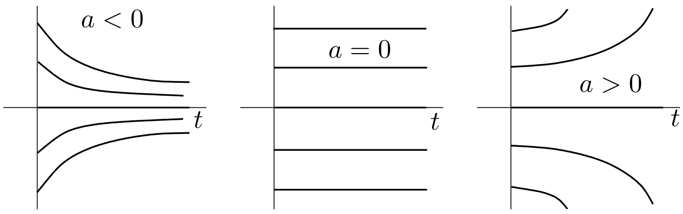

Figure 1.3: Exponential solutions $e^{ta}x_0$.

Thus the solutions all converge to zero, and moreover they converge to zero exponentially, that is, they there exist constants $M > 0$ and $\epsilon > 0$ such that the solutions satisfy an inequality of the type $|x(t)| \leq Me^{-\epsilon t}$ for all $t \geq 0$. (Note that not every decaying solution of an ODE has to converge exponentially fast–see the example on page 50). See Figure 1.3.

$\underline{2}°$ $a = 0$. In this case,

$$x(t) = e^{t0}x(0) = 1x(0) = x(0) \text{ for all } t \geq 0.$$

Thus the solutions are constants, the constant value being the initial value it starts from. See Figure 1.3.

$\underline{3}°$ $a > 0$. In this case, if the initial condition is zero, the solution is the constant function taking value 0 everywhere. If the initial condition is nonzero, then the solutions 'blow up'. See Figure 1.3.

We would like to have a similar idea about the qualitative behaviour of solutions, but when we have a system of linear differential equations. It turns out that for the system

$$x'(t) = Ax(t),$$

the behaviour of the solutions depends on the eigenvalues of the matrix A. In order to find out why this is so, we first give an expression for the solution of such a linear ODE in the next two sections. We find that the solution is notationally the same as the scalar case discussed in this section: $x(t) = e^{tA}x(0)$, with the little 'a' now replaced by the matrix 'A'! But what do we mean by the exponential of a matrix, e^{tA}? We first introduce this concept in the next section, and subsequently, we will show how it enables us to solve the system $x' = Ax$.

1.5 The matrix exponential

In this section we introduce the exponential of a square matrix A, which is useful for obtaining explicit solutions to the linear system $x'(t) = Ax(t)$. We begin with a few preliminaries concerning vector-valued functions.

A *vector-valued function* $t \mapsto x(t)$ is a vector whose entries are functions of t. Similarly, a *matrix-valued function* $t \mapsto A(t)$ is a matrix whose entries are functions:

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \dots & a_{mn}(t) \end{bmatrix}.$$

The calculus operations of taking limits, differentiating, and so on are extended to vector-valued and matrix-valued functions by performing the operations on each entry separately. Thus by definition,

$$\lim_{t\to t_0} x(t) = \begin{bmatrix} \lim_{t\to t_0} x_1(t) \\ \vdots \\ \lim_{t\to t_0} x_n(t) \end{bmatrix}.$$

So this limit exists iff $\lim_{t\to t_0} x_i(t)$ exists for all $i \in \{1, \dots, n\}$. Similarly, the derivative of a vector-valued or matrix-valued function is the function obtained by differentiating each entry separately:

$$\frac{dx}{dt}(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \quad \frac{dA}{dt}(t) = \begin{bmatrix} a_{11}'(t) & \dots & a_{1n}'(t) \\ \vdots & & \vdots \\ a_{m1}'(t) & \dots & a_{mn}'(t) \end{bmatrix},$$

where $x_i'(t)$ is the derivative of $x_i(t)$, and so on. So $\frac{dx}{dt}$ is defined iff each of the functions $x_i(t)$ is differentiable. The derivative can also be described in vector notation, as

$$\frac{dx}{dt}(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}. \tag{1.12}$$

Here $x(t+h) - x(t)$ is computed by vector addition and the h in the denominator stands for scalar multiplication by h^{-1}. The limit is obtained by evaluating the limit of each entry separately, as above. So the entries of (1.12) are the derivatives $x_i(t)$. The same is true for matrix-valued functions.

Suppose that analogous to

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots, \quad a \in \mathbb{R},$$

we define

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots, \quad A \in \mathbb{R}^{n\times n}. \tag{1.13}$$

In this section, we will study this matrix exponential, and show that the matrix-valued function

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^2 + \dots$$

(where t is a variable scalar) can be used to solve the system $x'(t) = Ax(t)$, $x(0) = x_0$: indeed, the solution is given by $x(t) = e^{tA}x_0$.

We begin by stating the following result, which shows that the series in (1.13) converges for any given square matrix A.

Theorem 1.5.1 *The series (1.13) converges for any given square matrix A.*

We have collected the proofs together at the end of this section in order to not break up the discussion.

Since matrix multiplication is relatively complicated, it isn't easy to write down the matrix entries of e^A directly. In particular, the entries of e^A are usually *not* obtained by exponentiating the entries of A. However, one case in which the exponential is easily computed, is when A is a diagonal matrix, say with diagonal entries λ_i. Inspection of the series shows that e^A is also diagonal in this case and that its diagonal entries are e^{λ_i}.

The exponential of a matrix A can also be determined when A is *diagonalizable* , that is, whenever we know a matrix P such that $P^{-1}AP$ is a diagonal matrix D. Then $A = PDP^{-1}$, and using $(PDP^{-1})^k = PD^kP^{-1}$, we obtain

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \dots \\ &= PIP^{-} + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \dots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \dots\right)P^{-1} \\ &= Pe^DP^{-1} \\ &= P\begin{bmatrix} e^{\lambda_1} & & \\ 0 & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}P^{-1}, \end{aligned}$$

where $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of A.

Exercise. $(**)$ The set of diagonalizable $n \times n$ complex matrices is *dense* in the set of all $n \times n$ complex matrices, that is, given any $A \in \mathbb{C}^{n \times n}$, there exists a $B \in \mathbb{C}^{n \times n}$ arbitrarily close to A (meaning that $|b_{ij} - a_{ij}|$ can be made arbitrarily small for all $i, j \in \{1, \dots, n\}$) such that B has n distinct eigenvalues.

HINT: Use the fact that every complex $n \times n$ matrix A can be 'upper-triangularized': that is, there exists an invertible complex matrix P such that PAP^{-1} is upper triangular. Clearly the diagonal entries of this new upper triangular matrix are the eigenvalues of A.

In order to use the matrix exponential to solve systems of differential equations, we need to extend some of the properties of the ordinary exponential to it. The most fundamental property is $e^{a+b} = e^a e^b$. This property can be expressed as a formal identity between the two infinite series which are obtained by expanding

$$\begin{aligned} e^{a+b} &= 1 + \tfrac{(a+b)}{1!} + \tfrac{(a+b)^2}{2!} + \dots \quad \text{and} \\ e^a e^b &= \left(1 + \tfrac{a}{1!} + \tfrac{a^2}{2!} + \dots\right)\left(1 + \tfrac{b}{1!} + \tfrac{b^2}{2!} + \dots\right). \end{aligned} \tag{1.14}$$

We cannot substitute matrices into this identity because the commutative law is needed to obtain equality of the two series. For instance, the quadratic terms of (1.14), computed without the commutative law, are $\frac{1}{2}(a^2 + ab + ba + b^2)$ and $\frac{1}{2}a^2 + ab + \frac{1}{2}b^2$. They are not equal unless $ab = ba$. So there is no reason to expect e^{A+B} to equal $e^A e^B$ in general. However, if two matrices A and B happen to commute, the formal identity can be applied.

Theorem 1.5.2 *If $A, B \in \mathbb{R}^{n \times n}$ commute (that is $AB = BA$), then $e^{A+B} = e^A e^B$.*

The proof is at the end of this section. Note that the above implies that e^A is always invertible and in fact its inverse is e^{-A}: Indeed $I = e^{A-A} = e^A e^{-A}$.

Exercises.

1. Give an example of 2×2 matrices A and B such that $e^{A+B} \neq e^A e^B$.

2. Compute e^A, where A is given by
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$
HINT: $A = 2I + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$.

We now come to the main result relating the matrix exponential to differential equations. Given an $n \times n$ matrix, we consider the exponential e^{tA}, t being a variable scalar, as a matrix-valued function:
$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots.$$

Theorem 1.5.3 *e^{tA} is a differentiable matrix-valued function of t, and its derivative is Ae^{tA}.*

The proof is at the end of the section.

Theorem 1.5.4 (Product rule.) *Let $A(t)$ and $B(t)$ be differentiable matrix-valued functions of t, of suitable sizes so that their product is defined. Then the matrix product $A(t)B(t)$ is differentiable, and its derivative is*
$$\frac{d}{dt}(A(t)B(t)) = \frac{dA(t)}{dt}B(t) + A(t)\frac{dB(t)}{dt}.$$

The proof is left as an exercise.

Theorem 1.5.5 *The first-order linear differential equation*
$$\frac{dx}{dt}(t) = Ax(t), \quad t \in \mathbb{R}, \quad x(0) = x_0 \tag{1.15}$$
has the unique solution $x(t) = e^{tA}x_0$.

Proof We have
$$\frac{d}{dt}(e^{tA}x_0) = Ae^{tA}x_0,$$
and so $t \mapsto e^{tA}x_0$ solves $\frac{dx}{dt}(t) = Ax(t)$. Furthermore, $x(0) = e^{0A}x_0 = Ix_0 = x_0$.

Finally we show that the solution is unique. Let x be a solution to (1.15). Using the product rule, we differentiate the matrix product $e^{-tA}x(t)$:
$$\frac{d}{dt}(e^{-tA}x(t)) = -Ae^{-tA}x(t) + e^{-tA}Ax(t).$$
From the definition of the exponential, it can be seen that A and e^{tA} commute, and so the derivative of $e^{tA}x(t)$ is zero. Therefore, $e^{tA}x(t)$ is a constant column vector, say C, and $x(t) = e^{tA}C$. As $x(0) = x_0$, we obtain that $x_0 = e^{0A}C$, that is, $C = x_0$. Consequently, $x(t) = e^{tA}x_0$. ■

Thus the matrix exponential enables us to solve the differential equation (1.15). Since direct computation of the exponential can be quite difficult, the above theorem may not be easy to apply in a concrete situation. But if A is a diagonalizable matrix, then the exponential can be computed: $e^A = Pe^DP^{-1}$. To compute the exponential explicitly in all cases requires putting the matrix into

Jordan form. But in the next section, we will learn yet another way of computing e^{tA} by using Laplace transforms.

We now go back to prove Theorems 1.5.1, 1.5.2, and 1.5.3.

For want of a more compact notation, we will denote the i,j-entry of a matrix A by A_{ij} here. So $(AB)_{ij}$ will stand for the entry of the matrix product matrix AB, and $(A^k)_{ij}$ for the entry of A^k. With this notation, the i,j-entry of e^A is the sum of the series

$$(e^A)_{ij} = I_{ij} + A_{ij} + \frac{1}{2!}(A^2)_{ij} + \frac{1}{3!}(A^3)_{ij} + \dots .$$

In order to prove that the series for the exponential converges, we need to show that the entries of the powers A^k of a given matrix do not grow too fast, so that the absolute values of the i,j-entries form a bounded (and hence convergent) series. Consider the following function $\|\cdot\|$ on $\mathbb{R}^{n\times n}$:

$$\|A\| := \max\{|A_{ij}| \mid 1 \le i, j \le n\}. \tag{1.16}$$

Thus $|A_{ij}| \le \|A\|$ for all i,j. This is one of several possible "norms" on $\mathbb{R}^{n\times n}$, and it has the following property.

Lemma 1.5.6 *If $A, B \in \mathbb{R}^{n\times n}$, then $\|AB\| \le n\|A\|\|B\|$, and for all $k \in \mathbb{N}$, $\|A^k\| \le n^{k-1}\|A\|^k$.*

Proof We estimate the size of the i,j-entry of AB:

$$|(AB)_{ij}| = \left|\sum_{k=1}^{n} A_{ik}B_{kj}\right| \le \sum_{k=1}^{n} |A_{ik}||B_{kj}| \le n\|A\|\|B\|.$$

Thus $\|AB\| \le n\|A\|\|B\|$. The second inequality follows from the first inequality by induction. ■

Proof (of Theorem 1.5.1:) To prove that the matrix exponential converges, we show that the series

$$I_{ij} + A_{ij} + \frac{1}{2!}(A^2)_{ij} + \frac{1}{3!}(A^3)_{ij} + \dots$$

is absolutely convergent, and hence convergent. Let $a = n\|A\|$. Then

$$\begin{aligned} |I_{ij}| + |A_{ij}| + \frac{1}{2!}|(A^2)_{ij}| + \frac{1}{3!}|(A^3)_{ij}| + \dots &\le 1 + \|A\| + \frac{1}{2!}n\|A\|^2 + \frac{1}{3!}n^2\|A\|^3 + \dots \\ &= 1 + \frac{1}{n}\left(a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \dots\right) = 1 + \frac{e^a - 1}{n}. \end{aligned}$$

■

Proof (of Theorem 1.5.2:) The terms of degree k in the expansions of (1.14) are

$$\frac{1}{k!}(A+B)^k = \frac{1}{k!}\sum_{r+s=k}\binom{k}{r}A^rB^s \quad \text{and} \quad \sum_{r+s=k}\frac{1}{r!}A^r\frac{1}{s!}B^s.$$

These terms are equal since for all k, and all r, s such that $r+s=k$,

$$\frac{1}{k!}\binom{k}{r} = \frac{1}{r!s!}.$$

Define

$$S_n(A) = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots + \frac{1}{n!}A^n.$$

Then

$$\begin{aligned} S_n(A)S_n(B) &= \left(I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n\right)\left(I + \frac{1}{1!}B + \frac{1}{2!}B^2 + \cdots + \frac{1}{n!}B^n\right) \\ &= \sum_{r,s=0}^{n} \frac{1}{r!}A^r \frac{1}{s!}B^s, \end{aligned}$$

while

$$\begin{aligned} S_n(A+B) &= I + \frac{1}{1!}(A+B) + \frac{1}{2!}(A+B)^2 + \cdots + \frac{1}{n!}(A+B)^n \\ &= \sum_{k=0}^{n} \sum_{r+s=k} \frac{1}{k!}\binom{k}{r} A^r B^s = \sum_{k=0}^{n} \sum_{r+s=k} \frac{1}{r!}A^r \frac{1}{s!}B^s. \end{aligned}$$

Comparing terms, we find that the expansion of the partial sum $S_n(A+B)$ consists of the terms in $S_n(A)S_n(B)$ such that $r+s \le n$. We must show that the sum of the remaining terms tends to zero as k tends to ∞.

Lemma 1.5.7 *The series*

$$\sum_k \sum_{r+s=k} \left| \left(\frac{1}{r!}A^r \frac{1}{s!}B^s\right)_{ij} \right|$$

converges for all i, j.

Proof Let $a = n\|A\|$ and $b = n\|B\|$. We estimate the terms in the sum using Lemma 1.5.6:

$$|(A^r B^s)_{ij}| \le \|A^r B^s\| \le n\|A^r\|\|B^s\| \le n(n^{r-1}\|A\|^r)(n^{s-1}\|B\|^s) \le a^r b^s.$$

Therefore

$$\sum_k \sum_{r+s=k} \left| \left(\frac{1}{r!}A^r \frac{1}{s!}B^s\right)_{ij} \right| \le \sum_k \sum_{r+s=k} \frac{a^r}{r!}\frac{b^s}{s!} = e^{a+b}.$$

The theorem follows from this lemma because, on the one hand, the i, j-entry of $(S_k(A)S_k(B) - S_k(A+B))_{ij}$ is bounded by

$$\sum_{r+s>k} \left| \left(\frac{1}{r!}A^r \frac{1}{s!}B^s\right)_{ij} \right|.$$

According to the lemma, this sum tends to 0 as k tends to ∞. And on the other hand, $S_k(A)S_k(B) - S_k(A+B)$ tends to $e^A e^B - e^{A+B}$. ■

This completes the proof of Theorem 1.5.2. ■

Proof (of Theorem 1.5.3:) By definition,

$$\frac{d}{dt}e^{tA} = \lim_{h \to 0} \frac{1}{h}(e^{(t+h)A} - e^{tA}).$$

Since the matrices tA and hA commute, we have

$$\frac{1}{h}(e^{(t+h)A} - e^{tA}) = \left(\frac{1}{h}(e^{hA} - I)\right) e^{tA}.$$

So our theorem follows from this lemma:

Lemma 1.5.8 $\lim_{h\to 0} \frac{1}{h}(e^{hA}-I) = A$.

Proof The series expansion for the exponential shows that

$$\frac{1}{h}(e^{hA}-I)-A = \frac{h}{2!}A^2 + \frac{h^2}{3!}A^3 + \dots. \tag{1.17}$$

We estimate this series. Let $a = |h|n\|A\|$. Then

$$\begin{aligned}
\left|\left(\frac{h}{2!}A^2 + \frac{h^2}{3!}A^3 + \dots\right)_{ij}\right| &\leq \left|\frac{h}{2!}(A^2)_{ij}\right| + \left|\frac{h^2}{3!}(A^3)_{ij}\right| + \dots \\
&\leq \frac{1}{2!}|h|n\|A\|^2 + \frac{1}{3!}|h|^2n^2\|A\|^3 + \dots \\
&= \|A\|\left(\frac{1}{2!}a + \frac{1}{3!}a^2 + \dots\right) = \frac{\|A\|}{a}(e^a - 1 - a) = \|A\|\left(\frac{e^a-1}{a} - 1\right).
\end{aligned}$$

Note that $a \to 0$ as $h \to 0$. Since the derivative of e^x is e^x, $\lim_{a\to 0} \frac{e^a - 1}{a} = \left.\frac{d}{dx}e^x\right|_{x=0} = e^0 = 1.$

So (1.17) tends to 0 with h. ∎

This completes the proof of Theorem 1.5.3. ∎

Exercises.

1. (∗) If $A \in \mathbb{R}^{n\times n}$, then show that $\|e^A\| \leq e^{n\|A\|}$. (In particular, for all $t \geq 0$, $\|e^{tA}\| \leq e^{tn\|A\|}$.)
2. (a) Let $n \in \mathbb{N}$. Show that there exists a constant C (depending only on n) such that if $A \in \mathbb{R}^{n\times n}$, then for all $v \in \mathbb{R}^n$, $\|Av\| \leq C\|A\|\|v\|$.
   (b) Show that if λ is an eigenvalue of A, then $|\lambda| \leq n\|A\|$.
3. (a) (∗) Show that if λ is an eigenvalue of A and v is a corresponding eigenvector, then v is also an eigenvector of e^A corresponding to the eigenvalue e^λ of e^A.
   (b) Solve $x'(t) = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} x(t)$, $x(0) = x_0$, when
   (i) $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (ii) $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and (iii) $x_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.
   HINT: In parts (i) and (ii), observe that the initial condition is an eigenvector of the square matrix in question, and in part (iii), express the initial condition as a combination of the initial conditions from the previous two parts.
4. (∗) Prove that $e^{tA^\top} = (e^{tA})^\top$. (Here $M^\top$ denotes the transpose of the matrix M.)
5. (∗) Let $A \in \mathbb{R}^{n\times n}$, and let $\mathscr{S} = \{x : \mathbb{R} \to \mathbb{R}^n \mid \forall t \in \mathbb{R},\ x'(t) = Ax(t)\}$. In this exercise we will show that $\mathscr{S}$ is a finite dimensional vector space with dimension n.
   (a) Let $C(\mathbb{R};\mathbb{R}^n)$ denote the vector space of all functions $f : \mathbb{R} \to \mathbb{R}^n$ with pointwise addition and scalar multiplication. Show that $\mathscr{S}$ is a subspace of $C(\mathbb{R};\mathbb{R}^n)$.
   (b) Let $e_1, \dots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. By Theorem 1.5.5, we know that for each $k \in \{1, \dots, n\}$, there exists a unique solution to the initial value problem $x'(t) = Ax(t)$, $t \in \mathbb{R}$, $x(0) = e_k$. Denote this unique solution by f_k. Thus we obtain the set of functions $f_1, \dots, f_n \in \mathscr{S}$. Prove that $\{f_1, \dots, f_n\}$ is linearly independent.
   HINT: Set $t = 0$ in $\alpha_1 f_1 + \dots + \alpha_n f_n = 0$.
   (c) Show that $\mathscr{S} = \operatorname{span}\{f_1, \dots, f_n\}$, and conclude that $\mathscr{S}$ is a finite dimensional vector space of dimension n.

1.6 Computation of e^{tA}

In the previous section, we saw that the computation of e^{tA} is easy if the matrix A is diagonalizable. However, not all matrices are diagonalizable. For example, consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are both 0, and so if it were diagonalizable, then the diagonal form will be the zero matrix, but then if there did exist an invertible P such that $P^{-1}AP$ is this zero matrix, then clearly A should be zero, which it is not!

In general, however, every matrix has what is called a *Jordan canonical form*, that is, there exists an invertible P such that $P^{-1}AP = D + N$, where D is diagonal, N is *nilpotent* (that is, there exists a $n \geq 0$ such that $N^n = 0$), and D and N commute. Then one can compute the exponential of A:

$$e^{tA} = Pe^{tD}e^{tD}\left(I + N + \frac{1}{2!}N^2 + \cdots + \frac{1}{n!}N^n\right)P^{-1}.$$

However, the algorithm for computing the P taking A to the Jordan form requires some sophisticated linear algebra. So we give a different procedure for calculating e^{tA} below, using Laplace transforms. First we will prove the following theorem.

Theorem 1.6.1 *For large enough s,* $\displaystyle\int_0^\infty e^{-st}e^{tA}dt = (sI - A)^{-1}$.

Proof First choose a s_0 large enough so that $s_0 > n\|A\|$. Then for all $s > s_0$, we have

$$\begin{aligned}
\int_0^\infty e^{-ts}e^{tA}dt &= \int_0^\infty e^{-t(sI-A)}dt \\
&= \int_0^\infty (sI-A)^{-1}(sI-A)e^{-t(sI-A)}dt \\
&= (sI-A)^{-1}\int_0^\infty (sI-A)e^{-t(sI-A)}dt \\
&= (sI-A)^{-1}\int_0^\infty -\frac{d}{dt}e^{-t(sI-A)}dt \\
&= (sI-A)^{-1}\left(-e^{-ts}e^{tA}\Big|_{t=0}^{t=\infty}\right) \\
&= (sI-A)^{-1}(0+I) \\
&= (sI-A)^{-1}.
\end{aligned}$$

(In the above, we used the Exercise 1 on page 15, which gives $\|e^{tA}\| \leq e^{tn\|A\|} \leq e^{ts_0} = e^{ts}e^{t(s_0-s)}$, and so $\|e^{-ts}e^{tA}\| \leq e^{t(s_0-s)}$. Also, we have used Exercise 2b, which gives invertibility of $sI - A$.)
■

If s is not an eigenvalue of A, then $sI - A$ is invertible, and *Cramer's rule*[1] says that,

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)}\operatorname{adj}(sI-A).$$

Here $\operatorname{adj}(sI-A)$ denotes the *adjoint* of the matrix $sI - A$, which is defined as follows: its (i, j)th entry is obtained by multiplying $(-1)^{i+j}$ and the determinant of the matrix obtained by deleting

[1]For a proof, see for instance Artin [3].

the jth row and ith column of $sI - A$. Thus we see that each entry of $\text{adj}(sI - A)$ is a polynomial in s whose degree is at most $n - 1$. (Here n denotes the size of A–that is, A is a $n \times n$ matrix.)

Consequently, each entry m_{ij} of $(sI - A)^{-1}$ is a rational function, in other words, it is ratio of two polynomials (in s) p_{ij} and $q := \det(sI - A)$:

$$m_{ij} = \frac{p_{ij}(s)}{q(s)}$$

Also from the above, we see that $\deg(p_{ij}) \leq \deg(q) - 1$. From the fundamental theorem of algebra, we know that the monic polynomial q can be factored as

$$q(s) = (s - \lambda_1)^{m_1} \dots (s - \lambda_k)^{m_k},$$

where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of $q(s) = \det(sI - A)$, with the algebraic multiplicities $m_1, \dots, m_k$.

By the "partial fraction expansion" one learns in calculus, it follows that one can find suitable coefficients for a decomposition of each rational entry of $(sI - A)^{-1}$ as follows:

$$m_{ij} = \sum_{l=1}^{k} \sum_{r=1}^{m_k} \frac{C_{l,r}}{(s - \lambda)^r}.$$

Thus if $f_{ij}(t)$ denotes the (i, j)th entry of e^{tA}, then its Laplace transform will be an expression of the type m_{ij} given above. Now it turns out that this determines the f_{ij}, and this is the content of the following result.

Theorem 1.6.2 *Let $a \in \mathbb{C}$ and $n \in \mathbb{N}$. If f is a continuous function defined on $[0, \infty)$, and if there exists a s_0 such that for all $s > s_0$,*

$$F(s) := \int_0^\infty e^{-st} f(t) dt = \frac{1}{(s - a)^n},$$

then

$$f(t) = \frac{1}{(n-1)!} t^{n-1} e^{ta} \quad \textit{for all } t \geq 0.$$

Proof The proof is beyond the scope of this course, but we refer the interested reader to Exercise 11.38 on page 342 of Apostol [1]. ■

So we have a procedure for computing e^{tA}: form the matrix $sI - A$, compute its inverse (as a rational matrix), perform a partial fraction expansion of each of its entry, and take the inverse Laplace transform of each elementary fraction. Sometimes, the partial fraction expansion may be avoided, by making use of the following corollary (which can be obtained from Theorem 1.6.2, by a partial fraction expansion!).

Corollary 1.6.3 *Let f be a continuous function defined on $[0, \infty)$, and let there exist a s_0 such that for all $s > s_0$, F defined by*

$$F(s) := \int_0^\infty e^{-st} f(t) dt,$$

is one of the functions given in the first column below. Then f is given by the corresponding entry in the second column.

F	f
$\dfrac{b}{(s-a)^2+b^2}$	$e^{ta}\sin(bt)$
$\dfrac{s-a}{(s-a)^2+b^2}$	$e^{ta}\cos(bt)$
$\dfrac{b}{(s-a)^2-b^2}$	$e^{ta}\sinh(bt)$
$\dfrac{s-a}{(s-a)^2-b^2}$	$e^{ta}\cosh(bt)$

Example. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$, and so

$$(sI-A)^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}.$$

By using Theorem 1.6.2 ('taking the inverse Laplace transform'), we obtain $e^{tA} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. $\Diamond$

Exercises.

1. Compute e^{tA}, when $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

2. Compute e^{tA}, for the 'Jordan block' $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$.

   Remark: In general, if

$$\text{if } A = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{bmatrix}, \text{ then } e^{tA} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n-1}}{(n-1)!} \\ & \ddots & \ddots & & \vdots \\ & & & & \vdots \\ & & & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix}.$$

3. (a) Compute e^{tA}, when $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

   (b) Find the solution to
$$x'' + kx = 0, \quad x(0) = 1, \quad x'(0) = 0.$$
   (Here k is a fixed positive constant.)
   HINT: Introduce the state variables $x_1 = \sqrt{k}x$ and $x_2 = x'$.
   Suppose that $k = 1$, and find $(x(t))^2 + (x'(t))^2$. What do you observe? If one identifies $(x(t), x'(t))$ with a point in the plane at time t, then how does this point move with time?

4. Suppose that A is a 2×2 matrix such that
$$e^{tA} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad t \in \mathbb{R}.$$
   Find A.

1.7 Stability considerations

Just as in the scalar example $x' = ax$, where we saw that the sign of (the real part of) a allows us to conclude the behaviour of the solution as $x \to \infty$, it turns out that by looking at the real parts of the eigenvalues of the matrix A one can say similar things in the case of the system $x' = Ax$. We will study this in this section.

We begin by proving the following result.

Lemma 1.7.1 *Suppose that $\lambda \in \mathbb{C}$ and k is a nonnegative integer. For every $\omega > \mathrm{Re}(\lambda)$, there exists a $M_\omega > 0$ such that for all $t \geq 0$, $|t^k e^{\lambda t}| \leq M_\omega e^{\omega t}$.*

Proof We have

$$e^{(\omega - \mathrm{Re}(\lambda))t} = \sum_{n=0}^{\infty} \frac{(\omega - \mathrm{Re}(\lambda))^n t^n}{n!} \geq \frac{(\omega - \mathrm{Re}(\lambda))^k t^k}{k!},$$

and so $t^k e^{(\mathrm{Re}(\lambda)-\omega)t} \leq M_\omega$, where

$$M_\omega := \frac{k!}{(\omega - \mathrm{Re}(\lambda))^k} > 0.$$

Consequently, for $t \geq 0$, $|t^k e^{\lambda t}| = t^k e^{\mathrm{Re}(\lambda)t} = t^k e^{(\mathrm{Re}(\lambda)-\omega)t} e^{\omega t} \leq M_\omega e^{\omega t}$. ■

In the sequel, we denote the set of eigenvalues of A by $\sigma(A)$, sometimes referred to as the *spectrum* of A.

Theorem 1.7.2 *Let $A \in \mathbb{R}^{n \times n}$.*

1. *Every solution of $x' = Ax$ tends to zero as $t \to \infty$ iff for all $\lambda \in \sigma(A)$, $\mathrm{Re}(\lambda) < 0$. Moreover, in this case, the solutions converge uniformly exponentially to 0, that is, there exist $\epsilon > 0$ and $M > 0$ such that for all $t \geq 0$, $\|x(t)\| \leq M e^{-\epsilon t} \|x(0)\|$.*
2. *If there exists a $\lambda \in \sigma(A)$ such that $\mathrm{Re}(\lambda) > 0$, then for every $\delta > 0$, there exists a $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$, such that the unique solution to $x' = Ax$ with initial condition $x(0) = x_0$ satisfies $\|x(t)\| \to \infty$ as $t \to \infty$.*

Proof 1. We use Theorem 1.6.1. From Cramer's rule, it follows that each entry in $(sI - A)^{-1}$ is a rational function with the denominator equal to the characteristic polynomial of A, and then by using a partial fraction expansion and Theorem 1.6.2, it follows that each entry in e^{tA} is a linear combination of terms of the form $t^k e^{\lambda t}$, where k is a nonnegative integer and $\lambda \in \sigma(A)$. By Lemma 1.7.1, we conclude that if each eigenvalue of A has real part < 0, then there exist positive constants M and ϵ such that for all $t \geq 0$, $\|e^{tA}\| < M e^{-\epsilon t}$.

On the other hand, if each solution tends to 0 as $t \to \infty$, then in particular, if $v \in \mathbb{R}^n$ is an eigenvector[2] corresponding to eigenvalue λ, then with initial condition $x(0) = v$, we have $x(t) = e^{tA} v = e^{\lambda t} v$, and so $\|x(t)\| = e^{\mathrm{Re}(\lambda)t} \|v\| \xrightarrow{t \to \infty} 0$, and so it must be the case that $\mathrm{Re}(\lambda) < 0$.

2. Let $\lambda \in \sigma(A)$ be such that $\mathrm{Re}(\lambda) > 0$, and let $v \in \mathbb{R}^n$ be a corresponding eigenvector[3]. Given $\delta > 0$, define $x_0 = \frac{\delta}{2\|v\|} v \in \mathbb{R}^n$. Then $\|x_0\| = \frac{\delta}{2} < \delta$, and the unique solution x to $x' = Ax$ with initial condition $x(0) = x_0$ satisfies $\|x(t)\| = \frac{\delta}{2} e^{\mathrm{Re}(\lambda)t} \to \infty$ as $t \to \infty$. ■

[2] With a complex eigenvalue, this vector is not in $\mathbb{R}^n$! But the proof can be modified so as to still yield the desired conclusion.

[3] See the previous footnote!

In the case when we have eigenvalues with real parts equal to zero, then a more careful analysis is required and the boundedness of solutions depends on the algebraic/geometric multiplicity of the eigenvalues with zero real parts. We will not give a detailed analysis, but consider two examples which demonstrate that the solutions may or may not remain bounded.

Examples. Consider the system $x' = Ax$, where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the system trajectories are constants $x(t) \equiv x(0)$, and so they are bounded.

On the other hand if

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$e^{tA} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

and so with the initial condition $x(0) = \delta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $\|x(t)\| = \delta\sqrt{1+t^2} \to \infty$ as $t \to \infty$ for all $\delta > 0$. So even if one starts arbitrarily close to the origin, the solution can become unbounded. $\Diamond$

Exercises.

1. Determine if all solutions of $x' = Ax$ are bounded, and if so if all solutions tend to 0 as $t \to \infty$.
   (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$
   (e) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$
   (f) $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
2. For what values of $\alpha \in \mathbb{R}$ can we conclude that all solutions of the system $x' = Ax$ will be bounded for $t \geq 0$, if $A = \begin{bmatrix} \alpha & 1+\alpha \\ -(1+\alpha) & \alpha \end{bmatrix}$?

Chapter 2

Phase plane analysis

2.1 Introduction

In the preceding chapter, we learnt how one can solve a system of linear differential equations. However, the equations that arise in most practical situations are inherently nonlinear, and typically it is impossible to solve these explicitly. Nevertheless, sometimes it is possible to obtain an idea of what its solutions look like (the "qualitative behaviour"), and we learn one such method in this chapter, called *phase plane analysis*.

Phase plane analysis is a graphical method for studying 2D autonomous systems. This method was introduced by mathematicians (among others, Henri Poincaré) in the 1890s.

The basic idea of the method is to generate in the state space motion trajectories corresponding to various initial conditions, and then to examine the qualitative features of the trajectories. As a graphical method, it allows us to visualize what goes on in a nonlinear system starting from various initial conditions, without having to solve the nonlinear equations analytically. Thus, information concerning stability and other motion patterns of the system can be obtained. In this chapter, we learn the basic tools of the phase plane analysis.

2.2 Concepts of phase plane analysis

2.2.1 Phase portraits

The phase plane method is concerned with the graphical study of 2 dimensional autonomous systems:

$$\begin{array}{rcl} x_1'(t) & = & f_1(x_1(t), x_2(t)), \\ x_2'(t) & = & f_2(x_1(t), x_2(t)), \end{array} \tag{2.1}$$

where x_1 and x_2 are the states of the system, and f_1 and f_2 are nonlinear functions from $\mathbb{R}^2$ to $\mathbb{R}$. Geometrically, the state space is a plane, and we call this plane the *phase plane*.

Given a set of initial conditions $x(0) = x_0$, we denote by x the solution to the equation (2.1). (We assume throughout this chapter that given an initial condition there exists a *unique* solution *for all* $t \geq 0$: this is guaranteed under mild assumptions on f_1, f_2, and we will learn more about this in Chapter 4.) With time t varied from 0 to ∞, the solution $t \mapsto x(t)$ can be represented geometrically as a curve in the phase plane. Such a curve is called a *(phase plane) trajectory*. A family of phase plane trajectories corresponding to various initial conditions is called a *phase portrait* of the system. From the assumption about the existence of solution, we know that from each point in the phase plane there passes a curve, and from the uniqueness, we know that there can be only one such curve. Thus no two trajectories in the phase plane can intersect, for if they did intersect at a point, then with that point as the initial condition, we would have two solutions[1], which is a contradiction!

To illustrate the concept of a phase portrait, let us consider the following simple system.

Example. Consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1. \end{aligned}$$

Thus the system is a linear ODE $x' = Ax$ with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$, and so if the initial condition expressed in polar coordinates is

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} r_0 \cos\theta_0 \\ r_0 \sin\theta_0 \end{bmatrix},$$

then it can be seen that the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} r_0 \cos(\theta_0 - t) \\ r_0 \sin(\theta_0 - t) \end{bmatrix}, \quad t \geq 0. \tag{2.2}$$

We note that

$$(x_1(t))^2 + (x_2(t))^2 = r_0^2,$$

which represents a circle in the phase plane. Corresponding to different initial conditions, circles of different radii can be obtained, and from (2.2), it is easy to see that the motion is clockwise. Plotting these circles on the phase plane, we obtain a phase portrait as shown in the Figure 2.1.

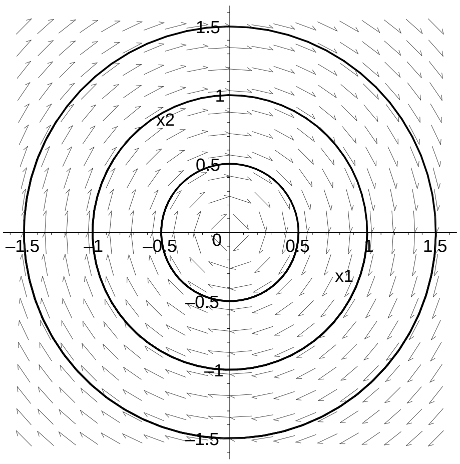

Figure 2.1: Phase portrait.

We see that the trajectories neither converge to the origin nor diverge to infinity. They simply circle around the origin. ♢

[1]Here we are really running the differential equation *backwards* in time, but then we can make the change of variables $\tau = -t$.

2.2.2 Singular points

An important concept in phase plane analysis is that of a singular point.

Definition. A *singular point* of the system $\left\{\begin{array}{rcl} x_1' & = & f_1(x_1, x_2) \\ x_2' & = & f_2(x_1, x_2), \end{array}\right.$ is a point (x_{1*}, x_{2*}) in the phase plane such that $f_1(x_{1*}, x_{2*}) = 0$ and $f_2(x_{1*}, x_{2*}) = 0$.

Such a point is also sometimes called an *equilibrium point* (see Chapter 3), that is, a point where the system states can stay forever: if we start with this initial condition, then the unique solution is $x_1(t) = x_{1*}$ and $x_2(t) = x_{2*}$. So through that point in the phase plane, only the 'trivial curve' comprising just that point passes.

For a linear system $x' = Ax$, if A is invertible, then the only singular point is $(0, 0)$, and if A is not invertible, then all the points from the kernel of A are singular points. So in the case of linear systems, either there is only one equilibrium point, or infinitely many singular points, none of which is then isolated. But in the case of nonlinear systems, there can be more than one isolated singular point, as demonstrated in the following example.

Example. Consider the system

$$\begin{array}{rcl} x_1' & = & x_2, \\ x_2' & = & -\dfrac{1}{2}x_2 - 2x_1 - x_1^2, \end{array}$$

whose phase portrait is shown in Figure 2.2. The system has two singular points, one at $(0, 0)$,

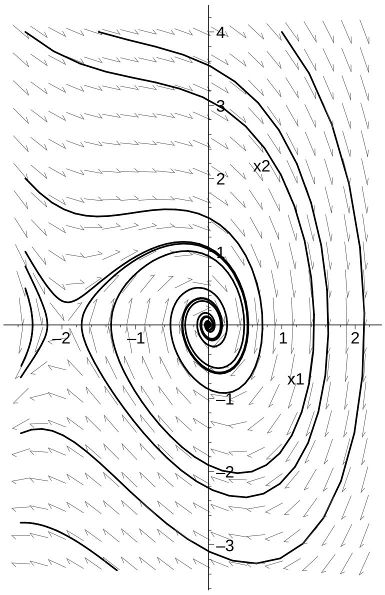

Figure 2.2: Phase portrait.

and the other at $(-2, 0)$. The motion patterns of the system trajectories starting in the vicinity of the two singular points have different natures. The trajectories move towards the point $(0, 0)$, while they move away from $(-2, 0)$. $\Diamond$

One may wonder why an equilibrium point of a 2D system is called a singular point. To answer this, let us examine the slope of the phase trajectories. The slope of the phase trajectory at time t is given by

$$\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.$$

When both f_1 and f_2 are zero at a point, this slope is undetermined, and this accounts for the adjective 'singular'.

Singular points are important features in the phase plane, since they reveal some information about the system. For nonlinear systems, besides singular points, there may be more complex features, such as limit cycles. These will be discussed later in this chapter.

Note that although the phase plane method is developed primarily for 2D systems, it can be also applied to the analysis of nD systems in general, but the graphical study of higher order systems is computationally and geometrically complex. On the other hand with 1D systems, the phase "plane" is reduced to the real line. We consider an example of a 1D system below.

Example. Consider the system $x' = -x + x^3$.

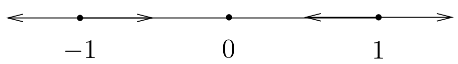

Figure 2.3: Phase portrait of the system $x' = -x + x^3$.

The singular points are determined by the equation

$$-x + x^3 = 0,$$

which has three real solutions, namely -1, 0 and 1. The phase portrait is shown in Figure 2.3. Indeed, for example if we consider the solution to

$$x'(t) = -x(t) - (x(t))^3, \quad t \geq t_0, \quad x(t_0) = x_0,$$

with $0 < x_0 < 1$, then we observe that

$$x'(t_0) = -x_0 - x_0^3 = -\underbrace{x_0}_{>0}\underbrace{(1 - x_0^2)}_{>0} < 0,$$

and this means that $t \mapsto x(t)$ is decreasing, and so the "motion" starting from x_0 is towards the left. This explains the direction of the arrow for the region $0 < x < 1$ in Figure 2.3. $\Diamond$

Exercises.

1. Locate the singular points of the following systems.

(a) $\begin{cases} x_1' &= x_2, \\ x_2' &= \sin x_1. \end{cases}$

(b) $\begin{cases} x_1' &= x_1 - x_2, \\ x_2' &= x_2^2 - x_1. \end{cases}$

(c) $\begin{cases} x_1' &= x_1^2(x_2 - 1), \\ x_2' &= x_1 x_2. \end{cases}$

(d) $\begin{cases} x_1' &= x_1^2(x_2 - 1), \\ x_2' &= x_1^2 - 2x_1 x_2 - x_2^2. \end{cases}$

(e) $\begin{cases} x_1' &= x_1 - x_2^2, \\ x_2' &= x_1^2 - x_2^2. \end{cases}$

(f) $\begin{cases} x_1' &= \sin x_2, \\ x_2' &= \cos x_1. \end{cases}$

2. Sketch the following parameterized curves in the phase plane.
   (a) $(x_1, x_2) = (a\cos t, b\sin t)$, where $a > 0$, $b > 0$.
   (b) $(x_1, x_2) = (ae^t, be^{-2t})$, where $a > 0$, $b > 0$.
3. Draw phase portraits of the following 1D systems.
   (a) $x' = x^2$.
   (b) $x' = e^x$.
   (c) $x' = \cosh x$.
   (d) $x' = \sin x$.
   (e) $x' = \cos x - 1$.
   (f) $x' = \sin(2x)$.
4. Consider a 2D autonomous system for which there exists a unique solution for every initial condition in $\mathbb{R}^2$ for all $t \in \mathbb{R}$.
   (a) Show that if (x_1, x_2) is a solution, then for any $T \in \mathbb{R}$, the shifted functions (y_1, y_2) given by
   $$\begin{aligned} y_1(t) &= x_1(t+T), \\ y_2(t) &= x_2(t+T), \end{aligned}$$
   ($t \in \mathbb{R}$) is also a solution.
   (b) (∗) Can $x(t) = \left(\frac{2\cos t}{1+(\sin t)^2}, \frac{\sin(2t)}{1+(\sin t)^2}\right)0$, $t \in \mathbb{R}$, be the solution of such a 2D autonomous system?
   HINT: By the first part, we know that $t \mapsto y_1(t) := x(t+\pi/2)$ and $t \mapsto y_2(t) := x(t+3\pi/2)$ are also solutions. Check that $y_1(0) = y_2(0)$. Is $y_1 \equiv y_2$? What does this say about uniqueness of solutions starting from a given initial condition?
   (c) Using Maple, sketch the curve
   $$t \mapsto \left(\frac{2\cos t}{1+(\sin t)^2}, \frac{\sin(2t)}{1+(\sin t)^2}\right).$$
   (This curve is called the *lemniscate*.)
5. Consider the ODE (1.6) from Exercise 2 on page 6.
   (a) Using Maple, find the singular points (approximately).
   (b) Draw a phase portrait in the region $x \geq 0$.
6. A simple model for a national economy is given by
   $$\begin{aligned} I' &= I - \alpha C \\ C' &= \beta(I - C - G), \end{aligned}$$
   where

I	denotes the national income,
C	denotes the rate of consumer spending, and
G	denotes the rate of government expenditure.

   The model is restricted to $I, C, G \geq 0$, and the constants α, β satisfy $\alpha > 1$, $\beta \geq 1$.

(a) Suppose that the government expenditure is related to the national income according to $G = G_0 + kI$, where G_0 and k are positive constants. Find the range of positive k's for which there exists an equilibrium point such that I, C, G are nonnegative.

(b) Let $k = 0$, and let (I_0, C_0) denote the equilibrium point. Introduce the new variables $I_1 = I - I_0$ and $C_1 = C - C_0$. Show that (I_1, C_1) satisfy a linear system of equations:

$$\begin{bmatrix} I_1' \\ C_1' \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} I_1 \\ C_1 \end{bmatrix}.$$

If $\beta = 1$ and $\alpha = 2$, then conclude that in fact the economy oscillates.

2.3 Constructing phase portraits

Phase portraits can be routinely generated using computers, and this has spurred many advances in the study of complex nonlinear dynamic behaviour. Nevertheless, in this section, we learn a few techniques in order to be able to roughly sketch the phase portraits. This is useful for instance in order to verify the plausibility of computer generated outputs. We describe two methods: one involves the analytic solution of differential equations. If an analytic solution is not available, the other tool, called the method of isoclines, is useful.

2.3.1 Analytic method

There are two techniques for generating phase portraits analytically. One is to first solve for x_1 and x_2 explicitly as functions of t, and then to eliminate t, as we had done in the example on page 22.

The other analytic method does not involve an explicit computation of the solutions as functions of time, but instead, one solves the differential equation

$$\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}.$$

Thus given a trajectory $t \mapsto (x_1(t), x_2(t))$, we eliminate the t by setting up a differential equation for the derivative of the second function 'with respect to the first one', not involving the t, and by solving this differential equation. We illustrate this in the same example.

Example. Consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1. \end{aligned}$$

We have $\dfrac{dx_2}{dx_1} = \dfrac{-x_1}{x_2}$, and so $x_2 \dfrac{dx_2}{dx_1} = -x_1$. Thus $\dfrac{d}{dx_1}\left(\dfrac{1}{2}x_2^2\right) = x_2\dfrac{dx_2}{dx_1} = -x_1$.

Integrating with respect to x_1, and using the fundamental theorem of calculus, we obtain $x_2^2 + x_1^2 = C$. This equation describes a circle in the (x_1, x_2)-plane. Thus the trajectories satisfy

$$(x_1(t))^2 + (x_2(t))^2 = C = (x_1(0)^2 + (x_2(0))^2, \quad t \geq 0,$$

and they are circles. We note that when $x_1(0)$ belongs to the right half plane, then $x_2'(0) = -x_1(0) < 0$, and so $t \mapsto x_2(t)$ should be decreasing. Thus we see that the motion is clockwise, as shown in Figure 2.1. $\Diamond$

Exercises.

1. Sketch the phase portrait of $\begin{cases} x_1' &= x_2, \\ x_2' &= x_1. \end{cases}$

2. Sketch the phase portrait of $\begin{cases} x_1' &= -2x_2, \\ x_2' &= x_1. \end{cases}$

3. (a) Sketch the curve curve $y(x) = x(A + B \log |x|)$ where A, B are constants and $B > 0$.

   (b) (∗) Sketch the phase portrait of $\begin{cases} x_1' &= x_1 + x_2, \\ x_2' &= x_2. \end{cases}$

   Hint: Solve the system, and try eliminating t.

2.3.2 The method of isoclines

At a point (x_1, x_2) in the phase plane, the slope of the tangent to the trajectory is $\frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}$. An *isocline* is a curve in $\mathbb{R}^2$ defined by $\frac{f_2(x_1,x_2)}{f_1(x_1,x_2)} = \alpha$, where α is a real number. This means that if we look at all the trajectories that pass through various points on the same isocline, then all of these trajectories have the same slope (equal to α) on the points of this isocline. To obtain trajectories from the isoclines, we assume that the tangent slopes are locally constant. The method of constructing the phase portrait using isoclines is thus the following:

Step 1. For various values of α, construct the corresponding isoclines. Along an isocline, draw small line segments with slope α. In this manner a field of directions is obtained.

Step 2. Since the tangent slopes are locally constant, we can construct a phase plane trajectory by connecting a sequence line segments.

We illustrate the method by means of two examples.

Example. Consider the system $\begin{cases} x_1' &= x_2, \\ x_2' &= -x_1. \end{cases}$ The slope is given by $\frac{dx_2}{dx_1} = \frac{-x_1}{x_2}$, and so the isocline corresponding to slope α is $x_1 + \alpha x_2 = 0$, and these points lie on a straight line. By taking

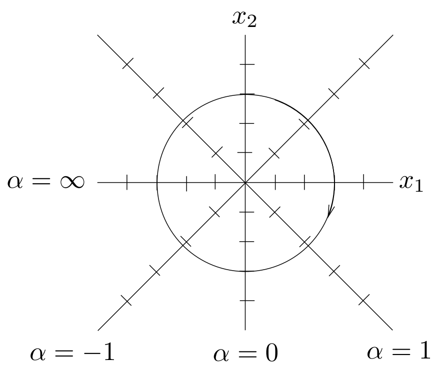

Figure 2.4: Method of isoclines.

different values for α, a set of isoclines can be drawn, and in this manner a field of directions of tangents to the trajectories are generated, as shown in Figure 2.4, and the trajectories in the phase portrait are circles. If $x_1 > 0$, then $x_2'(0) = -x_1(0) < 0$, and so the motion is clockwise. ◊

Let us now use the method of isoclines to study a nonlinear equation.

Example. *(van der Pol equation)* Consider the differential equation

$$y'' + \mu(y^2 - 1)y' + y = 0. \tag{2.3}$$

By introducing the variables $x_1 = y$ and $x_2 = y'$, we obtain the following 2D system:

$$\begin{array}{rcl} x_1' &=& x_2, \\ x_2' &=& -\mu(x_1^2 - 1)x_2 - x_1. \end{array}$$

An isocline of slope α is defined by $\dfrac{-\mu(x_1^2-1)x_2 - x_1}{x_2} = \alpha$, that is, the points on the curve $x_2 = \dfrac{x_1}{(\mu - \mu x_1^2) - \alpha}$ all correspond to the same slope α of tangents to trajectories.

We take the value of $\mu = \frac{1}{2}$. By taking different values for α, different isoclines can be obtained, and short line segments can be drawn on the isoclines to generate a field of directions, as shown in Figure 2.5. The phase portrait can then be obtained, as shown.

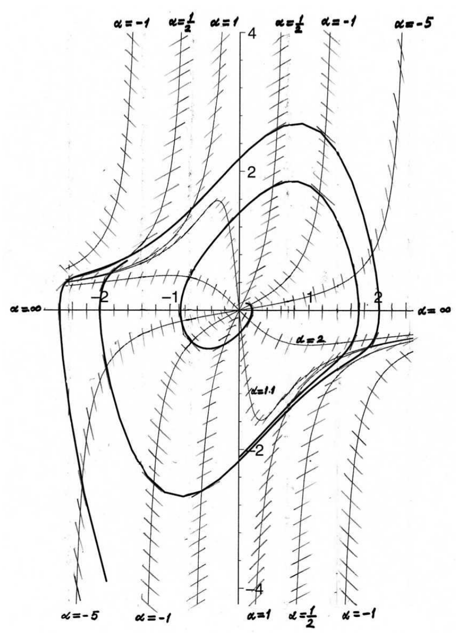

Figure 2.5: Method of isoclines.

It is interesting to note that from the phase portrait, one is able to guess that there exists a closed curve in the phase portrait, and the trajectories starting from both outside and inside seem to converge to this curve[2]. ◊

Exercise. Using the method of isoclines, sketch a phase portrait of the system $\left\{ \begin{array}{rcl} x_1' &=& x_2, \\ x_2' &=& x_1. \end{array} \right.$

[2]This is also expected based on physical considerations: the van der Pol equation arises from electric circuits containing vacuum tubes, where for small ocsillations, energy is fed into the system, while for large oscillations, energy is taken out of the system–in other words, large oscillations will be damped, while for small oscillations, there is 'negative damping' (that is energy is fed into the system). So one can expect that such a system will approach some periodic behaviour, which will appear as a closed curve in the phase portrait.

2.3.3 Phase portraits using Maple

We consider a few examples in order to illustrate how one can make phase portraits using Maple.

Consider the ODE system: $x_1'(t) = x_2(t)$ and $x_2(t) = -x_2(t)$. We can plot x_1 against x_2 by using `DEplot`. Consider for example:

```
> with(DEtools):
> ode3a := diff(x1(t), t) = x2(t); ode3b := diff(x2(t), t) = −x1(t);
> DEplot({ode3a, ode3b}, {x1(t), x2(t)}, t = 0..10, x1 = −2..2, x2 = −2..2,
  [[x1(0) = 1, x2(0) = 0]], stepsize = 0.01, linecolour = black);
```

The resulting plot is shown in Figure 2.6. The arrows show the direction field.

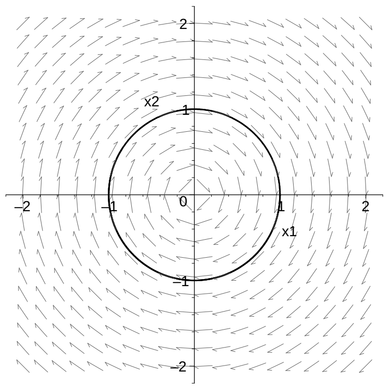

Figure 2.6: Phase portrait for the ODE system $x_1' = x_2$ and $x_2' = -x_1$.

By including some more trajectories, we can construct a phase portrait in a given region, as shown in the following example.

Example. Consider the system $\left\{ \begin{array}{lcl} x_1' & = & -x_2 + x_1(1 - x_1^2 - x_2^2), \\ x_2' & = & x_1 + x_1(1 - x_1^2 - x_2^2). \end{array} \right.$ Using the following Maple

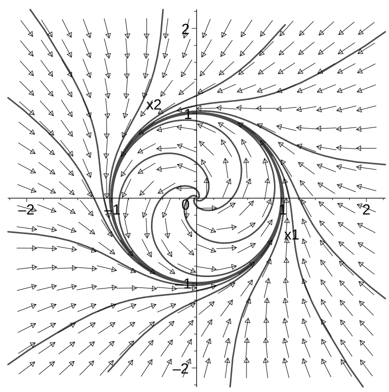

Figure 2.7: Phase portrait.

command, we can obtain the phase portrait shown in Figure 2.7.

```
> with(DEtools):
> ode1 := diff(x1(t), t) = −x2(t) + x1(t) ∗ (1 − x1(t)^2 − x2(t)^2);
> ode2 := diff(x2(t), t) = x1(t) + x1(t) ∗ (1 − x1(t)^2 − x2(t)^2);
> initvalues := seq(seq([x1(0) = i + 1/2, x2(0) = j + 1/2],
  i = −2..1), j = −2..1):
> DEplot({ode1, ode2}, [x1(t), x2(t)], t = −4..4, x1 = −2..2, x2 = −2..2, [initvalues],
  stepsize = 0.05, arrows = MEDIUM, colour = black, linecolour = red);
```

$\diamond$

Exercises.

1. Using Maple, construct phase portraits of the following systems:

   (a) $\begin{cases} x_1' &= x_2, \\ x_2' &= x_1. \end{cases}$

   (b) $\begin{cases} x_1' &= -2x_2, \\ x_2' &= x_1. \end{cases}$

   (c) $\begin{cases} x_1' &= x_1 + x_2, \\ x_2' &= x_2. \end{cases}$

2. Suppose a lake contains fish, which we simply call 'big fish' and 'small fish'. In the absence of big fish, the small fish population x_s evolves according to the law: $x_s' = ax_s$, where $a > 0$ is a constant. Indeed, the more the small fish, the more they reproduce. But big fish eat small fish, and so taking this into account, we have

$$x_s' = ax_s - bx_s x_b,$$

where $b > 0$ is a constant. The last term accounts for how often the big fish encounter the small fish–the more the small fish, the easier it becomes for the big fish to catch them, and the faster the population of the small fish decreases.

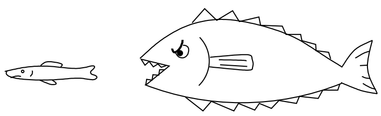

Figure 2.8: Big fish and small fish.

On the other hand, the big fish population evolution is given by

$$x_b' = -cx_b + dx_s x_b,$$

where $c, d > 0$ are constants. The first term has a negative sign which comes from the competition between these predators–the more the big fish, the fiercer the competition for survival. The second term accounts for the fact that the larger the number of small fish, the greater the growth in the numbers of the big fish.

   (a) *Singular points.* Show that the (x_s, x_b) ODE system has two singular points $(0, 0)$ and $\left(\frac{c}{d}, \frac{a}{b}\right)$. The point $(0, 0)$ corresponds to the extinction of both species–if both populations are 0, then they continue to remain so. The point $\left(\frac{c}{d}, \frac{a}{b}\right)$ corresponds to population levels at which both species sustain their current nonzero numbers indefinitely.

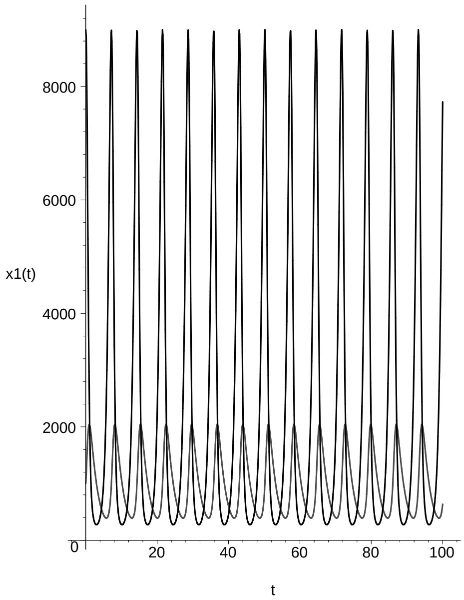

Figure 2.9: Periodic variation of the population levels.

(b) *Solution to the ODE system.* Use Maple to plot the population levels of the two species on the same plot, with the following data: $a = 2$, $b = 0.002$, $c = 0.5$, $d = 0.0002$, $x_s(0) = 9000$, $x_b(0) = 1000$, $t = 0$ to $t = 100$.

Your plot should show that the population levels vary in a periodic manner, and the population of the big fish lags behind the population of the small fish. This is expected since the big fish thrive when the small fish are plentiful, but ultimately outstrip their food supply and decline. As the big fish population is low, the small fish numbers increase again. So there is a a cycle of growth and decline.

(c) *Phase portrait.* With the same constants as before, plot a phase portrait in the region $x_s = 0$ to $x_s = 10000$ and $x_b = 0$ to 4000.

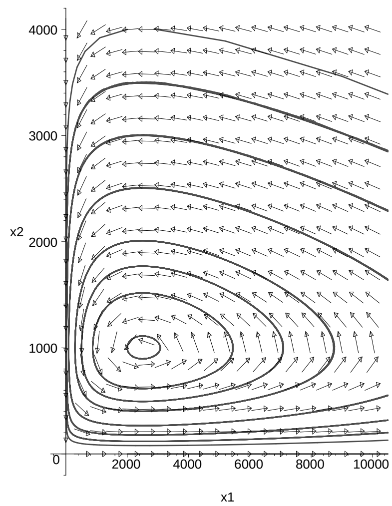

Figure 2.10: Phase portrait for the Lotka-Volterra ODE system.

Also plot in the same phase portrait the solution curves. What do you observe?

2.4 Phase plane analysis of linear systems

In this section, we describe the phase plane analysis of *linear* systems. Besides allowing us to visually observe the motion patterns of linear systems, this will also help the development of nonlinear system analysis in the next section, since similar motion patterns can be observed in the local behaviour of nonlinear systems as well.

We will analyze three simple types of matrices. It turns out that it is enough to consider these three types, since every other matrix can be reduced to such a matrix by an appropriate change of basis (in the phase portrait, this corresponds to replacing the usual axes by new ones, which may not be orthogonal). However, in this elementary first course, we will ignore this part of the theory.

2.4.1 Complex eigenvalues

Consider the system $x' = Ax$, where $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Then $e^{tA} = e^{ta} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$, and so if the initial condition $x(0)$ has polar coordinates (r_0, θ_0), then the solution is given by

$$x(t) = e^{ta} r_0 \cos(\theta_0 - bt) \quad \text{and} \quad y(t) = e^{ta} r_0 \sin(\theta_0 - bt), \quad t \geq 0,$$

so that the trajectories are spirals if a is nonzero, moving towards the origin if $a < 0$, and outwards if $a > 0$. If $a = 0$, the trajectories are circles. See Figure 2.11.

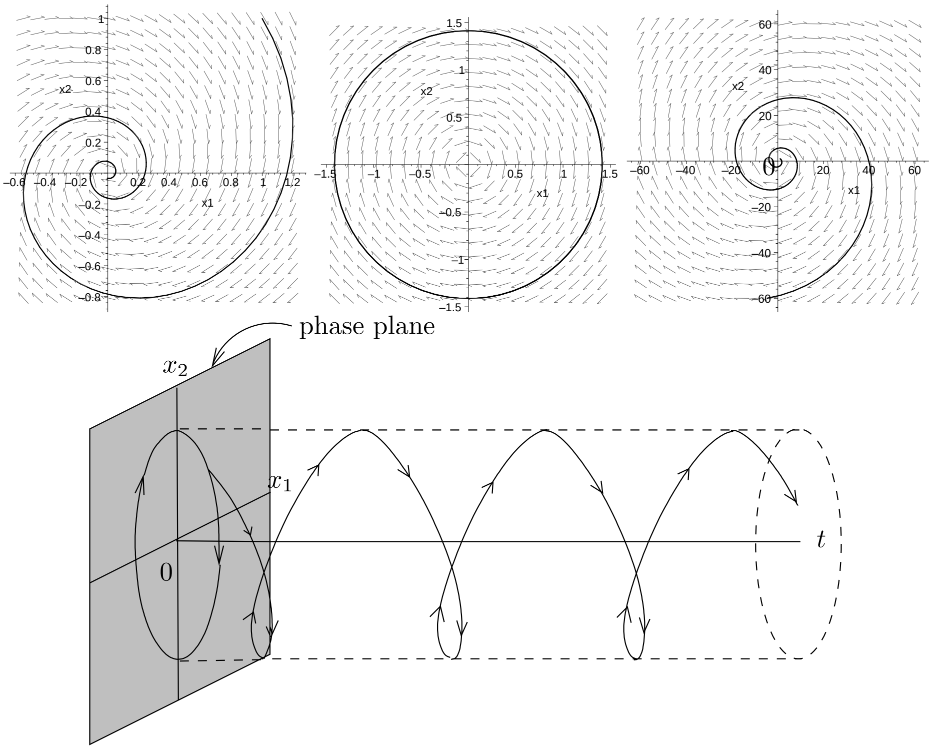

Figure 2.11: Case of complex eigenvalues. The last figure shows the phase plane trajectory as a projection of the curve $(t, x_1(t), x_2(t))$ in $\mathbb{R}^3$: case when $a = 0$, and $b > 0$.

2.4.2 Diagonal case with real eigenvalues

Consider the system $x' = Ax$, where

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are real numbers. The trajectory starting from the initial condition (x_{10}, x_{20}) is given by $x_1(t) = e^{\lambda_1 t} x_{10}$, $x_2(t) = e^{\lambda_2 t} x_{20}$. We also see that $Ax_1^{\lambda_2} = Bx_2^{\lambda_2}$ with appropriate values for the constants A and B. See the topmost figure in Figure 2.12 for the case when λ_1, λ_2 are both negative. In general, we obtain the phase portraits shown in Figure 2.12, depending on the signs of λ_1 and λ_2.

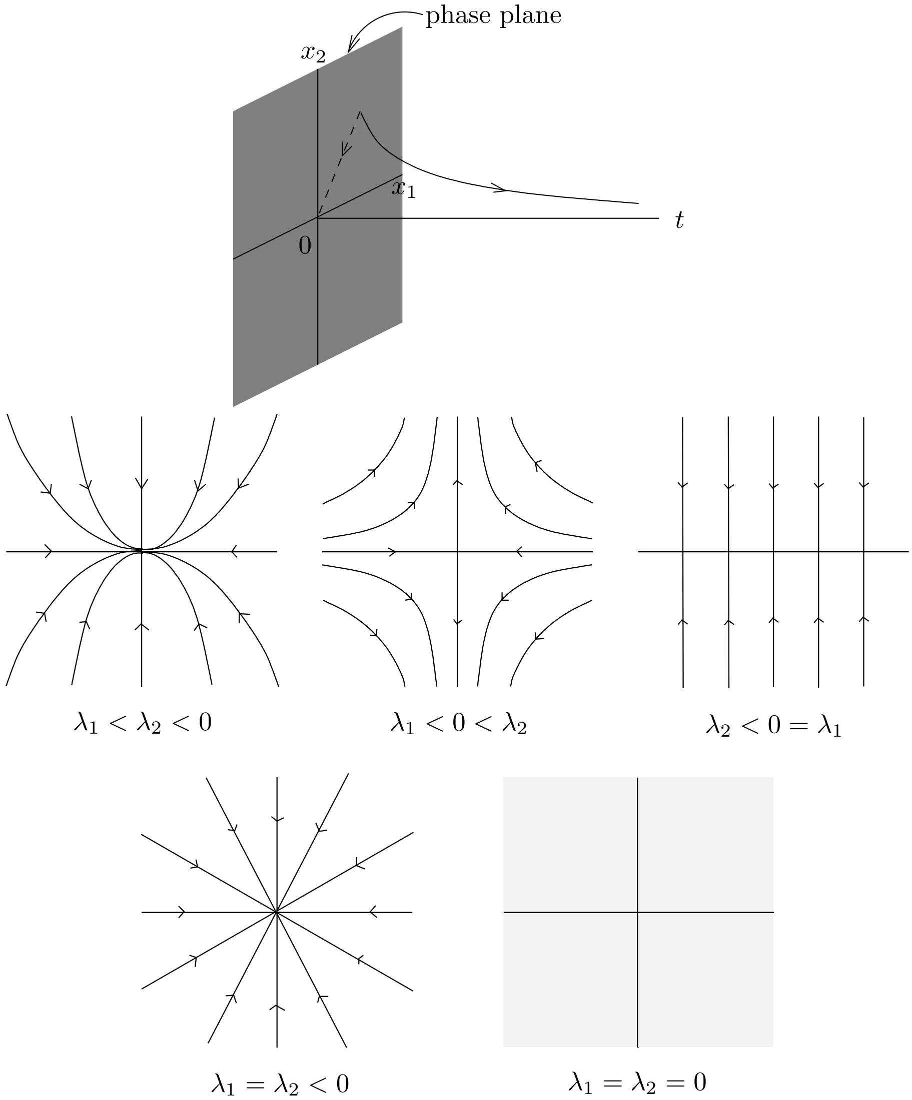

Figure 2.12: The topmost figure shows the phase plane trajectory as a projection of the curve $(t, x_1(t), x_2(t))$ in $\mathbb{R}^3$: diagonal case when both eigenvalues are negative. The other figures are phase portraits in the case when A is diagonal with real eigenvalues case. In the case when the eigenvalues have opposite signs, the singular point is called a *saddle point*.

2.4.3 Nondiagonal case

Consider the system $x' = Ax$, where

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where λ is a real number. It is easy to see that

$$e^{tA} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix},$$

so that the solution starting from the initial condition (x_{10}, x_{20}) is given by

$$x_1(t) = e^{\lambda t}(x_{10} + tx_{20}), \quad x_2(t) = e^{\lambda t}x_{20}.$$

Figure 2.13 shows the phase portraits for the three cases when $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

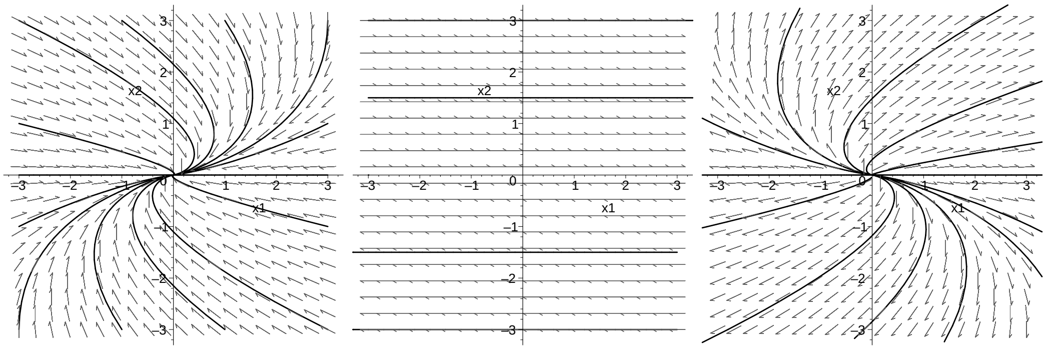

Figure 2.13: Nondiagonal case.

We note that the angle that a point on the trajectory makes with the x_1 axis is given by

$$\arctan\left(\frac{x_2(t)}{x_1(t)}\right) = \arctan\left(\frac{x_{20}}{x_{10} + tx_{20}}\right),$$

which tends to 0 or π as $t \to \infty$.

Exercises.

1. Draw the phase portrait for the system $\left\{ \begin{array}{rcl} x_1' & = & x_1 - 3x_2, \\ x_2' & = & -2x_2, \end{array} \right.$ using the following procedure:

   STEP 1. Find the eigenvectors and eigenvalues: Show that $A := \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$ has eigenvalues $1, -2$, with eigenvectors $v_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

   STEP 2. Set up the coordinate system in terms of the eigenvectors: Since v_1, v_2 form a bsis for $\mathbb{R}^2$, the solution vector $x(t)$ can be expressed as a linear combination of v_1, v_2: $x(t) = \alpha(t)v_1 + \beta(t)v_2$. Note that $\alpha(t)$ and $\beta(t)$ are the 'coordinates' of the point $x(t)$ in the directions v_1 and v_2, respectively. In other words, they are the 'projections' of the point $x(t)$ in the directions v_1 and v_2, as shown in the Figure 2.14.

   STEP 3. Eliminate t: Show that $(\alpha(t))^2\beta(t) = (\alpha(0))^2\beta(0)$, and using the 'distorted' coordinate system, draw a phase portrait for the system $x'(t) = Ax(t)$.

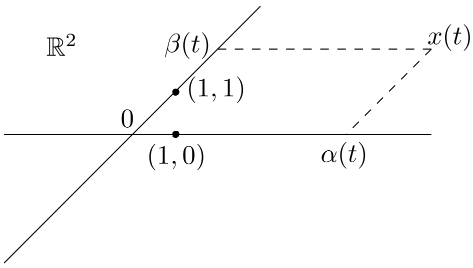

Figure 2.14: The distorted coordinate system.

2. $(*)$ Let $A \in \mathbb{R}^2$ have eigenvalues $a + ib$ and $a - ib$, where $a, b \in \mathbb{R}$, and $b \neq 0$.

   (a) If $v_1 := u + iv$ ($u, v \in \mathbb{R}^2$) is an eigenvector corresponding to the eigenvalue $a + ib$, then show that $v_2 := u - iv$ is an eigenvector corresponding to $a - ib$.

   (b) Using the fact that $b \neq 0$, conclude that v_1, v_2 are linearly independent in $\mathbb{C}^2$.

   (c) Prove that u, v are linearly independent as vectors in $\mathbb{R}^2$. Conclude that the matrix P with the columns u and v is invertible.

   (d) Verify that

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

2.5 Phase plane analysis of nonlinear systems

With the phase plane analysis of nonlinear systems, we should keep two things in mind. One is that the phase plane analysis is related to that of linear systems, because the local behaviour of a nonlinear system can be approximated by a linear system. And the second is that, despite this similarity with linear systems, nonlinear systems can display much more complicated patterns in the phase plane, such as multiple singular points and limit cycles. In this section, we will discuss these aspects. We consider the system

$$\begin{array}{rcl} x_1' &=& f_1(x_1, x_2), \\ x_2' &=& f_2(x_1, x_2), \end{array} \tag{2.4}$$

where we assume that f_1 and f_2 have continuous partial derivatives, and this assumption will be continued for the remainder of this chapter. We will learn later in Chapter 4 that a consequence of this assumption is that for the initial value problem of the system above, there will exist a unique solution. Moreover, we will also make the assumption that solutions exist for all times in $\mathbb{R}$.

2.5.1 Local behaviour of nonlinear systems

In order to see the similarity with linear systems, we decompose the the nonlinear system into a linear part and an 'error' part (which is small close to a singular point), using Taylor's theorem, as follows.

Let (x_{10}, x_{20}) be an isolated singular point of (2.4). Thus $f_1(x_{10}, x_{20}) = 0$ and $f_2(x_{10}, x_{20}) = 0$.

Then by Taylor's theorem, we have

$$x_1' = \left[\frac{\partial f_1}{\partial x_1}(x_{10}, x_{20})\right](x_1 - x_{10}) + \left[\frac{\partial f_1}{\partial x_2}(x_{10}, x_{20})\right](x_2 - x_{20}) + e_1(x_1 - x_{10}, x_2 - x_{20}), \quad (2.5)$$

$$x_2' = \left[\frac{\partial f_2}{\partial x_1}(x_{10}, x_{20})\right](x_1 - x_{10}) + \left[\frac{\partial f_2}{\partial x_2}(x_{10}, x_{20})\right](x_2 - x_{20}) + e_2(x_1 - x_{10}, x_2 - x_{20}), \quad (2.6)$$

where e_1 and e_2 are such that $e_1(0,0) = e_2(0,0) = 0$. We translate the singular point (x_{10}, x_{20}) to the origin by introducing the new variables $y_1 = x_1 - x_{10}$ and $y_2 = x_2 - x_{20}$. With

$$a := \frac{\partial f_1}{\partial x_1}(x_{10}, x_{20}), \qquad b := \frac{\partial f_1}{\partial x_2}(x_{10}, x_{20}),$$
$$c := \frac{\partial f_2}{\partial x_1}(x_{10}, x_{20}), \qquad d := \frac{\partial f_2}{\partial x_2}(x_{10}, x_{20}),$$

we can rewrite (2.5)-(2.6) as follows:

$$\begin{array}{rcl} y_1' & = & ay_1 + by_2 + e_1(y_1, y_2), \\ y_2' & = & cy_1 + dy_2 + e_2(y_1, y_2). \end{array} \quad (2.7)$$

We note that this new system has $(0,0)$ as a singular point. We will elaborate on the similarity between the phase portrait of the system (2.4) with the phase portrait of its linear part, that is, the system

$$\begin{array}{rcl} z_1' & = & az_1 + bz_2, \\ z_2' & = & cz_1 + dz_2. \end{array} \quad (2.8)$$

Before clarifying the relationship between (2.4) and (2.8), we pause to note some important differences. The system (2.4) may have *many* singular points; *one* of them has been selected and moved to the origin. If a different singular point would have been chosen, then the constants a, b, c, d in (2.8) would have been different. The important point is that any statement relating (2.4) and (2.8), is *local* in nature, in that they apply 'near' the singular point under consideration. By 'near' here, we mean in a sufficiently small neighbourhood or ball around the singular point. Totally different kinds of behaviour may occur in a neighbourhood of other critical points. The transformation above must be made, and the corresponding linear part must be analyzed, for each isolated singular point of the nonlinear system.

We now give the main theorem in this section about the local relationship between the nature of phase portraits of (2.4) and (2.8), but we will not prove this theorem.

Theorem 2.5.1 *Let (x_{10}, x_{20}) be an isolated singular point of (2.4), and let*

$$A := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_{10}, x_{20}) & \frac{\partial f_1}{\partial x_2}(x_{10}, x_{20}) \\ \frac{\partial f_2}{\partial x_1}(x_{10}, x_{20}) & \frac{\partial f_2}{\partial x_2}(x_{10}, x_{20}) \end{bmatrix}.$$

Then we have the following:

1. *If every eigenvalue of A has negative real parts, then all solutions of (2.4) starting in a small enough ball with centre (x_{10}, y_{10}) converge to (x_{10}, y_{10}) as $t \to \infty$. (This situation is abbreviated by saying that the equilibrium point (x_{10}, y_{10}) is 'asymptotically stable'; see §3.3.)*

2. *If the matrix A has an eigenvalue with a positive real part, then there exists a ball B such that for every ball B' of positive radius around (x_{10}, y_{10}), there exists a point in B' such that a solution x of (2.4) starting from that point leaves the ball B. (This situation is abbreviated by saying that the equilibrium point (x_{10}, y_{10}) is 'unstable'; see §3.2.)*

We illustrate the theorem with the following example.

Example. Consider the system

$$\begin{aligned} x_1' &= -x_1 + x_2 - x_1(x_2 - x_1), \\ x_2' &= -x_1 - x_2 + 2x_1^2 x_2. \end{aligned}$$

This nonlinear system has the singular points $(-1,-1)$, $(1,1)$ and $(0,0)$. If we linearize around the singular point $(0,0)$, we obtain the matrix

$$\begin{bmatrix} \left.\frac{\partial}{\partial x_1}(-x_1 + x_2 - x_1(x_2 - x_1))\right|_{(0,0)} & \left.\frac{\partial}{\partial x_2}(-x_1 + x_2 - x_1(x_2 - x_1))\right|_{(0,0)} \\ \left.\frac{\partial}{\partial x_1}(-x_1 - x_2 + 2x_1^2 x_2)\right|_{(0,0)} & \left.\frac{\partial}{\partial x_2}(-x_1 - x_2 + 2x_1^2 x_2)\right|_{(0,0)} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},$$

which has eigenvalues $-1+i$ and $-1-i$. Thus by Theorem 2.5.1, it follows that for the above nonlinear system, if we start close to $(0,0)$, then the solutions converge to $(0,0)$ as $t \to \infty$.

However, not all solutions of this nonlinear system converge to $(0,0)$. For example, we know that $(1,1)$ is also a singular point, and so if we start from there, then we stay there. $\diamondsuit$

The above example highlights the *local* nature of Theorem 2.5.1. How close is sufficiently close is generally a difficult question to answer.

Actually more than just similarity of convergence to the singular point can be said. It turns out that if the real parts of the eigenvalues are not equal to zero, then also the 'qualitative' structure is preserved. Roughly speaking, this means that there is a map T mapping a region Ω_1 around (x_{10}, x_{20}) to a region Ω_2 around $(0,0)$ such that

1. T is one-to-one and onto;
2. both T and T^{-1} are continuous;
3. if two points of Ω_1 lie on the same trajectory of (2.8), then their images under T lie on the same trajectory of (2.4);
4. if two points of Ω_2 lie on the same trajectory of (2.4), then their images under T^{-1} lie on the same trajectory of (2.8).

The mapping is shown schematically in Figure 2.15.

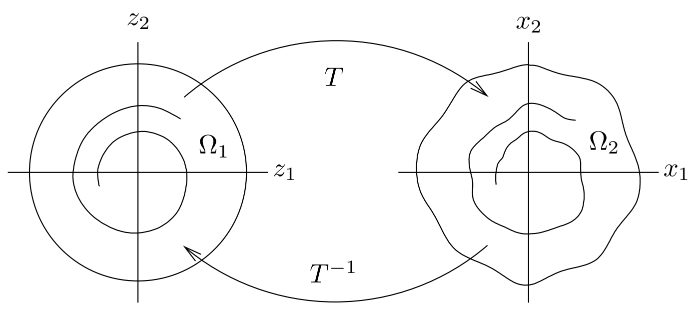

Figure 2.15: The mapping T.

The actual construction of such a mapping is not easy, but we demonstrate the plausibility of its existence by considering an example.

Example. Consider the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_1 - x_2 + x_1(x_1 - 2x_2). \end{aligned}$$

The singular points are solutions to

$$x_2 = 0 \quad \text{and} \quad x_1 - x_2 + x_1^2 - 2x_1x_2 = 0,$$

and so they are $(0,0)$ and $(-1,0)$. Furthermore,

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 0, & \frac{\partial f_1}{\partial x_2} &= 1, \\ \frac{\partial f_2}{\partial x_1} &= 1 + 2x_1 - 2x_2, & \frac{\partial f_1}{\partial x_2} &= -1 - 2x_1. \end{aligned}$$

At $(0,0)$, the matrix of the linear part is

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},$$

whose eigenvalues satisfy $\lambda^2 + \lambda - 1 = 0$. The roots are real and of opposite signs, and so the origin is a saddle point.

At $(-1,0)$, the matrix of the linear part is

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

whose eigenvalues satisfy $\lambda^2 - \lambda + 1 = 0$. The trajectories are thus outward spirals.

Figure 2.16 shows the phase portrait of the system. As expected we see that around the points $(0,0)$ and $(-1,0)$, the local picture is similar to the corresponding linearisations. $\Diamond$

Finally, we discuss the case when the eigenvalues of the linearisation have real part equal to 0. It turns out that in this case, the behaviour of the linearisation gives no information about the behaviour of the nonlinear system. For example, circles in the phase portrait may be converted into spirals. We illustrate this in the following example.

Example. Consider the system

$$\begin{aligned} x_1' &= -x_2 - x_1(x_1^2 + x_2^2), \\ x_2' &= x_1 - x_2(x_1^2 + x_2^2). \end{aligned}$$

The linearisation about the singular point $(0,0)$ gives rise to the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which has eigenvalues $-i$ and i. Thus the phase portrait of the linear part comprises *circles*.

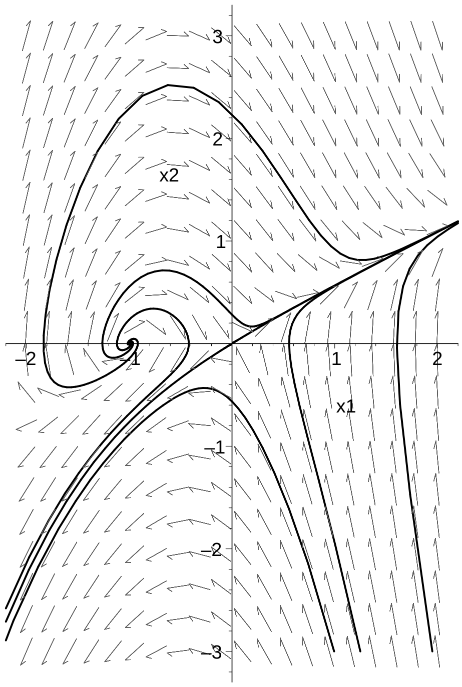

Figure 2.16: Phase portrait.

If we introduce the polar coordinates $r := \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$, then we have that

$$
\begin{array}{rcl}
r' & = & -r^3, \\
\theta' & = & 1.
\end{array}
$$

Thus we see that the trajectories approach the origin in *spirals*! $\Diamond$

Exercises. Use the linearisation theorem (Theorem 2.5.1), where possible, to describe the behaviour close to the equilibrium points of the following systems:

1. $\left\{ \begin{array}{rcl} x_1 & = & e^{x_1+x_2} - x_2, \\ x_2' & = & -x_1 + x_1 x_2. \end{array} \right.$

2. $\left\{ \begin{array}{rcl} x_1 & = & x_1 + 4x_2 + e^{x_1} - 1, \\ x_2' & = & -x_2 - x_2 e^{x_1}. \end{array} \right.$

3. $\left\{ \begin{array}{rcl} x_1 & = & x_2, \\ x_2' & = & -x_1^3. \end{array} \right.$

4. $\left\{ \begin{array}{rcl} x_1 & = & \sin(x_1 + x_2), \\ x_2' & = & x_2. \end{array} \right.$

5. $\left\{ \begin{array}{rcl} x_1 & = & \sin(x_1 + x_2), \\ x_2' & = & -x_2. \end{array} \right.$

2.5.2 Limit cycles and the Poincaré-Bendixson theorem

In the phase portrait of the van der Pol equation shown in Figure 2.5, we suspected that the system has a closed curve in the phase portrait, and moreover, trajectories starting inside that curve, as well as trajectories starting outside that curve, all tended towards this curve, while a motion starting on that curve would stay on it forever, circling periodically around the origin. Such a curve is called a "limit cycle", and we will study the exact definition later in this section. Limit cycles are a unique feature that can occur only in a nonlinear system. Although in the phase portrait in the middle of the top row of figures in Figure 2.11, we saw that if the real part of the eigenvalues is zero, we have periodic trajectories, these are not limit cycles, since now matter how close we start from such a periodic orbit, we can never approach it. We want to call those closed curves limit cycles such that for all trajectories starting close to it, they converge to it either as times increases to $+\infty$ or decreases to $-\infty$. In order to explain this further, we consider the following example.

Examples. Consider the system

$$\begin{aligned} x_1' &= x_2 - x_1(x_1^2 + x_2^2 - 1) \\ x_2' &= -x_1 - x_2(x_1^2 + x_2^2 - 1). \end{aligned}$$

By introducing polar coordinates $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$, the equations are transformed into

$$\begin{aligned} r' &= -r(r^2 - 1) \\ \theta' &= -1. \end{aligned}$$

When we are on the unit circle, we note that $r' = 0$, and so we stay there. Thus the unit circle is a periodic trajectory. When $r > 1$, then $r' < 0$, and so we see that if we start outside the unit circle, we tend towards the unit circle from the outside. On the other hand, if $r < 1$, then $r' > 0$, and so if we start inside the unit circle, we tend towards it from the inside. This can be made rigorous by examining the analytical solution, given by $r(t) = \left(\sqrt{1 + (1/r_0^2 - 1)e^{-2t}}\right)^{-1}$, $\theta(t) = \theta_0 - t$, where (r_0, θ_0) denotes the initial condition. So we have that all trajectories in the vicinity of the unit circle converge to the unit circle as $t \to \infty$.

Now consider the system

$$\begin{aligned} x_1' &= x_2 + x_1(x_1^2 + x_2^2 - 1) \\ x_2' &= -x_1 + x_2(x_1^2 + x_2^2 - 1). \end{aligned}$$

Again by introducing polar coordinates (r, θ) as before, we now obtain

$$\begin{aligned} r' &= r(r^2 - 1) \\ \theta' &= -1. \end{aligned}$$

When we are on the unit circle, we again have that $r' = 0$, and so we stay there. Thus the unit circle is a periodic trajectory. But now when $r > 1$, then $r' > 0$, and so we see that if we start outside the unit circle, we move away from the unit circle. On the other hand, if $r < 1$, then $r' < 0$, and so if we start inside the unit circle, we again move away from the unit circle. However, if we start with an initial condition (r_0, θ_0) with $r_0 < 1$, and go backwards in time, then we can show that the solution is given by $r(t) = \left(\sqrt{1 + (1/r_0^2 - 1)e^{2t}}\right)^{-1}$, $\theta(t) = \theta_0 - t$, and so we have that all trajectories in the vicinity of the unit circle from inside converge to the unit circle as $t \to -\infty$. $\Diamond$

In each of the examples considered above, we would like to call the unit circle a "limit cycle". This motivates the following definitions of ω- and α- limit points of a trajectory, and we will define limit cycles using these notions.

Definition. Let x be a solution of $x' = f(x)$. A point x_* is called an *ω-limit point of x* if there is a sequence of real numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} x(t_n) = x_*$. The set of all ω-limit points of x is denoted by $L_\omega(x)$.

A point x_* is called an *α-limit point of x* if there is a sequence of real numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = -\infty$ and $\lim_{n\to\infty} x(t_n) = x_*$. The set of all α-limit points of x is denoted by $L_\alpha(x)$.

For example, if x_* is a singular point, clearly $L_\omega(x_*) = L_\alpha(x_*) = x_*$. We consider a few more examples below.

Examples. Consider the system $\left\{\begin{array}{rcl} x_1' &=& -x_1 \\ x_2' &=& x_2, \end{array}\right.$ which has the origin as a saddle singular point. For any trajectory x starting on the x_1-axis (but not at the origin), we have $L_\omega(x) = (0,0)$, while $L_\alpha(x) = \emptyset$. On the other hand, for any trajectory x starting on the x_2-axis (but not at the origin), $L_\omega(x) = \emptyset$, and $L_\alpha(x) = (0,0)$. Finally, for any trajectory x that does not start on the x_1- or the x_2-axis, the sets $L_\omega(x) = L_\alpha(x) = \emptyset$.

Now consider the system $\left\{\begin{array}{rcl} x_1' &=& x_2 \\ x_2' &=& x_1, \end{array}\right.$ for which all trajectories are periodic and are circles in the phase portrait. For any trajectory starting from a point P, the sets $L_\omega(x)$, $L_\alpha(x)$ are both equal to the circle passing through P. $\Diamond$

We are now ready to define a limit cycle.

Definitions. A *periodic trajectory* is a nonconstant solution such that there exists a $T > 0$ such that $x(t) = x(t+T)$ for all $t \in \mathbb{R}$.

A *limit cycle* is a periodic trajectory that is contained in $L_\omega(x)$ or $L_\alpha(x)$ for some other trajectory x.

Limit cycles represent an important phenomenon in nonlinear systems. They can be found often in engineering and nature. For example, aircraft wing fluttering is an instance of a limit cycle frequently encountered which is sometimes dangerous. In an ecological system where two species share a common resource, the existence of a limit cycle would mean that none of the species becomes extinct. As one can see, limit cycles can be desirable in some cases, and undesirable in other situations. In any case, whether or not limit cycles exist is an important question, and we now study a few results concerning this. In particular, we will study an important result, called the Poincaré-Bendixson theorem, for which we will need a few topological preliminaries, which we list below.

Definitions. Two nonempty sets A and B in the plane $\mathbb{R}^2$ are said to be *separated* if there is no sequence of points $(p_n)_{n\in\mathbb{N}}$ contained in A such that $\lim_{n\to\infty} p_n \in B$, and there is no sequence $(q_n)_{n\in\mathbb{N}}$ contained in B such that $\lim_{n\to\infty} q_n \in A$.

A set that is not the union of two separated sets is said to be *connected*.

A set O is *open* if for every $x \in O$, there exists a $\epsilon > 0$ such that $B(x, \epsilon) \subset O$.

A set Ω is called a *region* if it is an open, connected set.

A set A is said to be bounded if there exists a $R > 0$ large enough so that $A \subset B(0, R)$.

For example, two disjoint circles in the plane are separated, while the quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ is not separated from the x-axis.

Although the definition of a connected set seems technical, it turns out that sets that we would intuitively think of as connected in a nontechnical sense are connected. For instance the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$ is connected, while the set $\mathbb{Z}^2$ is not.

Roughly speaking, an open set can be thought of as a set without its 'boundary'. For example, the unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is open.

The principal result is the following.

Theorem 2.5.2 (Poincaré-Bendixson) *Let* (x_1, x_2) *be a solution to* $\left\{ \begin{array}{rcl} x_1' & = & f_1(x_1, x_2), \\ x_2' & = & f_2(x_1, x_2), \end{array} \right.$ *such that for all* $t \geq t_0$*, the solution lies in a bounded region of the plane containing no singular points. Then either the solution is a periodic trajectory or its omega limit set is a periodic trajectory.*

The proof requires advanced mathematical techniques to prove, and the proof will be omitted. The Poincaré-Bendixson theorem is false for systems of dimension 3 or more. In the case of 2D systems, the proof depends heavily on a deep mathematical result, known as the Jordan curve theorem, which is valid only in the plane. Although the theorem sounds obvious, its proof is difficult. We state this theorem below, but first we should specify what we mean by a curve.

Definitions. A *curve* is a continuous function $f : [a, b] \to \mathbb{R}^2$. If for every $t_1, t_2 \in [a, b]$ such that $t_1 \neq t_2$, there holds that $f(t_1) \neq f(t_2)$, then the curve is called *simple.* A curve is called *closed* if $f(a) = f(b)$.

Theorem 2.5.3 (Jordan curve theorem) *A simple closed curve divides the plane into two regions, one of which is bounded, and the other is unbounded.*

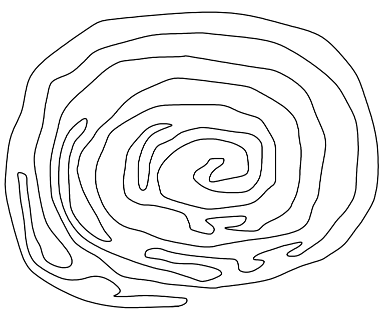

Figure 2.17: Jordan curve theorem.

Now that the inside of a curve is defined, the following result helps to clarify the type of region to seek in order to apply the Poincaré-Bendixson theorem.

Theorem 2.5.4 *Every periodic trajectory of* $\left\{ \begin{array}{lcl} x_1' & = & f_1(x_1,x_2) \\ x_2' & = & f_2(x_1,x_2) \end{array} \right.$ *contains a singular point in its interior.*

This theorem tells us that in order to apply the Poincaré-Bendixson theorem, the singular-point-free-region where the trajectory lies must have at least one hole in it (for the singular point).

We consider a typical application of the Poincaré-Bendixson theorem.

Example.($*$) Consider the system

$$\begin{array}{lcl} x_1' & = & x_1 + x_2 - x_1(x_1^2+x_2^2)[\cos(x_1^2+x_2^2)]^2 \\ x_2' & = & -x_1 + x_2 - x_2(x_1^2+x_2^2)[\cos(x_1^2+x_2^2)]^2. \end{array}$$

In polar coordinates, the equations are transformed into

$$\begin{array}{lcl} r' & = & r[1-r^2(\cos r^2)^2] \\ \theta' & = & -1. \end{array}$$

Consider a circle of radius $r_0 < 1$ about the origin. If we start on it, then all trajectories move outward, since

$$r'(t_0) = r_0[1-r_0^2(\cos r_0^2)^2] > r_0[1-r_0^2] > 0.$$

Also if $r(t_0) = \sqrt{\pi}$, then

$$r'(t_0) = \sqrt{\pi}[1-\pi] < 0,$$

and so trajectories starting on the circle with radius $\sqrt{\pi}$ (or close to it) move inwards. Then it can be shown that any trajectory starting inside the annulus $r_0 < r < \sqrt{\pi}$ stays there[3]. But it is clear that there are no singular points inside the annulus $r_1 < r < \sqrt{\pi}$. So this region must contain a periodic trajectory. Moreover, it is also possible to prove that a trajectory starting inside the unit circle is not a periodic trajectory (since it never returns to this circle), and hence its omega limit set is a limit cycle. $\Diamond$.

It is also important to know when there are no periodic trajectories. The following theorem provides a sufficient condition for the non-existence of periodic trajectories. In order to do that, we will need the following definition.

Definition. A region Ω is said to be *simply connected* if for any simple closed curve C lying entirely within Ω, all points inside C are points of Ω.

For example, the annulus $1 < r < 2$ is not simply connected, while the unit disk $r < 1$ is.

Theorem 2.5.5 *Let* Ω *be a simply connected region in* $\mathbb{R}^2$. *If the function* $\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ *is not identically zero in* Ω *and does not change sign in* Ω, *then there are no periodic trajectories of*

$$\begin{array}{lcl} x_1' & = & f_1(x_1,x_2) \\ x_2' & = & f_2(x_1,x_2) \end{array}$$

in the region Ω.

[3]Indeed, for example if the trajectory leaves the annulus, and reaches a point r_* inside the inner circle at time t_*, then go "backwards" along this trajectory and find the first point t_1 such that $r(t_1) = r_0$. We then arrive at the contradiction that $r_* - r_0 = r(t_*) - r(t_1) = \int_{t_1}^{t_*} r'(t)dt > 0$. The case of the trajectory leaving the outer circle of the annulus can be handled similarly.

Proof *(Sketch.)* Assume that a periodic trajectory exists with period T, and denote the curve by C. Using Green's theorem, we have

$$\iint \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = \int_C f_2 dx_1 - f_1 dx_2.$$

But

$$\int_C f_2 dx_1 - f_1 dx_2 = \int_0^T (f_2(x_1(t), x_2(t)) x_1'(t) - f_1(x_1(t), x_2(t)) x_2') dt = 0,$$

a contradiction. ■

Example. Consider the system

$$\begin{aligned} x_1' &= x_2 + x_1 x_2^2, \\ x_2' &= -x_1 + x_1^2 x_2. \end{aligned}$$

Since

$$\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = x_1^2 + x_2^2,$$

which is always strictly positive (except at the origin), the system does not have any periodic trajectories in the phase plane. ♢.

Exercises.

1. Show that the sets $A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ are separated.
2. Show that the system

$$\begin{aligned} x_1' &= 1 - x_1 x_2 \\ x_2' &= x_1 \end{aligned}$$

   has no limit cycles.

   Hint: Are there any singular points?
3. Show that the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - (1 + x_1^2) x_2 \end{aligned}$$

   has no periodic trajectories in the phase plane.
4. Prove that if the system

$$\begin{aligned} x_1' &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\ x_2' &= x_1 + x_2(1 - x_1^2 - x_2^2) + \frac{3}{4}, \end{aligned}$$

   has a periodic trajectory starting inside the circle C $x_1^2 + x_2^2 = \frac{1}{2}$, then it will either intersect C.

   Hint: Consider the simply connected region $x_1^2 + x_2^2 < \frac{1}{2}$.

Chapter 3

Stability theory

Given a differential equation, an important question is that of the stability. In everyday language, we say that "something is unstable" if a small deviation from the present state produces a major change in the state. A familiar example is that of a pendulum balanced vertically upwards. A small change in the position produces a major change–the pendulum falls. However, if on the other hand, the pendulum is at its lowest position, for small changes in the position, the resulting motion keeps the pendulum around the down position. In this chapter we will make these, and related ideas, precise in the context of solutions of systems of differential equations. Roughly speaking, a system is described as stable if by starting somewhere near its 'equilibrium' point implies that it will stay around that point ever after. In the case of the motions of a pendulum, the equilibrium points are the vertical up and down positions, and these are examples of unstable and stable points, respectively.

Having defined the notions of stable and unstable equilibrium points, we will begin with some elementary stability considerations, namely the stability of linear systems $x' = Ax$, which can be characterized in terms of signs of the real parts the eigenvalues of the matrix A.

In the case of general nonlinear systems, such a neat characterization is not possible. But a useful approach for studying stability was introduced in the late 19th century by the Russian mathematician Alexander Lyapunov, in which stability is determined by constructing a scalar "energy-like" function for the differential equation, and examining its time variation. In this chapter we will also study the basics of Lyapunov theory.

3.1 Equilibrium point

It is possible for a solution to correspond to only a single point in the phase portrait. Such a point is called an equilibrium point. Stability will be formulated with respect to an equilibrium point, that is, it is a particular equilibrium point of a differential equation for which the adjective *stable*, *unstable* etc. applies.

Definition. A point $x_e \in \mathbb{R}^n$ is called an *equilibrium point* of the differential equation

$$x'(t) = f(x(t)),$$

if $f(x_e) = 0$.

Recall that in the context of phase plane analysis of 2D autonomous systems, we had referred to equilibrium points also as 'singular points'.

Note that $x(t) = x_e$ for all $t \geq 0$ is a solution to the equation

$$x'(t) = f(x(t)), \quad x(0) = x_e,$$

that is, if we start from x_e, then we stay there for all future time, and hence the name 'equilibrium' (which usually means 'state of balance' in ordinary language).

Example. *(Linear system)* If $A \in \mathbb{R}^{n\times n}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f(x) = Ax$, $x \in \mathbb{R}^n$, then there are two possible cases:

$\underline{1}^\circ$ A is invertible. The only equilibrium point in this case is the zero vector, that is, $x_e = 0$.

$\underline{2}^\circ$ A is not invertible. In this case, there are infinitely many equilibrium points. Indeed, the set of equilibrium points is precisely the set of points in the kernel of A. $\Diamond$

A nonlinear system can have finite or infinitely many isolated equilibrium points. In the following example we consider the familiar physical system of the pendulum.

Example. *(The pendulum)* Consider the pendulum shown in Figure 3.1, whose dynamics is given by the following nonlinear equation:

$$ml^2\theta'' + k\theta' + mgl \sin\theta = 0,$$

where k is a friction coefficient, m is the mass of the bob, l is the length of the pendulum, and g

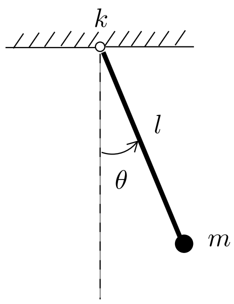

Figure 3.1: The pendulum.

is the acceleration due to gravity. Defining $x_1 = \theta$ and $x_2 = \theta'$, we obtain the state equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{ml^2}x_2 - \frac{g}{l}\sin x_1. \end{aligned}$$

Therefore the equilibrium points are given by

$$x_2 = 0, \quad \sin x_1 = 0,$$

which leads to the points $(n\pi, 0)$, $n \in \mathbb{Z}$. Physically, these points correspond to the pendulum resting exactly at the vertically up (n odd) and down (n even) positions. $\Diamond$

Exercise. $(*)$ Determine the number of equilibrium points of the system $x' = x - \cos x$.

3.2 Stability and instability

Let us introduce the basic concepts of stability and instability.

Definitions. An equilibrium point x_e is said to be *stable* if for any $R > 0$, there exists a $r > 0$ such that if $\|x(0) - x_e\| \leq r$, then for all $t \geq 0$, $\|x(t) - x_e\| \leq R$.

Otherwise, the equilibrium point x_0 is called *unstable*.

Essentially stability means that the trajectories can be kept arbitrarily close to the equilibrium point by starting sufficiently close to it. It does not mean that if we start close to the equilibrium point, then the solution approaches the equilibrium point, and the Figure 3.2 emphasizes this point. This other stronger version of stability when trajectories do tend to the equilibrium point is called *asymptotic stability*, which we will define in the next section.

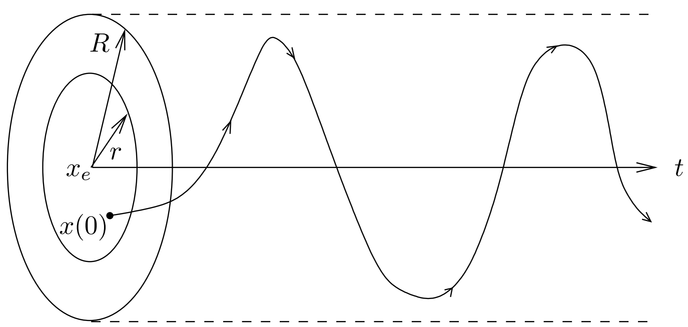

Figure 3.2: Stable equilibrium.

The definition of a stable equilibrium point says that no matter what small ball $B(x_e, R)$ is specified around the point x_e, we can always find a somewhat smaller ball with radius r (which might depend on the radius R of the big ball), such that if we start from within the small ball $B(x_e, r)$ we are guaranteed to stay in the big ball for all future times.

On the other hand, an equilibrium point is unstable if there exists *at least one* ball $B(x_e, R)$ such that for *every* $r > 0$, no matter how small, it is always possible to start from somewhere within the small ball $B(x_e, r)$ and eventually leave the ball $B(x_e, R)$.

It is important to point out the qualitative difference between instability and the intuitive notion of "blowing up". In the latter, one expects the trajectories close to the equilibrium to move further and further away. In linear systems, instability is indeed equivalent to blowing up, because eigenvalues in the right half plane always lead to growth of the system states in some direction. However, for nonlinear systems, blowing up is only one way of instability. The following example illustrates this point.

Example. *(van der Pol oscillator)* This example shows that

$$\boxed{\text{Unstable is not the same as "blowing up".}}$$

The van der Pol oscillator is described by

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 + (1 - x_1^2)x_2. \end{aligned}$$

The system has an equilibrium point at the origin.

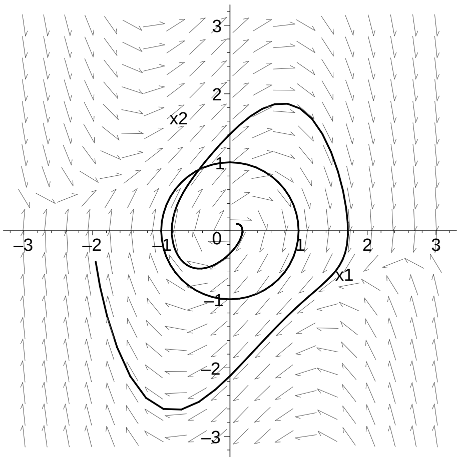

Figure 3.3: Unstable equilibrium of the van der Pol equation.

The solution trajectories starting from any non-zero initial states all asymptotically approach a limit cycle. Furthermore, the ball $B(0, 1)$ can be shown to be within the phase-plane region enclosed by the limit cycle. Therefore, solution trajectories starting from an arbitrarily small ball $B(0, r)$ will eventually get out of the ball $B(0, 1)$ to approach the limit cycle; see Figure 3.3. This implies that the origin is an unstable equilibrium point.

Thus, even though the solutions starting from points near the equilibrium point do not blow up (in fact they do remain close to the equilibrium point), they do not remain *arbitrarily* close to it. This is the fundamental distinction between stability and instability. $\Diamond$

Exercise. For what values of a is 0 a stable equilibrium point for the system $x' = ax$?

3.3 Asymptotic and exponential stability

Sometimes it can happen that trajectories actually approach the equilibrium point. This motivates the following definition.

Definition. An equilibrium point x_e is called *asymptotically stable* if it is stable and there exists a $r > 0$ such that if $\|x(0) - x_e\| \leq r$, then $\lim\limits_{t\to\infty} x(t) = 0$.

Asymptotic stability means that the equilibrium is stable, and that in addition, if we start close to x_e, the solution actually converges to x_e as the time tends to ∞; see Figure 3.4.

One may question the need for the explicit stability requirement in the definition above. One might think: Doesn't convergence to the equilibrium already imply that the equilibrium is stable? The answer is no. It is possible to build examples showing that solution convergence to an equilibrium point does not imply stability. For example, a simple system studied by Vinograd has trajectories of the form shown in Figure 3.5. All the trajectories from nonzero initial points within the unit disk first reach the curve C before converging to the origin. Thus the origin is unstable, despite the convergence. Calling such an equilibrium point unstable is quite reasonable,

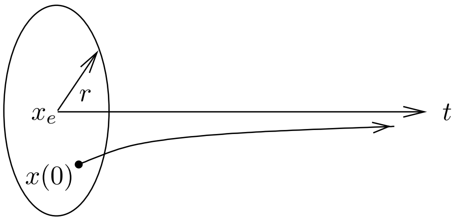

Figure 3.4: Asymptotically stable equilibrium.

since a curve such as C might be *outside* the region where the model is valid.

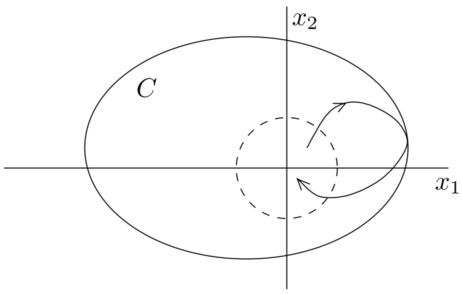

Figure 3.5: Asymptotically stable, but not stable, equilibrium.

In some applications, it is not enough to know that solutions converge to the equilibrium point, but there is also a need to estimate *how fast* this happens. This motivates the following concept.

Definition. An equilibrium point x_e is called *exponentially stable* if there exist positive numbers M, ϵ and r such that for all $\|x(0) - x_e\| \leq r$, we have

$$\text{for all } t \geq 0, \quad \|x(t) - x_e\| \leq M e^{-\epsilon t}. \tag{3.1}$$

In other words, the solutions converge to x_e faster than the exponential function, and (3.1) provides an explicit bound on the solution.

Example. The point 0 is an exponentially stable equilibrium point for the system

$$x' = -(1 + (\sin x)^2)x.$$

Indeed it can be seen that the solution satisfies

$$x(t) = x(0) \exp\left(-\int_0^t \left[1 + (\sin(x(\tau)))^2\right] d\tau\right)$$

(check this by differentiation!). Therefore, $|x(t)| \leq |x(0)|e^{-t}$. ◊

Note that exponential stability implies asymptotic stability. However, asymptotic stability does not guarantee exponential stability, as demonstrated by the following example.

Example. Consider the equation $x' = -x^3$. It can be shown that this has the solution

$$x(t) = \begin{cases} \frac{1}{\sqrt{2t+\frac{1}{(x(0))^2}}} & \text{if } x(0) \geq 0, \\ -\frac{1}{\sqrt{2t+\frac{1}{(x(0))^2}}} & \text{if } x(0) < 0, \end{cases}$$

for $t \geq 0$. It can be seen that 0 is an asymptotically stable equilibrium point. But we now prove that it is not exponentially stable. Suppose, on the contrary, that there exist positive M, ϵ and r such that for all $|x(0)| \leq r$, the corresponding solution x is such that for all $t \geq 0$, $|x(t)| \leq Me^{-\epsilon t}$. Then with $x(0) = r$, we would have that

$$\forall t \geq 0, \quad \frac{1}{\sqrt{2t+\frac{1}{r^2}}} \leq Me^{-\epsilon t}. \tag{3.2}$$

Since for $t \geq 0$ we have $e^{\epsilon t} = 1 + \epsilon t + (\epsilon t)^2/2 + \cdots \geq \epsilon t$, it follows that $e^{-\epsilon t} \leq 1/\epsilon t$. From (3.2), it follows that

$$\forall t > 0, \quad 1 \leq \frac{M^2}{\epsilon^2}\left(\frac{2}{t} + \frac{1}{r^2 t^2}\right).$$

Passing the limit as $t \to \infty$, we obtain $1 \leq 0$, a contradiction. ◊

Exercise. For what values of a is 0 an asymptotically stable equilibrium point for the system $x' = ax$? When is it exponentially stable?

3.4 Stability of linear systems

In the scalar example $x' = ax$ discussed in Section 1.4, we observe that if $\mathrm{Re}(a) < 0$, then the equilibrium point 0 is exponentially stable, while if $\mathrm{Re}(a) > 0$, then the equilibrium point is unstable. Analogously, from Theorem 1.7.2, we obtain the following corollary.

Corollary 3.4.1 *The equilibrium point 0 of the system $x' = Ax$ is exponentially stable if for all $\lambda \in \sigma(A)$, $\mathrm{Re}(\lambda) < 0$. The equilibrium point 0 of the system $x' = Ax$ is unstable if there exists a $\lambda \in \sigma(A)$ such that $\mathrm{Re}(\lambda) > 0$.*

Using Theorem 2.5.1, we can sometimes deduce the stability of an equilibrium point of 2D nonlinear system by linearisation.

As illustrated in the Examples on page 20, in the case when the real part of the eigenvalues is zero, then the stability depends on the number of independent eigenvectors associated with these eigenvalues.

Definition. Let $A \in \mathbb{R}^{n \times n}$, and let $\lambda \in \sigma(A)$. The $\dim(\ker(\lambda I - A))$ is called the *geometric multiplicity* of the eigenvalue λ. The multiplicity of λ as a root of the characteristic polynomial of A is called the *algebraic multiplicity* of the eigenvalue λ.

Example. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then the geometric multiplicity of the eigenvalue 0 is equal to 1, while its algebraic multiplicity is 2.

If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then the geometric multiplicity and the algebraic multiplicity of the eigenvalue 0 are both equal to 2. ◊

We state the following criterion for stability of 0 for the system $x' = Ax$ without proof.

Theorem 3.4.2 *Suppose that all the eigenvalues of A have nonnegative real parts. The equilibrium point 0 of the system $x' = Ax$ is stable iff all eigenvalues with zero real part have their algebraic multiplicity equal to the geometric multiplicity.*

Exercises.

1. Determine the stability of the system $x' = Ax$, where A is

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

   (e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   (f) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

2. For the following systems, find the equilibrium points, and determine their stability. Indicate whether the stability is asymptotic.

   (a) $\left\{ \begin{array}{rcl} x_1' & = & x_1 + x_2 \\ x_2' & = & -x_2 + x_1^3. \end{array} \right.$

   (b) $\left\{ \begin{array}{rcl} x_1' & = & -x_1 + x_2 \\ x_2' & = & -x_2 + x_1^3. \end{array} \right.$

3.5 Lyapunov's method

The basic idea behind Lyapunov's method is the mathematical extension of a fundamental physical observation:

If the energy of a system is continuously dissipated,
then the system eventually settles down to an equilibrium point.

Thus, we may conclude the stability of a system by examining the variation of its energy, which is a *scalar* function.

In order to illustrate this, consider for example a nonlinear spring-mass system with a damper, whose dynamic equation is

$$mx'' + bx'|x'| + k_0 x + k_1 x^3 = 0,$$

where

x denotes the displacement,

$bx'|x'|$ is the term corresponding to nonlinear damping,

$k_0x + k_1x^3$ is the nonlinear spring force.

Assume that the mass is pulled away by some distance and then released. Will the resulting motion be stable? It is very difficult to answer this question using the definitions of stability, since the general solution of the nonlinear equation is unavailable. However, examination of the energy of the system can tell us something about the motion pattern. Let E denote the mechanical energy, given by

$$E = \frac{1}{2}m(x')^2 + \int_0^x (k_0x + k_1x^3)dx = \frac{1}{2}m(x')^2 + \frac{1}{2}k_0x^2 + \frac{1}{4}k_1x^4.$$

Note that zero energy corresponds to the equilibrium point ($x = 0$ and $x' = 0$):

$$E = 0 \text{ iff } [x = 0 \text{ and } x' = 0].$$

Asymptotic stability implies the convergence of E to 0. Thus we see that the magnitude of a scalar quantity, namely the mechanical energy, indirectly says something about the magnitude of the state vector. In fact we have

$$E' = mx'x'' + (k_0x + k_1x^3)x' = x'(-bx'|x'|) = -b|x'|^3,$$

which shows that the energy of the system is continuously decreasing from its initial value. If it has a nonzero limit, then we note that x' must tend to 0, and then from physical considerations, it follows that x must also tend to zero, for the mass is subjected to a nonzero spring force at any position other than the natural length.

Lyapunov's method is a generalization of the method used in the above spring-mass system to more complex systems. Faced with a set of nonlinear differential equations, the basic procedure of Lyapunov's method is to generate a scalar 'energy-like' function for the system, and examine the time-variation of that scalar function. In this way, conclusions may be drawn about the stability of the set of differential equations without using the difficult stability definitions or requiring explicit knowledge of solutions.

3.6 Lyapunov functions

We begin this section with this remark: In the following analysis, for notational simplicity, we assume that the system has been transformed so that equilibrium point under consideration is the origin. We can do this as follows. Suppose that x_e is the specific equilibrium point of the system $x' = f(x)$ under consideration. Then we introduce the new variable $y = x - x_e$, and we obtain the new set of equations in y given by

$$y' = f(y + x_e).$$

Note that this new system has 0 as an equilibrium point. As there is a one-to-one correspondence between the solutions of the two systems, we develop the following stability analysis assuming that the equilibrium point of interest is 0.

The energy function has two properties. The first is a property of the function itself: it is strictly positive unless the state variables are zero. The second property is associated with the dynamics of the system: the function is decreasing when we substitute the state variables into the function, and view the overall function as a function of time. We give the precise definition below.

Definition. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *Lyapunov function* for the differential equation

$$x' = f(x) \tag{3.3}$$

if there exists a $R > 0$ such that in the ball $B(0, R)$, V has the following properties:

L1. *(Local positive definiteness)* For all $x \in B(0, R)$, $V(x) \geq 0$.
If $x \in B(0, R)$ and $V(x) = 0$, then $x = 0$.
$V(0) = 0$.

L2. V is continuously differentiable, and for all $x \in B(0, R)$, $\nabla V(x) \cdot f(x) \leq 0$.

If in addition to L1 and L2, a function satisfies

L3. $\nabla V(x) \cdot f(x) = 0$ iff $x = 0$. Here $\nabla V := \begin{bmatrix} \frac{\partial V}{\partial x_1} \dots \frac{\partial V}{\partial x_n} \end{bmatrix}$.

then V is called a *strong Lyapunov function.*

For example, the function V given by

$$V(x_1, x_2) = \frac{1}{2} m l^2 x_2^2 + mgl(1 - \cos x_1), \tag{3.4}$$

which is the mechanical energy of the pendulum (see the Example on page 46), is *locally positive definite*, that is, it satisfies L1 above.

If the R in L1 can be chosen to be arbitrarily large, then the function is said to be *globally positive definite.* For instance, the mechanical energy of the spring-mass system considered in the previous section is globally positive definite. Note that for this system, the kinetic energy

$$\frac{1}{2} m x_2^2 = \frac{1}{2} m (x')^2$$

is not positive definite by itself, because it can be equal to zero for nonzero values of the position $x_1 = x$.

Let us describe the geometrical meaning of locally positive definite functions on $\mathbb{R}^2$. If we plot $V(x_1, x_2)$ versus x_1, x_2 in 3-dimensional space, it typically corresponds to a surface looking like an upward cup, and the lowest point of the cup is located at the origin. See Figure 3.6.

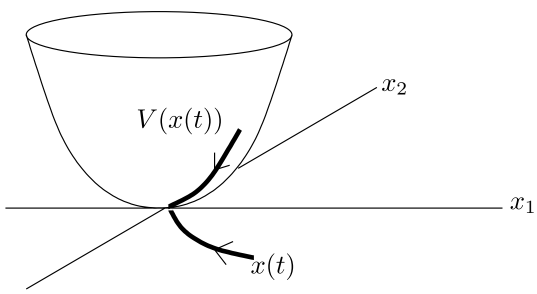

Figure 3.6: Lyapunov function.

We now observe that L2 implies that the 'energy' decreases with time. Define $v : [0, \infty) \to \mathbb{R}$ by $v(t) = V(x(t))$, $t \geq 0$, where x is a solution to (3.3). Now we compute the derivative with respect to time of v (that is, of the map $t \mapsto V(x(t))$), where x is a solution to $x' = f(x)$. By the chain rule we have

$$v'(t) = \frac{d}{dt}(V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \nabla V(x(t)) \cdot f(x(t)).$$

Since L2 holds, we know that if $x(t)$ lies within $B(0, R)$, then for all $t \geq 0$, $v'(t) \leq 0$. In Figure 3.6, we see that corresponding to a point $x(t)$ in the phase plane, we have the value $v(t) = V(x(t))$ on the inverted cup. As time progresses, the point moves along the surface of the inverted cup. But as $v(t)$ decreases with time (in $B(0, R)$), the point on the surface is forced to move to the origin if we start within $B(0, R)$. In this manner we can prove stability and asymptotic stability. We do this in the next section.

Exercise. Verify that (3.4) is a Lyapunov function for the system in the Example on page 46.

3.7 Sufficient condition for stability

Theorem 3.7.1 *Let the system $x' = f(x)$ have an equilibrium point at* 0.

1. *If there exists a Lyapunov function V for this system, then the equilibrium point* 0 *is stable.*
2. *If there exists a strong Lyapunov function, then* 0 *is asymptotically stable.*

Proof *(Sketch)* 1. To show stability we must show that given any $R > 0$, there exists a $r > 0$ such that any trajectory starting inside $B(0, r)$ remains inside the ball $B(0, R)$ for all future time.

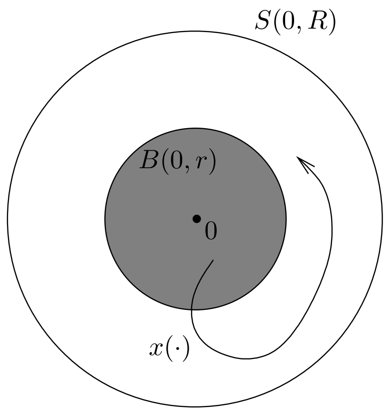

Figure 3.7: Proof of stability.

Let m be the minimum of V on the sphere $S(0, R) = \{x \in \mathbb{R}^n \mid \|x\| = R\}$. Since V is continuous (and $S(0, R)$ is compact), the minimum exists, and as V is positive definite, m is positive. Furthermore, since $V(0) = 0$, there exists a $r > 0$ such that for all $x \in B(0, r)$, $V(x) < m$; see Figure 3.7.

Consider now a trajectory whose initial point $x(0)$ is within the ball $B(0, r)$. Since $t \mapsto V(x(t))$ is nonincreasing, $V(x(t))$ remains strictly smaller than m, and therefore, the trajectory cannot possibly cross the outside sphere $S(0, R)$. Thus, any trajectory starting inside the ball $B(0, r)$ remains inside the ball $B(0, R)$, and therefore stability of 0 is guaranteed.

2. Let us now assume that $x \mapsto \nabla V(x) \cdot f(x)$ is negative definite, and show asymptotic stability by contradiction. Consider a trajectory starting in some ball $B(0,r)$ as constructed above corresponding to some R where the negative definiteness holds. Then the trajectory will remain in the ball $B(0,R)$ for all future time. Since V is bounded below and $t \mapsto V(x(t))$ decreases continually, $V(x(t))$ tends to a limit L, such that for all $t \geq 0$, $V(x(t)) \geq L$. Assume that this limit is not 0, that is, $L > 0$. Then since V is continuous and $V(0) = 0$, there exists a ball $B(0, r_0)$ that the solution never enters; see Figure 3.8.

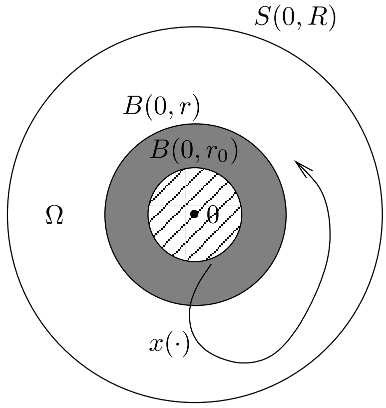

Figure 3.8: Proof of asymptotic stability.

But then, since $x \mapsto -\nabla V(x) \cdot f(x)$ is continuous and positive definite, and since the set $\Omega = \{x \in \mathbb{R}^n \mid r_0 \leq \|x\| \leq R\}$ is compact, $x \mapsto -\nabla V(x) \cdot f(x) : \Omega \to \mathbb{R}$ has some minimum value $L_1 > 0$.

This is a contradiction, because it would imply that $v(t) := V(x(t))$ decreases from its initial value $v(0) = V(x(0))$ to a value strictly smaller than L, in a finite time $T_0 > \frac{v(0)-L}{L_1} \geq 0$: indeed by the fundamental theorem of calculus, we have

$$v(t) - v(0) = \int_0^{T_0} v'(t)dt = \int_0^{T_0} \nabla V(x(t)) \cdot f(x(t))dt \leq \int_0^{T_0} -L_1 dt = -L_1 T_0 < L - v(0)$$

and so $v(t) < L$. Hence all trajectories starting in $B(0,r)$ asymptotically converge to the origin. ■

Example. A simple pendulum with viscous damping is described by

$$\theta'' + \theta' + \sin\theta = 0.$$

Defining $x_1 = \theta$ and $x_2 = \theta'$, we obtain the state equations

$$\begin{aligned} x_1' &= x_2 && (3.5) \\ x_2' &= -x_2 - \sin x_1. && (3.6) \end{aligned}$$

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x_1, x_2) = (1 - \cos x_1) + \frac{1}{2} x_2^2.$$

Thsi function is locally positive definite (Why?). As a matter of fact, this function represents the total energy of the pendulum, composed of the sum of the potential energy and the kinetic energy.

We observe that

$$\nabla V(x) \cdot f(x) = \begin{bmatrix} \sin x_1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ -x_2 - \sin x_1 \end{bmatrix} = -x_2^2 \leq 0.$$

(This is expected since the damping term absorbs energy.) Thus by invoking the above theorem, one can conclude that the origin is a stable equilibrium point. However, with this Lyapunov function, one cannot draw any conclusion about whether the equilibrium point is asymptotically stable, since V is not a strong Lyapunov function. But by considering yet another Lyapunov function, one can show that 0 is an asymptotically stable equilibrium point, and we do this in Exercise 1 below. ◊

Exercises.

1. Prove that the function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}(x_1 + x_2)^2 + 2(1 - \cos x_1)$$

is also a Lyapunov function for the system (3.5)-(3.6) (although this has no obvious physical meaning). Prove that this function is in fact a strong Lyapunov function. Conclude that the origin is an asymptotically stable equilibrium point for the system.

2. Consider the system

$$x_1' = x_2 - x_1(x_1^2 + x_2^2) \tag{3.7}$$
$$x_2' = -x_1 - x_2(x_1^2 + x_2^2). \tag{3.8}$$

Let $V : \mathbb{R}^2 \to \mathbb{R}$ given by $V(x_1, x_2) = x_1^2 + x_2^2$.

(a) Verify that V is a Lyapunov function. Is it a strong Lyapunov function?

(b) Prove that the origin is an asymptotically stable equilibrium point for the system (3.7)-(3.8).

Chapter 4

Existence and uniqueness

4.1 Introduction

A major theoretical question in the study of ordinary differential equations is: When do solutions exist? In this chapter, we study this question, and also the question of uniqueness of solutions.

To begin with, note that in Chapter 1, we have shown the existence and uniqueness of the solution to

$$x' = Ax, \quad x(t_0) = x_0.$$

Indeed, the unique solution is given by $x(t) = e^{(t-t_0)A}x_0$.

It is too much to expect that one can show existence by actually giving a formula for the solution in the general case:

$$x' = f(x,t), \quad x(0) = x_0.$$

Instead one can prove a theorem that asserts the existence of a unique solution if the function f is not 'too bad'. We will prove the following "baby version" of such a result.

Theorem 4.1.1 *Consider the differential equation*

$$x' = f(x,t), \quad x(t_0) = C, \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is such that there exists a constant L such that for all $x_1, x_2 \in \mathbb{R}$, and all $t \geq t_0$,

$$|f(x_1,t) - f(x_2,t)| \leq L|x_1 - x_2|.$$

Then there exists a $t_1 > t_0$ and a $x \in C^1[t_0,t_1]$ such that $x(t_0) = x_0$ and $x'(t) = f(x(t),t)$ for all $t \in [t_0,t_1]$.

We will state a more general version of the above theorem later on (which is for *nonscalar* f, that is, for a *system* of equations). The proof of this more general theorem is very similar to the proof of Theorem 4.1.1.

The next few sections of this chapter will be spent in proving Theorem 4.1.1. Before we get down to gory detail, though, we should discuss the method of proof.

Let us start with existence. The simplest way to prove that an equation has a solution is to write down a solution. Unfortunately, this seems impossible in our case. So we try a variation.

We write down functions which, while not solutions, are very good approximations to solutions: they miss solving the differential equations by less and less. Then we try to take the limit of these approximations and show that it is an actual solution.

Since all those words may not be much help, let's try an example. Suppose that we want to solve $x^2 - 2 = 0$. Here's a method: pick a number $x_0 > 0$, and let

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right), \\ x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right), \\ x_3 &= \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right), \end{aligned}$$

and so on. We get a sequence of numbers whose squares get close to 2 rather rapidly. For instance, if $x_0 = 1$, then the sequence goes

$$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{656657}{470832}, \dots,$$

and the squares are

$$1, 2\frac{1}{4}, 2\frac{1}{144}, 2\frac{1}{166464}, 2\frac{1}{221682772224}, \dots.$$

Presumably the x_n's approach a limit, and this limit is $\sqrt{2}$. Proving this has two parts. First, let's see that the numbers approach a limit. Notice that

$$x_n^2 - 2 = \frac{1}{4}\left(x_{n-1} + \frac{2}{x_{n-1}}\right)^2 - 2 = \frac{1}{4}\left(x_{n-1} - \frac{2}{x_{n-1}}\right)^2 \geq 0,$$

and so $x_n^2 \geq 2$ for all $n \geq 1$. Clearly $x_n > 0$ for all n (recall that $x_0 > 0$). But then

$$x_n - x_{n-1} = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right) - x_{n-1} = \frac{1}{2x_{n-1}}(2 - x_{n-1}^2) < 0.$$

Hence $(x_n)_{n\geq 1}$ is decreasing, but is also bounded below (by 0), and therefore it has a limit.

Now we need to show that the limit is $\sqrt{2}$. This is easy. Suppose the limit is L. Then as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),$$

taking[1] limits, we get $L = \frac{1}{2}\left(L + \frac{2}{L}\right)$, and so $L^2 = 2$. So L does indeed equal $\sqrt{2}$, that is, we have constructed a solution to the equation $x^2 - 2 = 0$.

The iterative rule $x_{n+1} = (x_n + 2/x_n)/2$ may seem mysterious, and in part, it is constructed by running the last part of the proof backwards. Suppose $x^2 - 2 = 0$. Then $x = 2/x$, or $2x = x + 2/x$, or $x = (x + 2/x)/2$. Now think of this equation not as an equation, but as a formula for producing a sequence, and we are done.

Analogously, in order to prove Theorem 4.1.1, we will proceed in 3 steps:

1. Take the equation, manipulate it cleverly, and turn it into a rule for producing a sequence of approximate solutions to the equation.
2. Show that the sequence converges.
3. Show that the limit solves the equation.

[1] To be precise, this is justified (using the algebra of limits) provided that $\lim_{n\to\infty} x_n \neq 0$. But since $x_n^2 > 2$, $x_n > 1$ for all n, so that surely $L \geq 1$.

4.2 Analytic preliminaries

In order to prove Theorem 4.1.1, we need to develop a few facts about calculus in the vector space $C[a,b]$. In particular, we need to know something about when two vectors from $C[a,b]$ (which are really two functions!) are "close". This is needed, since only then can we talk about a a *convergent* sequence of *approximate* solutions, and carry out the plan mentioned in the previous section.

It turns out that just as Chapter 1, in order to prove convergence of the sequence of matrices, we used the matrix norm given by (1.16), we introduce the following "norm" in the vector space $C[a,b]$: it is simply the function $\|\cdot\| : C[a,b] \to \mathbb{R}$ defined by

$$\|f\| = \sup_{t\in[a,b]} |f(t)|,$$

for $f \in C[a,b]$. (By the Extreme Value Theorem, the "sup" above can be replaced by "max", since f is continuous on $[a,b]$.) With the help of the above norm, we can discuss distances in $C[a,b]$. We think of $\|f-g\|$ as the distance between functions f and g in $C[a,b]$. So we can also talk about convergence:

Definition. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C[a,b]$. The series $\displaystyle\sum_{k=1}^{\infty} f_k$ is said to be *convergent* if there exists a $f \in C[a,b]$ such that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$, there holds that

$$\left\| \sum_{k=1}^{n} f_k - f \right\| < \epsilon.$$

The series $\displaystyle\sum_{n=1}^{\infty} f_n$ is said to be *absolutely convergent* if the real series $\displaystyle\sum_{k=1}^{\infty} \|f_k\|$ converges, that is, there exists a $S \in \mathbb{R}$ so that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$, there holds that

$$\left| \sum_{k=1}^{n} \|f_k\| - S \right| < \epsilon.$$

Now we prove the following remarkable result, we will will use later in the next section to prove our existence theorem about differential equations.

Theorem 4.2.1 *Absolutely convergent series in $C[a,b]$ converge, that is, if $\displaystyle\sum_{k=1}^{\infty} \|f_k\|$ converges in $\mathbb{R}$, then $\displaystyle\sum_{k=1}^{\infty} f_k$ converges in $C[a,b]$.*

Note the above theorem gives a lot for very little. Just by having convergence of a real series, we get a much richer converge–namely that of a sequence of functions in $\|\cdot\|$–which in particular gives pointwise convergence in $[a,b]$, that is we get convergence of an infinite family of sequences! Indeed this remarkable proof works since it is based on the notion of "uniform" convergence of the functions–convergence in the norm gives a uniform rate of convergence at all points t, which is stronger than simply saying that at each t the sequence of partial sums converge.

Proof Let $t \in [a,b]$. Then $|f_k(t)| \le \|f_k\|$. So by the Comparison Test, it follows that the real series $\displaystyle\sum_{k=1}^{\infty} |f_k(t)|$ converges, and so the series $\displaystyle\sum_{k=1}^{\infty} f_k(t)$ also converges, and let $f(t) = \displaystyle\sum_{k=1}^{\infty} f_k(t)$. So we obtain a function $t \mapsto f(t)$ from $[a,b]$ to $\mathbb{R}$. We will show that f is continuous on $[a,b]$ and that $\sum_{k=1}^{\infty} f_k$ converges to f in $[a,b]$.

The real content of the theorem is to prove that f is a continuous function on $[a,b]$. First let us see what we have to prove. To prove that f is continuous on $[a,b]$, we must show that it is continuous at each point $c \in [a,b]$. To prove that f is continuous at c, we must show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $t \in [a,b]$ satisfies $|t-c| < \delta$, then $|f(t) - f(c)| < \epsilon$. We prove this by finding a continuous function near f. Assume that c and ϵ have been picked. Since $\sum_{k=1}^{\infty} \|f_k\|$ is finite, we can choose an $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \|f_k\| < \frac{\epsilon}{3}$. Let $s_N(t) = \sum_{k=1}^{N} f_k(t)$. Then s_N is continuous, since it is the sum of finitely many continuous functions, and

$$|s_N(t) - f(t)| = \left| \sum_{k=N+1}^{\infty} f_k(t) \right| \leq \sum_{k=N+1}^{\infty} |f_k(t)| \leq \sum_{k=N+1}^{\infty} \|f_k\| < \frac{\epsilon}{3},$$

regardless of t. (This last part is the crux of the proof.) Since s_N is continuous, we can pick $\delta > 0$ so that if $|t-c| < \delta$, then $|s_N(t) - s_N(c)| < \frac{\epsilon}{3}$. This is the delta we want: if $|t-c| < \delta$, then

$$|f(t) - f(c)| \leq |f(t) - s_N(t)| + |s_N(t) - s_N(c)| + |s_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves the continuity of f.

The rest of the proof is straightforward. We have:

$$\left| \sum_{k=1}^{n} f_k(t) - f(t) \right| = \left| \sum_{k=n+1}^{\infty} f_k(t) \right| \leq \sum_{k=n+1}^{\infty} |f_k(t)| \leq \sum_{k=n+1}^{\infty} \|f_k\| \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that $\sum_{k=1}^{\infty} f_k$ converges to f in $C[a,b]$. ■

Theorem 4.2.2 *Suppose $\sum_{k=1}^{\infty} f_k$ converges absolutely to f in $C[a,b]$. Then*

$$\sum_{k=1}^{\infty} \int_a^b f_k(t)dt = \int_a^b f(t)dt.$$

Proof We need to show that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| \sum_{k=1}^{n} \int_a^b f_k(t)dt - \int_a^b f(t)dt \right| < \epsilon.$$

Choose N such that if $n \geq N$, then $\sum_{k=n+1}^{\infty} \|f_k\| < \frac{\epsilon}{b-a}$. So

$$\left| \sum_{k=1}^{n} f_k(t) - f(t) \right| = \left| \sum_{k=n+1}^{\infty} f_k(t) \right| \leq \sum_{k=n+1}^{\infty} |f_k(t)| \leq \sum_{k=n+1}^{\infty} \|f_k\| < \frac{\epsilon}{b-a}.$$

But then

$$\left| \sum_{k=1}^{n} \int_a^b f_k(t)dt - \int_a^b f(t)dt \right| = \left| \int_a^b (\sum_{k=1}^{n} f_k(t) - f(t))dt \right| \leq \int_a^b |\sum_{k=1}^{n} f_k(t) - f(t)|dt \leq \int_a^b \frac{\epsilon}{b-a} dt = \epsilon.$$

■

4.3 Proof of Theorem 4.1.1

4.3.1 Existence

STEP 1. The first step in the proof is to change the equation into one which we can use for creating a sequence. The principle for doing this is a useful and important one. We want to arrange matters so that successive terms are close together. For this, integration is much better than differentiation. Two functions that are close together have their integrals close together[2], but their derivatives can be far apart[3].

So we should change (4.1) into something involving integrals. The easiest way to do this is to integrate both sides. Taking into account the initial condition, we see that we get

$$x(t) - C = \int_{t_0}^{t} f(x(t), t)dt,$$

that is,

$$x(t) = \int_{t_0}^{t} f(x(t), t)dt + C.$$

Now we can construct our sequence.

We begin with

$$x_0(t) = C,$$

and define $x_1, x_2, x_3, \ldots$ inductively:

$$\begin{aligned} x_1(t) &= \int_{t_0}^{t} f(x_0(t), t)dt + C, \\ x_2(t) &= \int_{t_0}^{t} f(x_1(t), t)dt + C, \\ &\vdots \\ x_{k+1}(t) &= \int_{t_0}^{t} f(x_k(t), t)dt + C, \end{aligned}$$

and so on. All the functions $x_0, x_1, \ldots$ are continuous functions. This is the sequence we will work with.

STEP 2. We want to show that the sequence of functions $x_0, x_1, x_2, \ldots$ converges. This sequence is the sequence of partial sums of the series

$$x_0 + (x_1 - x_0) + (x_2 - x_1) + \ldots.$$

So we will prove that this series, namely

$$x_0 + \sum_{k=0}^{\infty} (x_{k+1} - x_k)$$

converges. In order to do this, we show that it converges absolutely, and then by Theorem 4.2.1, we would be done.

[2]At least for a while. As the interval of integration gets large, the functions drift apart.

[3]For instance, let $f(x) = \frac{1}{10^{10}} \sin(10^{100}x)$, a function close to 0. If we integrate, then we get $-\frac{1}{10^{110}} \cos(10^{100}x)$, which is really tiny; if we differentiate, we get $10^{90} \cos(10^{100}x)$, which can get very large.

Thus we need to look at $x_{k+1} - x_k$:

$$x_{k+1}(t) - x_k(t) = \int_{t_0}^{t} [f(x_k(t), t) - f(x_{k-1}(t), t)]dt.$$

Since we know that

$$|f(x_k(t), t) - f(x_{k-1}(t), t)| \le L|x_k(t) - x_{k-1}(t)|,$$

where L is a number not depending on k or t. Then

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &\le \int_{t_0}^{t} |f(x_{k-1}(t), t) - f(x_{k-1}(t), t)|dt \\ &\le \int_{t_0}^{t} L|x_k(t) - x_{k-1}(t)|dt \\ &\le \int_{t_0}^{t} L\|x_k - x_{k-1}\|dt \\ &= L(t - t_0)\|x_k - x_{k-1}\|. \end{aligned}$$

So if we work in the interval $t_0 \le t \le t_0 + \frac{1}{2L}$, we have

$$\|x_{k+1} - x_k\| \le L\frac{1}{2L}\|x_k - x_{k-1}\| = \frac{1}{2}\|x_k - x_{k-1}\|.$$

Then, as one may check using induction,

$$\|x_{k+1} - x_k\| \le \frac{1}{2^k}\|x_1 - x_0\| \tag{4.2}$$

for all k. Thus

$$\|x_0\| + \sum_{k=0}^{\infty} \|x_{k+1} - x_k\| \le \|x_0\| + \sum_{k=0}^{\infty} \frac{1}{2^k}\|x_1 - x_0\| < \infty,$$

and the series $x_0 + \sum_{k=0}^{\infty}(x_{k+1} - x_k)$ converges absolutely. By Theorem 4.2.1, it converges and has a limit, which we denote by x.

Step 3. Now we need to know that x satisfies (4.1). We begin by taking limits in

$$x_{k+1}(t) = \int_{t_0}^{t} f(x_k(t), t)dt + C.$$

By Theorem 4.2.2, taking limits inside the integral can be justified (see the Exercise on page 63 below), and so we get

$$x(t) = \int_{t_0}^{t} f(x(t), t)dt + C. \tag{4.3}$$

We see that $x(t_0) = C$ because the integral from t_0 to t_0 is 0. Also by the Fundamental Theorem of Calculus, x can be differentiated (since it is given as an integral), and

$$x'(t) = f(x(t), t).$$

This proves the existence.

4.3.2 Uniqueness

Finally, we prove uniqueness. Let y_1 and y_2 be two solutions. Then y_1 and y_2 satisfy the "integrated" equation (4.3) as well.

Now we repeat some of STEP 2 of the proof of the existence above. Let $z = y_1 - y_2$. Then using (4.3), we get

$$\begin{aligned}
|z(t)| &= |y_1(t) - y_2(t)| \\
&= \left| \int_{t_0}^{t} (f(y_1(t), t) - f(y_2(t), t)) dt \right| \\
&\leq \int_{t_0}^{t} |f(y_1(t), t) - f(y_2(t), t)| dt \\
&\leq \int_{t_0}^{t} L|y_1(t) - y_2(t)| dt = \int_{t_0}^{t} L|z(t)| dt \\
&\leq \int_{t_0}^{t} L\|z\| dt = L(t - t_0)\|z\|.
\end{aligned}$$

Thus on the interval $t_0 \leq t \leq t_0 + \frac{1}{2L}$, we have

$$\|z\| \leq \frac{1}{2}\|z\|,$$

which gives $\|z\| \leq 0$, and so $z \equiv 0$, that is, $y_1 = y_2$. This completes the proof of the theorem.

Exercise. ($*$) Justify taking limits inside the integral in STEP 3 of the proof.

4.4 The general case. Lipschitz condition.

We begin by introducing the class of Lipschitz functions.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *locally Lipschitz* if for every $r > 0$ there exists a constant L such that for all $x, y \in B(0, r)$,

$$\|f(x) - f(y)\| \leq L\|x - y\|. \tag{4.4}$$

If there exists a constant L such that (4.4) holds for all $x, y \in \mathbb{R}^n$, then f is said to be *globally Lipschitz*.

For $f : \mathbb{R} \to \mathbb{R}$, we observe that the following implications hold:

$$f \text{ is continuously differentiable} \;\Rightarrow\; f \text{ is locally Lipschitz} \;\Rightarrow\; f \text{ is continuous.}$$

That the inclusions of these three classes are strict can be seen by observing that $f(x) = |x|$ is globally Lipschitz, but not differentiable at 0, and the function $f(x) = \sqrt{x}$ is continuous, but not locally Lipschitz. (See Exercise 1c below.)

Just like Theorem 4.1.1, the following theorem can be proved, which gives a sufficient condition for the unique existence of a solution to the initial value problem of an ODE.

Theorem 4.4.1 *If there exists an $r > 0$ and a constant L such that the function f satisfies*

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\| \text{ for all } x, y \in B(0,r) \text{ and all } t \geq t_0, \tag{4.5}$$

then there exists a $t_1 > t_0$ such that the differential equation

$$x'(t) = f(x(t),t), \quad x(t_0) = x_0,$$

has a unique solution for all $t \in [t_0, t_1]$.

If the condition (4.5) holds, then f is said to be *locally Lipschitz in x uniformly with respect to t.*

The existence theorem above is of a local character, in the sense that the existence of a solution x_* is guaranteed only in a small interval $[t_0, t_1]$. We could, of course, take this solution and examine the new initial value problem

$$x'(t) = f(x(t),t), \quad t \geq t_1, \quad x(t_1) = x_*(t_1).$$

The existence theorem then guarantees a solution in a further small neighbourhood, so that the solution can be "extended". The process can then be repeated. However, it might happen that the lengths of the intervals get smaller and smaller, so that we cannot really say that such an extension will yield a solution for all times $t \geq 0$. We illustrate this by considering the following example.

Example. Consider the initial value problem

$$x' = 1 + x^2, \quad t \geq 0, \quad x(0) = 0.$$

Then it can be shown that $f(x) = 1 + x^2$ is locally Lipschitz in x (trivially uniformly in t). So a unique solution exists in a small time interval. In fact, we can explicitly solve the above equation and find out that the solution is given by

$$x(t) = \tan t, \quad t \in [0, \frac{\pi}{2}).$$

The solution cannot be extended to an interval larger than $[0, \frac{\pi}{2})$. ◊

Exercises.

1. (a) Prove that every continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.
   Hint: Mean value theorem.
   (b) Prove that every locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
   (c) ($*$) Show that $f(x) = \sqrt{|x|}$ is not locally Lipschitz.
2. ($*$) Find a Lipschitz constant for the following functions
   (a) $\sin x$ on $\mathbb{R}$.
   (b) $\frac{1}{1+x^2}$ on $\mathbb{R}$.
   (c) $e^{-|x|}$ on $\mathbb{R}$.
   (d) $\arctan(x)$ on $(-\pi, \pi)$.
3. ($*$) Show that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality

   $$|f(x) - f(y)| \leq L|x - y|^2 \text{ for all } x, y \in \mathbb{R},$$

   then show that f is continuously differentiable on $\mathbb{R}$.

4.5 Existence of solutions

Here is an example of a differential equation with more than one solution for a given initial condition.

Example. *An equation with multiple solutions.* Consider the equation

$$x' = 3x^{\frac{2}{3}}$$

with the initial condition $x(0) = 0$. Two of its solutions are $x(t) \equiv 0$ and $x(t) = t^3$. $\Diamond$

In light of this example, one can hope that there may be also theorems applying to more general situations than Theorem 4.4.1, which state that solutions exist (and say nothing about uniqueness). And there are. The basic one is:

Theorem 4.5.1 *Consider the differential equation*

$$x' = f(x, t),$$

with initial condition $x(t_0) = x_0$. If the function f is continuous (but not necessarily Lipschitz in x uniformly in t), then there exists a $t_1 > t_0$ and a $x \in C^1[t_0, t_1]$ such that $x(t_0) = x_0$ and $x'(t) = f(x(t), t)$ for all $t \in [t_0, t_1]$.

The above theorem says that the continuity of f is sufficient for the local *existence* of solutions. However, it does not guarantee the *uniqueness* of the solution. We will not give a proof of this theorem.

4.6 Continuous dependence on initial conditions

In this section, we will consider the following question: If we change the initial condition, then how does the solution change? The initial condition is often a measured quantity (for instance the estimated initial population of a species of fish in a lake at a certain starting time), and we would like our differential equation to be such that the solution varies 'continuously' as the initial condition changes. Otherwise we cannot be sure if the solution we have obtained is close to the real situation at hand (since we might have incurred some measurement error in the initial condition). We prove the following:

Theorem 4.6.1 *Let f be globally Lipschitz in x (with constant L) uniformly in t. Let x_1, x_2 be solutions to the equation $x' = f(x, t)$, for $t \in [t_0, t_1]$, with initial conditions $x_{0,1}$ and $x_{0,2}$, respectively. Then for all $t \in [t_0, t_1]$,*

$$\|x_1(t) - x_2(t)\| \le e^{L(t-t_0)} \|x_{0,1} - x_{0,2}\|.$$

Proof Let $f(t) := \|x_1(t) - x_2(t)\|^2$, $t \in [t_0, t_1]$. If $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, we have

$$\begin{aligned} f'(t) &= 2\langle x_1'(t) - x_2'(t), x_1(t) - x_2(t)\rangle = 2\langle f(x_1, t) - f(x_2, t), x_1(t) - x_2(t)\rangle \\ &\le 2\|f(x_1, t) - f(x_2, t)\| \|x_1(t) - x_2(t)\| \quad \text{(by the Cauchy-Schwarz inequality)} \\ &\le 2L\|x_1(t) - x_2(t)\|^2 = 2Lf(t). \end{aligned}$$

In other words,

$$\frac{d}{dt}(e^{-2Lt}f(t)) = e^{-2Lt}f'(t) - 2Le^{-2Lt}f(t) = e^{-2Lt}(f'(t) - 2Lf(t)) \leq 0.$$

Integrating from t_0 to $t \in [t_0, t_1]$ yields

$$e^{-2Lt}f(t) - e^{-2Lt_0}f(t_0) = \int_{t_0}^{t} \frac{d}{d\tau}e^{-2L\tau}f(\tau)d\tau \leq 0,$$

that is, $f(t) \leq e^{2L(t-t_0)}f(0)$. Taking square roots, we obtain $\|x_1(t)-x_2(t)\| \leq e^{L(t-t_0)}\|x_{0,1}-x_{0,2}\|$. ■

Exercises.

1. Prove the Cauchy-Schwarz inequality: if $x, y \in \mathbb{R}^n$, then $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Cauchy-Schwarz

HINT: If $\alpha \in \mathbb{R}$, and $x, y \in \mathbb{R}^n$, then we have $0 \leq \langle x+\alpha y, x+\alpha y\rangle = \langle x, x\rangle + 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle$, and so it follows that the discriminant of this quadratic expression is ≤ 0, which gives the desired inequality.

2. $(*)$ This exercise could have come earlier, but it's really meant as practice for the next one.

We know that the solution to

$$x'(t) = x(t), \quad x(0) = a,$$

is $x(t) = e^t a$. Now suppose that we knew about differential equations, but not about exponentials. We thus know that the above equation has a unique solution, but we are hampered in our ability to solve them by the fact that we have never come across the function e^t! So we declare E to be the function defined as the unique solution to

$$E'(t) = E(t), \quad E(0) = 1.$$

(a) Let $\tau \in \mathbb{R}$. Show that $t \mapsto E(t+\tau)$ solves the same differential equation as E (but with a different initial condition). Show from this that for all $t \in \mathbb{R}$, $E(t+\tau) = E(t)E(\tau)$.

(b) Show that for all $t \in \mathbb{R}$, $E(t)E(-t) = 1$.

(c) Show that $E(t)$ is never 0.

3. $(**)$ This is similar to the previous exercise, but requires more work. This time, imagine that we know about existence and uniqueness of solutions for second order differential equations of the type

$$x''(t) + x(t) = 0, \quad x(0) = a, \;\; x'(0) = b,$$

but nothing about trigonometric functions. (For example from Theorem 1.5.5, we know that this equations has a unique solutions, and we can see this by introducing the state vector comprising x and x'.) We define the functions S and C as the unique solutions, respectively, to

$$S''(t) + S(t) = 0, \quad S(0) = 0, \; S'(0) = 1;$$
$$C''(t) + C(t) = 0, \quad C(0) = 1, \; C'(0) = 0.$$

(Privately, we know that $S(t) = \sin t$ and $C(t) = \cos t$.) Now show that for all $t \in \mathbb{R}$,

(a) $S'(t) = C(t)$, $C'(t) = -S(t)$.

(b) $(S(t))^2 + (C(t))^2 = 1$.
HINT: What is the derivative?

(c) $S(t+\tau) = S(t)C(\tau) + C(t)S(\tau)$ and $C(t+\tau) = C(t)C(\tau) - S(t)S(\tau)$, all $\tau \in \mathbb{R}$.

(d) $S(-t) = -S(t)$, $C(-t) = C(t)$.

(e) There is a a number $\alpha > 0$ such that $C(\alpha) = 0$. (That is, $C(x)$ is not always positive. If we call the smallest such number $\pi/2$, we have a definition of π from differential equations.)

Chapter 5

Underdetermined ODEs

5.1 Control theory

The basic objects of study in control theory are *underdetermined* differential equations. This means that there is some *freeness* in the variables satisfying the differential equation. An example of an underdetermined *algebraic* equation is $x + u = 10$, where x, u are positive integers. There is freedom in choosing, say u, and once u is chosen, then x is uniquely determined. In the same manner, consider the *differential* equation

$$\frac{dx}{dt}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{5.1}$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$. So if written out, equation (5.1) is the set of equations

$$\begin{aligned} \frac{dx_1}{dt}(t) &= f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \quad x_1(0) = x_{0,1} \\ &\vdots \\ \frac{dx_n}{dt}(t) &= f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \quad x_n(0) = x_{0,n}, \end{aligned}$$

where $f_1, \dots, f_n$ denote the components of f. In (5.1), u is the free variable, called the *input*, which is assumed to be continuous.

A *control system* is an equation of the type (5.1), with input u and state x. Once the input u and the initial state $x(0) = x_0$ are specified, the state x is determined. So one can think of a control system as a box, which given the input u and initial state $x(0) = x_0$, manufactures the state according to the law (5.1); see Figure 5.1.

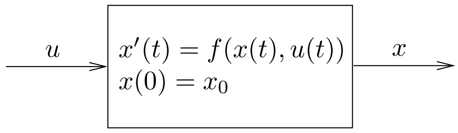

Figure 5.1: A control system.

Example. Suppose the population $x(t)$ at time t of fish in a lake evolves according to the differential equation:

$$x'(t) = f(x(t)),$$

where f is some complicated function which is known to model the situation reasonable accurately. A typical example is the Verhulst model, where

$$f(x) = rx\left(1 - \frac{x}{M}\right).$$

(This model makes sense, since first of all the rate of increase in the population should increase with more numbers of fish–the more there are fish, the more they reproduce, and larger is the population. However, if there are too many fish, there is competition for the limited food resource, and then the population starts declining, which is captured by the term $1 - \frac{x}{M}$.)

Now suppose that we harvest the fish at a harvesting rate h. Then the population evolution is described by

$$x'(t) = f(x(t)) - h(t).$$

But the harvesting rate depends on the harvesting effort u:

$$h(t) = x(t)u(t).$$

(The harvesting effort can be thought in terms of the amount of time used for fishing, or the number of fishing nets used, and so on. Then the above equation makes sense, as the harvesting rate is clearly proportional to the number of fish–the more the fish in the lake, the better the catch.)

Hence we arrive at the underdetermined differential equation

$$x'(t) = f(x(t)) - x(t)u(t).$$

This equation is underdetermined, since the u can be decided by the fisherman. This is the input, and once this has been chosen, then the population evolution is determined by the above equation, given some initial population level x_0 of the fish. $\Diamond$

If the function f is linear, that is, if

$$f(\xi, v) = A\xi + Bv$$

for some $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, then the control system is said to be *linear*. We will study this important class of systems in the rest of this chapter.

5.2 Solutions to the linear control system

In this section, we give a formula for the solution of a linear control system.

Theorem 5.2.1 *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. If $u \in (C[0,T])^m$, then the differential equation*

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0 \tag{5.2}$$

has the unique solution x on $[0,+\infty)$ given by

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau. \tag{5.3}$$

Proof We have

$$\begin{aligned}
\frac{d}{dt}\left(e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau\right) &= \frac{d}{dt}\left(e^{tA}x_0 + e^{tA}\int_0^t e^{-\tau A}Bu(\tau)d\tau\right) \\
&= Ae^{tA}x_0 + Ae^{tA}\int_0^t e^{-\tau A}Bu(\tau)d\tau + e^{tA}e^{-tA}Bu(t) \\
&= A\left(e^{tA}x_0 + e^{tA}\int_0^t e^{-\tau A}Bu(\tau)d\tau\right) + e^{tA-tA}Bu(t) \\
&= A\left(e^{tA}x_0 + e^{tA}\int_0^t e^{-\tau A}Bu(\tau)d\tau\right) + Bu(t),
\end{aligned}$$

and so it follows that $x(\cdot)$ given by (5.3) satisfies $x'(t) = Ax(t) + Bu(t)$. Furthermore,

$$e^{0A}x_0 + \int_0^0 e^{(0-\tau)A}Bu(\tau)d\tau = Ix_0 + 0 = x_0.$$

Finally we show uniqueness. If x_1, x_2 are both solutions to (5.2), then it follows that $x := x_1 - x_2$ satisfies $x'(t) = Ax(t)$, $x(0) = 0$, and so from Theorem 1.5.5 it follows that $x(t) = 0$ for all $t \geq 0$, that is $x_1 = x_2$. ∎

Exercises.

1. Suppose that $p \in C^1[0,T]$ is such that for all $t \in [0,T]$, $p(t) + \alpha \neq 0$, and it satisfies the scalar Riccati equation
$$p'(t) = \gamma(p(t)+\alpha)(p(t)+\beta).$$
Prove that q given by
$$q(t) := \frac{1}{p(t)+\alpha}, \quad t \in [0,T],$$
satisfies
$$q'(t) = \gamma(\alpha-\beta)q(t) - \gamma, \quad t \in [0,T].$$
2. Find $p \in C^1[0,1]$ such that $p'(t) = (p(t))^2 - 1$, $t \in [0,1]$, $p(1) = 0$.
3. It is useful in control theory to be able to make estimates on the size of the solution without computing it precisely. Show that if for all $t \geq 0$, $|u(t)| \leq M$, then the solution x to $x' = ax + bu$ ($a \neq 0$), $x(0) = x_0$ satisfies
$$|x(t) - e^{ta}x_0| \leq \frac{M|b|}{a}[e^{ta} - 1].$$

That is, the solution differs from that of $x' = ax$, $x(0) = x_0$ by at most $\frac{M|b|}{a}[e^{ta} - 1]$. What happens to the bound as $a \to 0$?

4. Newton's law of cooling says that the rate of change of temperature is proportional to the difference between the temperature of the object and the environmental temperature:

$$\Theta' = \kappa(\Theta - \Theta_e),$$

where κ denotes the proportionality constant and Θ_e denotes the environmental temperature. In the episode of the TV series CSI, the body of a murder victim was discovered at 11:00 a.m. The medical examiner arrived at 11:30 a.m., and found the temperature of the body was 94.6°F. The temperature of the room was 70°F. One hour later, in the same room, she took the body temperature again and found that it was 93.4°F. Estimate the time of death, using the fact that the body temperature of any living human being is around 98.6°F.

5.3 Controllability of linear control systems

A characteristic of underdetermined equations is that one can choose the free variable in a way that some desirable effect is produced on the other dependent variable.

For example, if with our underdetermined algebraic equation $x + u = 10$ we wish to make $x < 5$, then we can achieve this by choosing the free variable u to be strictly larger than 5.

Control theory is all about doing similar things with differential equations of the type (5.1). The state variables x comprise the 'to-be-controlled' variables, which depend on the free variables u, the inputs. For example, in the case of an aircraft, the speed, altitude and so on are the to-be-controlled variables, while the angle of the wing flaps, the speed of the propeller and so on, which the pilot can specify, are the inputs. So one of the basic questions in control theory is then the following:

$$\boxed{\text{How do we choose the control inputs to achieve regulation of the state variables?}}$$

For instance, one may wish to drive the state to zero or some other desired value of the state at some time instant T. This brings us naturally to the notion of controllability which, roughly speaking, means that any state can be driven to any other state using an appropriate control.

For the sake of simplicity, we restrict ourselves to linear systems: $x'(t) = Ax(t) + Bu(t)$, $t \geq 0$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We first give the definition of controllability for such a linear control system.

Example. Suppose a lake contains two species of fish, which we simply call 'big fish' and 'small fish', which form a predator-prey pair. Suppose that the evolution of their populations x_b and x_s are reasonably accurately modelled by

$$\begin{bmatrix} x_b'(t) \\ x_s'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_b(t) \\ x_s(t) \end{bmatrix}.$$

Now suppose that one is harvesting these fish at harvesting rates h_b and h_s (which are inputs, since they can be decided by the fisherman). The model describing the evolution of the populations then becomes:

$$\begin{bmatrix} x_b'(t) \\ x_s'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_b(t) \\ x_s(t) \end{bmatrix} - \begin{bmatrix} h_b(t) \\ h_s(t) \end{bmatrix}.$$

The goal is to harvest the species of fish over some time period $[0, T]$ in such a manner starting from the initial population levels

$$\begin{bmatrix} x_b(0) \\ x_s(0) \end{bmatrix} = \begin{bmatrix} x_{b,i} \\ x_{s,i} \end{bmatrix},$$

we are left with the desired population levels

$$\begin{bmatrix} x_b(T) \\ x_s(T) \end{bmatrix} = \begin{bmatrix} x_{b,f} \\ x_{s,f} \end{bmatrix}.$$

For example, if one of the species of fish is nearing extinction, it might be important to maintain some critical levels of the populations of the predator versus the prey. Thus we see that controllability problems arise quite naturally from applications. $\Diamond$

Definition. The system

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t), \quad t \in [0, T] \tag{5.4}$$

is said to be *controllable at time* T if for every pair of vectors x_0, x_1 in $\mathbb{R}^n$, there exists a control $u \in (C[0, T])^m$ such that the solution x of (5.4) with $x(0) = x_0$ satisfies $x(T) = x_1$.

Examples.

1. *(A controllable system)* Consider the system

$$x'(t) = u(t), \quad t \in [0, T],$$

so that $A = 0$, $B = 1$. Given $x_0, x_1 \in \mathbb{R}$, define $u \in C[0, T]$ to be the constant function

$$u(t) = \frac{x_1 - x_0}{T}, \quad t \in [0, T].$$

By the fundamental theorem of calculus,

$$x(T) = x(0) + \int_0^T x'(\tau)d\tau = x_0 + \int_0^T u(\tau)d\tau = x_0 + \frac{x_1 - x_0}{T}(T - 0) = x_1.$$

2. *(An uncontrollable system)* Consider the system

$$\begin{aligned} x_1'(t) &= x_1(t) + u(t), && (5.5) \\ x_2'(t) &= x_2(t), && (5.6) \end{aligned}$$

so that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The equation (5.6) implies that $x_2(t) = e^t x_2(0)$, and so if $x_2(0) > 0$, then $x_2(t) > 0$ for all $t \geq 0$. So a final state with the x_2-component negative is never reachable by any control. $\Diamond$

We would like to characterize the property of controllability in terms of the matrices A and B. For this purpose, we introduce the notion of reachable space at time T:

Definition. The *reachable space* of (5.4) at time T, denoted by $\mathscr{R}_T$, is defined as the set of all $x \in \mathbb{R}^n$ for which there exists a control $u \in (C[0, T])^m$ such that

$$x = \int_0^T e^{(T-\tau)A} Bu(\tau)d\tau. \tag{5.7}$$

Note that the above simply says that if we run the differential equation (5.4) with the input u, and with initial condition $x(0) = 0$, then x is the set of all points in the state-space that are 'reachable' at time T starting from 0 by means of some input u.

We now prove that $\mathscr{R}_T$ is a subspace of $\mathbb{R}^n$.

Lemma 5.3.1 *$\mathscr{R}_T$ is a subspace of $\mathbb{R}^n$.*

Proof We verify that $\mathscr{R}_T$ is nonempty, and closed under addition and scalar multiplication.

S1 If we take $u = 0$, then

$$\int_0^T e^{(T-\tau)A}Bu(\tau)d\tau = 0,$$

and so $0 \in \mathscr{R}_T$.

S2 If $x_1, x_2 \in \mathscr{R}_T$, then there exist u_1, u_2 in $(C[0,T])^m$ such that

$$x_1 = \int_0^T e^{(T-\tau)A}Bu_1(\tau)d\tau \quad \text{and} \quad x_2 = \int_0^T e^{(T-\tau)A}Bu_2(\tau)d\tau.$$

Thus $u := u_1 + u_2 \in (C[0,T])^m$ and

$$\int_0^T e^{(T-\tau)A}Bu(\tau)d\tau = \int_0^T e^{(T-\tau)A}Bu_1(\tau)d\tau + x_2 = \int_0^T e^{(T-\tau)A}Bu_2(\tau)d\tau = x_1 + x_2.$$

Consequently $x_1 + x_2 \in \mathscr{R}_T$.

S3 If $x \in \mathscr{R}_T$, then there exists a $u \in (C[0,T])^m$ such that

$$x = \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau.$$

If $\alpha \in \mathbb{R}$, then $\alpha \cdot u \in (C[0,T])^m$ and

$$\int_0^T e^{(T-\tau)A}B(\alpha u)(\tau)d\tau = \alpha \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau = \alpha x.$$

Consequently $\alpha x \in \mathscr{R}_T$.

Thus $\mathscr{R}_T$ is a subspace of $\mathbb{R}^n$. ∎

We now prove Theorem 5.3.3, which will yield Corollary 5.3.4 below on the characterization of the property of controllability. In order to prove Theorem 5.3.3, we will use the Cayley-Hamilton theorem, and for the sake of completeness, we have included a sketch of proof of this result here.

Theorem 5.3.2 (Cayley-Hamilton) *If $A \in \mathbb{C}^{n\times n}$ and $p(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$ is its characteristic polynomial, then $p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0$.*

Proof *(Sketch)* This is easy to see if A is diagonal, since

$$p\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix} = 0.$$

It is also easy to see if A is diagonalizable, since if $A = PDP^{-1}$, then

$$p(A) = p(PDP^{-1}) = Pp(D)P^{-1} = P0P^{-1} = 0.$$

As $\det : \mathbb{C}^{n\times n} \to \mathbb{C}$ is a continuous function, it follows that the coefficients of the characteristic polynomial are continuous functions of the matrix entries. Using the fact that the set of diagonalizable matrices is dense in $\mathbb{C}^{n\times n}$, we see that the result extends to all complex matrices by continuity. ∎

Theorem 5.3.3 $\mathscr{R}_T = \mathbb{R}^n$ *iff* $\operatorname{rank}\left[\ B \mid AB \mid \dots \mid A^{n-1}B\ \right] = n$.

Proof IF: If $\mathscr{R}_T \neq \mathbb{R}^n$, then there exists a $x_0 \neq 0$ in $\mathbb{R}^n$ such that for all $x \in \mathscr{R}_T$, $x_0^\top x = 0$. Consequently,

$$x_0^\top \int_0^T e^{(T-\tau)A} Bu(\tau)d\tau = 0 \quad \forall u \in (C[0,T])^m.$$

In particular, u_0 defined by $u_0(t) = B^\top e^{(T-\tau)A^\top} x_0$, $t \in [0,T]$, belongs to $(C[0,T])^m$, and so

$$\int_0^T x_0^\top e^{(T-\tau)A} BB^\top e^{(T-\tau)A^\top} x_0 d\tau = 0,$$

and so it can be seen that

$$x_0^\top e^{(T-\tau)A} B = 0, \quad t \in [0,T]. \tag{5.8}$$

(Why?) With $t = T$, we obtain $x_0^\top B = 0$. Differentiating (5.8), we obtain $x_0^\top e^{(T-t)A}AB = 0$, $t \in [0,T]$, and so with $t = T$, we have $x_0^\top AB = 0$. Proceeding in this manner (that is, successively differentiating (5.8) and setting $t = T$), we see that $x_0^\top A^k B = 0$ for all $k \in \mathbb{N}$, and so in particular,

$$x_0^\top \left[\ B \mid AB \mid \dots \mid A^{n-1}B\ \right] = 0.$$

As $x_0 \neq 0$, we obtain $\operatorname{rank}\left[\ B \mid AB \mid \dots \mid A^{n-1}B\ \right] < n$.

ONLY IF: Let $\mathscr{C} := \operatorname{rank}\left[\ B \mid AB \mid \dots \mid A^{n-1}B\ \right] < n$. Then there exists a nonzero $x_0 \in \mathbb{R}^n$ such that

$$x_0^\top \left[\ B \mid AB \mid \dots \mid A^{n-1}B\ \right] = 0. \tag{5.9}$$

By the Cayley-Hamilton theorem, it follows that

$$x_0^\top A^n B = x_0^\top \left[\alpha_0 I + \alpha_1 A + \dots + \alpha_n A^{n-1}\right] B = 0.$$

By induction,

$$x_0^\top A^k B = 0 \quad \forall k \geq n. \tag{5.10}$$

From (5.9) and (5.10), we obtain $x_0^\top A^k B = 0$ for all $k \geq 0$, and so $x_0^\top e^{tA} B = 0$ for all $t \in [0,T]$. But this implies that $x_0 \notin \mathscr{R}_T$, since otherwise if some $u \in C[0,T]$,

$$x_0 = \int_0^T e^{(T-\tau)A} Bu(\tau)d\tau,$$

then

$$x_0^\top x_0 = \int_0^T x_0 e^{(T-\tau)A} Bu(\tau)d\tau = \int_0^T 0u(\tau)d\tau = 0,$$

which yields $x_0 = 0$, a contradiction. ∎

The following result gives an important characterization of controllability.

Corollary 5.3.4 *Let $T > 0$. The system (5.4) is controllable at T iff*

$$\operatorname{rank} \left[\begin{array}{c|c|c|c} B & AB & \dots & A^{n-1}B \end{array} \right] = n,$$

where n denotes the dimension of the state space.

Proof ONLY IF: Let $x'(t) = Ax(t) + Bu(t)$ be controllable at time T. Then with $x_0 = 0$, all the states $x_1 \in \mathbb{R}^n$ can be reached at time T. So $\mathscr{R}_T = \mathbb{R}^n$. Hence by Theorem 5.3.3, $\operatorname{rank} \left[\begin{array}{c|c|c|c} B & AB & \dots & A^{n-1}B \end{array} \right] = n$.

IF: Let $\operatorname{rank} \left[\begin{array}{c|c|c|c} B & AB & \dots & A^{n-1}B \end{array} \right] = n$. Then by Theorem 5.3.3, $\mathscr{R}_T = \mathbb{R}^n$. Given $x_0, x_1 \in \mathbb{R}^n$, we have $x_1 - e^{TA}x_0 \in \mathbb{R}^n = \mathscr{R}_T$, and so there exists a $u \in (C[0,T])^m$ such that

$$x_1 - e^{TA}x_0 = \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau, \text{ that is, } x_1 = e^{TA}x_0 + \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau.$$

In other words $x(T) = x_1$, where $x(\cdot)$ denotes the unique solution to $x'(t) = Ax(t) + Bu(t)$, $t \in [0,T]$, $x(0) = x_0$. ■

We remark that the test:

$$\operatorname{rank} \left[\begin{array}{c|c|c|c} B & AB & \dots & A^{n-1}B \end{array} \right] = n$$

is independent of T, and so it follows that if $T_1, T_2 > 0$, then the system $x'(t) = Ax(t) + Bu(t)$ is controllable at T_1 iff it is controllable at T_2. So for the system $x'(t) = Ax(t) + Bu(t)$, we usually talk about 'controllability' instead of 'controllability at $T > 0$'.

Examples. Consider the two examples on page 73.

1. *(Controllable system)* In the first example,

$$\operatorname{rank} \left[\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array} \right] \stackrel{(n=1)}{=} \operatorname{rank} \left[\begin{array}{c} B \end{array} \right] = \operatorname{rank} \left[\begin{array}{c} 1 \end{array} \right] = 1 = n,$$

   the dimension of the state space ($\mathbb{R}$).

2. *(Uncontrollable system)* In the second example, note that

$$\operatorname{rank} \left[\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array} \right] \stackrel{(n=2)}{=} \operatorname{rank} \left[\begin{array}{cc} B & AB \end{array} \right] = \operatorname{rank} \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right] = 1 \neq 2 = n,$$

   the dimension of the state space ($\mathbb{R}^2$). ♢

Exercises.

1. For what values of α is the system (5.4) controllable, if

$$A = \left[\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array} \right], \quad B = \left[\begin{array}{c} 1 \\ \alpha \end{array} \right]?$$

2. $(*)$ Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$. Prove that if the system (5.4) is controllable, then every matrix commuting with A is a polynomial in A.

3. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Show that the reachable subspace $\mathscr{R}_T$ at time T of $x'(t) = Ax(t) + Bu(t)$, $t \geq 0$, is equal to
$$\operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ that is, the set } \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}.$$

4. A nonzero vector $v \in \mathbb{R}^{1\times n}$ is called a *left eigenvector* of $A \in \mathbb{R}^{n\times n}$ if there exists a $\lambda \in \mathbb{R}$ such that $vA = \lambda v$.

   Show that if the system described by $x'(t) = Ax(t) + Bu(t)$, $t \geq 0$, is controllable, then for every left eigenvector v of A, there holds that $vB \neq 0$.

   Hint: Observe that if $vA = \lambda v$, then $vA^k = \lambda^k v$ for all $k \in \mathbb{N}$.

Solutions

Chapter 1: Linear equations

Solutions to the exercises on page 2

1. (a) Nonautonomous.
   (b) Autonomous; nonlinear.
   (c) Nonautonomous.
   (d) Autonomous; nonlinear.
   (e) Autonomous; linear.

2. (a) $x'(t) = 0$, and
   $e^{\sin(x(t))} + \cos(x(t)) = e^{\sin \pi} + \cos(\pi) = e^0 + -1 = 1 + -1 = 0.$
   (b) $x'(t) = \frac{d}{dt}(e^{ta}x_0) = \frac{d}{dt}(e^{ta})x_0 = ae^{ta}x_0,$
   $ax(t) = ae^{ta}x_0$, and
   $x(0) = e^{0a}x_0 = e^0 x_0 = 1x_0 = x_0.$
   (c) $x_1'(t) = \frac{d}{dt}\sin(2t) = 2\cos(2t) = 2x_2(t),$
   $x_2'(t) = \frac{d}{dt}\cos(2t) = -2\sin(2t) = -2x_1(t).$
   (d) $x_1'(t) = \frac{d}{dt}\left(\frac{1}{1-t^2}\right) = \frac{-1}{(1-t^2)^2}(-2t) = 2t\left(\frac{1}{1-t^2}\right)^2 = 2t(x_1(t))^2.$
   $x_2'(t) = 0 = 2t(0)^2 = 2t(x_2(t))^2.$

3. If $x(t) = t^m$ is a solution to $2tx'(t) = x(t)$, then $2t(mt^{m-1}) = t^m$, and so $(2m-1)t^m = 0$. As $t \geq 1$, we obtain $2m - 1 = 0$, and so $m = 1/2$.

   Conversely, with $x(t) = \sqrt{t}$, we see that $x'(t) = 1/(2\sqrt{t})$, and so for all $t > 0$,

$$2tx'(t) = 2t\frac{1}{2\sqrt{t}} = \sqrt{t} = x(t).$$

4. Let x be a solution on (a, b), and let $t_1, t_2 \in (a, b)$ be such that $t_1 < t_2$. Then we have by the fundamental theorem of calculus that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(t)dt = \int_{t_1}^{t_2} [(x(t))^2 + 1]dt \geq \int_{t_1}^{t_2} dt = t_2 - t_1 > 0.$$

   Thus $x(t_2) > x(t_1)$.

Solutions to the exercises on page 5

1.
```
> with(DEtools):
> ode1a := diff(x(t),t)=x(t)+(x(t))^3;
> DEplot(ode1a,x(t),t=-1..0.3,[[x(0)=1]],stepsize=0.01, linecolour=black);
```

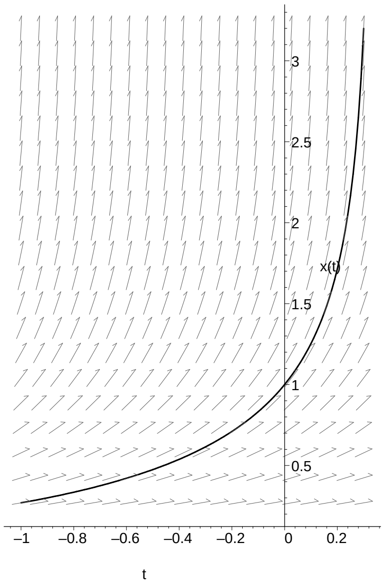

Figure 5.2: Solution to $x' = x + x^3$ with $x(0) = 1$.

2.
```
> with(DEtools):
> ode1b := diff(x(t),t,t)=-x(t)+(1/2)*cos(t);
> DEplot(ode1b,x(t),t=-20..20,[[x(0)=1,D(x)(0)=1]],stepsize=0.01, linecolour=black);
```

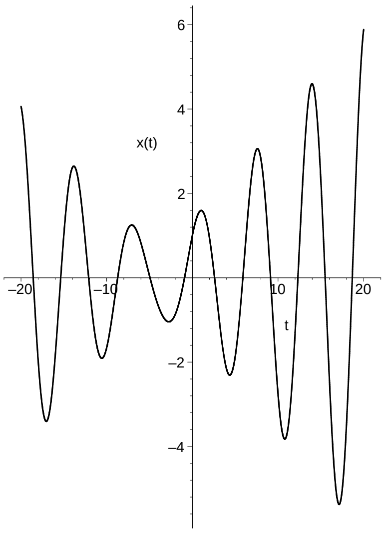

Figure 5.3: Solution to $x'' + x = \frac{1}{2}\cos t$ with $x(0) = 1$ and $x'(0) = 1$.

3.
```
> with(DEtools):
```

```
> ode1cA := diff(x1(t),t)=-x1(t)+x2(t);
> ode1cB := diff(x2(t),t)=x1(t)+x2(t)+t;
> DEplot(ode1cA,ode1cB,x1(t),x2(t),t=-1..1,[[x1(0)=0,x2(0)=0]],stepsize=0.01,
linecolour=black);
```

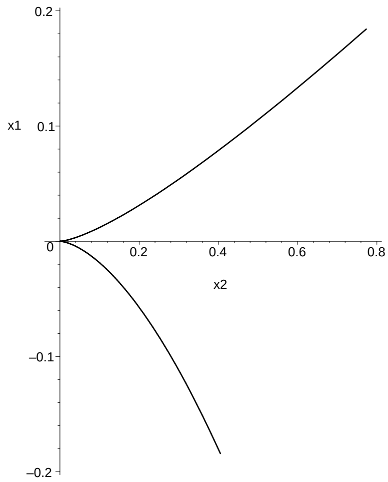

Figure 5.4: Plot of x_1 versus x_2.

```
> with(plots):
> plot1cA:=DEplot(ode1cA,ode1cB,x1(t),x2(t),t=-1..1,[[x1(0)=0,x2(0)=0]],
scene=[t,x1(t)],stepsize=0.01, linecolour=black);
> plot1cB:=DEplot(ode1cA,ode1cB,x1(t),x2(t),t=-1..1,[[x1(0)=0,x2(0)=0]],
scene=[t,x2(t)],stepsize=0.01, linecolour=black);
> display(plot1cA,plot1cB);
```

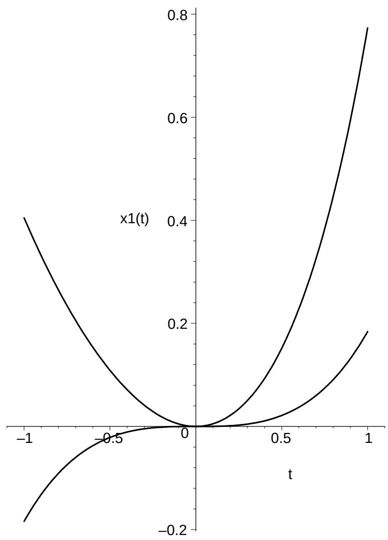

Figure 5.5: Plots of x_1 and x_2 versus time.

4. (a) `> with(DEtools):`

```
> ode2a := diff(x(t),t)=x(t)*(2-(1/5)*x(t)-(5*x(t))/(2+(x(t))^2));
> DEplot(ode2a,x(t),t=0..10,[[x(0)=0],[x(0)=1],[x(0)=2],[x(0)=3],
[x(0)=4], [x(0)=5], [x(0)=6],[x(0)=7],[x(0)=8],[x(0)=9],[x(0)=10]],
stepsize=0.01,linecolour=black);
```

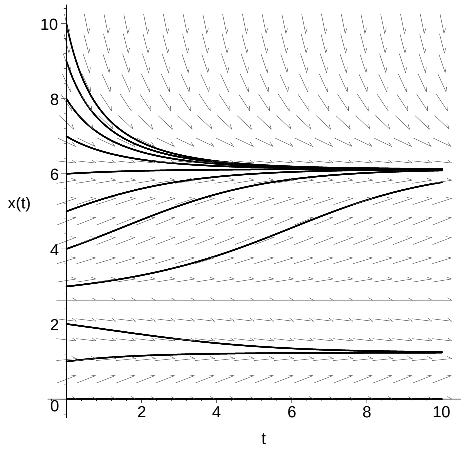

Figure 5.6: Population evolution of the budworm for various initial conditions.

(b) and (c): We can plot the function

$$f(x) = x\left(2 - \frac{1}{5}x - \frac{5x}{2+x^2}\right),$$

in the range $x \in [0, 10]$ using the Maple command:

```
> plot(x*( 2-(1/5)*x -5*x/(2+x^2)),x=0..10);
```

and the result is displayed in Figure 5.14.

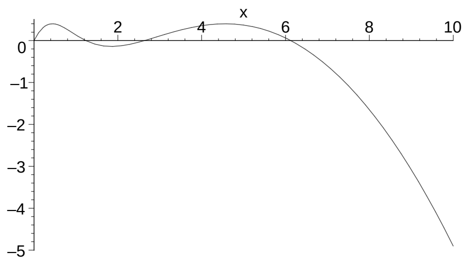

Figure 5.7: Graph of the function f.

From this plot we observe that the function has zeros at (approximately) $0, 1.24, 2.64$ and 6.125, and also we have

$$\begin{aligned} x &\in (0, 1.24) & f(x) &> 0, \\ x &\in (1.25, 2.63) & f(x) &< 0, \\ x &\in (2.65, 6.12) & f(x) &> 0, \\ x &\in (6.13, \infty) & f(x) &< 0. \end{aligned}$$

Thus for instance if $x(t)$ is in one of the regions above, say the region $(1.25, 2.63)$, then $x'(t) = f(x(t)) < 0$, and so the function $t \mapsto x(t)$ is decreasing, and this explains the behaviour of the curve that starts with an initial condition in this region; see Figure 5.6.

On the other hand, in some regions the rate of change of $x(t)$ is positive–for example for initial conditions in the region $(2.63, 6.12)$. This demarcation between the regions is highlighted in Figure 5.8, and we summarize the behaviour below:

$$\begin{aligned}
x(0) &= 0 && x(t) = 0 \quad \forall t \geq 0; \\
x(0) &\in (0, 1.24) && x(t) \nearrow 1.24 \text{ as } t \to \infty; \\
x(0) &\approx 1.24 && x(t) \approx 1.24 \quad \forall t \geq 0; \\
x(0) &\in (1.25, 2.63) && x(t) \searrow 1.24 \text{ as } t \to \infty; \\
x(0) &\approx 2.64 && x(t) \approx 2.64 \quad \forall t \geq 0; \\
x(0) &\in (2.65, 6.12) && x(t) \nearrow 6.125 \text{ as } t \to \infty; \\
x(0) &\approx 6.125 && x(t) \approx 6.125 \quad \forall t \geq 0; \\
x(0) &\in (6.13, \infty) && x(t) \searrow 6.125 \text{ as } t \to \infty.
\end{aligned}$$

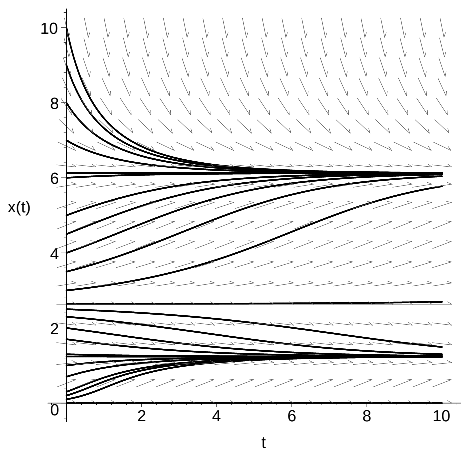

Figure 5.8: Population evolution of the budworm for various initial conditions.

Solutions to the exercises on page 8

1. With $x_1 := x$ and $x_2 := x'$, we have

$$\begin{aligned} x_1' &= x' = x_2, \\ x_2' &= x'' = -\omega^2 x = -\omega^2 x_1, \end{aligned}$$

   that is,

$$\left\{ \begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega^2 x_1. \end{aligned} \right.$$

2. Define $w_1 := x$, $w_2 := x'$, $w_3 = y$, $w_4 = y'$. Then we have

$$\begin{aligned} w_1' &= x' = w_2, \\ w_2' &= x'' = -x = -w_1, \\ w_3' &= y' = w_4, \\ w_4' &= y'' = -y' - y = -w_4 - w_3, \end{aligned}$$

   that is,

$$\left\{ \begin{aligned} w_1' &= w_2 \\ w_2' &= -w_1 \\ w_3' &= w_4 \\ w_4' &= -w_3 - w_4. \end{aligned} \right.$$

Solutions to the exercise on page 11

Given $A \in \mathbb{C}^{n\times n}$, there exists an invertible matrix P such that

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Now choose $\epsilon_1, \dots, \epsilon_n \in (0, \delta)$, where $\delta > 0$ (chosen suitable small eventually), and $\lambda_1 + \epsilon_1, \dots, \lambda_n + \epsilon_n$ are all distinct. Define

$$B = P^{-1} \begin{bmatrix} \lambda_1 + \epsilon_1 & & * \\ & \ddots & \\ 0 & & \lambda_n + \epsilon_n \end{bmatrix} P.$$

Then B has distinct eigenvalues $\lambda_1+\epsilon_1, \dots, \lambda_n+\epsilon_n$, and so it is diagonalizable (indeed eigenvectors corresponding to distinct eigenvalues are linearly independent!). But

$$\|A - B\| = \left\| P^{-1} \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_n \end{bmatrix} P \right\|,$$

and this can be made arbitrarily small by choosing δ small enough.

In the above solution, we used the following result:

Theorem 5.1.5 (Triangular form) *For every $A \in \mathbb{C}^{ntimesn}$, there exists an invertible $P \in \mathbb{C}^{n\times n}$ such that PAP^{-1} is an upper triangular matrix:*

$$PAP^{-1} = \begin{bmatrix} * & \dots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix}.$$

Furthermore, the diagonal entries of PAP^{-1} are the eigenvalues of A.

Proof By the fundamental theorem of algebra, every polynomial has at least one complex root, and so A has at least one eigenvector v_1. Construct a basis for $\mathbb{C}^n$ starting from v_1. Thus there exists an invertible $\widetilde{P} \in \mathbb{C}^{n\times n}$ such that

$$\widetilde{P}A\widetilde{P}^{-1} = \left[\begin{array}{c|ccc} \lambda_1 & * & \dots & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right],$$

where $B \in \mathbb{C}^{(n-1)\times(n-1)}$. Now we induct on n. By induction, we may assume the existence of an invertible $Q \in \mathbb{C}^{(n-1)\times(n-1)}$ such that QBQ^{-1} is upper triangular. Let

$$P := \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & Q & \\ 0 & & & \end{array}\right] \widetilde{P}.$$

Then

$$PAP^{-1} := \left[\begin{array}{c|ccc} \lambda_1 & * & \dots & * \\ \hline 0 & & & \\ \vdots & & \mathrm{QBQ}^{-1} & \\ 0 & & & \end{array}\right],$$

which is upper triangular.

Since

$$\begin{aligned} \det(tI - PAP^{-1}) &= \det(tPIP^{-1} - PAP^{-1}) = \det(P(tI - A)P^{-1}) \\ &= \det(P)\det(tI - A)\det(P^{-1}) = \det(P)\det(tI - A)(\det(P))^{-1} \\ &= \det(tI - A), \end{aligned}$$

it follows that the diagonal entries of PAP^{-1} are the roots of $\det(tI - A)$, the characteristic polynomials of A, and so they are the eigenvalues of A. ■

Solutions to the exercises on page 11

1. (Since for commuting matrices A and B, we know that $e^{A+B} = e^A e^B$, we must choose noncommuting A and B.) let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = BA,$$

and so A and B do not commute. Since $A^2 = B^2 = 0$, we obtain

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \cdots = I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \\ e^B &= I + B + \frac{1}{2!}B^2 + \cdots = I + B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \end{aligned}$$

and so

$$e^A e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

On the other hand,

$$A + B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which has eigenvalues i and $-i$, so that

$$A + B = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}.$$

Hence

$$e^{A+B} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^i & 0 \\ 0 & e^{-i} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} = \begin{bmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix},$$

and so we see that $e^{A+B} \neq e^A e^B$.

2. We have

$$A = 2I + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix},$$

and since $2I$ commutes with every matrix, it follows that

$$\begin{aligned} e^A &= e^{2I + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}} = e^{2I} e^{\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}} \\ &= e^2 I \left(I + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}^2 + \dots \right) \\ &= e^2 I \left(I + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} + 0 + 0 + \dots \right) \\ &= e^2 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Solutions to the exercises on page 15

1. Define $S_k(A) = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{k!}A^k$ $(k \in \mathbb{N})$. Then we have

$$\begin{aligned}
|(S_k(A))_{ij}| &= \left|(I)_{ij} + \frac{1}{1!}(A)_{ij} + \frac{1}{2!}(A^2)_{ij} + \cdots + \frac{1}{k!}(A^k)_{ij}\right| \\
&\leq |(I)_{ij}| + \frac{1}{1!}|(A)_{ij}| + \frac{1}{2!}|(A^2)_{ij}| + \cdots + \frac{1}{k!}|(A^k)_{ij}| \quad \text{(triangle inequality)} \\
&\leq \|I\| + \frac{1}{1!}\|A\| + \frac{1}{2!}\|A^2\| + \cdots + \frac{1}{k!}\|A^k\| \quad \text{(definition of } \|\cdot\|\text{)} \\
&\leq 1 + \frac{1}{1!}\|A\| + \frac{1}{2!}n\|A\|^2 + \cdots + \frac{1}{k!}n^{k-1}\|A\|^k \quad \text{(Lemma 1.5.6)} \\
&\leq 1 + \frac{1}{1!}n\|A\| + \frac{1}{2!}n^2\|A\|^2 + \cdots + \frac{1}{k!}n^k\|A\|^k \\
&\leq 1 + \frac{1}{1!}n\|A\| + \frac{1}{2!}n^2\|A\|^2 + \cdots + \frac{1}{k!}n^k\|A\|^k + \cdots = e^{n\|A\|}.
\end{aligned}$$

Passing the limit as $k \to \infty$, we obtain that $|(e^A)_{ij}| \leq e^{n\|A\|}$. As the choice of i and j was arbitrary, it follows that $\|e^A\| \leq e^{n\|A\|}$.

2. (a) We have

$$|(Av)_i| = \left|\sum_{j=a}^{n}(A)_{ij}v_j\right| \leq \sum_{j=a}^{n}|(A)_{ij}||v_j| \leq \|A\|\sum_{j=1}^{n}|v_j| \leq n\|A\|\|v\|,$$

and so $\|Av\|^2 \leq n(n\|A\|\|v\|)^2$. Thus for all $v \in \mathbb{R}^n$, $\|Av\| \leq n^{3/2}\|A\|\|v\|$.

(b) If λ is an eigenvalue and v is a corresponding eigenvector, then we have $Av = \lambda v$, and so

$$|\lambda||v_i| = |(\lambda v)_i| = |(Av)_i| = \left|\sum_{j=a}^{n}(A)_{ij}v_j\right| \leq \sum_{j=a}^{n}|(A)_{ij}||v_j| \leq \|A\|\sum_{j=1}^{n}|v_j|.$$

Adding the n inequalities (for $i = 1, \ldots, n$), we obtain

$$|\lambda|\sum_{i}^{n}|v_i| \leq n\|A\|\sum_{j=1}^{n}|v_j|.$$

As $v \neq 0$, we can divide by $\sum_{i=1}^{n}|v_i|$, and this yields that $|\lambda| \leq n\|A\|$.

3. (a) Define $S_n(A) = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n$ $(n \in \mathbb{N})$. Since $Av = \lambda v$, it is easy to show by induction that $A^k v = \lambda^k v$ for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
S_n(A)v &= \left(I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n\right)v \\
&= v + \frac{1}{1!}Av + \frac{1}{2!}A^2v + \cdots + \frac{1}{n!}A^nv \\
&= v + \frac{1}{1!}\lambda v + \frac{1}{2!}\lambda^2 v + \cdots + \frac{1}{n!}\lambda^n v \\
&= \underbrace{\left(1 + \frac{1}{1!}\lambda + \frac{1}{2!}\lambda^2 + \cdots + \frac{1}{n!}\lambda^n\right)}_{=:S_n(\lambda)}v.
\end{aligned}$$

Let $i \in \{1, \ldots, n\}$,

$$S_n(\lambda)v_i = (S_n(\lambda)v)_i = (S_n(A)v)_i = \sum_{j=1}^{n}(S_n(A))_{ij}v_j.$$

Since $\lim_{n\to\infty}(S_n(A))_{ij} = (e^A)_{ij}$ and $\lim_{n\to\infty} S_n(\lambda) = e^\lambda$, by passing the limit as $n \to \infty$ in the above, we obtain that

$$(e^\lambda v)_i = e^\lambda v_i = \sum_{j=1}^{n}(e^A)_{ij}v_j = (e^A v)_i.$$

As the choice of i was arbitrary, it follows that $e^A v = e^\lambda v$.

(b) Let $A := \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

i. Since $tAx_0 = 7tx_0$, we know that x_0 is an eigenvector of tA corresponding to the eigenvalue $7t$. So by the previous exercise, v is also an eigenvector of e^{tA} corresponding to the eigenvalue e^{7t}. So the solution is

$$x(t) = e^{tA}x(0) = e^{tA}x_0 = e^{7t}x_0 = \begin{bmatrix} e^{7t} \\ e^{7t} \end{bmatrix}.$$

ii. Since $tAx_0 = tx_0$, we know that x_0 is an eigenvector of tA corresponding to the eigenvalue t. So v is also an eigenvector of e^{tA} corresponding to the eigenvalue e^t. So the solution is

$$x(t) = e^{tA}x(0) = e^{tA}x_0 = e^{t}x_0 = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}.$$

iii. We observe that

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and so

$$\begin{aligned} x(t) &= e^{tA}x(0) = e^{tA}x_0 \\ &= e^{tA}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = e^{tA}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\ &= e^{tA}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{tA}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{7t} \\ e^{7t} \end{bmatrix} + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} \quad \text{(by the previous two parts of the exercise)} \\ &= \begin{bmatrix} e^{7t} + e^t \\ e^{7t} - e^t \end{bmatrix}. \end{aligned}$$

4. Define $S_n(A) = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n$ $(n \in \mathbb{N})$. Then we have

$$\begin{aligned} (S_n(A))^\top &= \left(I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n\right)^\top \\ &= I^\top + \frac{1}{1!}A^\top + \frac{1}{2!}(A^2)^\top + \cdots + \frac{1}{n!}(A^n)^\top \\ &= I^\top + \frac{1}{1!}A^\top + \frac{1}{2!}(A^\top)^2 + \cdots + \frac{1}{n!}(A^\top)^n \\ &= S_n(A^\top). \end{aligned}$$

Thus

$$[S_n(A)]_{ji} = [(S_n(A))^\top]_{ij} = [S_n(A^\top)]_{ij}$$

and by passing the limit as $n \to \infty$, we obtain

$$\left([(e^A)^\top]_{ij} =\right) \;\; [e^A]_{ji} = [e^{A^\top}]_{ij}.$$

Since the choice of i and j was arbitrary, it follows that $(e^A)^\top = e^{A^\top}$.

5. (a) We verify that $\mathscr{S}$ is nonempty and is closed under addition and scalar multiplication.

S0. Clearly the zero function $f \equiv 0$ belongs to $\mathscr{S}$.

S1. If $f_1, f_2 \in \mathscr{S}$, then $f_1' = Af_1$ and $f_2' = Af_2$, and so $(f_1 + f_2)' = f_1' + f_2' = Af_1 + Af_2 = A(f_1 + f_2)$. Thus $f_1 + f_2 \in \mathscr{S}$.

S2. If $f \in \mathscr{S}$, then $f' = Af$. Suppose that $\alpha \in \mathbb{R}$. Then $(\alpha \cdot f)' = \alpha \cdot f' = \alpha \cdot (Af) = A(\alpha \cdot f)$, and so $\alpha \cdot f \in \mathscr{S}$.

Thus $\mathscr{S}$ is a subspace of $C(\mathbb{R}; \mathbb{R}^n)$.

Remark. $\mathscr{S}$ is thus a vector space with the operations of pointwise addition and scalar multiplication.

(b) Suppose there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\alpha_1 \cdot f_1 + \cdots + \alpha_n \cdot f_n = 0$. Then for all $t \in \mathbb{R}$, we have that $\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0$. In particular, with $t = 0$, we obtain that $\alpha_1 f_1(0) + \cdots + \alpha_n f_n(0) = 0$, that is, $\alpha_1 f e_1 + \cdots + \alpha_n e_n = 0$. By independence of the standard basis vectors in $\mathbb{R}^n$, it follows that $\alpha_1 = \cdots = \alpha_n = 0$. So $f_1, \dots, f_n$ are linearly independent in the vector space $\mathscr{S}$.

(c) We know that $\operatorname{span}\{f_1, \dots, f_n\} \subset \mathscr{S}$. We now show that the reverse inclusion holds as well. Suppose that f belongs to $\mathscr{S}$, and so $f' = Af$. As $f(0) \in \mathbb{R}^n$, and since $e_1, \dots, e_n$ form a basis for $\mathbb{R}6n$, we can find scalars $\alpha_1, \dots, \alpha_n$ such that $f(0) = \alpha_1 e_1 + \cdots + \alpha_n e_n$. Let $\widetilde{f} := \alpha_1 \cdot f_1 + \cdots + \alpha_n \cdot f_n$. Then we have

$$\begin{aligned}
\widetilde{f}' &= (\alpha_1 \cdot f_1 + \cdots + \alpha_n \cdot f_n)' = \alpha_1 \cdot f_1' + \cdots + \alpha_n \cdot f_n' \\
&= \alpha_1 \cdot (Af_1) + \cdots + \alpha_n \cdot (Af_n) = A(\alpha_1 \cdot f_1 + \cdots + \alpha_n \cdot f_n) \\
&= A\widetilde{f}.
\end{aligned}$$

Moreover, $\widetilde{f}(0) = \alpha_1 \cdot f_1(0) + \cdots + \alpha_n \cdot f_n(0) = \alpha_1 \cdot e_1 + \cdots + \alpha_n \cdot e_n = f(0)$. By (the uniqueness part of) Theorem 1.5.5, it follows that $\widetilde{f} = f$. Since $\widetilde{f} \in \operatorname{span}\{f_1, \dots, f_n\}$, we obtain $f \in \mathscr{S}$.

Solutions to the exercises on page 18

1. We have that
$$sI - A = \begin{bmatrix} s-3 & 1 \\ -1 & s-1 \end{bmatrix},$$
and so
$$\det(sI - A) = (s-3)(s-1) + 1 = s^2 - 4s + 4 = (s-2)^2.$$
Thus
$$\begin{aligned}(sI-A)^{-1} &= \frac{1}{(s-2)^2}\begin{bmatrix} s-1 & -1 \\ 1 & s-3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s-1}{(s-2)^2} & -\frac{1}{(s-2)^2} \\ \frac{1}{(s-2)^2} & \frac{s-3}{(s-2)^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{s-2+1}{(s-2)^2} & -\frac{1}{(s-2)^2} \\ \frac{1}{(s-2)^2} & \frac{s-2-1}{(s-2)^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s-2} + \frac{1}{(s-2)^2} & -\frac{1}{(s-2)^2} \\ \frac{1}{(s-2)^2} & \frac{1}{s-2} - \frac{s-2-1}{(s-2)^2} \end{bmatrix}.\end{aligned}$$
Using Theorem 1.6.2 ('taking the inverse Laplace transform'), we obtain
$$e^{tA} = \begin{bmatrix} e^{2t} + te^{2t} & -te^{2t} \\ te^{2t} & e^{2t} - te^{2t} \end{bmatrix}.$$

2. We have that
$$sI - A = \begin{bmatrix} s-\lambda & -1 & 0 \\ 0 & s-\lambda & -1 \\ 0 & 0 & s-\lambda \end{bmatrix},$$
and so $\det(sI - A) = (s-\lambda)^3$. A short computation gives
$$\operatorname{adj}(sI - A) = \begin{bmatrix} (s-\lambda)^2 & s-\lambda & 1 \\ 0 & (s-\lambda)^2 & s-\lambda \\ 0 & 0 & (s-\lambda)^2 \end{bmatrix}.$$
By Cramer's rule
$$(sI - A)^{-1} = \frac{1}{\det(sI-A)}\operatorname{adj}(sI - A) = \begin{bmatrix} \frac{1}{s-\lambda} & \frac{1}{(s-\lambda)^2} & \frac{1}{(s-\lambda)^3} \\ 0 & \frac{1}{s-\lambda} & \frac{1}{(s-\lambda)^2} \\ 0 & 0 & \frac{1}{s-\lambda} \end{bmatrix}.$$
By using Theorem 1.6.2 ('taking the inverse Laplace transform'), we obtain
$$e^{tA} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

3. (a) We have that
$$sI - A = \begin{bmatrix} s-a & -b \\ b & s-a \end{bmatrix},$$
and so
$$(sI - A)^{-1} = \frac{1}{(s-a)^2 + b^2}\begin{bmatrix} s-a & b \\ -b & s-a \end{bmatrix} = \begin{bmatrix} \frac{s-a}{(s-a)^2+b^2} & \frac{b}{(s-a)^2+b^2} \\ \frac{-b}{(s-a)^2+b^2} & \frac{s-a}{(s-a)^2+b^2} \end{bmatrix}.$$

By using Theorem 1.6.2 ('taking the inverse Laplace transform'), we obtain

$$e^{tA} = \begin{bmatrix} e^{ta}\cos(bt) & e^{ta}\sin(bt) \\ -e^{ta}\sin(bt) & e^{ta}\cos(bt) \end{bmatrix}.$$

(b) With $x_1 := \sqrt{k}x$ and $x_2 := x'$, we obtain

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \sqrt{k}x' \\ -kx \end{bmatrix} = \begin{bmatrix} \sqrt{k}x_2 \\ -\sqrt{k}x_1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{k} \\ -\sqrt{k} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus we obtain

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= e^{t\begin{bmatrix} 0 & \sqrt{k} \\ -\sqrt{k} & 0 \end{bmatrix}}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\sqrt{k}t) & \sin(\sqrt{k}t) \\ -\sin(\sqrt{k}t) & \cos(\sqrt{k}t) \end{bmatrix}\begin{bmatrix} \sqrt{k}x(0) \\ x'(0) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\sqrt{k}t) & \sin(\sqrt{k}t) \\ -\sin(\sqrt{k}t) & \cos(\sqrt{k}t) \end{bmatrix}\begin{bmatrix} \sqrt{k}1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{k}\cos(\sqrt{k}t) \\ -\sqrt{k}\sin(\sqrt{k}t) \end{bmatrix}. \end{aligned}$$

Thus

$$x(t) = \frac{1}{\sqrt{k}}x_1(t) = \cos(\sqrt{k}t), \quad t \geq 0.$$

We have (with $k = 1$)

$$(x(t))^2 + (x'(t))^2 = (\cos t)^2 + (-\sin t)^2 = 1.$$

Thus the point $(x(t), x'(t))$ lies on a circle of radius 1 in the plane. Also it is easy to see that the point $(x(t), x'(t)) = (\cos t, -\sin t)$ moves clockwise on this circle with increasing t; see Figure 5.9.

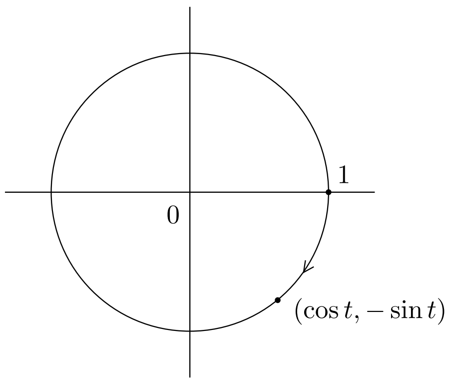

Figure 5.9: Locus of the point $(x(t), x'(t)) = (\cos t, -\sin t)$.

4. From Theorem 1.6.1 and Corollary 1.6.3, we conclude that for large enough real s,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2-1} & \frac{1}{s^2-1} \\ \frac{1}{s^2-1} & \frac{s}{s^2-1} \end{bmatrix},$$

and so

$$sI - A = ((sI - A)^{-1})^{-1} = \begin{bmatrix} \frac{s}{s^2-1} & \frac{1}{s^2-1} \\ \frac{1}{s^2-1} & \frac{s}{s^2-1} \end{bmatrix}^{-1} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}.$$

Consequently, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Alternately, one could simply observe that

$$A = \frac{d}{dt} e^{tA} \bigg|_{t=0} = \frac{d}{dt} \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \bigg|_{t=0} = \begin{bmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{bmatrix} \bigg|_{t=0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solutions to the exercises on page 20

1. (a) The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{bmatrix} = \lambda^2 - 5\lambda - 2,$$

which has the roots

$$\lambda_1 = \frac{5 + \sqrt{25 + 8}}{2} \text{ and } \lambda_2 = \frac{5 - \sqrt{25 + 8}}{2}.$$

Since $\lambda_1 > 0$, it follows from Theorem 1.7.2 that not all solutions remain bounded as $t \to \infty$.

(b) The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{bmatrix} = (\lambda - 1)(\lambda + 1),$$

which has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. Since $\lambda_1 > 0$, it follows from Theorem 1.7.2 that not all solutions remain bounded as $t \to \infty$.

(c) The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda + 2 & 1 \\ 0 & 0 & \lambda + 1 \end{bmatrix} = (\lambda - 1)(\lambda + 2)(\lambda + 1),$$

which has the roots $1, -2, -1$. Since $1 > 0$, it follows from Theorem 1.7.2 that not all solutions remain bounded as $t \to \infty$.

(d) The characteristic polynomial is

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda & -1 \\ 0 & 0 & \lambda + 2 \end{bmatrix} = (\lambda + 2) \det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{bmatrix} \\ &= (\lambda + 2)(\lambda(\lambda - 1) - 1) = (\lambda + 2)(\lambda^2 - \lambda - 1), \end{aligned}$$

which has the roots

$$\lambda_1 = -2, \quad \lambda_2 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_3 = \frac{1 - \sqrt{5}}{2}.$$

Since $\lambda_2 > 0$, it follows from Theorem 1.7.2 that not all solutions remain bounded as $t \to \infty$.

(e) The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda + 1 & 0 & 0 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda + 1 \end{bmatrix} = (\lambda + 1)(\lambda + 2)(\lambda + 1),$$

which has the roots $-1, -2, -1$. Since they are all < 0, it follows from Theorem 1.7.2 that all solutions remain bounded, and in fact they tend to 0, as $t \to \infty$.

(f) The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda + 1 & 0 & 1 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda \end{bmatrix} = (\lambda + 2)(\lambda^2 + \lambda + 1),$$

which has the roots

$$\lambda_1 = -2, \quad \lambda_2 = \frac{-1+\sqrt{3}i}{2} \quad \text{and} \quad \lambda_3 = \frac{-1-\sqrt{3}i}{2}.$$

Since they are all have real parts < 0, it follows from Theorem 1.7.2 that all solutions remain bounded, and in fact they tend to 0, as $t \to \infty$.

2. The characteristic polynomial is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-\alpha & -(1+\alpha) \\ 1+\alpha & \lambda-\alpha \end{bmatrix} = (\lambda-\alpha)^2 + (1+\alpha)^2$$

which has the roots $\alpha + i(1+\alpha)$ and $\alpha - i(1+\alpha)$. Thus we have the following cases:

$\underline{1}^\circ$ $\alpha > 0$. Then one of the eigenvalues has positive real part, and so not all solutions are bounded as $t \to \infty$.

$\underline{2}^\circ$ $\alpha = 0$. Then the eigenvalues are i and $-i$, and a short computation shows that

$$e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

so all solutions are bounded for $t \geq 0$.

$\underline{3}^\circ$ $\alpha < 0$. Then both the eigenvalues have negative real parts, and so all solutions are bounded for $t \geq 0$.

Consequently, the solutions are bounded iff $\alpha \leq 0$.

Chapter 2: Phase plane analysis

Solutions to the exercises on page 24

1. (a) We have $[x_2 = 0$ and $\sin(x_1) = 0]$ iff $[x_1 \in \{n\pi \mid n \in \mathbb{Z}\}$ and $x_2 = 0]$. Thus the singular points are at $(n\pi, 0)$, $n \in \mathbb{Z}$.

   (b) We have $[x_1 - x_2 = 0$ and $x_2^2 - x_1 = 0]$ iff $[x_1 = x_2$ and $x_1 = x_2^2 = x_1^2]$. So $x_1 = x_2$ and $x_1(1 - x_1) = 0$. Consequently, the singular points are at $(0, 0)$ and $(1, 1)$.

   (c) We have $[x_1^2(x_2 - 1) = 0$ and $x_1 x_2 = 0]$ iff $[(x_1 = 0$ or $x_2 = 1)$ and $(x_1 = 0$ or $x_2 = 0)]$. Thus the singular points are at $(0, a)$, $a \in \mathbb{R}$.

   (d) If $x_1^2(x_2 - 1) = 0$ and $x_1^2 - 2x_1x_2 - x_2^2 = 0$, then we obtain $[x_1 = 0$ or $x_2 = 1]$ and $[x_1^2 - 2x_1x_2 - x_2^2 = 0]$. Thus we have the following two cases:

   $\underline{1}^\circ$ $x_1 = 0$ and $x_2^2 = 0$.

   $\underline{2}^\circ$ $x_1 \neq 0$. Then $x_2 = 1$. Also, $x_1^2 - 2x_1 - 1 = 0$, and so $x_1 \in \{1 + \sqrt{2}, 1 - \sqrt{2}\}$.

   Hence the singular points are at $(0, 0)$, $(1 + \sqrt{2}, 1)$ and $(1 - \sqrt{2}, 1)$.

   (e) $\sin x_2 = 0$ iff $x_2 \in \{n\pi \mid n \in \mathbb{Z}\}$. $\cos x_1 = 0$ iff $x_1 \in \{(2m + 1)\pi/2 \mid m \in \mathbb{Z}\}$. Thus the singular points are at $((2m + 1)\pi/2, n\pi)$, where $m, n \in \mathbb{Z}$.

2. (a) Since $x_1^2/a^2 + x_2^2/b^2 = (\cos t)^2 + (\sin t)^2 = 1$, we see that that $(x_1(t), x_2(t))$ traces out an ellipse (with axes lengths a and b). Since at $t = 0$, $(x_1, x_2) = (a, 0)$ and $(x_1', x_2') = (0, b)$, we see that the motion is anticlockwise. The curve is plotted in Figure 5.10.

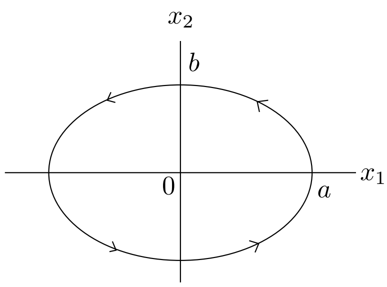

Figure 5.10: The curve $t \mapsto (a \cos t, b \sin t)$.

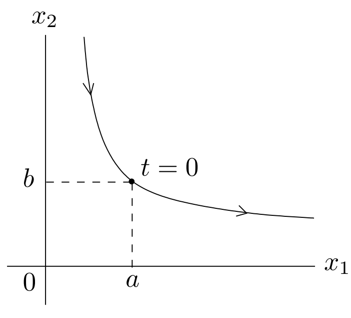

Figure 5.11: The curve $t \mapsto (ae^t, be^{-2t})$.

   (b) We observe that $x_1/a = e^t$ and $x_2/b = e^{-2t}$. Thus $(x_1/a)^2(x_2/b) = e^{2t} \cdot e^{-2t} = 1$. Since $x_1 = ae^t > 0$, we have $x_2 = (a^2b)/(x_1^2)$. At $t = 0$, $(x_1, x_2) = (a, b)$ and for $t \geq 0$, $t \mapsto x_a(t) = ae^t$ is increasing, while $t \mapsto x_2(t) = be^{-2t}$ is decreasing. The curve is shown in Figure 5.11.

3. (a) The only singular point is at 0. If the initial condition $x(t_0)$ at time t_0 is such that $x(t_0) \neq 0$, then $x'(t_0) = (x(t_0))^2 > 0$. The phase portrait is sketched in Figure 5.12.

   (b) The system has no singular point since for all real x, $e^x > 0$. Moreover, we note that $x'(t) = e^{x(t)} > 0$. The phase portrait is sketched in Figure 5.12.

   (c) The system has no singular point, since $\cosh x = (e^x + e^{-x})/2 > e^x/2 > 0$ for all $x \in \mathbb{R}$. Moreover, we note that $x'(t) = \cosh(x(t)) > 0$. The phase portrait is sketched in Figure 5.12.

   (d) The system has singular points at $n\pi$, $n \in \mathbb{Z}$. If $x(t_0)$ belongs to $(2m\pi, (2m+1)\pi)$ for some $m \in \mathbb{Z}$, then $x'(t_0) = \sin(x(t_0)) > 0$, while if $x(t_0) \in ((2m+1)\pi, (2m+2)\pi)$, then $x'(t_0) = \sin(x(t_0)) < 0$. The phase portrait is sketched in Figure 5.12.

   (e) The system has singular points at $2n\pi$, $n \in \mathbb{Z}$. Since $\cos x < 1$ for all $x \in \mathbb{R} \setminus \{2n\pi \mid n \in \mathbb{Z}\}$, we have that if the initial condition $x(t_0)$ at time t_0 does not belong to the set $\{2n\pi \mid n \in \mathbb{Z}\}$, then $x'(t_0) = \cos(x(t_0)) - 1 < 0$. The phase portrait is sketched in Figure 5.12.

   (f) The system has singular points at $n\pi/2$, $n \in \mathbb{Z}$. Since $\sin(2x) < 0$ for all $x \in (2m\pi/2, (2m+1)\pi/2)$, $m \in \mathbb{Z}$, and $\sin(2x) > 0$ for all $x \in ((2m+1)\pi/2, (2m+2)\pi/2)$, $m \in \mathbb{Z}$, we obtain the phase portrait as sketched in Figure 5.12.

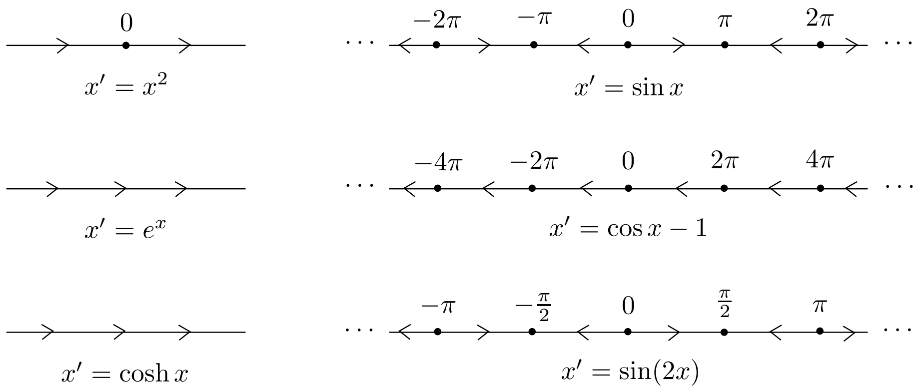

Figure 5.12: Phase portraits.

(a) Let the 2D autonomous system be given by

$$\left.\begin{array}{rcl} x_1' & = & f_1(x_1, x_2), \\ x_2' & = & f_2(x_1, x_2). \end{array}\right\} \tag{5.11}$$

For $t \in \mathbb{R}$, we have

$$\begin{aligned} y_1'(t) &= \left(\frac{d}{dt} x_1(\cdot + T)\right)(t) \\ &= x_1'(t+T)\frac{d}{dt}(t+T) \quad \text{(chain rule)} \\ &= x_1'(t+T) \cdot 1 = x_1'(t+T) \\ &= f_1(x_1(t+T), x_2(t+T)) \\ &= f_1(y_1(t), y_2(t)). \end{aligned}$$

Similarly, $y_2'(t) = f_2(y_1(t), y_2(t))$. Thus (y_1, y_2) also satisfies (5.11).

(b) By the first part, we know that y_1 given by

$$\begin{aligned} y_1(t) &:= x(t+\frac{\pi}{2}) \\ &= \left(\frac{2\cos(t+\frac{\pi}{2})}{1+(\sin(t+\frac{\pi}{2}))^2}, \frac{\sin(2(t+\frac{\pi}{2}))}{1+(\sin(t+\frac{\pi}{2}))^2}\right) \\ &= \left(\frac{-2\sin t}{1+(\cos t)^2}, \frac{-\sin(2t)}{1+(\cos t)^2}\right) \end{aligned}$$

is a solution. Also y_2 given by

$$\begin{aligned} y_2(t) &:= x(t+\frac{3\pi}{2}) \\ &= \left(\frac{2\cos(t+\frac{3\pi}{2})}{1+(\sin(t+\frac{3\pi}{2}))^2}, \frac{\sin(2(t+\frac{3\pi}{2}))}{1+(\sin(t+\frac{3\pi}{2}))^2}\right) \\ &= \left(\frac{2\sin t}{1+(\cos t)^2}, \frac{-\sin(2t)}{1+(\cos t)^2}\right) \end{aligned}$$

is a solution. We observe that $y_1(0) = (0,0) = y_2(0)$. However $y_1 \not\equiv y_2$ since (for example) $y_1(\pi/2) = (2,0) \neq (02,0) = y_2(\pi/2)$. This would mean that starting from the initial condition $(0,0)$, there are two different solutions y_1 and y_2, which contradicts our assumption about the 2D system.

(c) Using the Maple command

```
> plot([2*cos(t)/(1+(sin(t))^2),sin(2*t)/(1+(sin(t))^2),t=0..10);
```

we have sketched the curve in Figure 5.13.

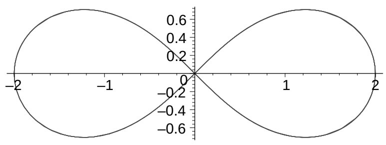

Figure 5.13: The lemniscate.

4. (a) Clearly 0 is a singular point. In order to find other singular points, we use Maple. Using the commands:

```
> eq:= 2-1/5*x-5*x/(2+x^2)=0;
> sols:=solve(eq,x);
> sols:=evalf(sols);
```

we find that the nonzero singular points are given approximately by $6.125, 1.238, 2.637$. Thus the set of singular points is $\{0, 1.238, 2.637, 6.125\}$.

(b) We can plot the function

$$f(x) = x\left(2 - \frac{1}{5}x - \frac{5x}{2+x^2}\right),$$

in the range $x \in [0,10]$ using the Maple command:

```
> plot(x*( 2-(1/5)*x -5*x/(2+x^2)),x=0..10);
```

and the result is displayed in Figure 5.14.

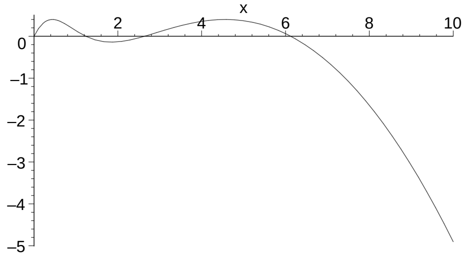

Figure 5.14: Graph of the function f.

From this plot we observe that

$$\begin{array}{rl} x \in (0, 1.24) & f(x) > 0, \\ x \in (1.25, 2.63) & f(x) < 0, \\ x \in (2.65, 6.12) & f(x) > 0, \\ x \in (6.13, \infty) & f(x) < 0. \end{array}$$

Thus the resulting phase portrait is as shown in Figure 5.15.

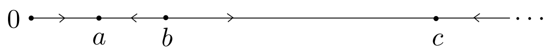

Figure 5.15: Phase portrait for $x' = f(x)$. Here a, b, c are approximately $1.238, 2.637, 6.125$, respectively.

5. (a) We have

$$\begin{aligned} I' &= I - \alpha C \\ C' &= \beta(I - C - (G_0 + kI)) = \beta((1-k)I - C - G_0). \end{aligned}$$

Clearly, $f_1(I, C) = I - \alpha C = 0$ iff $I = \alpha C$. If $I = \alpha C$, then $f_2(I, C) = \beta((1-k)\alpha C - C - G_0)$, which is zero iff

$$C = \frac{G_0}{(1-k)\alpha - 1}.$$

So the system has an equilibrium point at

$$(I_0, C_0) := \left(\frac{\alpha G_0}{(1-k)\alpha - 1}, \frac{G_0}{(1-k)\alpha - 1} \right).$$

Since $G_0 > 0$ and $\alpha > 0$, it follows that C_0 and I_0 are both positive iff $(1-k)\alpha - 1 > 0$, or equivalently iff $k < 1 - 1/\alpha$. Thus an equilibrium point for which I, C, G are nonnegative exists iff $k < 1 - 1/\alpha$.

(b) If $k =$), then the equilibrium point is at

$$(I_0, C_0) = \left(\frac{\alpha G_0}{\alpha - 1}, \frac{G_0}{\alpha - 1} \right).$$

We have

$$I_1' = I' = I - \alpha C = \left(I_1 + \frac{\alpha G_0}{\alpha - 1} \right) - \alpha \left(C_1 + \frac{G_0}{\alpha - 1} \right) = I_1 - \alpha C_1,$$

and

$$C_1' = C' = \beta(I - C - G_0) = \beta \left(I_1 + \frac{\alpha G_0}{\alpha - 1} - C_1 - \frac{G_0}{\alpha - 1} - G_0 \right) = \beta(I_1 - C_1),$$

and so we obtain

$$\begin{bmatrix} I_1' \\ C_1' \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} I_1 \\ C_1 \end{bmatrix}.$$

If $\alpha = 2$ and $\beta = 1$, then we obtain

$$\begin{bmatrix} I_1' \\ C_1' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}}_{=:A} \begin{bmatrix} I_1 \\ C_1 \end{bmatrix}.$$

We have

$$sI - A = \begin{bmatrix} s-1 & 2 \\ -1 & s+1 \end{bmatrix},$$

and so

$$(sI-A)^{-1} = \frac{1}{s^2+1} \begin{bmatrix} s+1 & -2 \\ 1 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} + \frac{1}{s^2+1} & -2\frac{1}{s^2+1} \\ \frac{1}{s^2+1} & \frac{1}{s^2+1} - \frac{1}{s^2+1}s \end{bmatrix}.$$

Thus

$$e^{tA} = \begin{bmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{bmatrix},$$

and so

$$\begin{aligned} I_1(t) &= (\cos t + \sin t)I_1(0) - 2(\sin t)C_1(0) = I_1(0)\cos t + (I_1(0) - 2C_1(0))\sin t, \\ C_1(t) &= (\sin t)I_1(0) + (\cos t - \sin t)C_1(0) = C_1(0)\cos t + (C_1(0) - I_1(0))\sin t. \end{aligned}$$

Consequently I_1, C_1 oscillate (if $(I_1(0), C_1(0)) \neq (0,0)$) and since I, C differ from I_1, C_1 by constants, we conclude that I, C oscillate as well.

Solutions to the exercises on page 27

1. We solve the differential equation
$$\frac{dx_2}{dx_1} = \frac{x_1}{x_2}.$$
Thus
$$\frac{d}{x_1}\left(\frac{1}{2}x_2^2\right) = x_2\frac{dx_2}{dx_1} = x_1,$$
and by integrating with respect to x_1, and using the fundamental theorem of calculus, we obtain
$$x_2^2 - x_1^2 = C.$$
This equation describes a hyperbola in the (x_1, x_2)-plane. Thus the trajectories $t \mapsto (x_1(t), x_2(t))$ are hyperbolas. We note that when $x_1(0)$ belongs to the right half plane, then $x_2'(0) = x_1(0) > 0$, and so $t \mapsto x_2(t)$ should be increasing, while if $x_1(0)$ belongs to the left half plane, then $x_2'(0) = x_1(0) < 0$, and so $t \mapsto x_2(t)$ should be decreasing. The phase portrait is shown in Figure 5.16.

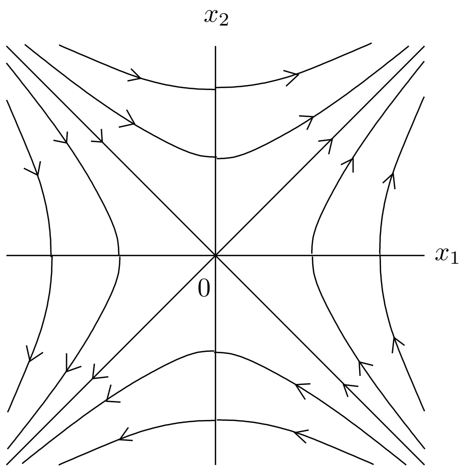

Figure 5.16: Phase portrait.

2. We solve the differential equation
$$\frac{dx_2}{dx_1} = \frac{x_1}{-2x_2}.$$
Thus
$$\frac{d}{x_1}(x_2^2) = 2x_2\frac{dx_2}{dx_1} = -x_1,$$
and by integrating with respect to x_1, and using the fundamental theorem of calculus, we obtain
$$x_2^2 + \frac{x_1^2}{2} = C.$$
This equation describes an ellipse in the (x_1, x_2)-plane. Thus the trajectories $t \mapsto (x_1(t), x_2(t))$ are ellipses. We note that when $x_1(0)$ belongs to the right half plane, then $x_2'(0) = x_1(0) > 0$, and so $t \mapsto x_2(t)$ should be increasing. Hence the motion is anticlockwise. The phase portrait is shown in Figure 5.17.

3. (a) We note that the graph of $A + B\log|x|$ looks qualitatively the same as the graph of $B\log|x|$, since the constant A shifts the graph of $B\log|x|$ by A. If we multiply this result by x, then we obtain the graph as shown in Figure 5.18.

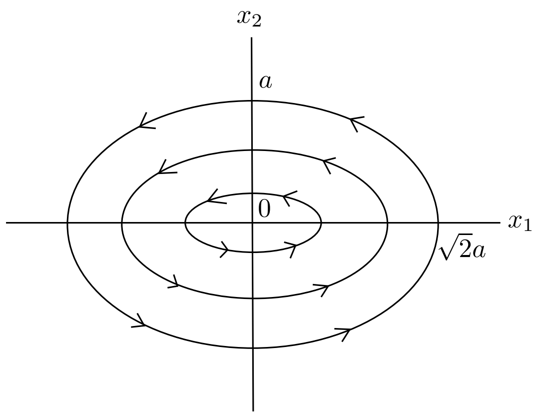

Figure 5.17: Phase portrait.

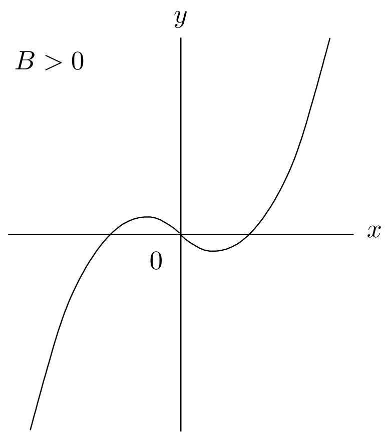

Figure 5.18: Graph of $y = x(A + B \log |x|)$.

(b) If $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $sI - A = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$, and so

$$(sI - A)^{-1} = \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}.$$

Consequently, $e^{tA} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$. Hence

$$\begin{aligned} x_1(t) &= e^t(x_1(0) + tx_2(0)) \\ x_2(t) &= e^t x_2(0). \end{aligned}$$

We consider the following two cases:

$\underline{1}^\circ$ $x_2(0) = 0$. Then $x_2(t) = 0$, and $x_1(t) = e^t x_1(0)$. Thus on the x_1 axis, the motion is away from the origin.

$\underline{2}^\circ$ $x_2 \neq 0$. If $t \geq 0$, we have $e^t \geq 1$, and $|x_2(t)| \geq |x_2(0)|$, and $t = \log |x_2(t)/x_2(0)|$. Thus

$$x_1(t) = \frac{x_2(t)}{x_2(0)} \left(x_1(0) + x_2(0) \log \left| \frac{x_2(t)}{x_2(0)} \right| \right) = x_2(t)(A + B \log |x_2(t)|),$$

where

$$A := \frac{x_1(0)}{x_2(0)} - \log|x_2(0)|, \qquad \text{and} \qquad B := 1 > 0.$$

Thus the trajectories are curves of the type found in the previous part of this exercise. Also, if $x_2(0) > 0$, then $x_2(t) \to \infty$ as $t \to \infty$, and if $x_2(0) < 0$, then $x_2(t) \to -\infty$ as $t \to \infty$.

The phase portrait is shown in Figure 5.19.

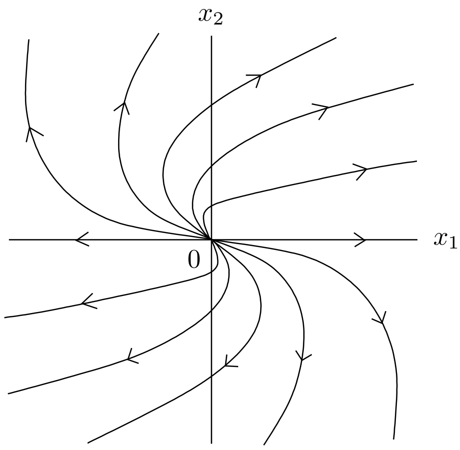

Figure 5.19: Phase portrait.

Solutions to the exercises on page 28

The isocline corresponding to slope α is given by

$$\frac{x_1}{x_2} = \alpha,$$

and so

$$x_2 = \frac{1}{\alpha} x_1,$$

and these points lie on a straight line with slope $1/\alpha$. By taking different values of α, a set of isoclines can be drawn, and in this manner a field of tangents to the trajectories are generated, and so a phase portrait can be constructed, as shown in Figure 5.20.

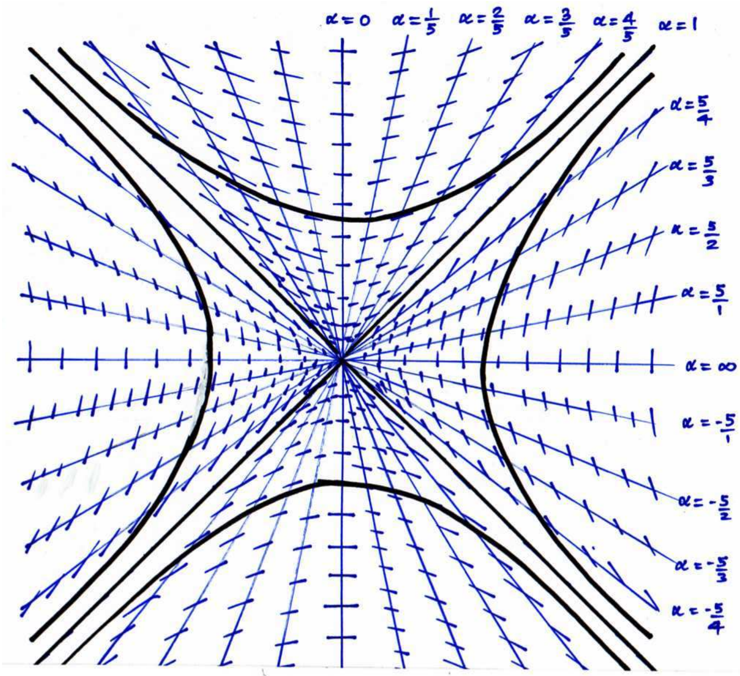

Figure 5.20: Phase portrait using isoclines.

Solutions to the exercises on page 30

1. (a) Using the following Maple commands, we obtain Figure 5.21.

> $>$ `with(DEtools) :`
> $>$ `ode1aA := diff(x1(t), t) = x2(t);`
> $>$ `ode1aB := diff(x2(t), t) = x1(t);`
> $>$ `initvalues := seq(seq([x1(0) = i + 1/2, x2(0) = j + 1/2],`
> `i = −2..1), j = −2..1) :`
> $>$ `DEplot({ode1aA, ode1aB}, [x1(t), x2(t)], t = −10..10, x1 = −2..2, x2 = −2..2,`
> `[initvalues], stepsize = 0.05, arrows = MEDIUM, colour = black,`
> `linecolour = red);`

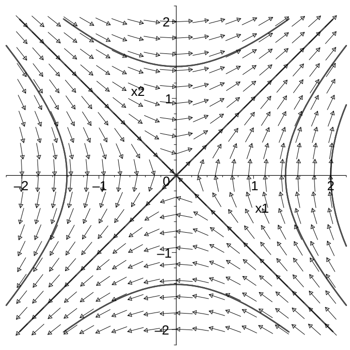

Figure 5.21: Phase portrait.

(b) Using the following Maple commands, we obtain Figure 5.22.

> $>$ `with(DEtools) :`
> $>$ `ode1bA := diff(x1(t), t) = −2 * x2(t);`
> $>$ `ode1bB := diff(x2(t), t) = x1(t);`
> $>$ `initvalues := seq(seq([x1(0) = i + 1/2, x2(0) = j + 1/2],`
> `i = −2..1), j = −2..1) :`
> $>$ `DEplot({ode1bA, ode1bB}, [x1(t), x2(t)], t = −10..10, x1 = −2..2, x2 = −2..2,`
> `[initvalues], stepsize = 0.05, arrows = MEDIUM, colour = black,`
> `linecolour = red);`

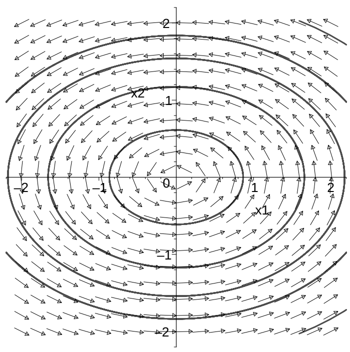

Figure 5.22: Phase portrait.

(c) Using the following Maple commands, we obtain Figure 5.23.

$>$ `with(DEtools) :`
$>$ `ode1cA := diff(x1(t), t) = x1(t) + x2(t);`
$>$ `ode1cB := diff(x2(t), t) = x2(t);`
$>$ `initvalues := seq(seq([x1(0) = i + 1/2, x2(0) = j + 1/2],`
`i = −2..1), j = −2..1) :`
$>$ `DEplot({ode1cA, ode1cB}, [x1(t), x2(t)], t = −10..10, x1 = −2..2, x2 = −2..2,`
`[initvalues], stepsize = 0.05, arrows = MEDIUM, colour = black,`
`linecolour = red);`

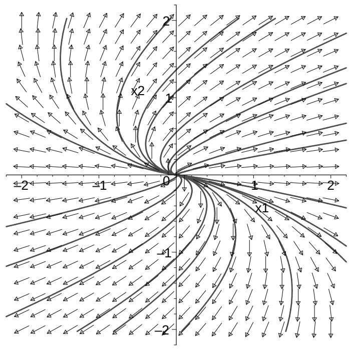

Figure 5.23: Phase portrait.

2. (a) Clearly $f_1(x_s, x_b) = x_s(a - bx_b) = 0$ iff $[x_s = 0$ or $x_b = a/b]$. If $x_s = 0$, then $f_2(x_s, x_b) = -cx_b$, and this is 0 iff $x_b = 0$. On the other hand, if $x_b = a/b$, then $f_2(x_s, x_b) = -ca/b + x_s da/b$, and this is 0 iff $x_s = c/d$. So the only singular points are $(c/d, a/b)$.

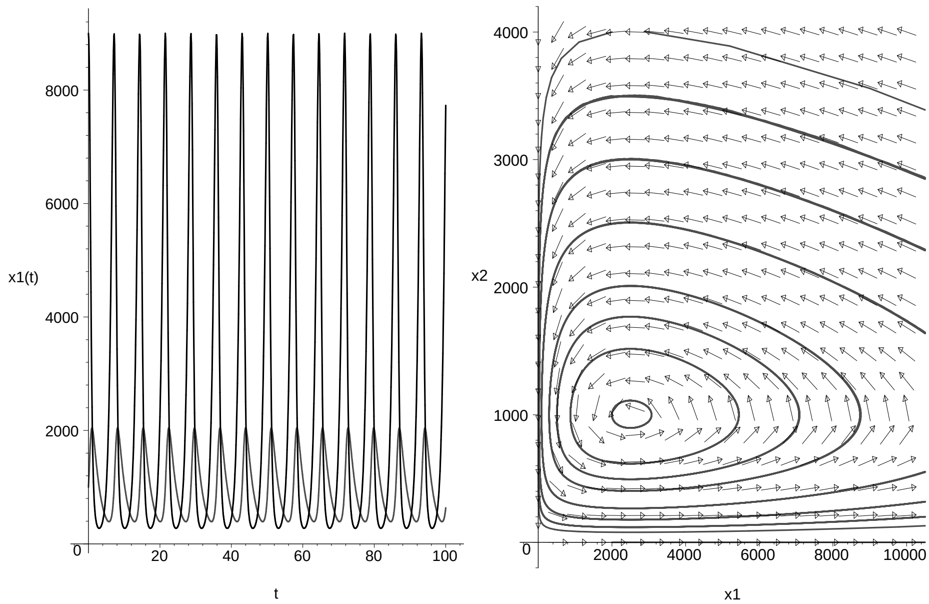

Figure 5.24: Plots of the solution curves and phase portrait for the Lotka-Volterra ODE system.

(b) Using the following Maple commands, we obtain Figure 5.24.

```
> with(DEtools):
> ode2bA := diff(x1(t), t) = x1(t) + x2(t);
> ode2bB := diff(x2(t), t) = x2(t);
> with(plots):
> plot1 := DEplot({ode2bA, ode2bB}, x1(t), x2(t), t = 0..100,
  [[x1(0) = 9000, x2(0) = 1000]], scene = [t, x1(t)], stepsize = 0.1):
> plot2 := DEplot({ode2bA, ode2bB}, x1(t), x2(t), t = 0..100,
  [[x1(0) = 9000, x2(0) = 1000]], scene = [t, x2(t)], stepsize = 0.1):
> display(plot1, plot2);
```

(c) Using the following Maple commands, we obtain Figure 5.24.

```
> initvalues := seq([x1(0) = 2000, x2(0) = 500 * j], j = 0..8):
> DEplot({ode2bA, ode2bB}, [x1(t), x2(t)], t = 0..100, x1 = 0..10000, x2 = 0..4000,
  [initvalues, [x1(0) = 9000, x2(0) = 1000]], stepsize = 0.1,
  arrows = MEDIUM, colour = black, linecolour = red);
```

We observe that the solution curve from the previous part is a closed curve, indicating the periodic rise and fall of the populations.

Solutions to the exercises on page 34

1. Since v_1 and v_2 are eigenvectors corresponding to the eigenvalues 1 and -2, respectively, we have $e^{tA}v_1 = e^t v_1$, and $e^{tA}v_2 = e^{-2t}v_2$. Hence in the direction of the eigenvector v_1, the motion is away from 0, while in the direction of the eigenvector v_2, the motion is towards 0.

   If $x(t) = \alpha(t)v_1 + \beta(t)v_2$, we have

$$\begin{aligned}\alpha(t)v_1 + \beta(t)v_2 = x(t) = e^{tA}x(0) &= e^{tA}(\alpha(0)v_1 + \beta(0)v_2)\\ &= \alpha(0)e^{tA}v_1 + \beta(0)e^{tA}v_2\\ &= \alpha(0)e^t v_1 + \beta(0)e^{-2t}v_2,\end{aligned}$$

   and by the independence of v_1 and v_2, it follows that $\alpha(t) = e^t\alpha(0)$ and $\beta(t) = e^{-2t}\beta(0)$. Clearly then we have

$$(\alpha(t))^2\beta(t) = e^{2t}(\alpha(0))^2 e^{-2t}\beta(0) = (\alpha(0))^2\beta(0).$$

   So we obtain the phase portrait as shown in Figure 5.25.

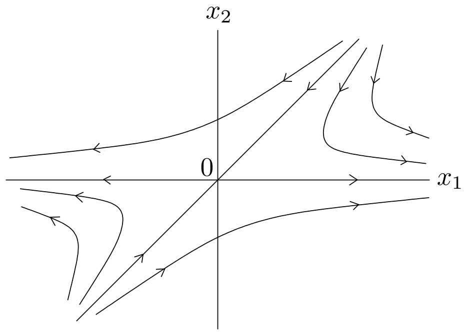

Figure 5.25: Phase portrait.

2. (a) Since $A(u+iv) = (a+ib)(u+iv)$, by taking entry-wise complex conjugates, we obtain $A(u-iv) = (a-ib)(u-iv)$. Moreover, since $u+iv$ is an eigenvector, $u+iv \neq 0$, and so $u-iv \neq 0$ as well. Consequently $u-iv$ is an eigenvector corresponding to the eigenvalue $a-ib$.

   (b) More generally, we prove that if v_1, v_2 are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively, of a linear transformation $T : X \to X$, then v_1, v_2 are linearly independent in the complex vector space X. For if

$$\alpha v_1 + \beta v_2 = 0, \tag{5.12}$$

   then upon operating by T, we obtain

$$\alpha\lambda_1 v_1 + \beta\lambda_2 v_2 = 0. \tag{5.13}$$

   Now if we subtract λ_1 times the equation (5.12) from equation (5.13), then we obtain $\beta(\lambda_2 - \lambda_1)v_2 = 0$. Since v_2 is an eigenvector, $v_1 \neq 0$, and so we conclude that $\beta(\lambda_2 - \lambda_1) = 0$. But $\lambda_1 \neq \lambda_2$, and so $\beta = 0$. Substituting this in (5.12) gives $\alpha v_1 = 0$, and so $\alpha = 0$.

   If $b \neq 0$, the eigenvalues $a+ib$ and $a-ib$ are distinct and applying the above, we can conclude that v_1, v_2 are independent vectors in $\mathbb{C}^2$.

(c) We note that $u = (v_1 + v_2)/2$ and $v = (v_1 - v_2)/2i$. Thus if $\alpha, \beta \in \mathbb{R}$ are such that $\alpha u + \beta v = 0$, then we have

$$\alpha \left(\frac{v_1 + v_2}{2} \right) + \beta \left(\frac{v_1 - v_2}{2i} \right) = 0,$$

and so $(\beta + i\alpha)v_1 + (i\alpha - \beta)v_2 = 0$. By the independence of v_1, v_2, it follows that $\beta + i\alpha = 0$, and so $\alpha = 0$ and $\beta = 0$. So u, v are linearly independent vectors in $\mathbb{R}^2$.

(d) We have $Au + iAv = A(u + iv) = (a + ib)(u + iv) = au - bv + i(av + bu)$, and so it follows that

$$\begin{array}{rcl} Au &=& au - bv \\ Av &=& av + bu, \end{array}$$

that is,

$$AP = A \left[\begin{array}{cc} u & v \end{array} \right] = \left[\begin{array}{cc} u & v \end{array} \right] \left[\begin{array}{cc} a & b \\ -b & a \end{array} \right] = P \left[\begin{array}{cc} a & b \\ -b & a \end{array} \right],$$

and since P is invertible, by premultiplying by P^{-1}, we obtain the desired result.

Solutions to the exercises on page 39

1. We have $f_1(x_1, x_2) = x_1(1 - x_2) = 0$ iff $[x_1 = 0$ or $x_2 = 1]$. But if $x_1 = 0$, then $f_2(x_1, x_2) = e^{x_1+x_2} - x_2 = e^{x_2} - x_2 > 0$ for $x_2 \in \mathbb{R}$. On the other hand, if $x_2 = 1$, then $f_2(x_1, x_2) = e^{x_1+1} - 1$. Clearly $e^{x_1+1} - 1 = 0$ iff $x_1 + 1 = 0$, that is, $x_1 = -1$. So the only singular point is $(-1, 1)$.

   Linearisation about the point $(-1, 1)$ gives

$$\left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} = e^{x_1+x_2} & \frac{\partial f_1}{\partial x_2} = e^{x_1+x_2} - 1 \\ \frac{\partial f_2}{\partial x_1} = -1 + x_2 & \frac{\partial f_2}{\partial x_2} = x_1 \end{array}\right]\Bigg|_{(x_1,x_2)=(-1,1)} = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right].$$

   The eigenvalues are 1 and -1. Since one of the eigenvalues is positive, it follows from the linearisation theorem that the equilibrium point $(-1, 1)$ is unstable.

2. We note that $f_2(x_1, x_2) = -x_2 - x_2 e^{x_1} = -x_2(1 + e^{x_1}) = 0$ iff $x_2 = 0$. And if $x_2 = 0$, then $f_1(x_1, x_2) = x_1 + e^{x_1} - 1 > 0$ for all $x_1 \in \mathbb{R}$. So the system does not have any singular points, and the linearisation theorem is not applicable.

3. Clearly the only singular point is $(0, 0)$. Linearisation about the point $(0, 0)$ gives

$$\left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} = 0 & \frac{\partial f_1}{\partial x_2} = 1 \\ \frac{\partial f_2}{\partial x_1} = -3x_1^2 & \frac{\partial f_2}{\partial x_2} = 0 \end{array}\right]\Bigg|_{(x_1,x_2)=(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right],$$

   which has both eigenvalues equal to 0, and so the linearisation theorem is not applicable.

4. We have $f_2(x_1, x_2) = x_2 = 0$ iff $x_2 = 0$. Also, if $x_2 = 0$, then $f_1(x_1, x_2) = sin x_1 = 0$ iff $x_1 \in \{n\pi \mid n \in \mathbb{Z}\}$. Thus the singular points are $(n\pi, 0)$, $n \in \mathbb{Z}$, and each one of them is isolated.

   Linearisation about the point $(n\pi, 0)$ gives

$$\left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} = \cos(x_1 + x_2) & \frac{\partial f_1}{\partial x_2} = \cos(x_1 + x_2) \\ \frac{\partial f_2}{\partial x_1} = 0 & \frac{\partial f_2}{\partial x_2} = 1 \end{array}\right]\Bigg|_{(x_1,x_2)=(n\pi,0)} = \left[\begin{array}{cc} \cos(n\pi) & \cos(n\pi) \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} (-1)^n & (-1)^n \\ 0 & 1 \end{array}\right].$$

   The eigenvalues are $(-1)^n$ and 1. Since 1 is always an eigenvalue, by the linearisation theorem we can conclude that each equilibrium point is unstable.

5. Again, the singular points are $(n\pi, 0)$, $n \in \mathbb{Z}$, but now the linearisation about the point $(n\pi, 0)$ gives

$$\left[\begin{array}{cc} (-1)^n & (-1)^n \\ 0 & -1 \end{array}\right],$$

   which has eigenvalues $(-1)^n$ and -1. We now consider the two possible cases:

   1° n even: Then $(-1)^n = 1$, and so this equilibrium point is unstable.

   2° n odd: Then $(-1)^n = -1$, and so this equilibrium point is aysmptotically stable.

   By the linearisation theorem we conclude that the equilibrium points $(2m\pi, 0)$, $m \in \mathbb{Z}$, are all unstable, while the equilibrium points $((2m+1)\pi, 0)$, $m \in \mathbb{Z}$, are all asymptotically stable.

Solutions to the exercises on page 44

1. Let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence of points in A that is convergent to (x, y). Then for all $n \in \mathbb{N}$, $x_n y_n = 1$, and so

$$xy = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} 1 = 1 \neq 0,$$

   and so $(x, y) \notin B$.

   On the other hand, if $((x_n, y_n))_{n\in\mathbb{N}}$ is a sequence of points in B that is convergent to (x, y), then for all $n \in \mathbb{N}$, $x_n y_n = 0$, and so

$$xy = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} 0 = 0 \neq 1,$$

   and so $(x, y) \notin A$. Thus A and B are separated.

   Remark. Note that although A and B are separated, given any $\epsilon > 0$, we can choose a point from A and a point from B such that their distance is smaller than ϵ. In this sense the "distance" between the two sets is 0.

2. The system has no singular points since $f_2(x_1, x_2) = x_1 = 0$ implies that $f_1(x_1, x_2) = 1 - 0 \cdot x_2 = 1 \neq 0$. By Theorem 2.5.4, it follows that the system has no periodic trajectory, and in particular, no limit cycle.

3. We have

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - (1 + x_1^2) < -1 < 0.$$

   By Theorem 2.5.5 (with $\Omega = \mathbb{R}^2$!), it follows that the system has no periodic trajectories.

4. Suppose that the system has a periodic trajectory starting inside C and which does not intersect C. We have

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2 - 4(x_1^2 + x_2^2),$$

   which is positive inside the simply connected region Ω given by $x_1^2 + x_2^2 < 1/2$. Thus by Theorem 2.5.5, this region cannot wholly contain the periodic trajectory. So this periodic trajectory has to intersect the circle C.

Chapter 3: Stability theory

Solution to the exercise on page 46

We observe that $x = \cos x$ has only one real solution. This can be seen by sketching the graphs of the functions x and $\cos x$. Since $|\cos x| \leq 1$ for all real x, it follows that the graph of x does not intersect the graph of $\cos x$ when $|x| \geq \pi/2 > 1$. In the interval $(-\pi/2, \pi/2)$, the graph of the function x intersects the graph of $\cos x$ in precisely one point. See Figure 5.26. Thus there is only one equilibrium point.

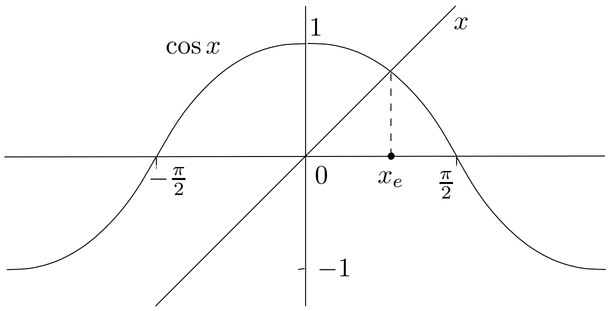

Figure 5.26: Graphs of x and $\cos x$.

Solution to the exercise on page 48

If $a > 0$, then 0 is an unstable equilibrium point. Indeed, if $R = 1$, then for every $r > 0$, if we take $x(0) = r$, we have that $x(t) = e^{ta}x(0) = re^{ta} \to \infty$ as $t \to \infty$. Thus $x(t)$ eventually goes outside the ball $B(0, 1)$. Hence 0 is an unstable equilibrium point.

On the other hand, if $a \leq 0$, then 0 is a stable equilibrium point. Let $R > 0$. Then with $r := R$, we have that for every initial condition $x(0$ satisfying $|x(0) - 0| < r = R$, there holds that $|x(t) - 0| = |e^{ta}x(0)| = e^{ta}|x(0)| \leq 1 \cdot R = R$. So 0 is a stable equilibrium point.

Hence we conclude that 0 is a stable equilibrium point for $x' = ax$ iff $a \leq 0$.

Solution to the exercise on page 50

We have already seen in a previous exercise that 0 is a stable equilibrium point iff $a \leq 0$.

If $a = 0$, then 0 is not an asymptotically stable equilibrium point. Indeed, for every $x(0) \neq 0$, the solution is $x(t) = x(0)$ and this does not tend to 0 as $t \to \infty$.

If $a < 0$, then with $M := 1$, $\epsilon := |a|$ and $r := 1$, we have for all initial conditions $x(0)$ satisfying $|x(0) - 0| \leq r = 1$ that $|x(t) - 0| = |x(t)| = |e^{ta}x(0)| \leq e^{ta} \cdot 1 = 1 \cdot e^{-t|a|} = Me^{-\epsilon t}$. So 0 is an exponentially stable equilibrium point (and in particular, it is asymptotically stable).

Hence we conclude that the following are equivalent:

1. 0 is an asymptotically stable equilibrium point for $x' = ax$.
2. 0 is an exponentially stable equilibrium point for $x' = ax$.
3. $a < 0$.

Solutions to the exercises on page 51

1. (a) The eigenvalues are 1 and -1. Since one of the eigenvalues of A has a positive real part, we conclude that 0 is an unstable equilibrium point.

   (b) The eigenvalues are $1, -2, 1$. Since one of the eigenvalues of A has a positive real part, we conclude that 0 is an unstable equilibrium point.

   (c) The eigenvalues are $(1+\sqrt{5})/2, (1-\sqrt{5})/2, -2$. Since one of the eigenvalues of A has a positive real part, we conclude that 0 is an unstable equilibrium point.

   (d) The eigenvalues are $-1, -2, -1$. Since all the eigenvalues of A have negative real parts, we conclude that 0 is an exponentially stable equilibrium point.

   (e) The eigenvalue is 0, and its geometric multiplicity is 1, while the algebraic multiplicity is 3. So 0 is an unstable equilibrium point.

   (f) The eigenvalues are $i, -i$. In each case the geometric and algebraic multiplicities are both equal to 1, and so 0 is a stable equilibrium point.

2. (a) The only equilibrium point is 0. Linearisation about the point 0 gives the matrix

$$\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} = 1 & \frac{\partial f_1}{\partial x_2} = 1 \\ \frac{\partial f_2}{\partial x_1} = 3x_1^2 & \frac{\partial f_2}{\partial x_2} = -1 \end{bmatrix}\right|_{(x_1,x_2)=(0,0)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

   which has the eigenvalues $1, -1$. Since one of the eigenvalues has a positive real part, we conclude that 0 is an unstable equilibrium point.

   (b) The equilibrium points are $(0,0)$, $(1,1)$ and $(-1,-1)$. We have

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3x_1^2 & -1 \end{bmatrix}.$$

   Linearisation at $(0,0)$ gives the matrix

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

   and since all its eigenvalues have a negative real part, we conclude that $(0,0)$ is a stable equilibrium point.
   Linearisation at $(1,1)$ or $(-1,-1)$ gives the matrix

$$\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix},$$

   which has the eigenvalues $\sqrt{3}-1$ and $-\sqrt{3}-1$. Since $\sqrt{3}-1 > 0$, it follows that $(1,1)$ and $(-1,-1)$ are unstable equilibrium points.

Solution to the exercise on page 54

We check that L1 and L2 hold.

L1. Since $|\cos\theta| \leq 1$ for real θ, it follows that $mgl(1-\cos x_1) \geq 0$. Thus $V(x) \geq 0$ for all $x \in \mathbb{R}^2$.

If $V(x_1, x_2) = 0$, then $x_2 = 0$ and $x_1 = 2n\pi$, $n \in \mathbb{Z}$. In order to have "$V(x_1, x_2) = 0$ only if $(x_1, x_2) = 0$", we must restrict x_2, for instance in $[-\pi, \pi]$. So with the choice $R = \pi$, we have: If $x \in B(0, R)$ and $V(x) = 0$, then $x = 0$.

Finally, $V(0,0) = ml^2 0^2/2 + mgl(1 - \cos 0) = 0$.

L2. We have

$$\nabla V(x) \cdot f(x) = \begin{bmatrix} mgl\sin x_1 & ml^2 x_2 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ -\frac{k}{ml^2}x_2 - \frac{g}{l}\sin x_1 \end{bmatrix} = -kx_2^2 \leq 0.$$

Thus V is a Lyapunov function for the system.

Solutions to the exercises on page 56

1. We verify that L1, L2, L3 hold.

   L1. Since $|\cos\theta| \leq 1$ for real θ, it follows that $2(1-\cos x_1) \geq 0$. Thus $V(x) \geq 0$ for all $x \in \mathbb{R}^2$.
   If $V(x_1, x_2) = 0$, then $x_2 = 0$, $x_1 + x_2 = 0$ and $\cos x_1 = 1$. Hence $x_1 = x_2 = 0$.
   Finally, $V(0,0) = 0^2/2 + (0+0)^2/2 + 2(1-\cos 0) = 0$.

   L2. We have

   $$\nabla V(x) \cdot f(x) = \begin{bmatrix} x_1 + x_2 + 2\sin x_1 & x_1 + 2x_2 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ -x_2 - \sin x_1 \end{bmatrix} = -x_1 \sin x_1 - 2x_2^2.$$

   We note that $x_1 \sin x_1$ is not always nonnegative, but it is nonnegative for instance in the interval $[-\pi/2, \pi/2]$. So with $R = \pi/2$, we have that for all $x \in B(0,R)$, $\nabla V(x) \cdot f(x) \leq 0$.

   L3. Clearly $\nabla V(0) \cdot f(0) = 0$. On the other hand if $x \in B(0, \pi/2)$ and $\nabla V(x) \cdot f(x) = 0$, then we have $x_1 \sin x_1 + 2x_2^2 = 0$. Since $x_1 \in [-\pi/2, \pi/2]$, we know that $x_1 \sin x_1 \geq 0$. Hence we can conclude that $x_1 \sin x_1 = 0$ and $x_2 = 0$. Again using the fact that $x_1 \in [-\pi/2, \pi/2]$, we conclude that $x_1 = 0$. So $(x_1, x_2) = (0,0)$.

   Thus V is a strong Lyapunov function for the system. Consequently, 0 is an asymptotically stable equilibrium point for the system.

2. (a) We verify that L1, L2, L3 hold.

   L1. For all $x \in \mathbb{R}^2$, $V(x_1, x_2) = x_1^2 + x_2^2 \geq 0$.
   If $V(x_1, x_2) = x_1^2 + x_2^2 = 0$, then $x_1 = x_2 = 0$.
   Finally, $V(0,0) = 0^2 + 0^2 = 0$.

   L2. We have

   $$\nabla V(x) \cdot f(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \cdot \begin{bmatrix} x_2 - x_1(x_1^2 + x_2^2) \\ -x_1 - x_2(x_1^2 + x_2^2) \end{bmatrix} = -2(x_1^2 + x_2^2)^2 \leq 0.$$

   L3. Clearly $\nabla V(0) \cdot f(0) = 0$. On the other hand if $\nabla V(x) \cdot f(x) = 0$, then we have $2(x_1^2 + x_2^2)^2 = 0$, and so $x_1 = x_2) = 0$.

   Thus V is a strong Lyapunov function for the system.

   (b) Since V is a strong Lyapunov function for the system, it follows that 0 is an asymptotically stable equilibrium point.

Chapter 4: Existence and uniqueness

Solution to the exercise on page 63

Define the continuous function g_n $(n = 0, 1, 2, \dots)$ as follows:

$$g_n(t) = f(x_n(t), t).$$

Then the sequence $g_0, g_1, g_2, \dots$ is the sequence of partial sums of the series

$$g_0 + \sum_{k=1}^{n} (g_{k+1} - g_k). \tag{5.14}$$

We have

$$|g_{k+1}(t) - g_k(t)| = |f(x_{k+1}(t), t) - f(x_k(t), t)| \leq L|x_{k+1}(t) - x_k(t)| \leq L\|x_{k+1} - x_k\| \leq \frac{1}{2^k}\|x_1 - x_0\|,$$

where we have used (4.2) to obtain the last inequality. Thus $\|g_{k+1} - g_k\| \leq L\|x_1 - x_0\|/2^k$, and so (5.14) converges absolutely to a function g. We have

$$g(t) = \lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} f(x_n(t), t) = f(x(t), t).$$

By Theorem 4.2.2,

$$\lim_{n\to\infty} \int_{t_0}^{t} f(x_n(t), t)dt = \lim_{n\to\infty} \int_{t_0}^{t} (g_0(t) + \sum_{k=0}^{n-1} (g_{k+1}(t) - g_k(t)))dt = \int_{t_0}^{t} f(x(t), t)dt.$$

Solutions to the exercises on page 64

1. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Given $r > 0$, if $x, y \in B(0, r) = [-r, r]$, then by the mean value theorem, there exists a $c \in [-r, r]$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

Hence if M is the maximum of the continuous function $|f'|$ in the closed interval $[-r, r]$ (extreme value theorem!), it follows that

$$|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \le M|x - y|,$$

and so f is locally Lipschitz (with M serving as a Lipschitz constant in $[-r, r]$).

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Let $c \in \mathbb{R}$, and let L be a Lipschitz constant in $[-(|c|+1), |c|+1]$. Given $\epsilon > 0$, let $\delta := \min\{1, \epsilon/L\}$. Let $x \in \mathbb{R}$ be such that $|x - c| < \delta$. Then $|x| = |x - c + c| \le |x - c| + |c| \le \delta + |c| \le 1 + |c|$, and so $x \in [-(|c| + 1), |c| + 1]$. Clearly $c \in [-(|c| + 1), |c| + 1]$. Thus we have

$$|f(x) - f(c)| \le L|x - c| \le L\delta \le L\frac{\epsilon}{L} = \epsilon.$$

Hence f is continuous at c. Since the choice of c was arbitrary, it follows that f is continuous.

(c) Let $r > 0$, and let there exist a L such that for all $x, y \in [0, r]$,

$$|\sqrt{x} - \sqrt{y}| \le L|x - y|.$$

Thus

$$|\sqrt{x} - \sqrt{y}| \le L|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|,$$

and for distinct x and y, we obtain

$$1 \le L|\sqrt{x} + \sqrt{y}|.$$

Now choose N large enough so that $1/N^2 < r$ (Archimedean principle!). Then by taking $x := 1/n^2$ and $y := 0$, where $n > N$, we obtain

$$1 \le L \cdot \frac{1}{n},$$

and by passing the limit as $n \to \infty$, we obtain that $1 \le 0$, a contradiction.

2. (a) By the mean value theorem,

$$\sin x - \sin y = (\cos c)(x - y),$$

for some real c. Since $|\cos c| \le 1$, we see that for all $x, y \in \mathbb{R}$, $|\sin x - \sin y| \le 1 \cdot |x - y|$, and so $L = 1$ serves as a Lipschitz constant.

(b) By the mean value theorem,

$$\frac{1}{1 + x^2} - \frac{1}{1 + y^2} = \left(\frac{-2c}{(1 + c^2)^2}\right)(x - y),$$

for some real c. Using elementary calculus, it can be shown that the function $c \mapsto \frac{2|c|}{(1+c^2)^2}$ has a maximum at $c = \frac{1}{2}$, and so we have that for all $x, y \in \mathbb{R}$,

$$\left|\frac{1}{1 + x^2} - \frac{1}{1 + y^2}\right| \le \frac{2 \cdot \frac{1}{2}}{(1 + \frac{1}{4})^2} \cdot |x - y|,$$

and so $L = 16/25$ serves as a Lipschitz constant.

(c) Without loss of generality, we may assume that $|x| > |y|$. Then $a := -|x| + |y| < 0$, and by the mean value theorem, $e^a - 1 = e^c a$ for some $c < 0$, and so $|e^a - 1| = |a||e^c| < |a| \cdot 1 = |a|$. Hence

$$\begin{aligned}
|e^{-|x|} - e^{-|y|}| &= |(e^{-|x|+|y|} - 1)e^{-|y|}| \\
&= |e^{-|x|+|y|} - 1||e^{-|y|}| \\
&\leq |e^{-|x|+|y|} - 1| \cdot 1 \\
&= |e^a - 1| \\
&< |a| = |-|x| + |y|| \\
&\leq |x - y|,
\end{aligned}$$

where the very last inequality is simply the triangle inequality. Thus $L = 1$ serves as a Lipschitz constant.

(d) By the mean value theorem,

$$\arctan x - \arctan y = \left(\frac{1}{1+c^2}\right)(x - y),$$

for some real c. Since $|1/(1+c^2)| \leq 1$, we see that for all $x, y \in \mathbb{R}$, $|\arctan x - \arctan y| \leq 1 \cdot |x - y|$, and so $L = 1$ serves as a Lipschitz constant.

3. f is in fact a constant! We have for distinct $x, y \in \mathbb{R}$ that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq L|x - y|.$$

Hence with $y = c$ (fixed) and $x = c + h$ ($h \neq 0$), we have

$$\left|\frac{f(c+h) - f(c)}{h}\right| \leq L|h|,$$

that is,

$$-h \leq \frac{f(c+h) - f(c)}{h} \leq h,$$

and so by the Sandwich theorem,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and it equals $\lim\limits_{h \to 0} h = 0$. In other words, $f'(c)$ exists and it equals 0. Since the choice of c was arbitrary, it follows that f is differentiable, and the derivative at each point is 0.

Solutions to the exercises on page 66

1. If $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, then with $z := x + \alpha y$, we have

$$0 \le \langle z, z\rangle = \langle x + \alpha y, x + \alpha y\rangle = \langle x, x\rangle + 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle. \tag{5.15}$$

We assume now that $\langle y, y\rangle \neq 0$. (If $\langle y, y\rangle = 0$, then the Cauchy-Schwarz inequality is obvious, since in that case $y = 0$, and both sides of the inequality are 0!) If $\langle y, y\rangle \neq 0$, then with $\alpha = -\langle x, y\rangle/\langle y, y\rangle$, we obtain from (5.15) that

$$\langle x, x\rangle - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} \ge 0,$$

and so $|\langle x, y\rangle|^2 \le \|x\|^2\|y\|^2$. Taking square roots, we have $|\langle x, y\rangle| \le \|x\|\|y\|$.

2. (a) Let $f(t) = E(t + \tau)$, $t \in \mathbb{R}$. Then $f'(t) = E'(t + \tau) = E(t + \tau) = f(t)$. We have $f(0) = E(\tau)$. It is easy to check that $x(t) = aE(t)$ satisfies the initial value problem $x'(t) = x(t)$, $x(0) = a$, and since we know that the solution is unique, we know that this is *the* solution. In light of this observation, we can conclude that $f(t) = f(0)E(t) = E(\tau)E(t)$. Consequently, $E(t + \tau) = E(t)E(\tau)$.

   (b) With $\tau = -t$, we have $E(t)E(-t) = E(t - t) = E(0) = 1$.

   (c) Since for all $t \in \mathbb{R}$, $E(t)E(-t) = 1$, $E(t)$ can never be 0.

3. (a) We have $S''(t) + S(t) = 0$, and upon differentiation we obtain that $S'''(t) + S'(t) = 0$. Thus with $f := S'$, we see that f satisfies the equation $f''(t) + f(t) = 0$, and moreover, we have that $f(0) = S'(0) = 1$, and $f'(0) = S''(0) = -S(0) = -0 = 0$. But the equation $f''(t) + f(t) = 0$ with $f(0) = 1$ and $f'(0) = 0$ has the unique solution (which we have decided to denote by C). Hence $(S' =)f = C$.
   Furthermore, $-S = S'' = (S')' = C'$.

   (b) We have

$$((S(t))^2 + (C(t))^2)' = 2S(t)S'(t) + 2C(t)C'(t) = 2S(t)C(t) + 2C(t)(-S(t)) = 0.$$

   Integrating, we have $(S(t))^2 + (C(t))^2 - ((S(0))^2 + (C(0))^2) = 0$, that is, $(S(t))^2 + (C(t))^2 = (S(0))^2 + (C(0))^2 = 0^2 + 1^2 = 1$.

   (c) Let $x(t) = aC(t) + bS(t)$. Then $x''(t) = aC''(t) + bS''(t) = -aC(t) - bS(t) = -x(t)$, and so x satisfies the equation $x''(t) + x(t) = 0$. Moreover,

$$x(0) = aC(0)+bS(0) = a\cdot 1+b\cdot 0 = a, \quad \text{and} \quad x'(0) = -aS(0)+bC(0) = -a\cdot 0+b\cdot 1 = b.$$

   Hence x satisfies the initial value problem $x''(t) + x(t) = 0$, $x(0) = a$, $x'(0) = b$. But since we know that the solution is unique, this is *the* solution.
   Now let $f(t) = S(t + \tau)$, $t \in \mathbb{R}$. Then

$$f''(t) + f(t) = S''(t + \tau) + S(t + \tau) = 0.$$

   Moreover, $f(0) = S(\tau)$ and $f'(0) = C(\tau)$. From the above, we can conclude that

$$f(t) = \underbrace{S(\tau)}_{a}C(t) + \underbrace{C(\tau)}_{b}S(t).$$

   Thus we have $S(t + \tau) = S(\tau)C(t) + C(\tau)S(t)$. Differentiating with respect to t gives $C(t + \tau) = -S(t)S(\tau) + C(t)C(\tau)$.

(d) Let $f(t) = S(-t)$, $t \in \mathbb{R}$. Then $f'(t) = -C(-t)$, and $f''(t) = -(-S(-t)\cdot(-1)) = -S(-t) = -f(t)$, and so $f''(t) + f(t) = 0$. Moreover, $f(0) = S(-0) = S(0) = 0$, and $f'(0) = -C(-0) = -C(0) = -1$. Hence $f(t) = 0 \cdot C(t) + (-1)\cdot S(t) = -S(t)$, that is, $S(-t) = -S(t)$. Differentiating we obtain $C(-t)\cdot(-1) = -C(t)$, and so $C(-t) = C(t)$.

(e) We will use the intermediate value theorem to conclude that such an α exists. In order to do this, we need to show that $C(x)$ is not always positive. We will do this by bounding C above by a function that becomes negative.

Since $(S(t))^2 + (C(t))^2 = 1$, we see that $C(t) \le 1$. Thus $S'(t) = C(t) \le 1$. Integrating, we obtain $S(t) - S(0) = S(t) \le t$. Hence $C'(t) = -S(t) \ge -t$. Integrating, we have $C(t) - C(0) = C(t) - 1 \ge -t^2/2$, and so $C(t) \ge 1 - t^2/2$. So $S'(t) = C(t) \ge 1 - t^2/2$. Integrating, we obtain $S(t) - S(0) = S(t) - 0 \ge t - t^3/6$, and so $S(t) \ge t - t^3/6$. Thus $C'(t) = -S(t) \le t - t^3/6$. Integrating, we have $C(t) - 1 \le -t^2/2 + t^4/24$. Consequently, $C(t) \le 1 - t^2/2 + t^4/24$.

Now $C(0) = 1 > 0$ and $C(\sqrt{3}) \le 1 - (\sqrt{3})^2/2 + (\sqrt{3})^4/24 = -1/8 < 0$. By the intermediate value theorem applied to the continuous function $C : [0, \sqrt{3}] \to \mathbb{R}$, it follows that there exists an $\alpha \in (0, \sqrt{3})$ such that $C(\alpha) = 0$.

Chapter 5: Underdetermined equations

Solutions to the exercises on page 71

1. By the chain rule,

$$\begin{aligned} q'(t) &= -\frac{1}{(p(t)+\alpha)^2}p'(t) = -\frac{\gamma(p(t)+\alpha)(p(t)+\beta)}{(p(t)+\alpha)^2} = -\frac{\gamma(p(t)+\beta)}{p(t)+\alpha} \\ &= -\gamma\frac{p(t)+\alpha+\beta-\alpha}{p(t)+\alpha} = -\gamma\left[\frac{p(t)+\alpha}{p(t)+\alpha} + (\beta-\alpha)\frac{1}{p(t)+\alpha}\right] \\ &= -\gamma[1+(\beta-\alpha)q(t)] = \gamma(\alpha-\beta)q(t)-\gamma, \end{aligned}$$

for all $t \in [0,T]$.

2. If $p(t)+1 \neq 0$ for all $t \in [0,1]$, then with

$$q(t) := \frac{1}{p(t)+1}, \quad t \in [0,1],$$

then by the previous exercise, we have that $q'(t) = 1(1-(-1))q(t) - 1 = 2q(t) - 1$, so that

$$\begin{aligned} q(t) &= e^{t2}q(0) + e^{2t}\int_0^t e^{-\tau 2}(-1)1 d\tau = e^{2t}q(0) - e^{2t}\int_0^t e^{-2\tau}d\tau \\ &= e^{2t}q(0) + \frac{e^{2t}}{2}(e^{-2t}-1) = e^{2t}q(0) + \frac{1-e^{2t}}{2}. \end{aligned}$$

As $q(1) = 1/(p(1)+1) = 1$, we obtain $1 = e^2 q(0) + (1 = e^2)/2$, and so $q(0) = (e^{-2}+1)/2$. Consequently,

$$\frac{1}{p(t)+1} = e^{2t}\left(\frac{e^{-2}+1}{2}\right) + \frac{1-e^{2t}}{2},$$

and so

$$p(t) = \frac{1-e^{2(t-1)}}{1+e^{2(t-1)}}, \quad t \in [0,1].$$

(It is easy to check that for all $t \in [0,1]$, indeed $p(t)+1 \neq 0$.)

3. The solution to $x' = ax + bu$, $x(0) = x_0$ is given by

$$x(t) = e^{ta}x_0 + \int_0^t e^{(t-\tau)a}bu(\tau)d\tau, \quad t \geq 0,$$

and so

$$\begin{aligned} |x(t) - e^{ta}x_0| &= \left|\int_0^t e^{(t-\tau)a}bu(\tau)d\tau\right| \\ &\leq \int_0^t |e^{(t-\tau)a}bu(\tau)|d\tau = \int_0^t e^{(t-\tau)a}|b||u(\tau)|d\tau \\ &\leq \int_0^t e^{(t-\tau)a}|b|M d\tau \\ &\leq M|b|e^{ta}\int_0^t e^{-\tau a}d\tau = M|b|e^{ta}\frac{1}{-a}e^{\tau a}|_0^t \\ &= M|b|\frac{(e^{ta}-1)}{a}. \end{aligned}$$

We have
$$\lim_{a\to 0} M|b|\frac{(e^{ta}-1)}{a} = M|b|t \lim_{a\to 0}\frac{(e^{ta}-1)}{ta} = M|b|t.$$
This bound can also be obtained directly: by the fundamental theorem of calculus,
$$x(t)-x(0)=\int_0^t x'(\tau)d\tau = \int_0^t bu(\tau)d\tau,$$
and so
$$|x(t)-x_0| = \left|\int_0^t bu(\tau)d\tau\right| \le \int_0^t |bu(\tau)|d\tau \le \int_0^t |b|Md\tau = |b|Mt.$$

4. Note that the solution to
$$\Theta'(t) = \kappa(\Theta(t)-\Theta_e), \quad t\ge 0,\ \Theta(0)=\Theta_0$$
is given by
$$\begin{aligned}\Theta(t) &= e^{t\kappa}\Theta_0 + \int_0^t e^{(t-\tau)\kappa}(-\kappa\Theta_e)d\tau\\ &= e^{t\kappa}\Theta_0 + e^{t\kappa}(-\kappa\Theta_e)\int_0^t e^{-\tau\kappa}d\tau\\ &= e^{t\kappa}\Theta_0 + e^{t\kappa}(-\kappa\Theta_e)\frac{1}{-\kappa}[e^{-t\kappa}-1]\\ &= e^{t\kappa}\Theta_0 + \Theta_e[1-e^{t\kappa}]\\ &= e^{t\kappa}[\Theta_0-\Theta_e]+\Theta_e.\end{aligned}$$
Thus $\Theta(t)-\Theta_e = e^{t\kappa}[\Theta_0-\Theta_e]$.

Suppose the time of murder is chosen as the origin, and at this time, the body temperature was 98.6°F. At 11:30 a.m. (corresponding to a time lapse of T hours from the origin), the body temperature is 94.6°F, and at 12:30 a.m. (a time lapse of $T+1$ hours from the origin), the body temperature is 93.4°F. With this data we obtain,
$$94.6-70 = e^{T\kappa}(98.6-70),$$
and so $e^{T\kappa} = 24.6/28.6$. Furthermore,
$$93.4-70 = e^{(T+1)\kappa}(98.6-70),$$
and this yields that
$$e^{\kappa} = \frac{1}{e^{T\kappa}}\frac{93.4-70}{98.6-70} = \frac{23.4}{28.6}\cdot\frac{28.6}{24.6} = \frac{23.4}{24.6}.$$
Consequently,
$$\kappa = \log\left(\frac{23.4}{24.6}\right) \approx -0.05001042.$$
Thus
$$T = \frac{1}{\kappa}\log\left(\frac{24.6}{28.6}\right) = 3.0125777 \text{ hours } \approx 3 \text{ hours.}$$
So the time of murder was 3 hours before 11:30 a.m., that is at about 8:30 a.m.

Solutions to the exercises on page 76

1. We have
$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \stackrel{(n=2)}{=} \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2+\alpha \\ \alpha & \alpha \end{bmatrix}.$$
Thus
$$\det \begin{bmatrix} 1 & 2+\alpha \\ \alpha & \alpha \end{bmatrix} = \alpha - \alpha(2+\alpha) = -\alpha(1+\alpha),$$
which is 0 iff $\alpha \in \{0, -1\}$. Consequently
$$\operatorname{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \stackrel{(n=2)}{=} \begin{bmatrix} B & AB \end{bmatrix} = n = 2$$
iff $\alpha \notin \{0, -1\}$. Hence the system $x'(t) = Ax(t) + Bu(t)$ is controllable iff $\alpha \in \mathbb{R} \setminus \{0, -1\}$.

2. Let C commute with A. As $B, AB, \dots, A^{n-1}B$ is a basis for $\mathbb{R}^n$, it follows that there exist scalars $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$ such that
$$CB = \alpha_0 B + \alpha_1 AB + \dots + \alpha_{n-1} A^{n-1} B. \tag{5.16}$$
Define $\chi(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$. We claim that $C = \chi(A)$. From (5.16), we have that $CB = \chi(A)B$. As $A^k \chi(A) = \chi(A) A^k$, we also have
$$\begin{aligned} C(A^k B) &= (CA^k)B \\ &= (A^k C)B \quad (\text{since } AC = CA) \\ &= A^k(CB) = A^k(\chi(A)B) = \chi(A)(A^k B), \end{aligned}$$
for all $k \in \mathbb{N}$. As the action of C and $\chi(A)$ on the basis $\{B, AB, \dots, A^{n-1}B\}$ is identical, $C = \chi(A)$.

3. Let $u \in C[0, T]$. Then
$$\begin{aligned} x &= \int_0^T e^{(T-\tau)A} B u(\tau) d\tau = \int_0^T \begin{bmatrix} e^{T-\tau} & 0 \\ 0 & e^{T-\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(\tau) d\tau = \int_0^T \begin{bmatrix} e^{T-\tau} \\ 0 \end{bmatrix} u(\tau) d\tau \\ &= \begin{bmatrix} \int_0^T e^{T-\tau} u(\tau) d\tau \\ 0 \end{bmatrix} = \left(\int_0^T e^{T-\tau} u(\tau) d\tau \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$
Define $u(t) = \alpha e^{t-T}/T$, $t \in [0, T]$. Then $u \in C[0, T]$. We have
$$\begin{aligned} \int_0^T e^{(T-\tau)A} B u(\tau) d\tau &= \int_0^T \begin{bmatrix} e^{T-\tau} & 0 \\ 0 & e^{T-\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{\alpha}{T} e^{\tau - T} d\tau \\ &= \int_0^T \begin{bmatrix} e^{T-\tau} \\ 0 \end{bmatrix} \frac{\alpha}{T} e^{\tau-T} d\tau = \int_0^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{\alpha}{T} d\tau = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \end{aligned}$$
and so $\begin{bmatrix} \alpha \\ 0 \end{bmatrix} \in \mathscr{R}_T$. Consequently $\mathscr{R}_T = \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

4. Let v be a left eigenvector of A. Then there exists a $\lambda \in \mathbb{R}$ such that $vA = \lambda v$. Using induction, we see that for all $k \in \mathbb{N}$, $vA^k = \lambda^k v$. Since the system is controllable, we have $\operatorname{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$. If $vB = 0$, then
$$\begin{aligned} v \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} &= \begin{bmatrix} vB & vAB & \dots & vA^{n-1}B \end{bmatrix} \\ &= \begin{bmatrix} vB & \lambda vB & \dots & \lambda^{n-1} vB \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}, \end{aligned}$$
and as $v \neq 0$, this contradicts the fact that $\operatorname{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$. So $vB \neq 0$.

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