Forward Guidance in the Yield Curve: Short Rates versus Bond Supply

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December 7, 2015*

Abstract

We present a model of the yield curve in which the central bank can provide market participants with forward guidance on both future short rates and on future Quantitative Easing (QE) operations, which affect bond supply. Forward guidance on short rates works through the expectations hypothesis, while forward guidance on QE works through expected future bond risk premia. If a QE operation is expected to be undone in the near term, then its announcement will have a hump-shaped effect on the yield and forward-rate curves; otherwise the effect may be increasing with maturity. Humps associated to QE announcements typically occur at maturities longer than those associated to short-rate announcements, even when the effects of the former are expected to last over a shorter horizon. We use our model to re-examine the empirical evidence on QE announcements in the US.

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1 Introduction

Since late 2008, when short-term interest rates reached their zero lower bound, central banks have been conducting monetary policy through two primary instruments: quantitative easing (QE), in which they buy long-term government bonds and other long-term securities, and so-called "forward guidance," in which they guide market expectations about the path of future short rates. Because QE alters the maturity structure of the government debt that is available to the public, it changes the amount of duration risk that market participants must bear, thereby impacting bond risk premia and long-term interest rates. Forward guidance may also impact long rates because it contains information about the central bank's willingness to keep short rates low in the future.

Although the term "forward guidance" is normally used in reference to central-bank policy on future short rates, QE operations typically involve some forward guidance as well. This is because announcements that the central bank will purchase long-term securities are made well in advance of the actual purchases, which are spread out over a period of months or years. For example, on March 18, 2009, the Federal Open Market Committee (FOMC) announced that to "help improve conditions in private credit markets," the Federal Reserve would increase the scale of its previously announced asset purchase program from $600 billion to $1.75 trillion and that these purchases would be carried out over the next six to twelve months. At the same time, the FOMC provided forward guidance on short rates, stating that it "anticipates that economic conditions are likely to warrant exceptionally low levels of the Federal Funds rate for an extended period." The impact of announcements such as these on the yield curve has been substantial. Following the March 18, 2009 announcement, for example, ten-year zero-coupon Treasury yields fell by 51 basis points over the course of two days.

How should forward guidance on short rates and forward guidance on QE be reflected in the yield curve? Policymakers have taken the implicit view that forward guidance on short rates is easy to interpret. If the expectations hypothesis of the yield curve holds, then the expected future path of short rates coincides with the curve of instantaneous forward rates. But forward guidance on QE is inherently more difficult to assess because it depends on how future bond risk premia change in response to QE and how these changes are incorporated into current bond prices. For example, suppose that market participants believe the central bank plans to acquire large amounts of long-term government bonds, but then plans to sell these bonds in five years. How should these beliefs impact long rates today? What if the market revises its expectations about how long the central bank will maintain its elevated holdings of long-term bonds?

To make these questions concrete, consider the so-called "Taper Tantrum" of May-June 2013, a period in which market participants feared that the Fed might reduce the pace of future bond purchases. On May 22, 2013, Federal Reserve Chairman Bernanke testified in front of Congress that the Fed would slow or "taper" its QE program if the economy showed signs of improving. Within a week, yields of ten-year government bonds had increased by 21 basis points. On June 19, 2013, bond yields increased further following a Federal Reserve press conference, as markets feared an end to the Fed's balance-sheet expansion.

Figure 1 shows the evolution of the zero-coupon Treasury yield curve between May 21 and June 28, 2013 (nine days after the Fed press conference). Panel (a) shows yields and Panel (b) shows changes in yields between the two dates. The peak increase in yields occurred at a maturity of seven years, where the yield to maturity increased by a total of 60 basis points (bps). Panel (c) and Panel (d) show the same information for the forward rate curve. The peak increase in forward rates occurred at five years to maturity: the one-year yield four-year ahead increased by over 100bps between the two dates. The change in forward rates was large even as far as ten years into the future.

How should we interpret the yield curve changes in Figure 1? Were they mainly driven by market participants' revised expectations about the path of future short rates? If so, then under the expectations hypothesis of the yield curve, expectations were revised the most about short rates five years into the future, and revisions were significant even over a ten-year horizon. Were the changes in the yield curve instead driven by expectations about future purchases of long-term bonds by the Fed? If so, then over what horizon did expectations have to change to generate the observed yield curve changes?

In this paper, we build a no-arbitrage model of the yield curve that allows us to characterize and compare the effects of forward guidance on short rates and forward guidance on QE. Among other results, we show that forward guidance on QE tends to impact longer maturities than forward guidance on short rates, even when expectations about bond purchases by the central bank concern a shorter horizon than expectations about future short rates. Using our model we interpret reactions of the US yield curve to policy announcements during the QE period.

Our model builds on Vayanos and Vila (2009) and Greenwood and Vayanos (2014). There is a continuum of default-free, zero-coupon bonds that are available in positive supply. For simplicity, we consolidate the central bank and the fiscal authority, so that the only relevant quantity is the supply of bonds that must be held by the public. The marginal holders of the bonds are risk-averse arbitrageurs with short investment horizons. These arbitrageurs demand a risk premium for holding bonds, because of the possibility that unexpected shocks will cause the bonds to underperform relative to the short rate. Following a long line of research on the portfolio-balance channel (Tobin (1958), Tobin (1969)), declines in bond supply lower the amount of duration risk that is borne by arbitrageurs, reducing bond risk premia and raising bond prices.

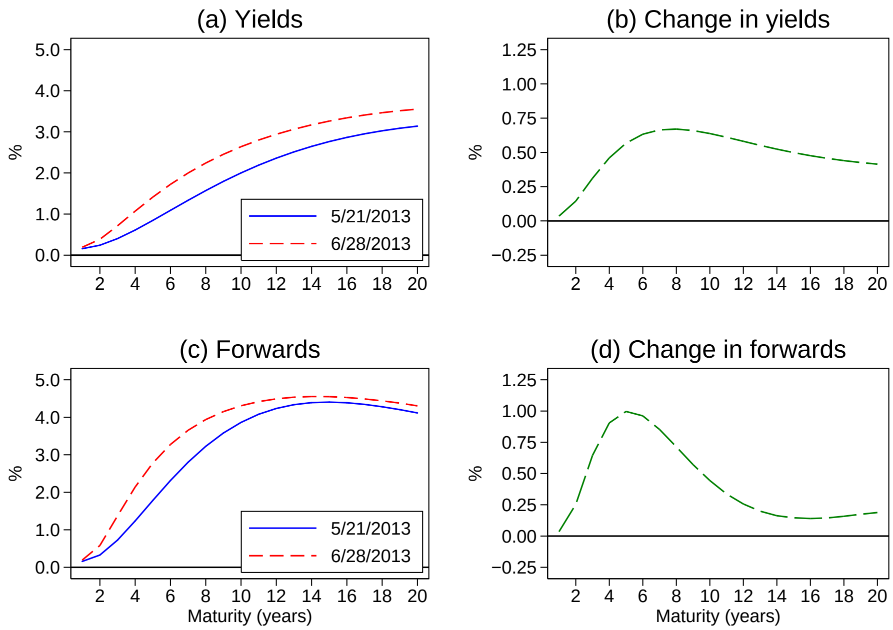

Figure 1: **Changes in US yields and forwards during the 2013 "Taper Tantrum."** Panels (a) and (c) plot zero-coupon Treasury yields and one-year forward rates prior to and following the Taper Tantrum (May 21 to June 28, 2013). Panels (b) and (d) plot cumulative changes during the Taper Tantrum. Yields and forward rates are computed using the continuously compounded yield curve fitted by Gurkaynak, Sack, and Wright (2007).

Relative to previous work, our key theoretical innovation is that we allow for news about both the future path of short rates and the future supply of bonds. Specifically, the short rate in our model evolves stochastically. However, holding fixed the current level of the short rate, we also allow for shocks to the expected path of future short rates. Similarly, the supply of bonds evolves stochastically. But, holding fixed current supply, we also allow for shocks to the expected path of future supply. Shocks to the expected path of future short rates and future supply can be interpreted as policy announcements that provide forward guidance on these variables.

After deriving the equilibrium yield curve, we describe the impact of forward guidance. Forward guidance on short rates in our model works through the expectations hypothesis. Suppose, for example, that arbitrageurs' expectation of the short rate three years from now declines by 100bps. This is reflected directly in a 100bps decline in the instantaneous forward rate three years from now. The expectations hypothesis describes the effects of shocks to expected future short rates because these shocks do not affect the positions that arbitrageurs hold in equilibrium and hence do not affect bond risk premia.

Forward guidance on supply works through expected future bond risk premia. Suppose, for example, that the central bank announces that it will buy ten-year bonds one year from now. After the purchase occurs, arbitrageurs will be holding a smaller position in ten-year bonds and be bearing less duration risk. Hence, the premium associated to that risk will decrease and bond prices will increase. The anticipation of this happening in one year causes an immediate rise in the prices of all bonds with maturity longer than one year. Note that the price increase is not confined to the bonds that the central bank announces it will purchase; in fact, other bonds may be more heavily affected. This is because—as in Vayanos and Vila (2009) and Greenwood and Vayanos (2014)—supply effects do not operate locally, but globally through changes in the prices of risk.

Announcements about expected future short rates have a hump-shaped effect on the yield and forward-rate curves. This is because current short rates are not affected, nor are expected short rates far in the future. The location of the hump on the forward-rate curve coincides with that in expected future short rates because of the expectation hypothesis.

Announcements about future supply can also have a hump-shaped effect on the yield and forward-rate curves. The intuition can be seen by noting that the impact of a supply shock on a bond's yield is the average on the shock's effect on the bond's instantaneous expected return over the bond's lifetime. When comparing the effect across bonds of different maturities, there are two opposing forces. On one hand, the supply shock has a larger impact on the current expected return that arbitrageurs require to hold the longer-term bond. On the other hand, if the shock is expected to revert quickly, required returns are expected to remain elevated over a larger portion of the shorter-term bond's life. The combination of these effects means that a supply shock that is expected to revert quickly has a hump-shaped effect on the yield curve. Moreover, the more quickly the shock is expected to revert, the shorter is the maturity where the hump is located. If the shock is expected to revert slowly, its effect is increasing with maturity (i.e., the hump is located at infinity).

A key difference between shocks to future supply and shocks to future short rates is that the former can impact yields and forward rates at maturities much longer than the time by which the shocks are expected to die out. And likewise, the humps on the yield and forward-rate curves associated to supply shocks typically occur at maturities longer than those associated to short-rate shocks, even when the former are expected to revert more quickly. Consider, for example, the impact of a supply shock on the one-year forward rate in nine years. We show that it can be written as the sum of the shock's impact on the difference between expected returns on ten- and nine-year bonds over the next year, plus the impact on the difference between expected returns on nine- and eight-year bonds over the year after, and so on. Even a temporary shock can have a significantly larger effect on the current expected return on ten-year bonds relative to nine-year, and hence impact the one-year rate forward rate in nine years.

After developing the theoretical results, we re-examine the empirical evidence on QE announcements in the US. Existing studies of QE have computed changes in bond yields around major policy announcements in the US and elsewhere. We add to these studies by computing changes in forward rates along the entire curve and considering a large set of announcement dates. We show that the cumulative effect of all expansionary announcements up to 2013 was hump-shaped with a maximum effect at the ten-year maturity for the yield curve and the seven-year maturity for the forward-rate curve. Explaining this evidence through changing expectations about short rates would mean that expectations were revised the most drastically for short rates seven years into the future, while revisions one to four years out were much more modest. This seems unlikely. On the other hand, the evidence is more consistent with changing expectations about supply: according to our model, the maximum revision in supply expectations would have to be only one year into the future.

Our findings accord nicely with those of Swanson (2015), who decomposes the effect of FOMC announcements from 2009-2015 into a component that reflects news about the future path of short rates (forward guidance) and a component that reflects news about future asset purchases (QE). Consistent with our model, Swanson (2015) finds that both QE-related and forward-guidance-related announcements have hump-shaped effects on the yield curve. Moreover, the hump for the former announcements occurs at a longer maturity than for the latter: QE announcements have their largest impact at around the ten-year maturity, while forward-guidance announcements have their largest impact at two to five years.

Our paper builds on a recent literature that seeks to characterize how shocks to supply and demand affect the yield curve (Vayanos and Vila (2009); Greenwood and Vayanos (2014); Hanson (2014); Malkhozov, Mueller, Vedolin, and Venter (forthcoming)). It is also related to a number of event studies that analyze the behavior of the yield curve and prices of other securities around QE-related events. Modigliani and Sutch (1966), Ross (1966), Wallace (1967), and Swanson (2011) study the impact of the 1962-1964 Operation Twist program. More recent event studies of QE in the wake of the Great Recession include Gagnon, Raskin, Remache, and Sack (2011), Krishnamurthy and Vissing-Jorgensen (2011), D'Amico, English, Lopez-Salido, and Nelson (2012), D'Amico and King (2013), Mamaysky (2014), and Swanson (2015) for the US, and Joyce, Lasaosa, Stevens, and Tong (2011) for the UK.[1]

The paper proceeds as follows. Section 2 presents the model. Section 3 derives the equilibrium yield curve. Section 4 describes the impact of announcements on the yield and forward-rate curves. Section 5 re-examines the empirical evidence on QE in light of our model. Section 6 concludes.

[1] See also Bernanke, Reinhart, and Sack (2004) for a broader analysis of QE programs, and Joyce, Myles, Scott, and Vayanos (2012) for a survey of the theoretical and empirical literature on QE.

2 Model

The model is set in continuous time. The yield curve at time t consists of a continuum of default-free zero-coupon bonds with maturities in the interval $(0, T]$ and face value one. We denote by $P_t^{(\tau)}$ the price of the bond with maturity τ at time t, and by $y_t^{(\tau)}$ the bond's yield. The yield $y_t^{(\tau)}$ is the spot rate for maturity τ. We denote by $f_t^{(\tau-\Delta\tau,\tau)}$ the forward rate between maturities $\tau - \Delta\tau$ and τ at time t. The spot rate and the forward rate are related to bond prices through

$$y_t^{(\tau)} = -\frac{\log P_t^{(\tau)}}{\tau}, \tag{1}$$

$$f_t^{(\tau-\Delta\tau,\tau)} = -\frac{\log\left(\frac{P_t^{(\tau)}}{P_t^{(\tau-\Delta\tau)}}\right)}{\Delta\tau}, \tag{2}$$

respectively. The short rate is the limit of $y_t^{(\tau)}$ when τ goes to zero, and we denote it by r_t. The instantaneous forward rate for maturity τ is the limit of $f_t^{(\tau-\Delta\tau,\tau)}$ when $\Delta\tau$ goes to zero, and we denote it by $f_t^{(\tau)}$. We sometimes refer to $f_t^{(\tau)}$ simply as the forward rate for maturity τ.

We treat the short rate r_t as exogenous, and assume that it follows the process

$$dr_t = \kappa_r(\bar{r}_t - r_t)dt + \sigma_r dB_{r,t}, \tag{3}$$

where

$$d\bar{r}_t = \kappa_{\bar{r}}(\bar{r} - \bar{r}_t)dt + \sigma_{\bar{r}} dB_{\bar{r},t}, \tag{4}$$

$(\kappa_r, \sigma_r, \bar{r}, \kappa_{\bar{r}}, \sigma_{\bar{r}})$ are positive constants, and $(B_{r,t}, B_{\bar{r},t})$ are Brownian motions that are independent of each other. The short rate r_t reverts to a target $\bar{r}_t$, which is itself mean-reverting. The assumption that the diffusion coefficients $(\sigma_r, \sigma_{\bar{r}})$ are positive is without loss of generality since we can switch the signs of $(B_{r,t}, B_{\bar{r},t})$. We refer to $\bar{r}_t$ as the "target short rate." To emphasize the distinction with r_t, we sometimes refer to the latter as the "current short rate." Shocks to $\bar{r}_t$ can be interpreted as policy announcements by the central bank that provide forward guidance on the future path of the short rate. The process (3) and (4) for the short rate has been used in the term-structure literature (e.g., Chen (1996), Balduzzi, Das, and Foresi (1998)) and is known as a stochastic-mean process.[2]

[2] Although we refer to $\bar{r}_t$ as the "target short rate," this should be interpreted as the central banks intermediate-term policy target (e.g., at a 1- to 2-year horizon) and not as the current operating target for the short rate (e.g., the current target for the federal funds rate set by the FOMC).

Bonds are issued by the government and are traded by arbitrageurs and other investors. We consolidate the central bank and the fiscal authority, so that only the net supply coming out of the two institutions matters. This means, for example, that a QE policy in which the central bank expands the size of its balance sheet, issuing interest-bearing reserves (i.e., overnight government debt) to purchase long-term government bonds, is equivalent to direct reduction in the average maturity of government debt issued by the fiscal authority. For simplicity, we treat the net supply coming out of the government as exogenous and price-inelastic. We do the same for the demand of investors other than arbitrageurs, and model explicitly only the arbitrageurs. Hence, the relevant supply in our model is that held by arbitrageurs, and reflects the combined effects of central bank purchases, issuance by the fiscal authority, and demand by other investors in the economy.

We assume that arbitrageurs choose a bond portfolio to trade off the instantaneous mean and variance of changes in wealth. Denoting their time-t wealth by W_t and their dollar investment in the bond with maturity τ by $x_t^{(\tau)}$, their budget constraint is

$$dW_t = \int_0^T x_t^{(\tau)} \frac{dP_t^{(\tau)}}{P_t^{(\tau)}} d\tau + \left(W_t - \int_0^T x_t^{(\tau)} d\tau \right) r_t dt. \tag{5}$$

The first term in (5) is the arbitrageurs' return from investing in bonds, and the second term is their return from investing their remaining wealth in the short rate. The arbitrageurs' optimization problem is

$$\max_{\{x_t^{(\tau)}\}_{\tau \in (0,T]}} \left[E_t(dW_t) - \frac{a}{2} Var_t(dW_t) \right], \tag{6}$$

where a is a risk-aversion coefficient.

We model the supply of bonds in a symmetric fashion to the short rate, so to be able to capture forward guidance on bond supply. Specifically, we assume that the net supply coming out of the central bank, the fiscal authority, and the other investors is described by a one-factor model: the dollar value of the bond with maturity τ supplied to arbitrageurs at time t is

$$s_t^{(\tau)} = \zeta(\tau) + \theta(\tau)\beta_t, \tag{7}$$

where $\zeta(\tau)$ and $\theta(\tau)$ are deterministic functions of τ, and β_t is a stochastic supply factor. Intuitively, it may be useful to think of β_t as proportional to the amount of *ten-year bond equivalents*, meaning duration-adjusted dollars of long-term debt. See Greenwood, Hanson, Rudolph, and Summers (2015) for a calculation along these lines for US government debt.

The factor β_t follows the process

$$d\beta_t = \kappa_\beta(\bar{\beta}_t - \beta_t)dt + \sigma_\beta dB_{\beta,t}, \tag{8}$$

where

$$d\bar{\beta}_t = -\kappa_{\bar{\beta}}\bar{\beta}_t dt + \sigma_{\bar{\beta}} dB_{\bar{\beta},t}, \tag{9}$$

$(\kappa_\beta, \sigma_\beta, \kappa_{\bar{\beta}}, \sigma_{\bar{\beta}})$ are positive constants, and $(B_{\beta,t}, B_{\bar{\beta},t})$ are Brownian motions that are independent of each other and of $(B_{r,t}, B_{\bar{r},t})$. The process (8) and (9) is a stochastic-mean process, analogous to that followed by the short rate r_t. The assumption that the diffusion coefficients $(\sigma_\beta, \sigma_{\bar{\beta}})$ are positive is without loss of generality since we can switch the signs of $(B_{\beta,t}, B_{\bar{\beta},t})$. We refer to $\bar{\beta}_t$ as the target supply. To emphasize the distinction with β_t, we sometimes refer to the latter as the current supply. Shocks to $\bar{\beta}_t$ can be interpreted as policy announcements by the central bank that provide forward guidance on future purchases or sales of bonds, which in our model affect bond yields.

Since the supply factor β_t has mean zero, the function $\zeta(\tau)$ measures the average supply for maturity τ. The function $\theta(\tau)$ measures the sensitivity of that supply to β_t. We assume that $\theta(\tau)$ has the following properties.

Assumption 1. *The function $\theta(\tau)$ satisfies:*

(i) $\int_0^T \theta(\tau)d\tau \geq 0$.

(ii) There exists $\tau^* \in [0, T)$ such that $\theta(\tau) < 0$ for $\tau < \tau^*$ and $\theta(\tau) > 0$ for $\tau > \tau^*$.

Part (i) of Assumption 1 requires that an increase in β_t does not decrease the total dollar value of bonds supplied to arbitrageurs. This is without loss of generality since we can switch the sign of β_t. Part (ii) of Assumption 1 allows for the possibility that the supply for some maturities decreases when β_t increases, even though the total supply does not decrease. The maturities for which supply can decrease are restricted to be at the short end of the yield curve. As we show in Section 3 below, Parts (i) and (ii) together ensure that an increase in β_t makes the overall portfolio that arbitrageurs hold in equilibrium more sensitive to movements in the short rate.

3 Equilibrium Yield Curve

Our model has four risk factors: the current short rate r_t, the target short rate $\bar{r}_t$, the current supply β_t, and the target supply $\bar{\beta}_t$. We next examine how shocks to these factors influence the bond prices $P_t^{(\tau)}$ that are endogenously determined in equilibrium. We solve for equilibrium in two steps: first solve the arbitrageurs' optimization problem for equilibrium bond prices of a conjectured form, and second use market clearing to verify the conjectured form of prices. We conjecture that equilibrium spot rates are affine functions of the risk factors. Bond prices thus take the form

$$P_t^{(\tau)} = e^{-\left[A_r(\tau)r_t + A_{\bar{r}}(\tau)\bar{r}_t + A_\beta(\tau)\beta_t + A_{\bar{\beta}}(\tau)\bar{\beta}_t + C(\tau)\right]} \tag{10}$$

for five functions $A_r(\tau)$, $A_{\bar{r}}(\tau)$, $A_\beta(\tau)$, $A_{\bar{\beta}}(\tau)$, and $C(\tau)$ that depend on maturity τ. The functions $A_r(\tau)$, $A_{\bar{r}}(\tau)$, $A_\beta(\tau)$, and $A_{\bar{\beta}}(\tau)$ characterize the sensitivity of bond prices to the current short rate r_t, the target short rate $\bar{r}_t$, the current supply β_t, and the target supply $\bar{\beta}_t$, respectively. Sensitivity to factor $i = r, \bar{r}, \beta, \bar{\beta}$ is defined as the percentage price drop per unit of factor increase.

Substituting (10) into (1) and (2), we can write spot rates and instantaneous forward rates as

$$y_t^{(\tau)} = \frac{A_r(\tau)r_t + A_{\bar{r}}(\tau)\bar{r}_t + A_\beta(\tau)\beta_t + A_{\bar{\beta}}(\tau)\bar{\beta}_t + C(\tau)}{\tau}, \tag{11}$$

$$f_t^{(\tau)} = A'_r(\tau)r_t + A'_{\bar{r}}(\tau)\bar{r}_t + A'_\beta(\tau)\beta_t + A'_{\bar{\beta}}(\tau)\bar{\beta}_t + C'(\tau), \tag{12}$$

respectively. Thus, the sensitivity of spot rates to factor $i = r, \bar{r}, \beta, \bar{\beta}$ is characterized by the function $\frac{A_i(\tau)}{\tau}$, and that of instantaneous forward rates by the function $A'_i(\tau)$.

Applying Ito's Lemma to (10) and using the dynamics of r_t in (3), $\bar{r}_t$ in (4), β_t in (8), and $\bar{\beta}_t$ in (9), we find that the instantaneous return of the bond with maturity τ is

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)}dt - A_r(\tau)\sigma_r dB_{r,t} - A_{\bar{r}}(\tau)\sigma_{\bar{r}} dB_{\bar{r},t} - A_\beta(\tau)\sigma_\beta dB_{\beta,t} - A_{\bar{\beta}}(\tau)\sigma_{\bar{\beta}} dB_{\bar{\beta},t}, \tag{13}$$

where

$$\begin{aligned}\mu_t^{(\tau)} \equiv & A'_r(\tau)r_t + A'_{\bar{r}}(\tau)\bar{r}_t + A'_\beta(\tau)\beta_t + A'_{\bar{\beta}}(\tau)\bar{\beta}_t + C'(\tau) \\ & + A_r(\tau)\kappa_r(r_t - \bar{r}_t) + A_{\bar{r}}(\tau)\kappa_{\bar{r}}(\bar{r}_t - \bar{r}) + A_\beta(\tau)\kappa_\beta(\beta_t - \bar{\beta}_t) + A_{\bar{\beta}}(\tau)\kappa_{\bar{\beta}}\bar{\beta}_t \\ & + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_{\bar{r}}(\tau)^2\sigma_{\bar{r}}^2 + \frac{1}{2}A_\beta(\tau)^2\sigma_\beta^2 + \frac{1}{2}A_{\bar{\beta}}(\tau)^2\sigma_{\bar{\beta}}^2\end{aligned} \tag{14}$$

denotes the instantaneous expected return. Substituting bond returns (13) into the arbitrageurs' budget constraint (5), we can solve the arbitrageurs' optimization problem (6).

Lemma 1. *The arbitrageurs' first-order condition is*

$$\mu_t^{(\tau)} - r_t = A_r(\tau)\lambda_{r,t} + A_{\bar{r}}(\tau)\lambda_{\bar{r},t} + A_\beta(\tau)\lambda_{\beta,t} + A_{\bar{\beta}}(\tau)\lambda_{\bar{\beta},t}, \tag{15}$$

where for $i = r, \bar{r}, \beta, \bar{\beta}$,

$$\lambda_{i,t} \equiv a\sigma_i^2 \int_0^T x_t^{(\tau)} A_i(\tau) d\tau. \tag{16}$$

According to (15), a bond's instantaneous expected return in excess of the short rate, $\mu_t^{(\tau)} - r_t$, is a linear function of the bond's sensitivities $A_i(\tau)$ to the factors $i = r, \bar{r}, \beta, \bar{\beta}$. The coefficients $\lambda_{i,t}$ of the linear function are the prices of risk associated to the factors: they measure the expected excess return per unit of sensitivity to each factor. Although we derive (15) from the optimization problem of arbitrageurs with mean-variance preferences, this equation is a more general consequence of the absence of arbitrage: the expected excess return per unit of factor sensitivity must be the same for all bonds (i.e., independent of τ), otherwise it would be possible to construct arbitrage portfolios.

Absence of arbitrage imposes essentially no restrictions on the prices of risk, and in particular on how they vary over time t and how they depend on bond supply. We determine these prices from market clearing. Equation (16) shows that the price of risk $\lambda_{i,t}$ for factor $i = r, \bar{r}, \beta, \bar{\beta}$ at time t depends on the overall sensitivity $\int_0^T x_t^{(\tau)} A_i(\tau) d\tau$ of arbitrageurs' portfolio to that factor. Intuitively, if arbitrageurs are highly exposed to a factor, they require that any asset they hold yields high expected return per unit of factor sensitivity. The portfolio that arbitrageurs hold in equilibrium is determined from the market-clearing condition

$$x_t^{(\tau)} = s_t^{(\tau)}, \tag{17}$$

which equates the arbitrageurs' dollar investment $x_t^{(\tau)}$ in the bond with maturity τ to the bond's dollar supply $s_t^{(\tau)}$. Substituting $\mu_t^{(\tau)}$ and $x_t^{(\tau)}$ from (7), (14) and (17) into (15), we find an affine equation in r_t, $\bar{r}_t$, β_t, and $\bar{\beta}_t$. Setting linear terms in r_t, $\bar{r}_t$, β_t, and $\bar{\beta}_t$ to zero yields four ordinary differential equations (ODEs) in $A_r(\tau)$, $A_{\bar{r}}(\tau)$, $A_\beta(\tau)$, and $A_{\bar{\beta}}(\tau)$ respectively. Setting constant terms to zero yields an additional ODE in $C(\tau)$. We solve the five ODEs in Theorem 1.

Theorem 1. *The functions $A_r(\tau)$, $A_{\bar{r}}(\tau)$, $A_\beta(\tau)$, and $A_{\bar{\beta}}(\tau)$ are given by*

$$A_r(\tau) = \frac{1-e^{-\kappa_r\tau}}{\kappa_r}, \tag{18}$$

$$A_{\bar{r}}(\tau) = \kappa_r \int_0^\tau A_r(\tau')e^{-\kappa_{\bar{r}}(\tau-\tau')}d\tau' = \frac{\kappa_{\bar{r}}(1-e^{-\kappa_r\tau}) - \kappa_r(1-e^{-\kappa_{\bar{r}}\tau})}{\kappa_{\bar{r}}(\kappa_{\bar{r}}-\kappa_r)}, \tag{19}$$

$$A_\beta(\tau) = Z_1\left(\frac{\gamma_2}{\gamma_1-\gamma_2}e^{-\gamma_1\tau} - \frac{\gamma_1}{\gamma_1-\gamma_2}e^{-\gamma_2\tau}+1\right) + Z_2\left(\frac{\gamma_2-\kappa_r}{\gamma_1-\gamma_2}e^{-\gamma_1\tau} - \frac{\gamma_1-\kappa_r}{\gamma_1-\gamma_2}e^{-\gamma_2\tau}+e^{-\kappa_r\tau}\right)$$
$$+ Z_3\left(\frac{\gamma_2-\kappa_{\bar{r}}}{\gamma_1-\gamma_2}e^{-\gamma_1\tau} - \frac{\gamma_1-\kappa_{\bar{r}}}{\gamma_1-\gamma_2}e^{-\gamma_2\tau}+e^{-\kappa_{\bar{r}}\tau}\right), \tag{20}$$

$$A_{\bar{\beta}}(\tau) = \kappa_\beta \int_0^\tau A_\beta(\tau')e^{-\kappa_{\bar{\beta}}(\tau-\tau')}d\tau', \tag{21}$$

respectively, where

$$Z_1 \equiv \frac{\kappa_{\bar{\beta}} a\left(\frac{\sigma_r^2 I_r}{\kappa_r} + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\kappa_{\bar{r}}}\right)}{\kappa_{\bar{\beta}}\kappa_\beta - \kappa_{\bar{\beta}} a\sigma_\beta^2 I_\beta - \kappa_\beta a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}}}, \tag{22}$$

$$Z_2 \equiv \frac{(\kappa_r-\kappa_{\bar{\beta}})a\left(\frac{\sigma_r^2 I_r}{\kappa_r} + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\kappa_{\bar{r}}-\kappa_r}\right)}{\kappa_r^2 - \kappa_r(\kappa_{\bar{\beta}}+\kappa_\beta - a\sigma_\beta^2 I_\beta) + \kappa_{\bar{\beta}}\kappa_\beta - \kappa_{\bar{\beta}} a\sigma_\beta^2 I_\beta - \kappa_\beta a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}}}, \tag{23}$$

$$Z_3 \equiv \frac{\frac{\kappa_r(\kappa_{\bar{\beta}}-\kappa_{\bar{r}})}{\kappa_{\bar{r}}(\kappa_{\bar{r}}-\kappa_r)} a\sigma_{\bar{r}}^2 I_{\bar{r}}}{\kappa_{\bar{r}}^2 - \kappa_{\bar{r}}(\kappa_{\bar{\beta}}+\kappa_\beta - a\sigma_\beta^2 I_\beta) + \kappa_{\bar{\beta}}\kappa_\beta - \kappa_{\bar{\beta}} a\sigma_\beta^2 I_\beta - \kappa_\beta a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}}}, \tag{24}$$

$$I_r \equiv \int_0^T A_r(\tau)\theta(\tau)d\tau, \tag{25}$$

$$I_{\bar{r}} \equiv \int_0^T A_{\bar{r}}(\tau)\theta(\tau)d\tau, \tag{26}$$

(γ_1, γ_2) are the solutions of the quadratic equation

$$\gamma^2 - \gamma(\kappa_{\bar{\beta}}+\kappa_\beta - a\sigma_\beta^2 I_\beta) + \kappa_{\bar{\beta}}\kappa_\beta - \kappa_{\bar{\beta}} a\sigma_\beta^2 I_\beta - \kappa_\beta a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}} = 0, \tag{27}$$

and $(I_\beta, I_{\bar{\beta}})$ solve the system of equations

$$I_\beta = \int_0^T A_\beta(\tau)\theta(\tau)d\tau, \tag{28}$$

$$I_{\bar{\beta}} = \int_0^T A_{\bar{\beta}}(\tau)\theta(\tau)d\tau, \tag{29}$$

in which the right-hand side is a function of $(I_\beta, I_{\bar{\beta}})$ through (20)-(27). A solution to the system of (28) and (29) exists if a is below a threshold $\bar{a} > 0$. The function $C(\tau)$ is given by (A.16) in Appendix A.

As in Greenwood and Vayanos (2014), an equilibrium with affine spot rates may fail to exist, and when it exists there can be multiplicity. Equilibrium exists if the arbitrageurs' risk-aversion coefficient a is below a threshold $\bar{a} > 0$. We focus on that case, and select the equilibrium that corresponds to the smallest value of I_β. When a converges to zero, that equilibrium converges to the unique equilibrium that exists for $a = 0$.

4 Shocks to the Yield Curve

In this section we examine how shocks to the four risk factors $r, \bar{r}, \beta, \bar{\beta}$ affect the equilibrium yield curve. We start with a numerical example that illustrates the main results. We next return to the analysis of the general model, and provide more complete characterizations and intuition.

4.1 Numerical Example

Table 1 summarizes the parameters used in our baseline numerical example. While we attempt to choose realistic values for the parameters, the example's main purpose is to illustrate general properties of the effects of the shocks rather than to provide exact quantitative estimates.

Table 1: Parameters for baseline numerical example

Parameter	Value
κ_r: Rate at which short rate r_t reverts to target short rate $\bar{r}_t$	1.3
σ_r: Volatility of shocks to short rate r_t	1.65%
$\kappa_{\bar{r}}$: Rate at which target short rate $\bar{r}_t$ reverts to long-run mean	0.2
$\sigma_{\bar{r}}$: Volatility of shocks to target short rate $\bar{r}_t$	2.15%
κ_β: Rate at which supply factor β_t reverts to target supply $\bar{\beta}_t$	2.5
σ_β: Volatility of shocks to supply factor β_t	0.18
$\kappa_{\bar{\beta}}$: Rate at which target supply $\bar{\beta}_t$ reverts to long-run mean	0.25
$\sigma_{\bar{\beta}}$: Volatility of shocks to target supply $\bar{\beta}_t$	0.18
T: Maximum bond maturity	20
a: Arbitrageur risk aversion	1.65

We choose values for κ_r, σ_r, $\kappa_{\bar{r}}$, and $\sigma_{\bar{r}}$ to match four time-series moments of the short rate. For the purposes of this exercise we identify the short rate with the one-year nominal yield, and use monthly Gurkaynak, Sack, and Wright (2007) data from June 1961 to September 2015. We match the variance ($\sqrt{\mathrm{Var}(r_t)} = 3.33\%$), the one-month autocorrelation ($\mathrm{Corr}(r_t, r_{t-1/12}) = 0.99$), the one-year autocorrelation ($\mathrm{Corr}(r_t, r_{t-1}) = 0.86$), and the three-year autocorrelation ($\mathrm{Corr}(r_t, r_{t-3}) = 0.59$). This yields $\kappa_r = 1.3$, $\sigma_r = 1.65\%$, $\kappa_{\bar{r}} = 0.2$, and $\sigma_{\bar{r}} = 2.15\%$. Under these values, 90% of the total variance of the short rate is driven by persistent shocks to the target short rate.[3] The half-life of the shocks to the target short rate is 3.46 years ($=\log(2)/\kappa_{\bar{r}}$) whereas the half-life of the shocks to the current short rate is only 0.53 years ($=\log(2)/\kappa_r$).

We choose the values of the remaining parameters to capture aspects of the Fed's QE program. We assume that the $\theta(\tau)$ function (which characterizes the sensitivity of the dollar supply of the bond with maturity τ to the supply factor β_t) satisfies $\int_0^T \theta(\tau) d\tau = 0$. Under this assumption, changes in β_t do not alter the total value of bonds that arbitrageurs hold in equilibrium, but affect only the duration of their portfolio. For simplicity, we assume that $\theta(\tau)$ depends linearly on τ. This yields the specification

$$\theta(\tau) = \theta_0 \left(1 - \frac{2\tau}{T}\right).$$

We normalize θ_0 to one, which is without loss of generality because only the product $\theta(\tau)\beta_t$ matters in the definition of bond supply.

We choose values for κ_β and $\kappa_{\bar{\beta}}$ to match plausible market expectations about the persistence of the Fed's balance-sheet operations. We assume that the Fed's initial announcement of large scale

[3]The variance of the short rate is $\mathrm{Var}(r_t) = \frac{\sigma_r^2}{2\kappa_r} + \frac{\kappa_r \sigma_{\bar{r}}^2}{2\kappa_{\bar{r}}(\kappa_r + \kappa_{\bar{r}})}$. The second term in this expression corresponds to the part of the variance that is driven by shocks to the target short rate.

asset purchases (LSAP) in 2008 and 2009 led market participants to expect a large reduction in bond supply over the next twelve months and a gradual increase in supply thereafter. Accordingly, we choose κ_β and $\kappa_{\bar{\beta}}$ so that the change in the expected supply factor $E_t(\beta_{t+\tau})$ at time $t+\tau$ following a shock to target supply $\bar{\beta}_t$ at time t is maximum after one year ($\tau = 1$) and decays to 50% of the maximum after the next three years ($\tau = 4$). This yields $\kappa_\beta = 2.5$ and $\kappa_{\bar{\beta}} = 0.25$. In Section 4.5 we examine the sensitivity of our results to a smaller value of $\kappa_{\bar{\beta}}$, under which the effect of a $\bar{\beta}_t$-shock on expected supply is maximum after a period longer than one year.

We assume that a unit shock to $\bar{\beta}_t$ corresponds to the announcement of a QE program that will reduce bond supply by \$3 trillion of ten-year bond equivalents. This is without loss of generality because it amounts to a re-normalization of the monetary units in which supply is measured. Figure 2 plots the change in the expected supply factor $E_t(\beta_{t+\tau})$ at time $t+\tau$ following a unit shock to $\bar{\beta}_t$ at time t. This change, which we denote by $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau})$, is a hump-shaped function of τ under any parameter values. Indeed, the effect of the $\bar{\beta}_t$-shock on $E_t(\beta_{t+\tau})$ is small for small τ because the shock does not affect β_t, increases with τ as $E_t(\beta_{t+\tau})$ catches up with the new value of $\bar{\beta}_t$, and decreases again to zero because $\bar{\beta}_t$ mean-reverts. Under our chosen values for κ_β and $\kappa_{\bar{\beta}}$, the hump occurs after one year, and the function reaches half of its maximum value after the next three years.

The change $\Delta_{\bar{r}} E_t(r_{t+\tau})$ in the expected short rate $E_t(r_{t+\tau})$ following a unit shock to $\bar{r}_t$ is similarly hump-shaped. Under our chosen values for κ_r and $\kappa_{\bar{r}}$, the hump occurs after 1.7 years. This is because we assume that supply shocks are less persistent than shocks to the short rate. The mean-reversion parameter for supply shocks is larger than for short-rate shocks both when comparing shocks to current supply β_t and the current short rate r_t ($\kappa_\beta > \kappa_r$), and when comparing shocks to target supply $\bar{\beta}_t$ and the target short rate $\bar{r}_t$ ($\kappa_{\bar{\beta}} > \kappa_{\bar{r}}$).

We set $\sigma_\beta = \sigma_{\bar{\beta}} = 0.18$. Under these values, the volatility $\sqrt{\mathrm{Var}(\beta_t)}$ of the supply factor is 0.25. We can compare this quantity to the change $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau})$ in the expected supply factor following a unit shock to $\bar{\beta}_t$. This change is 0.75 after one year ($\Delta_{\bar{\beta}} E_t(\beta_{t+1}) = 0.75$), which is three times the standard deviation of β_t. Thus, a unit shock to $\bar{\beta}_t$ is a rare and large shock to expected future supply, consistent with it being a QE program undertaken in a crisis.

Our final parameter is the arbitrageurs' risk-aversion coefficient a, and we choose its value to match the price effects of supply shocks. As noted by Greenwood, Hanson, Rudolph, and Summers (2015), the Fed's combined QE policies from late 2008 to mid-2014 cumulatively reduced the ten-year bond equivalents available to investors by roughly \$3 trillion. Following the meta-analysis of studies examining the impact of QE announcements in Wiliams (2014), we assume that an

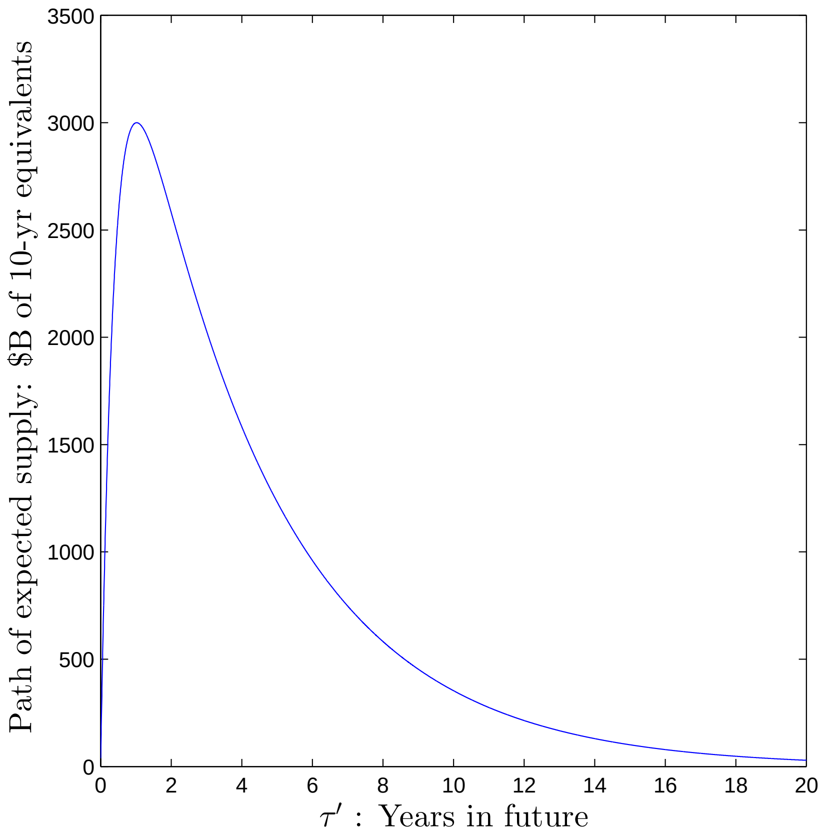

Figure 2: **Model-implied path of QE in ten-year bond equivalents**

announced purchase of $500 billion ten-year bond equivalents reduces ten-year yields by 25 bps. This suggests a total price impact for all QE announcements of 1.50%. Therefore, the value of a must be such that $A_{\bar{\beta}}(10)/10 = 1.50\%$. This yields $a = 1.65$.[4]

Figure 3 plots the effects of shocks to the four risk factors $r, \bar{r}, \beta, \bar{\beta}$ on the equilibrium yield curve and forward-rate curve. There are four plots, each describing the effect that a unit shock to one of the factors has on the yield and forward-rate curves, holding the remaining factors constant. Recall from (11) and (12) that the effect of a unit shock to factor $i = r, \bar{r}, \beta, \bar{\beta}$ on the yield for maturity τ is $\frac{A_i(\tau)}{\tau}$, and on the forward rate for that maturity is $A_i'(\tau)$. Plotting these functions reveals the "footprint" that shocks to factor i leave on the yield and forward-rate curves.

[4]In principle, one could use the simulated method of moments to estimate the parameters of our model. The parameters that govern the short rate process $(\kappa_r, \sigma_r, \kappa_{\bar{r}}, \sigma_{\bar{r}})$ could be identified as above by matching time-series moments of short rates. The parameters that govern the bond supply process $(\kappa_\beta, \sigma_\beta, \kappa_{\bar{\beta}}, \sigma_{\bar{\beta}})$ and arbitrageur risk aversion (a) could be identified by matching time-series moments of long-term bond yields of various maturities and the excess returns on long-term bonds. We do not pursue this approach because the supply and demand shocks that have driven bond risk premia over the past decades may have been of a different nature from the supply shocks generated by the Fed's QE policies since 2008.

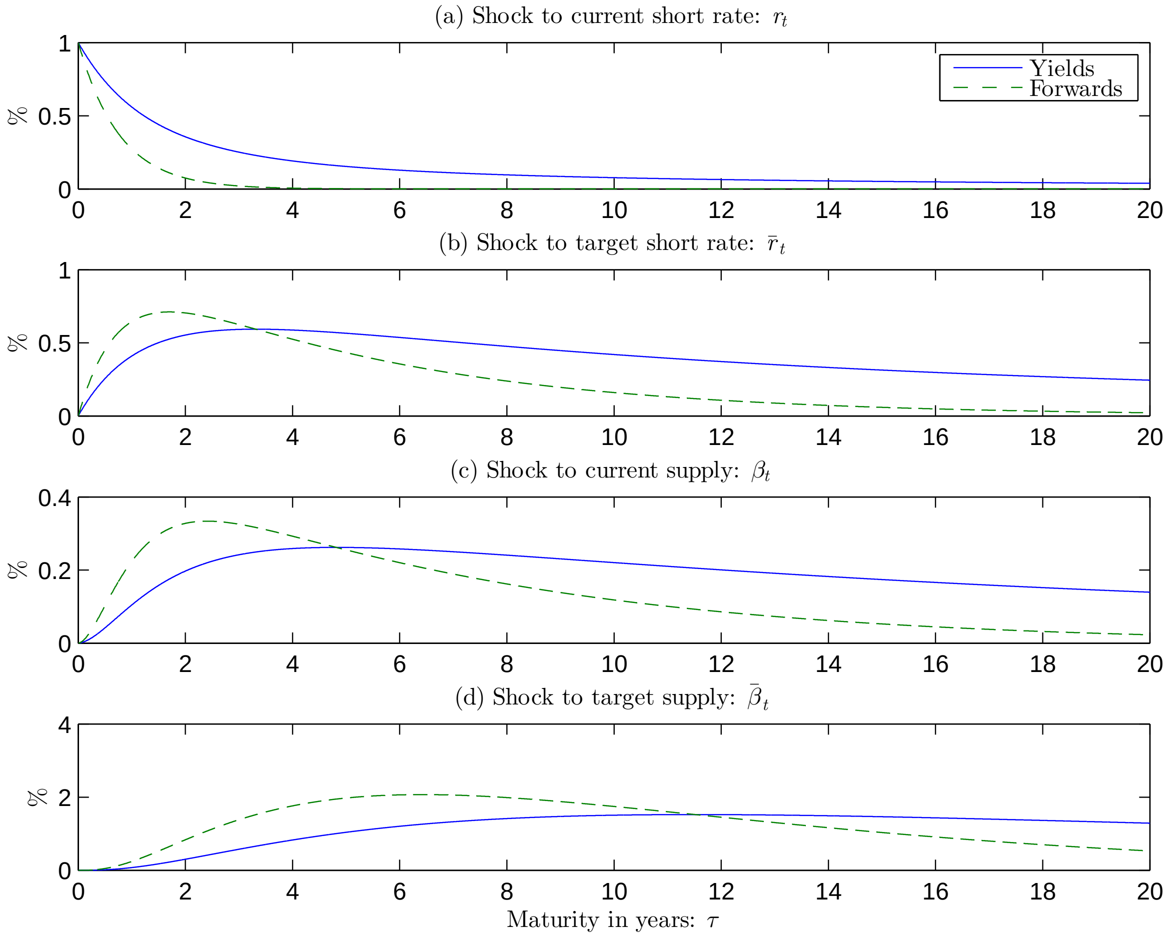

Figure 3: **The effects of a unit shock to each of the four risk factors $r, \bar{r}, \beta, \bar{\beta}$ on the equilibrium yield curve and forward-rate curve.** Panel (a) plots a shock to the current short rate r_t, Panel (b) a shock to the target short rate $\bar{r}_t$, Panel (c) a shock to current supply β_t, and Panel (d) a shock to target supply $\bar{\beta}_t$. For each factor $i = r, \bar{r}, \beta, \bar{\beta}$, the blue solid line represents the effect $\frac{A_i(\tau)}{\tau}$ on the yield curve and the green dashed line represents the effect $A_i'(\tau)$ on the forward-rate curve.

We make three observations regarding Figure 3. First, an increase in any of the factors raises all yields and forward rates. Thus, yields and forward rates for any maturity move up in response to increases in the current and in the target short rate. They also move up in response to increases in current and in target supply.

The second observation is that the effect of shocks to factors other than the current short rate is hump-shaped with maturity. Figure 3 thus suggests that policy announcements by the central bank that provide forward guidance on the short rate or on balance-sheet operations should have hump-shaped effects on the yield and forward-rate curves. This is consistent with the evidence on the Taper Tantrum, presented in the Introduction.

The third observation suggests a way to differentiate between the two types of forward guidance. The hump for shocks to target supply $\bar{\beta}_t$ occurs at a much longer maturity than for shocks to the target short rate $\bar{r}_t$: 11.5 years compared to 3.3 years for the yield curve, and 6.4 years compared to 1.7 years for the forward-rate curve. This result cannot be attributed to supply shocks being more persistent than shocks to the short rate: in our baseline numerical example they are actually *less* persistent. Figure 3 thus suggests that hump-shaped effects of forward guidance are more likely to concern guidance on supply rather than on the short rate when the hump is located at longer maturities.

Figure 3 accords nicely with the empirical findings of Swanson (2015), who decomposes the effect of FOMC announcements from 2009-2015 into a component that reflects news about the future path of short rates (forward guidance) and a component that reflects news about future asset purchases (QE). Swanson (2015) finds that both QE-related and forward-guidance-related announcements have hump-shaped effects on the yield curve. Moreover, QE announcements ($\bar{\beta}_t$-shocks in our model) have their largest impact at around the ten-year maturity, while forward-guidance announcements ($\bar{r}_t$-shocks) have their largest impact at two to five years.

In the remainder of this section we show that the these three observations hold more generally, and we explain the intuition behind them. Section 4.2 analyzes shocks to the current and the target short rate. Section 4.2 analyzes shocks to current and target supply. Section 4.4 compares the footprints left by shocks to target supply and shocks to the target short rate. Section 4.5 examines how the effects of the shocks depend on various parameters of the model.

4.2 Shocks to the Current and the Target Short Rate

Shocks to the current and the target short rate do not affect bond risk premia in our model. This is because premia depend only on the positions that arbitrageurs hold in equilibrium, and these depend only on the supply factor β_t. Since these shocks do not affect risk premia, their effects on yields and forward rates are only through expected future short rates, and are fully consistent with the expectations hypothesis. That is, the changes in forward rates caused by these shocks are equal to the changes in expected future short rates.

Proposition 1. *The expectations hypothesis holds for shocks to the current and the target short rate.*

- *Consider a unit shock to the current short rate r_t at time t, holding constant the remaining risk factors $(\bar{r}_t, \beta_t, \bar{\beta}_t)$. The change $A_r'(\tau)$ in the forward rate for maturity τ is equal to the change $\Delta_r E_t(r_{t+\tau})$ in the expected short rate at time $t+\tau$.*

- *Consider a unit shock to the target short rate $\bar{r}_t$ at time t, holding constant the remaining risk factors $(r_t, \beta_t, \bar{\beta}_t)$. The change $A'_{\bar{r}}(\tau)$ in the forward rate for maturity τ is equal to the change $\Delta_{\bar{r}} E_t(r_{t+\tau})$ in the expected short rate at time $t+\tau$.*

Using Proposition 1, we next determine how the effects of shocks to the current and the target short rate depend on maturity. The effect of shocks to the current short rate r_t decreases with maturity, and is hence strongest for short maturities. Indeed, because r_t mean-reverts, the effect of shocks to r_t on the expected future short rate $E_t(r_{t+\tau})$ is largest in the near future, i.e., for small τ. The same applies to the forward rate because of Proposition 1. On the other hand, the effect of shocks to the target short rate $\bar{r}_t$ is hump-shaped with maturity, and is hence strongest for intermediate maturities. Indeed, the effect of shocks to $\bar{r}_t$ on the expected future short rate $E_t(r_{t+\tau})$ is small for short maturities because the shocks do not affect r_t, increases with maturity as $E_t(r_{t+\tau})$ catches up with the new value of $\bar{r}_t$, and decreases again to zero because $\bar{r}_t$ mean-reverts. These results hold both for the yield curve and the forward-rate curve, and are consistent with our baseline numerical example.

Proposition 2. *The following results hold for both the yield curve and the forward-rate curve:*

- *An increase in the short rate r_t moves the curve upwards. The effect is decreasing with maturity, is equal to one for $\tau = 0$, and to zero for $\tau \to \infty$.*
- *An increase in the target short rate r_t moves the curve upwards. The effect is hump-shaped with maturity, and is equal to zero for $\tau = 0$ and $t \to \infty$.*

4.3 Shocks to Current and Target Supply

Shocks to current and target supply affect yields and forward rates only through bond risk premia. Proposition 3 expresses the effects of the shocks on a bond's price as an integral of risk premia over the life of the bond.

Proposition 3. *The effects of supply shocks can be expressed as follows:*

- *Consider a unit shock to current supply β_t at time t, holding constant the remaining risk factors $(r_t, \bar{r}_t, \bar{\beta}_t)$. The time-$t$ instantaneous expected return of the bond with maturity τ changes by*

$$URP(\tau) \equiv a\sigma_r^2 A_r(\tau) I_r + a\sigma_{\bar{r}}^2 A_{\bar{r}}(\tau) I_{\bar{r}} + a\sigma_\beta^2 A_\beta(\tau) I_\beta + a\sigma_{\bar{\beta}}^2 A_{\bar{\beta}}(\tau) I_{\bar{\beta}}. \tag{30}$$

The bond's price change in percentage terms is

$$A_{\beta}(\tau) = \int_0^{\tau} URP(\tau - \tau')\Delta_{\beta}E_t(\beta_{t+\tau'})d\tau', \tag{31}$$

where $\Delta_{\beta}E_t(\beta_{t+\tau'})$ *is the change in the expected supply factor* $E_t(\beta_{t+\tau'})$ *at time* $t + \tau'$*.*

- *Consider a unit shock to target supply* $\bar{\beta}_t$ *at time* t*, holding constant the remaining risk factors* $(r_t, \bar{r}_t, \beta_t)$*. The percentage price change of the bond with maturity* τ *is*

$$A_{\bar{\beta}}(\tau) = \int_0^{\tau} URP(\tau - \tau')\Delta_{\bar{\beta}}E_t(\beta_{t+\tau'})d\tau', \tag{32}$$

where $\Delta_{\bar{\beta}}E_t(\beta_{t+\tau'})$ *is the change in the expected supply factor* $E_t(\beta_{t+\tau'})$ *at time* $t + \tau'$*.*

A unit shock in current supply changes the instantaneous expected return of the bond with maturity τ by a quantity that we denote $URP(\tau)$. This acronym stands for "Unit Risk Premium," as it is a required compensation for *risk* resulting from a *unit* increase in supply. The unit risk premium for the bond with maturity τ is the product of the arbitrageurs' risk-aversion coefficient a times the change in the bond's instantaneous covariance with the arbitrageurs' portfolio. The covariance changes in response to the supply shock because arbitrageurs change their portfolio in equilibrium. The unit risk premium $URP(\tau)$ is small for bonds with short maturity τ because these bonds have small price sensitivity to the risk factors. As maturity increases, price sensitivity increases and so does $URP(\tau)$.

The impact of a shock to current or target supply on a bond's price derives from its effect on risk premia over the life of a bond. If, for example, the risk premia increase, then the price decreases. Equations (31) and (32) make this relationship precise by expressing the effect of a unit supply shock on the percentage price of a bond with maturity τ as a integral of unit risk premia over the bond's life, i.e., from t to $t + \tau$. The risk premium corresponding to time $t + \tau'$, when the bond reaches maturity $\tau - \tau'$, is proportional to the unit risk premium $URP(\tau - \tau')$. Since $URP(\tau - \tau')$ corresponds to a unit increase in the supply factor at $t + \tau'$, we need to multiply it by the actual increase in the expected supply factor. This is $\Delta_{\beta}E_t(\beta_{t+\tau'})$ in the case of a shock to current supply and $\Delta_{\bar{\beta}}E_t(\beta_{t+\tau'})$ in the case of a shock to target supply.

Using Proposition 3, we next characterize more fully the effects of shocks to current and target supply: the sign of the effects and how they depend on maturity. As for our analysis on short rates, the results are the same whether we are looking at the yield curve or the forward-rate curve. For the formal propositions that we show in the rest of this section, we assume $\sigma_{\bar{\beta}} = 0$, hence interpreting shocks to $\bar{\beta}_t$ as unanticipated and one-off. However, these formal results are consistent with our baseline numerical example as well as other examples that we have explored, all of which assume $\sigma_{\bar{\beta}} > 0$.

As in Greenwood and Vayanos (2014), an increase in current supply β_t moves the yield curve upwards. Moreover, this occurs even though Assumption 1 allows for the possibility that the supply of short-term bonds can decrease. Yields and supply for a given maturity can move in opposite directions because—as in Vayanos and Vila (2009) and Greenwood and Vayanos (2014)—supply effects do not operate locally, but globally through changes in the prices of risk. Equations (16) and (17) show that the prices of risk, $\lambda_{i,t}$ for $i = r, \bar{r}, \beta, \bar{\beta}$, depend on the supply of debt adjusted by measures of duration (the price sensitivities to the factors). An increase in the supply factor raises duration-adjusted supply and hence the prices of risk. Risk premia also increase, and bond prices decrease from Proposition 3. As with β_t, an increase in target supply $\bar{\beta}_t$ in our model moves the yield curve upwards.

We next examine how supply effects depend on maturity. Equation (31) implies that the effect of a unit shock to current supply β_t on the yield of a τ-year bond is

$$\frac{A_\beta(\tau)}{\tau} = \frac{\int_0^\tau URP(\tau - \tau') \times \Delta_\beta E_t(\beta_{t+\tau'}) d\tau'}{\tau}. \tag{33}$$

This is an average of risk premia over the bond's life. The premium corresponding to time $t + \tau'$, when the bond reaches maturity $\tau - \tau'$, is the product of the unit risk premium $URP(\tau - \tau')$ corresponding to that maturity, times the increase $\Delta_\beta E_t(\beta_{t+\tau'})$ in the expected supply factor at time $t + \tau'$.

Supply shocks have small effects on short-maturity bonds because these bonds carry small risk premia. This can be seen formally from (33): for small maturity τ the unit risk premia $URP(\tau - \tau')$ are small, and so is the average in (33). As maturity τ increases, the average in (33) increases because unit risk premia increase. A countervailing effect, however, is that because shocks to β_t mean-revert, unit risk premia corresponding to distant times $t + \tau'$ are multiplied by the increasingly smaller quantity $\Delta_\beta E_t(\beta_{t+\tau'})$. This pushes the average down. The countervailing effect is not present in the extreme case where there is no mean-reversion ($\kappa_\beta = 0$). In that case, the effect of shocks to β_t is increasing with τ, i.e., is strongest at the long end of the term structure. In the other extreme case where mean-reversion is high, only the terms for times $t + \tau'$ close to t matter in the average. Because unit risk premia increase less than linearly with τ (in particular, changes to r_t or $\bar{r}_t$ have a vanishing effect on spot rates for long maturities) dividing by τ makes the average converge to zero. The overall effect is hump-shaped and hence strongest for intermediate maturities. The same result holds for shocks to $\bar{\beta}_t$. The hump-shaped effects are consistent with our baseline numerical example.

Proposition 4. *Suppose that $\sigma_{\bar{\beta}} = 0$. An increase in current supply β_t or target supply $\bar{\beta}_t$ moves both the term structure of spot rates and that of instantaneous forward rates upwards. The effect is equal to zero for $\tau = 0$. For large enough values of κ_β, the effect is hump-shaped with maturity and is equal to zero for $\tau \to \infty$. Otherwise, the effect is increasing with maturity.*

To illustrate the effects of supply, we plot in Figure 4 the functions inside the integrals (31) and (32) in the context of our baseline numerical example. Panel (a) confirms that the unit risk premium $URP(\tau)$ is equal to zero for $\tau = 0$ and increases with τ. Panels (b)-(d) plot $URP(\tau - \tau')$, $\Delta_\beta E_t(\beta_{t+\tau'})$, and $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$ as a function of $\tau' \in [0, \tau]$ for three different bonds: a two-year bond ($\tau = 2$) in Panel (b), a ten-year bond ($\tau = 10$) in Panel (c), and a twenty-year bond ($\tau = 20$) in Panel (d). The function $\Delta_\beta E_t(\beta_{t+\tau'})$ is decreasing with τ': because β_t mean-reverts, the effect of shocks to β_t on the expected future supply factor $E_t(\beta_{t+\tau'})$ is largest in the near future, i.e., for small τ'. The function $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$ is hump-shaped, as explained in Section 4.1.

In the case of the two-year bond, unit risk premia are small, and so are the average values of $URP(\tau - \tau') \times \Delta_\beta E_t(\beta_{t+\tau'})$ and $URP(\tau - \tau') \times \Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$ over the interval $[0, 2]$. Hence, supply effects are small. In the case of the ten-year bond, unit risk premia are larger and so are supply effects. In the case of the twenty-year bond, unit risk premia are even larger, but the average values of $URP(\tau - \tau') \times \Delta_\beta E_t(\beta_{t+\tau'})$ and $URP(\tau - \tau') \times \Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$ over the interval $[0, 20]$ are smaller because of the declines in $\Delta_\beta E_t(\beta_{t+\tau'})$, and $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$. Hence supply effects are smaller, yielding the hump shape. Note that the smaller supply effect on the yield of the twenty-year bond masks a strong time-variation in expected return. The bond's instantaneous expected return is high (and higher than for the other bonds) in the short term, but the effect dies out in the longer term, resulting in a smaller average.

4.4 Forward Guidance on Supply vs. the Short Rate

We next compare the effects of shocks to target supply $\bar{\beta}_t$ and shocks to the target short rate $\bar{r}_t$. Interpreting these shocks as forward guidance by the central bank, we are effectively examining whether different types of forward guidance leave a different "footprint" on the yield and forward-rate curves. For simplicity, we focus on the forward-rate curve for the rest of this section.

In our baseline numerical example, shocks to target supply $\bar{\beta}_t$ have their maximum effect at a

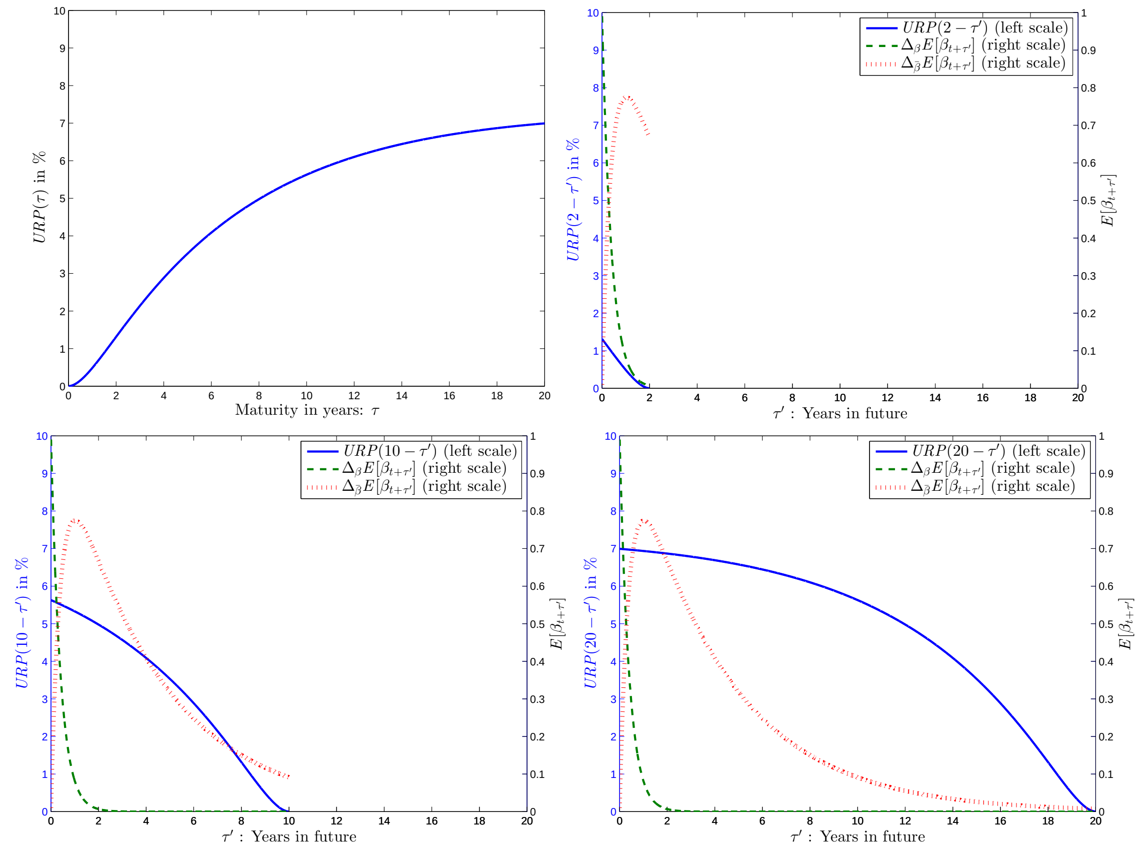

Figure 4: **Decomposition of the effect of supply shocks.** Panel (a) plots the unit risk premium $URP(\tau)$. Panels (b)-(d) plot $URP(\tau-\tau')$ (blue solid line), $\Delta_\beta E_t(\beta_{t+\tau'})d\tau'$ (green dashed line), and $\Delta_{\bar\beta} E_t(\beta_{t+\tau'})d\tau'$ (red dotted line) as a function of $\tau' \in [0,\tau]$ for three different bonds: Panel (b) for a two-year bond ($\tau = 2$), Panel (c) for a ten-year bond ($\tau = 10$), and Panel (d) for a twenty-year bond ($\tau = 20$).

longer maturity than shocks to the target short rate $\bar{r}_t$. While this is the "typical" outcome in our model, the result is not completely general: if the shocks to current and target supply mean-revert very rapidly, the comparison can reverse. Proposition 5 derives sufficient conditions for $\bar\beta_t$-shocks to have their maximum effect at a longer maturity than $\bar{r}_t$-shocks. The proposition compares the location of the humps associated to two types of shocks, with the convention if the effect of a shock is monotonically increasing with maturity then the hump is located at infinity.

Proposition 5. *Suppose that $\sigma_{\bar\beta} = 0$. If $\kappa_{\bar r} \geq \kappa_\beta$ or $\kappa_{\bar r} \geq \kappa_{\bar\beta}$, then the hump on the forward rate curve associated to shocks to $\bar\beta_t$ is located at a strictly longer maturity than the hump associated to shocks to $\bar r_t$.*

Shocks to $\bar\beta_t$ have their largest impact at longer maturities compared to shocks to $\bar r_t$ under the sufficient condition that the latter shocks do not mean-revert more slowly than the former shocks ($\kappa_{\bar r} \geq \kappa_{\bar\beta}$). Alternatively, $\bar r_t$-shocks can revert more slowly than $\bar\beta_t$-shocks, but then they must not mean-revert more slowly than β_t-shocks ($\kappa_{\bar{r}} \geq \kappa_{\beta}$). Note that under either sufficient condition, the hump associated to $\bar{\beta}_t$-shocks occurs at a strictly longer maturity than that associated to $\bar{r}_t$-shocks, even though the sufficient conditions are weak inequalities. Our baseline numerical example shows that the comparison between the two humps remains the same even when κ_β and $\kappa_{\bar{\beta}}$ are both significantly larger than $\kappa_{\bar{r}}$. (For very large values, however, the comparison can reverse.) Thus, the sufficient conditions in Proposition 5 are not tight, and the "typical" result is that shocks to target supply have their maximum impact at longer maturities than shocks to the target short rate.

The intuition why shocks to future supply tend to have their largest impact at longer maturities compared to shocks to the future short rate can be seen from (32). The impact of a $\bar{\beta}_t$-shock on the forward rate for maturity τ is

$$A'_{\bar{\beta}}(\tau) = \int_0^\tau \frac{\partial URP(\tau - \tau')}{\partial \tau} \Delta_{\bar{\beta}} E_t(\beta_{t+\tau'}) d\tau', \tag{34}$$

where (34) follows from (32) by differentiating with respect to τ and noting that $URP(0) = 0$. The impact on the forward rate can be thought of as the impact on the percentage price of the bond with maturity τ relative to the same effect for the bond with maturity $\tau - \Delta\tau$. The bond with maturity τ is impacted more heavily because for any given future time $t+\tau'$, the unit risk premium $URP(\tau - \tau')$ associated to that bond is larger than the corresponding premium $URP(\tau - \Delta\tau - \tau')$ associated to the bond with maturity $\tau - \Delta\tau$. The impact on the forward rate hence involves the derivative $\frac{\partial URP(\tau-\tau')}{\partial \tau}$, as (34) confirms. This derivative is multiplied by the increase $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$ in the expected supply factor at time $t + \tau'$, and the product is integrated from zero to τ.

Compare next the shock's impact on the forward rate for maturity τ and for maturity $\hat{\tau} > \tau$. The derivative $\frac{\partial URP(\tau-\tau')}{\partial \tau}$ that is present in the integral (34) for maturity τ is also present in the integral for maturity $\hat{\tau}$. But while in the former integral it corresponds to time $t + \tau'$ and is multiplied by $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$, in the latter integral it corresponds to the more distant time $t+\hat{\tau}-\tau+\tau'$ and is multiplied by $\Delta_{\bar{\beta}} E_t(\beta_{t+\hat{\tau}-\tau+\tau'})$. If $\hat{\tau} \leq \tau_{\bar{\beta}}$, where $t + \tau_{\bar{\beta}}$ denotes the location of the hump of $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$, then $\Delta_{\bar{\beta}} E_t(\beta_{t+\hat{\tau}-\tau+\tau'}) > \Delta_{\bar{\beta}} E_t(\beta_{t+\tau'})$ for all $\tau' \in [0, \tau]$. Therefore, the impact of a β_t-shock on the forward rate for maturity $\hat{\tau}$ is larger than for maturity τ, which means that the shock's maximum impact occurs at a maturity strictly longer than $\tau_{\bar{\beta}}$. On the other hand, Proposition 1 implies that the maximum impact of a $\bar{r}_t$-shock occurs exactly at $\tau_{\bar{r}}$, where $t + \tau_{\bar{r}}$ denotes the location of the hump of $\Delta_{\bar{r}} E_t(r_{t+\tau'})$. Therefore, if the shocks to $\bar{r}_t$ and $\bar{\beta}_t$ are symmetric in their persistence, then $\bar{\beta}_t$-shocks have their largest impact at longer maturities compared to $\bar{r}_t$-shocks.

We can also compare $\bar{r}_t$- and $\bar{\beta}_t$-shocks by focusing on the long end of the term structure rather than on the hump. Proposition 6 derives sufficient conditions for the effect of $\bar{\beta}_t$-shocks to decay more slowly with maturity than that of $\bar{r}_t$-shocks. Under these conditions, $\bar{\beta}_t$-shocks impact the long end of the term structure more than $\bar{r}_t$-shocks do.

Proposition 6. *Suppose that $\sigma_{\bar{\beta}} = 0$. If $\min\{\kappa_r, \kappa_{\bar{r}}\} \geq \kappa_{\bar{\beta}}$, then the effect of shocks to $\bar{\beta}_t$ on the forward rate curve decays with maturity at a slower rate than the effect of shocks to $\bar{r}_t$. If $\min\{\kappa_r, \kappa_{\bar{r}}\} \geq \kappa_{\beta}$, then the same comparison holds and is strict.*

The sufficient conditions in Proposition 6 have a similar flavor to those in Proposition 5. As with Proposition 5, the conditions are not tight. Our baseline numerical example illustrates this.

4.5 Comparative statics

Figure 5 examines how the effect of supply shocks depends on arbitrageur risk aversion. The figure plots the effect of shocks to current supply β_t and future supply $\bar{\beta}_t$ on the forward-rate curve in two numerical examples: our baseline one where the risk-aversion coefficient a is set to 1.65, and an example where a is set to 2.25 (and all other parameters remain the same). When arbitrageurs are more risk averse, they require a larger risk premium to accommodate supply shocks, and hence the shocks have a larger impact on yields and forward rates. Furthermore, the hump for both β_t- and $\bar{\beta}_t$-shocks occurs at longer maturities. For example, the location of the hump that $\bar{\beta}_t$-shocks generate on the forward-rate curve increases from 6.4 years in the baseline example where $a = 1.65$ to 9 years when $a = 2.25$. The hump occurs at a longer maturity because when arbitrageurs are more risk averse the unit risk premium $URP(\tau)$ increases proportionately more for long-term bonds, i.e., becomes a more convex function of τ. This is because with more risk-averse arbitrageurs, supply shocks have larger price effects, and the impact of these shocks on long-term bonds relative to short-term bonds is larger than that of short-rate shocks. For example, the impact of r_t-shocks is characterized by the increasing function $A_r(\tau)$, while the impact of β_t-shocks involves an integral of that function.

Figure 6 examines how the effect of supply shocks depends on the shocks' persistence. The figure plots the effect of shocks to current and future supply on the forward-rate curve in two numerical examples: our baseline one where the mean-reversion coefficient $\kappa_{\bar{\beta}}$ of $\bar{\beta}_t$-shocks is set to 0.25, and one where $\kappa_{\bar{\beta}}$ is set to 0.2 and hence shocks are more persistent. When $\kappa_{\bar{\beta}} = 0.2$, the effect of a $\bar{\beta}_t$-shock on the expected supply factor $E_t(\beta_{t+\tau})$ at time $t+\tau$ is maximum after 1.1 year ($\tau = 1.1$) and decays to 50% of the maximum after the next 3.9 years ($\tau = 5$). When shocks are more persistent, they have a larger impact on the yield and forward-rate curves. Furthermore, the hump for both β_t- and $\bar{\beta}_t$-shocks occurs at longer maturities. For example, the location of the hump

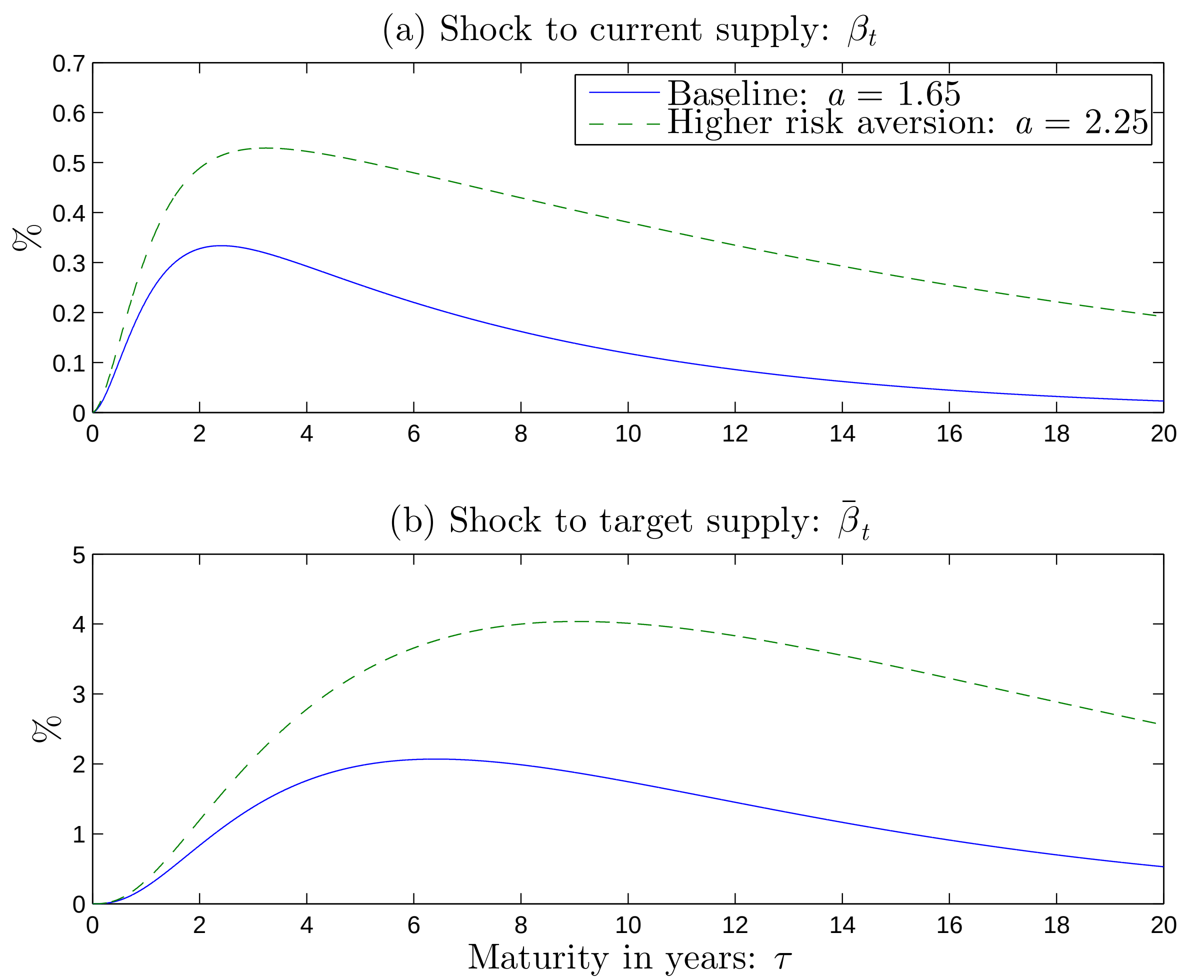

Figure 5: **Impact of supply shocks on the forward-rate curve under different values of arbitrageur risk aversion a. The blue solid lines correspond to our baseline numerical example where $a = 1.65$ and the green dashed lines to an example where $a = 2.25$ and all other parameters remain the same.**

that $\bar{\beta}_t$-shocks generate on the forward-rate curve increases from 6.4 years in the baseline example where $\kappa_{\bar{\beta}} = 0.25$ to 7.6 years when $\kappa_{\bar{\beta}} = 0.2$. While the shift of the hump to longer maturities may not be surprising in the case of $\bar{\beta}_t$-shocks, whose persistence increases, it may be more surprising in the case of β_t shocks, whose persistence does not change. The intuition for β_t-shocks is that higher persistence means that supply shocks have larger price effects, and this makes the unit risk premium $URP(\tau)$ a more convex function of τ.

5 Reassessing QE and the Taper Tantrum

Table 2 summarizes the reaction of the US Treasury yield and forward-rate curves to major QE announcements. The table shows the two-day change in zero-coupon Treasury yields and one-year

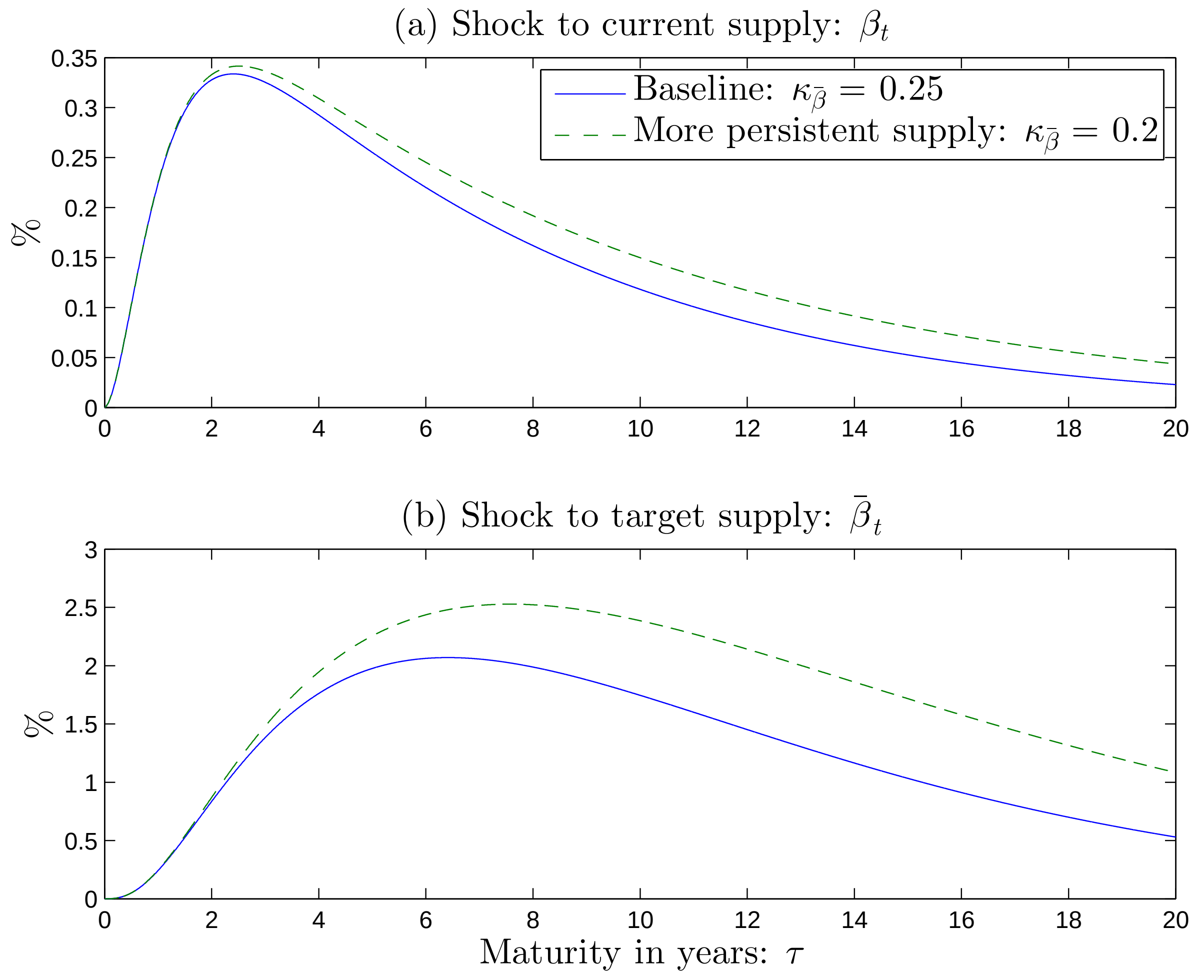

Figure 6: **Impact of supply shocks on the forward-rate curve under different values of the shocks' persistence. The blue solid lines correspond to our baseline numerical example where the mean-reversion coefficient of shocks to future supply is $\kappa_{\bar{\beta}} = 0.25$ and the green dashed lines to an example where $\kappa_{\bar{\beta}} = 0.2$ and all other parameters remain the same.**

forward rates around major policy announcements about the Fed's QE operations. We use two-day changes to allow for the possibility that market participants need time to digest news about LSAP programs. However, we obtain qualitatively similar results if we restrict attention to one-day changes. We obtain US Treasury yields and forward rates using the fitted nominal Treasury curve estimated by Gurkaynak, Sack, and Wright (2007). We use GSW's zero-coupon yields, and compute one-year forward rates from those yields: $f_t^{(\tau-1,\tau)} = \tau y_t^{(\tau)} - (\tau-1) y_t^{(\tau-1)}$. The one-year forward rates are close to the instantaneous forward rates estimated by GSW. We measure all variables in percentage points.

Our set of QE-related announcement dates is drawn from Fawley and Neely (2013), who provide a comprehensive list of FOMC policy announcements and speeches that contained major news about QE. We classify these events based on whether the announcement contained significant news indicating that the Fed would be expanding or contracting its asset purchases. Many of these events contain a mixture of news about future QE operations as well as the path of the short rate. For example, the list includes the March 18, 2009 FOMC announcement discussed in the Introduction, in which the Fed announced that it was expanding the scale of its long-term asset purchase program from $600 billion to $1.75 trillion and that it intended to hold rates at the zero lower bound for "an extended period."

Table 2 shows the change in yields and forward rates around each announcement date. It also shows yield and forward-rate changes aggregated across all expansionary and all contractionary announcements. Figure 7 plots the latter aggregates. As the table and the figure show, both expansionary and contractionary announcements had hump-shaped effects on both the yield and the forward-rate curve. In the case of expansionary announcements, the hump in yields occurred at the ten-year maturity. One-year yields dropped by 33 basis points (bps) on aggregate, two-year yields by 52bps, three-year yields by 86bps, four-year yields by 121bps, five-year yields by 153bps, seven-year yields by 195bps, ten-year yields by 211bps, fifteen-year yields by 179bps, and twenty-year yields by 144bps. The hump in forward rates occurred at the seven-year maturity, with the one-year forward rate seven years into the future dropping by 301bps. In the case of contractionary announcements, the hump in yields occurred at the seven-year maturity and that in forward rates at the five-year maturity.

What is the most natural interpretation of these changes? According to our model, the hump-shaped impact on yields and forward rates can be explained either by forward guidance on the path of future short rates or forward guidance on the path of future bond supply.

Forward guidance on short rates works through the expectations hypothesis. This means, in particular, that following expansionary announcements, yields and forward rates dropped because market participants revised downwards their expectations about future short rates. Moreover, expectations dropped the most for short rates seven years into the future, with the aggregate effect over all announcements being 301bps. That market participants revised so drastically their expectations about the short rate seven years into the future, while expectations one to four years out were revised much more modestly, seems unlikely.

Forward guidance on supply works through expected future risk premia. In contrast to forward guidance on short rates, humps in the yield and forward-rate curves that are consistent with the data could have been the results of changes in supply expectations concerning the near future. Indeed, in our baseline numerical example, shocks to target supply have their largest effect at the 11.5-year maturity in the yield curve and the 6.4-year maturity in the forward-rate curve. Yet, these shocks have their maximum effect on expectations about supply only one year into the future, with

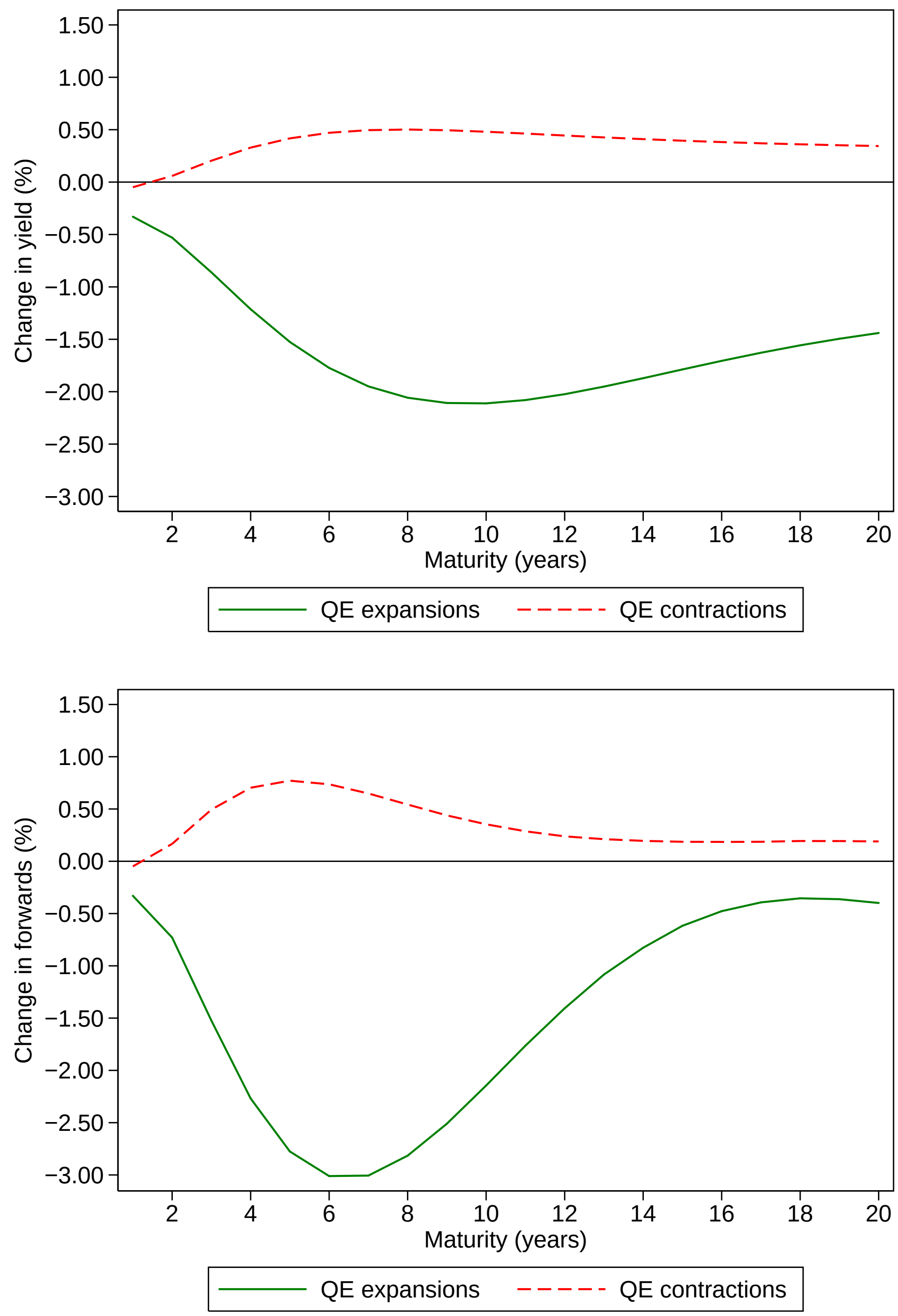

Figure 7: **Changes in Yields and Forward Rates Surrounding QE Announcement Dates.**

the effect four years out being only half of the maximum.

Corroborating evidence on the relative role of supply and short-rate expectations in driving the effects of QE comes from the work by Adrian, Crump, and Moench (2013). These authors construct a methodology for decomposing yields and forward rates into an expectations and a term-premium component. Drawing on their data for the same announcement dates, Figure 8 plots the changes in expected future short rates and in the term premia.[5] As with the previous figures, we show the results for both yields and forward rates. ACM's estimates attribute almost all of the impact of QE announcements to changes in term premia and almost none to changes in expected future short rates.

6 Conclusion

In this paper, we build a model to analyze the impact of forward guidance on the yield curve. Our model recognizes that in recent years forward guidance pertains not only to the future path of short-term interest rates, but also to the future size of the central bank's balance sheet.

We show that forward guidance on short-term interest rates is easy to interpret because it works through the expectations hypothesis. If, for example, the market expectation of the short rate three years from now declines by 100bps, this is reflected directly in a 100bps decline in the instantaneous forward rate three years from now. However, when the central bank provides forward guidance on supply, the effects are more subtle. In particular, yields and forward rates are impacted at maturities much longer than the time by which supply shocks are expected to die out. Moreover, while the effects of either type of forward guidance on the yield and forward-rate curves can be hump-shaped, the humps associated to supply shocks typically occur at maturities longer than those associated to short-rate shocks.

Using our model we re-examine the empirical evidence on QE announcements in the US. We show that the cumulative effect of all expansionary announcements up to 2013 was hump-shaped with a maximum effect at the ten-year maturity for the yield curve and the seven-year maturity for the forward-rate curve. This evidence is hard to square with changing expectations about short rates as the maximum change would have to concern short rates seven years into the future. On the other hand, the evidence is more consistent with changing expectations about supply, as the maximum change would have to be only one year into the future.

[5]The ACM model does not fit the GSW yields exactly. Thus, the two parts of the ACM curve do not perfectly sum to the GSW curve.

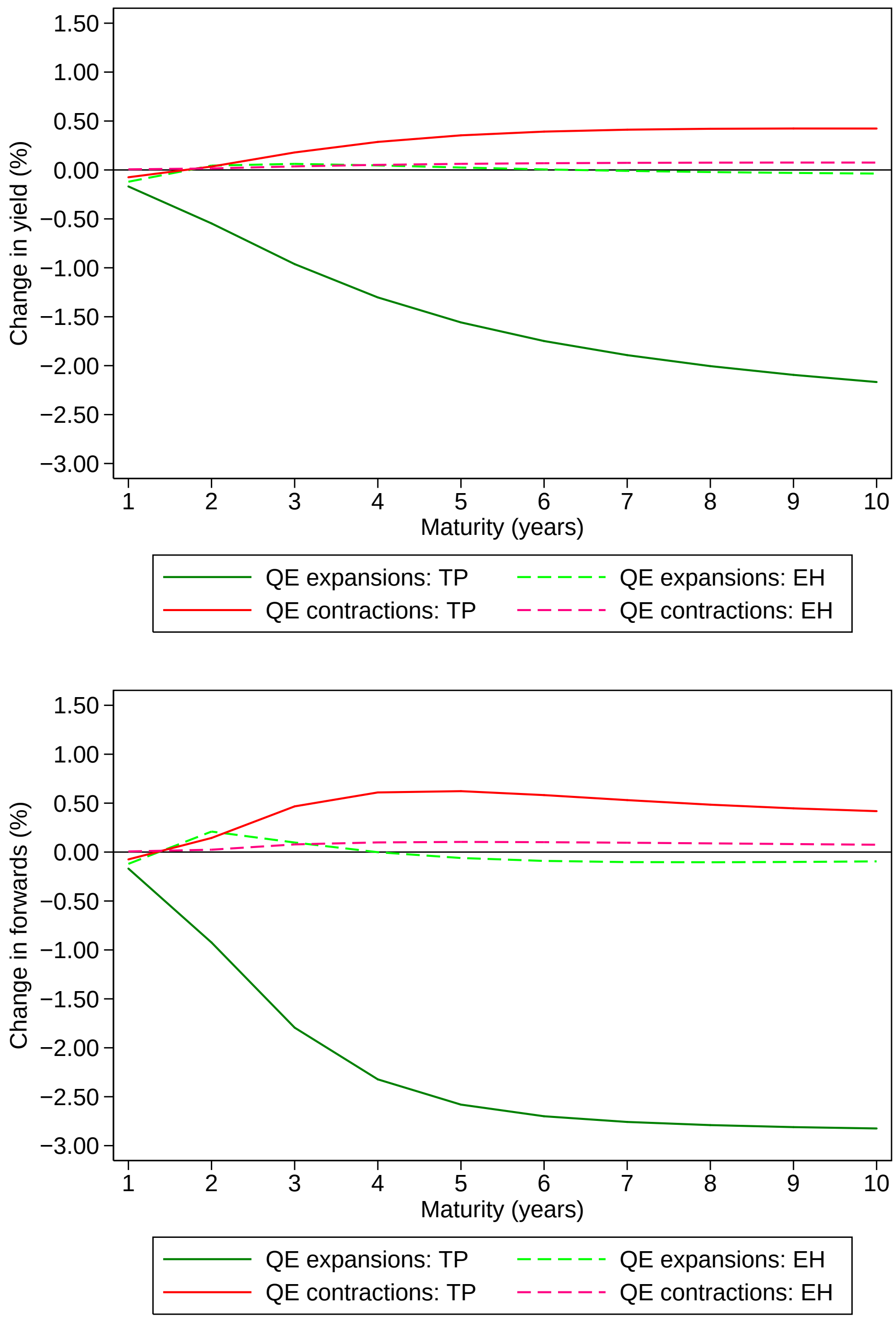

Figure 8: **Changes in expected future short rates and in the term premia surrounding QE announcement dates.** EH refers to the expectations component and TP to the term-premium component. The decomposition into EH and TP draws on data from Adrian, Crump, and Moench (2013).

Appendix

A Proofs of Theoretical Results

Proof of Lemma 1: Using (13), we can write (5) as

$$dW_t = \left(W_t r_t + \int_0^T x_t^{(\tau)}(\mu_t^{(\tau)} - r_t)d\tau\right)dt - \left(\int_0^T x_t^{(\tau)} A_r(\tau)d\tau\right)\sigma_r dB_{r,t}$$
$$- \left(\int_0^T x_t^{(\tau)} A_{\bar{r}}(\tau)d\tau\right)\sigma_{\bar{r}} dB_{\bar{r},t} - \left(\int_0^T x_t^{(\tau)} A_\beta(\tau)d\tau\right)\sigma_\beta dB_{\beta,t} - \left(\int_0^T x_t^{(\tau)} A_{\bar{\beta}}(\tau)d\tau\right)\sigma_{\bar{\beta}} dB_{\bar{\beta},t},$$

and (6) as

$$\max_{\{x_t^{(\tau)}\}_{\tau\in(0,T]}} \Bigg[\int_0^T x_t^{(\tau)}(\mu_t^{(\tau)} - r_t)d\tau - \frac{a\sigma_r^2}{2}\left(\int_0^T x_t^{(\tau)} A_r(\tau)d\tau\right)^2$$
$$-\frac{a\sigma_{\bar{r}}^2}{2}\left(\int_0^T x_t^{(\tau)} A_{\bar{r}}(\tau)d\tau\right)^2 - \frac{a\sigma_\beta^2}{2}\left(\int_0^T x_t^{(\tau)} A_\beta(\tau)d\tau\right)^2 - \frac{a\sigma_{\bar{\beta}}^2}{2}\left(\int_0^T x_t^{(\tau)} A_{\bar{\beta}}(\tau)d\tau\right)^2\Bigg]. \tag{A.1}$$

Point-wise maximization of (A.1) yields (15). ∎

Proof of Theorem 1: Substituting $x_t^{(\tau)}$ from (7) and (17) into (16), we find

$$\lambda_{i,t} = a\sigma_i^2 \int_0^T [\zeta(\tau) + \theta(\tau)\beta_t] A_i(\tau)d\tau. \tag{A.2}$$

Substituting $\mu_t^{(\tau)}$ and $\lambda_{i,t}$ from (14) and (A.2) into (15), we find an affine equation in $(r_t, \bar{r}_t, \beta_t, \bar{\beta}_t)$. Identifying terms in r_t yields

$$\kappa_r A_r(\tau) + A_r'(\tau) - 1 = 0, \tag{A.3}$$

identifying terms in $\bar{r}_t$ yields

$$-\kappa_r A_r(\tau) + \kappa_{\bar{r}} A_{\bar{r}}(\tau) + A_{\bar{r}}'(\tau) = 0, \tag{A.4}$$

identifying terms in β_t yields

$$\kappa_\beta A_\beta(\tau) + A'_\beta(\tau) = a\sigma_r^2 A_r(\tau) \int_0^T A_r(\tau)\theta(\tau)d\tau$$
$$+ a\sigma_{\bar{r}}^2 A_{\bar{r}}(\tau) \int_0^T A_{\bar{r}}(\tau)\theta(\tau)d\tau + a\sigma_\beta^2 A_\beta(\tau) \int_0^T A_\beta(\tau)\theta(\tau)d\tau + a\sigma_{\bar{\beta}}^2 A_{\bar{\beta}}(\tau) \int_0^T A_{\bar{\beta}}(\tau)\theta(\tau)d\tau, \tag{A.5}$$

identifying terms in $\bar{\beta}_t$ yields

$$-\kappa_\beta A_\beta(\tau) + \kappa_{\bar{\beta}} A_{\bar{\beta}}(\tau) + A'_{\bar{\beta}}(\tau) = 0, \tag{A.6}$$

and identifying constant terms yields

$$C'(\tau) - \kappa_{\bar{r}}\bar{r}A_{\bar{r}}(\tau) + \frac{\sigma_r^2}{2}A_r(\tau)^2 + \frac{\sigma_{\bar{r}}^2}{2}A_{\bar{r}}(\tau)^2 + \frac{\sigma_\beta^2}{2}A_\beta(\tau)^2 + \frac{\sigma_{\bar{\beta}}^2}{2}A_{\bar{\beta}}(\tau)^2$$
$$= a\sigma_r^2 A_r(\tau) \int_0^T A_r(\tau)\zeta(\tau)d\tau + a\sigma_{\bar{r}}^2 A_{\bar{r}}(\tau) \int_0^T A_{\bar{r}}(\tau)\zeta(\tau)d\tau$$
$$+ a\sigma_\beta^2 A_\beta(\tau) \int_0^T A_\beta(\tau)\zeta(\tau)d\tau + a\sigma_{\bar{\beta}}^2 A_{\bar{\beta}}(\tau) \int_0^T A_{\bar{\beta}}(\tau)\zeta(\tau)d\tau. \tag{A.7}$$

The ordinary differential equations (ODEs) (A.3)-(A.7) must be solved with the initial conditions $A_r(0) = A_{\bar{r}}(0) = A_\beta(0) = A_{\bar{\beta}}(0) = C(0) = 0$. The solution to (A.3) with the initial condition $A_r(0) = 0$ is (18). The solution to (A.4) with the initial condition $A_{\bar{r}}(0) = 0$ is (19). The solution to (A.6) with the initial condition $A_{\bar{\beta}}(0) = 0$ is (21). To solve (A.5), we write it as

$$\kappa_\beta A_\beta(\tau) + A'_\beta(\tau) = a\sigma_r^2 I_r A_r(\tau) + a\sigma_{\bar{r}}^2 I_{\bar{r}} A_{\bar{r}}(\tau) + a\sigma_\beta^2 I_\beta A_\beta(\tau) + a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}} A_{\bar{\beta}}(\tau), \tag{A.8}$$

using (25), (26), (28), and (29). Differentiating with respect to τ, we find

$$\kappa_\beta A'_\beta(\tau) + A''_\beta(\tau) = a\sigma_r^2 I_r A'_r(\tau) + a\sigma_{\bar{r}}^2 I_{\bar{r}} A'_{\bar{r}}(\tau) + a\sigma_\beta^2 I_\beta A'_\beta(\tau) + a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}} A'_{\bar{\beta}}(\tau). \tag{A.9}$$

Multiplying (A.8) by $\kappa_{\bar{\beta}}$, adding to (A.9), and using (18), (19), and (A.6), we find

$$\left(\kappa_{\bar{\beta}}\kappa_\beta - \kappa_{\bar{\beta}}a\sigma_\beta^2 I_\beta - \kappa_\beta a\sigma_{\bar{\beta}}^2 I_{\bar{\beta}}\right) A_\beta(\tau) + (\kappa_{\bar{\beta}} + \kappa_\beta - a\sigma_\beta^2 I_\beta)A'_\beta(\tau) + A''_\beta(\tau)$$
$$= a\sigma_r^2 I_r \left(\frac{\kappa_{\bar{\beta}}}{\kappa_r} + \frac{\kappa_r - \kappa_{\bar{\beta}}}{\kappa_r} e^{-\kappa_r\tau}\right) + a\sigma_{\bar{r}}^2 I_{\bar{r}} \left(\frac{\kappa_{\bar{\beta}}}{\kappa_{\bar{r}}} + \frac{\kappa_r - \kappa_{\bar{\beta}}}{\kappa_{\bar{r}} - \kappa_r} e^{-\kappa_r\tau} + \frac{\kappa_r(\kappa_{\bar{\beta}} - \kappa_{\bar{r}})}{\kappa_{\bar{r}}(\kappa_{\bar{r}} - \kappa_r)} e^{-\kappa_{\bar{r}}\tau}\right). \quad \text{(A.10)}$$

Equation (A.10) is a second-order linear ODE with constant coefficients. Its solution has the form

$$A_\beta(\tau) = \Gamma_1 e^{-\gamma_1\tau} + \Gamma_2 e^{-\gamma_2\tau} + \hat{A}_\beta(\tau), \quad \text{(A.11)}$$

where (γ_1, γ_2) are the solutions of the quadratic equation (27), and $\hat{A}_\beta(\tau)$ is one solution to (A.10). We look for $\hat{A}_\beta(\tau)$ of the form

$$\hat{A}_\beta(\tau) = Z_1 + Z_2 e^{-\kappa_r\tau} + Z_3 e^{-\kappa_{\bar{r}}\tau}.$$

Substituting into (A.10), we find that (Z_1, Z_2, Z_3) are given by (22)-(24), respectively. To determine (Γ_1, Γ_2) we use the initial conditions. The initial condition $A_\beta(0) = 0$ implies

$$\Gamma_1 + \Gamma_2 + Z_1 + Z_2 + Z_3 = 0. \quad \text{(A.12)}$$

The initial condition $A'_\beta(0) = 0$, which follows from (A.5) and $A_r(0) = A_\beta(0) = A_{\bar{\beta}}(0)$, implies

$$\gamma_1\Gamma_1 + \gamma_2\Gamma_2 + \kappa_r Z_2 + \kappa_{\bar{r}} Z_3 = 0. \quad \text{(A.13)}$$

Solving the linear system of (A.12) and (A.13) yields

$$\Gamma_1 = \frac{\gamma_2 Z_1 + (\gamma_2 - \kappa_r)Z_2 + (\gamma_2 - \kappa_{\bar{r}})Z_3}{\gamma_1 - \gamma_2}, \quad \text{(A.14)}$$

$$\Gamma_2 = -\frac{\gamma_1 Z_1 + (\gamma_1 - \kappa_r)Z_2 + (\gamma_1 - \kappa_{\bar{r}})Z_3}{\gamma_1 - \gamma_2}. \quad \text{(A.15)}$$

Substituting (Γ_1, Γ_2) from (A.14) and (A.15) into (A.11), we find (20).

The solution to (A.7) is

$$C(\tau) = Z_r \int_0^\tau A_r(\tau')d\tau' + Z_{\bar{r}} \int_0^\tau A_{\bar{r}}(\tau')d\tau' + Z_\beta \int_0^\tau A_\beta(\tau')d\tau' + Z_{\bar{\beta}} \int_0^\tau A_{\bar{\beta}}(\tau')d\tau'$$
$$- \frac{\sigma_r^2}{2} \int_0^\tau A_r(\tau')^2 d\tau' - \frac{\sigma_{\bar{r}}^2}{2} \int_0^\tau A_{\bar{r}}(\tau')^2 d\tau' - \frac{\sigma_\beta^2}{2} \int_0^\tau A_\beta(\tau')^2 d\tau' - \frac{\sigma_{\bar{\beta}}^2}{2} \int_0^\tau A_{\bar{\beta}}(\tau')^2 d\tau', \tag{A.16}$$

where

$$Z_r \equiv a\sigma_r^2 \int_0^T A_r(\tau)\zeta(\tau)d\tau,$$

$$Z_r \equiv \kappa_{\bar{r}}\bar{r} + a\sigma_{\bar{r}}^2 \int_0^T A_{\bar{r}}(\tau)\zeta(\tau)d\tau,$$

$$Z_\beta \equiv a\sigma_\beta^2 \int_0^T A_\beta(\tau)\zeta(\tau)d\tau,$$

$$Z_{\bar{\beta}} \equiv a\sigma_{\bar{\beta}}^2 \int_0^T A_{\bar{\beta}}(\tau)\zeta(\tau)d\tau.$$

For $a = 0$, the solutions of (27) are $(\gamma_1, \gamma_2) = (\kappa_\beta, \kappa_{\bar{\beta}})$, and the solution to the system of (28) and (29) is $(I_\beta, I_{\bar{\beta}}) = (0, 0)$. The existence of a solution to (28) and (29) for a close to zero follows from the implicit function theorem. ∎

Proof of Proposition 1: Consider first the unit shock to r_t. Taking expectations in (3), we find that the change $\Delta_r E_t(r_{t+\tau})$ in the expected short rate at time $t + \tau$ follows the dynamics

$$d[\Delta_r E_t(r_{t+\tau})] = -\kappa_r \Delta_r E_t(r_{t+\tau})d\tau.$$

With the initial condition $\Delta_r E_t(r_t) = 1$, these dynamics integrate to

$$\Delta_r E_t(r_{t+\tau}) = e^{-\kappa_r \tau} = A_r'(\tau),$$

where the second step in the first equation follows from (18).

Consider next the unit shock to $\bar{r}_t$. Taking expectations in (3) and (4), we find that the change $\Delta_{\bar{r}} E_t(r_{t+\tau})$ in the expected short rate and $\Delta_{\bar{r}} E_t(\bar{r}_{t+\tau})$ in the target short rate at time $t + \tau$ follow the dynamics

$$d[\Delta_{\bar{r}} E_t(r_{t+\tau})] = \kappa_r [\Delta_{\bar{r}} E_t(\bar{r}_{t+\tau}) - \Delta_{\bar{r}} E_t(r_{t+\tau})] d\tau,$$

$$d[\Delta_{\bar{r}} E_t(\bar{r}_{t+\tau})] = -\kappa_{\bar{r}} \Delta_{\bar{r}} E_t(\bar{r}_{t+\tau}) d\tau.$$

With the initial condition $(\Delta_{\bar{r}} E_t(r_t), \Delta_{\bar{r}} E_t(\bar{r}_t)) = (0, 1)$, these dynamics integrate to

$$\Delta_{\bar{r}} E_t(r_{t+\tau}) = \kappa_r \frac{e^{-\kappa_r \tau} - e^{-\kappa_{\bar{r}} \tau}}{\kappa_{\bar{r}} - \kappa_r} = A'_{\bar{r}}(\tau),$$

$$\Delta_{\bar{r}} E_t(\bar{r}_{t+\tau}) = e^{-\kappa_{\bar{r}} \tau},$$

where the second step in the first equation follows from (19). ∎

We next show a useful lemma.

Lemma A.1. *If a function $f(\tau)$ is positive and increasing, then $\int_0^T f(\tau)\theta(\tau)d\tau > 0$.*

Proof: We can write the integral $\int_0^T f(\tau)\theta(\tau)d\tau$ as

$$\begin{aligned}
\int_0^T f(\tau)\theta(\tau)d\tau &= \int_0^{\tau^*} f(\tau)\theta(\tau)d\tau + \int_{\tau^*}^T f(\tau)\theta(\tau)d\tau \\
&> f(\tau^*) \int_0^{\tau^*} \theta(\tau)d\tau + f(\tau^*) \int_{\tau^*}^T \theta(\tau)d\tau \\
&= f(\tau^*) \int_0^T \theta(\tau)d\tau \geq 0,
\end{aligned}$$

where the second step follows from Part (ii) of Assumption 1 and because $f(\tau)$ is increasing, and the last step follows from Part (i) of Assumption 1 and because $f(\tau)$ is positive. ∎

Proof of Proposition 2: The effect of an increase in r_t on the term structure of spot rates is described by the function $\frac{A_r(\tau)}{\tau}$, and the effect on the term structure of instantaneous forward rates by the function $A'_r(\tau)$. We will show that these functions have the following properties:

- $\frac{A_r(\tau)}{\tau} > 0$ and $A'_r(\tau) > 0$ for $\tau > 0$.
- $\lim_{\tau \to 0} \frac{A_r(\tau)}{\tau} = 1$ and $A'_r(0) = 1$.
- $\lim_{\tau \to \infty} \frac{A_r(\tau)}{\tau} = 0$ and $\lim_{\tau \to 0} A'_r(\tau) = 0$.

- $\frac{A_r(\tau)}{\tau}$ and $A_r'(\tau)$ are decreasing in τ.

Equation (18) implies that the function $A_r(\tau)$ is positive and increasing. Therefore, the functions $\frac{A_r(\tau)}{\tau}$ and $A_r'(\tau)$ are positive, which means that an increase in r_t shifts the term structure upwards. Moreover, both $\frac{A_r(\tau)}{\tau} = \frac{1-e^{-\kappa_r \tau}}{\kappa_r \tau}$ and $A_r'(\tau) = e^{-\kappa_r \tau}$ are equal to one for $\tau = 0$ and to zero for $\tau \to \infty$. Finally, $A_r'(\tau)$ is decreasing in τ, and the same is true for $\frac{A_r(\tau)}{\tau}$ because for a general function $g(\tau)$

$$\frac{d}{d\tau}\frac{g(\tau)}{\tau} = \frac{\tau g'(\tau) - g(\tau)}{\tau^2} = \frac{\int_0^\tau \tau' g''(\tau') d\tau'}{\tau^2}. \tag{A.17}$$

The effect of an increase in $\bar{r}_t$ on the term structure of spot rates is described by the function $\frac{A_{\bar{r}}(\tau)}{\tau}$ and that on the term structure of instantaneous forward rates by the function $A_{\bar{r}}'(\tau)$. We will show that these functions have the following properties:

- $\frac{A_{\bar{r}}(\tau)}{\tau} > 0$ and $A_{\bar{r}}'(\tau) > 0$ for $\tau > 0$.
- $\lim_{\tau\to 0} \frac{A_{\bar{r}}(\tau)}{\tau} = 0$ and $A_{\bar{r}}'(0) = 0$.
- $\lim_{\tau\to\infty} \frac{A_{\bar{r}}(\tau)}{\tau} = 0$ and $\lim_{\tau\to 0} A_{\bar{r}}'(\tau) = 0$.
- $\frac{A_{\bar{r}}(\tau)}{\tau}$ and $A_{\bar{r}}'(\tau)$ are hump-shaped in τ.

Since the function $A_r(\tau)$ is positive, (19) implies that the function $A_{\bar{r}}(\tau)$ is also positive. Moreover, $A_{\bar{r}}(\tau)$ is increasing because

$$\begin{aligned} A_{\bar{r}}'(\tau) &= \kappa_r \left(A_r(\tau) - \kappa_{\bar{r}} \int_0^\tau A_r(\tau') e^{-\kappa_{\bar{r}}(\tau-\tau')} d\tau' \right) \\ &\geq \kappa_r A_r(\tau) \left(1 - \kappa_{\bar{r}} \int_0^\tau e^{-\kappa_{\bar{r}}(\tau-\tau')} d\tau' \right) \\ &= \kappa_r A_r(\tau) e^{-\kappa_{\bar{r}}\tau} > 0, \end{aligned} \tag{A.18}$$

where the first step follows by differentiating (19) and the second because $A_r(\tau)$ is increasing. Since $A_{\bar{r}}(\tau)$ is positive and increasing, the functions $\frac{A_{\bar{r}}(\tau)}{\tau}$ and $A_{\bar{r}}'(\tau)$ are positive, which means that an increase in $\bar{r}_t$ shifts the term structure upwards. Since $A_r(0) = 0$, (19) implies that $\frac{A_{\bar{r}}(\tau)}{\tau}$ is equal to zero for $\tau = 0$, and (A.18) implies the same property for $A_{\bar{r}}'(\tau)$. Since $A_r(\tau)$ converges to the finite limit $\frac{1}{\kappa_r}$ for $\tau \to \infty$, (19) implies that $\frac{A_{\bar{r}}(\tau)}{\tau}$ converges to zero for $\tau \to \infty$, and (A.18) implies the same property for $A'_{\bar{r}}(\tau)$.

To show that $\frac{A_{\bar{r}}(\tau)}{\tau}$ and $A'_{\bar{r}}(\tau)$ are hump-shaped, it suffices to show this property for $A'_{\bar{r}}(\tau)$. Indeed, (A.17) would then imply that $\frac{A_{\bar{r}}(\tau)}{\tau}$ can either be increasing or increasing and then decreasing, and the first pattern is ruled out because $\frac{A_{\bar{r}}(\tau)}{\tau}$ is equal to zero for both $\tau = 0$ and $\tau \to \infty$. Differentiating (A.18), we find

$$A''_{\bar{r}}(\tau) = \kappa_r \left(A'_r(\tau) - \kappa_{\bar{r}} A_r(\tau) + \kappa_{\bar{r}}^2 \int_0^\tau A_r(\tau') e^{-\kappa_{\bar{r}}(\tau-\tau')} d\tau' \right). \tag{A.19}$$

The term in brackets has the same sign as

$$H_{\bar{r}}(\tau) \equiv \left[A'_r(\tau) - \kappa_{\bar{r}} A_r(\tau) \right] e^{\kappa_{\bar{r}}\tau} + \kappa_{\bar{r}}^2 \int_0^\tau A_r(\tau') e^{\kappa_{\bar{r}}\tau'} d\tau'.$$

The function $H_{\bar{r}}(\tau)$ is equal to $A'_r(0) = 1$ for $\tau = 0$, and its derivative is

$$H'_{\bar{r}}(\tau) = A''_r(\tau) e^{\kappa_{\bar{r}}\tau} < 0.$$

Therefore, $H_{\bar{r}}(\tau)$ is either positive or positive and then negative. This means that $A'_{\bar{r}}(\tau)$ is either increasing or increasing and then decreasing. The first pattern is ruled out because $A'_{\bar{r}}(\tau)$ is equal to zero for both $\tau = 0$ and $\tau \to \infty$. ∎

Proof of Proposition 3: Consider first the unit shock to β_t. Using the definition (30) of $URP(\tau)$, we can write (A.5) as

$$\kappa_\beta A_\beta(\tau) + A'_\beta(\tau) = URP(\tau). \tag{A.20}$$

Integrating (A.20) with the initial condition $A_\beta(\tau) = 0$, we find

$$A_\beta(\tau) = \int_0^\tau URP(\tau') e^{-\kappa_\beta(\tau-\tau')} d\tau' \tag{A.21}$$

$$= \int_0^\tau URP(\tau - \tau') e^{-\kappa_\beta \tau'} d\tau'. \tag{A.22}$$

Taking expectations in (8), we find that the change $\Delta_\beta E_t(\beta_{t+\tau})$ in the expected future supply factor at time $t+\tau$ follows the dynamics

$$d[\Delta_\beta E_t(\beta_{t+\tau})] = -\kappa_\beta \Delta_\beta E_t(\beta_{t+\tau}) d\tau.$$

With the initial condition $\Delta_\beta E_t(\beta_t) = 1$, these dynamics integrate to

$$\Delta_\beta E_t(\beta_{t+\tau}) = e^{-\kappa_\beta \tau}. \tag{A.23}$$

Using (A.23), we can write (A.22) as (31).

Consider next the unit shock to $\bar{\beta}_t$. Taking expectations in (8) and (9), we find that the changes $\Delta_{\bar{\beta}} E_t(\beta_{t+\tau})$ in the expected future supply factor and $\Delta_{\bar{\beta}} E_t(\bar{\beta}_{t+\tau})$ in the expected future target supply follow the dynamics

$$d[\Delta_{\bar{\beta}} E_t(\beta_{t+\tau})] = \kappa_\beta [\Delta_{\bar{\beta}} E_t(\bar{\beta}_{t+\tau}) - \Delta_{\bar{\beta}} E_t(\beta_{t+\tau})] d\tau,$$

$$d[\Delta_{\bar{\beta}} E_t(\bar{\beta}_{t+\tau})] = -\kappa_{\bar{\beta}} \Delta_{\bar{\beta}} E_t(\bar{\beta}_{t+\tau}) d\tau.$$

With the initial condition $(\Delta_{\bar{\beta}} E_t(\beta_t), \Delta_{\bar{\beta}} E_t(\bar{\beta}_t)) = (0,1)$, these dynamics integrate to

$$\Delta_{\bar{\beta}} E_t(\beta_{t+\tau}) = \kappa_\beta \frac{e^{-\kappa_\beta \tau} - e^{-\kappa_{\bar{\beta}} \tau}}{\kappa_{\bar{\beta}} - \kappa_b}, \tag{A.24}$$

$$\Delta_{\bar{\beta}} E_t(\bar{\beta}_{t+\tau}) = e^{-\kappa_{\bar{\beta}} \tau}.$$

Substituting (A.21) into (21), we find

$$\begin{aligned}
A_{\bar{\beta}}(\tau) &= \kappa_\beta \int_0^\tau \left(\int_0^{\tau'} URP(\tau'') e^{-\kappa_\beta(\tau'-\tau'')} d\tau'' \right) e^{-\kappa_{\bar{\beta}}(\tau-\tau')} d\tau' \\
&= \kappa_\beta \int_0^\tau \left(\int_{\tau''}^\tau e^{-\kappa_\beta(\tau'-\tau'')} e^{-\kappa_{\bar{\beta}}(\tau-\tau')} d\tau' \right) URP(\tau'') d\tau'' \\
&= \int_0^\tau URP(\tau') \kappa_\beta \frac{e^{-\kappa_\beta(\tau-\tau')} - e^{-\kappa_{\bar{\beta}}(\tau-\tau')}}{\kappa_{\bar{\beta}} - \kappa_\beta} d\tau' \\
&= \int_0^\tau URP(\tau-\tau') \kappa_\beta \frac{e^{-\kappa_\beta \tau'} - e^{-\kappa_{\bar{\beta}} \tau'}}{\kappa_{\bar{\beta}} - \kappa_\beta} d\tau'.
\end{aligned} \tag{A.25}$$

Using (A.24), we can write (A.25) as (32). ∎

Proof of Proposition 4: The effect of an increase in β_t on the term structure of spot rates is described by the function $\frac{A_\beta(\tau)}{\tau}$ and that on the term structure of instantaneous forward rates by the function $A'_\beta(\tau)$. For $\sigma_{\bar{\beta}} = 0$, (A.8) becomes

$$\kappa_\beta A_\beta(\tau) + A'_\beta(\tau) = a\sigma_r^2 I_r A_r(\tau) + a\sigma_{\bar{r}}^2 I_{\bar{r}} A_{\bar{r}}(\tau) + a\sigma_\beta^2 I_\beta A_\beta(\tau),$$

and integrates to

$$A_\beta(\tau) = a\sigma_r^2 I_r \int_0^\tau A_r(\tau')e^{-\hat{\kappa}_\beta(\tau-\tau')}d\tau' + a\sigma_{\bar{r}}^2 I_{\bar{r}} \int_0^\tau A_{\bar{r}}(\tau')e^{-\hat{\kappa}_\beta(\tau-\tau')}d\tau', \tag{A.26}$$

where

$$\hat{\kappa}_\beta \equiv \kappa_\beta - a\sigma_\beta^2 I_\beta.$$

We will show that the functions $\frac{A_\beta(\tau)}{\tau}$ and $A'_\beta(\tau)$ have the following properties:

- $\frac{A_\beta(\tau)}{\tau} > 0$ and $A'_\beta(\tau) > 0$ for $\tau > 0$.
- $\lim_{\tau\to 0} \frac{A_\beta(\tau)}{\tau} = 0$ and $A'_\beta(0) = 0$.
- For $\hat{\kappa}_\beta > 0$, $\lim_{\tau\to\infty} \frac{A_\beta(\tau)}{\tau} = 0$ and $\lim_{\tau\to 0} A'_\beta(\tau) = 0$.
- For $\hat{\kappa}_\beta > 0$, $\frac{A_\beta(\tau)}{\tau}$ and $A'_\beta(\tau)$ are hump-shaped in τ. For $\hat{\kappa}_\beta < 0$, $\frac{A_\beta(\tau)}{\tau}$ and $A'_\beta(\tau)$ are increasing in τ.

Since the functions $A_r(\tau)$ and $A_{\bar{r}}(\tau)$ are positive and increasing, Lemma A.1 implies that $(I_r, I_{\bar{r}})$ are positive. Hence, (A.26) implies that the function $A_\beta(\tau)$ is positive. To show that $A_\beta(\tau)$ is increasing, we differentiate (A.26):

$$A'_\beta(\tau) = a\sigma_r^2 I_r \left(A_r(\tau) - \hat{\kappa}_\beta \int_0^\tau A_r(\tau')e^{-\hat{\kappa}_\beta(\tau-\tau')}d\tau'\right) + a\sigma_{\bar{r}}^2 I_{\bar{r}} \left(A_{\bar{r}}(\tau) - \hat{\kappa}_\beta \int_0^\tau A_{\bar{r}}(\tau')e^{-\hat{\kappa}_\beta(\tau-\tau')}d\tau'\right). \tag{A.27}$$

If $\hat{\kappa}_\beta \leq 0$, then (A.27) and the positivity of $(A_r(\tau), A_{\bar{r}}(\tau))$ imply that $A'_\beta(\tau)$ is positive. If $\hat{\kappa}_\beta > 0$, then the same conclusion follows by proceeding as in the proof of the result in Proposition 2 that $A_{\bar{r}}(\tau)$ is increasing. Since $A_\beta(\tau)$ is positive and increasing, the functions $\frac{A_\beta(\tau)}{\tau}$ and $A'_\beta(\tau)$ are positive, which means that an increase in β_t shifts the term structure upwards. Since $(A_r(0), A_{\bar{r}}(0)) = 0$, (A.26) implies that $\frac{A_\beta(\tau)}{\tau}$ is equal to zero for $\tau = 0$, and (A.27) implies the same property for $A'_\beta(\tau)$. Since $(A_r(\tau), A_{\bar{r}}(\tau))$ converge to the finite limit $(\frac{1}{\kappa_r}, \frac{1}{\kappa_{\bar{r}}})$ for $\tau \to \infty$, (19) implies that when $\hat{\kappa}_\beta > 0$, $A_\beta(\tau)$ converges to a finite limit for $\tau \to \infty$. Therefore, when $\hat{\kappa}_\beta > 0$, $\frac{A_\beta(\tau)}{\tau}$ converges to zero for $\tau \to \infty$, and (A.18) implies the same property for $A'_\beta(\tau)$.

We next study the monotonicity of $\frac{A_\beta(\tau)}{\tau}$ and $A'_\beta(\tau)$. Differentiating (A.27), we find

$$\begin{aligned} A''_\beta(\tau) =& a\sigma_r^2 I_r \left(A'_r(\tau) - \hat{\kappa}_\beta A_r(\tau) + \hat{\kappa}_\beta^2 \int_0^\tau A_r(\tau') e^{-\hat{\kappa}_\beta(\tau-\tau')} d\tau' \right) \\ &+ a\sigma_{\bar{r}}^2 I_{\bar{r}} \left(A'_{\bar{r}}(\tau) - \hat{\kappa}_\beta A_{\bar{r}}(\tau) + \hat{\kappa}_\beta^2 \int_0^\tau A_{\bar{r}}(\tau') e^{-\hat{\kappa}_\beta(\tau-\tau')} d\tau' \right). \end{aligned} \tag{A.28}$$

If $\hat{\kappa}_\beta \leq 0$, then (A.28) and the positivity of $(A_r(\tau), A_{\bar{r}}(\tau))$ imply that $A''_\beta(\tau)$ is positive. Therefore, $A'_\beta(\tau)$ is increasing, and (A.17) implies that $\frac{A_\beta(\tau)}{\tau}$ is increasing. If $\hat{\kappa}_\beta > 0$, then we will show that $A''_\beta(\tau)$ is positive and then negative, and hence $A'_\beta(\tau)$ is hump-shaped. The hump-shape of $\frac{A_\beta(\tau)}{\tau}$ will follow by using (A.17) and noting that $\frac{A_\beta(\tau)}{\tau}$ is equal to zero for both $\tau = 0$ and $\tau \to \infty$.

The right-hand side of (A.28) has the same sign as

$$\begin{aligned} H_\beta(\tau) =& a\sigma_r^2 I_r \left\{ \left[A'_r(\tau) - \hat{\kappa}_\beta A_r(\tau) \right] e^{\hat{\kappa}_\beta \tau} + \hat{\kappa}_\beta^2 \int_0^\tau A_r(\tau') e^{\hat{\kappa}_\beta \tau'} d\tau' \right\} \\ &+ a\sigma_{\bar{r}}^2 I_{\bar{r}} \left\{ \left[A'_{\bar{r}}(\tau) - \hat{\kappa}_\beta A_{\bar{r}}(\tau) \right] e^{\hat{\kappa}_\beta \tau} + \hat{\kappa}_\beta^2 \int_0^\tau A_{\bar{r}}(\tau') e^{\hat{\kappa}_\beta \tau'} d\tau' \right\}. \end{aligned}$$

The function $H_\beta(\tau)$ is equal to $a\sigma_r^2 I_r A_r'(0) = a\sigma_r^2 I_r > 0$ for $\tau = 0$. Its derivative is

$$\begin{aligned}
H_\beta'(\tau) &= a\sigma_r^2 I_r A_r''(\tau) e^{\hat{\kappa}_\beta \tau} + a\sigma_{\bar{r}}^2 I_{\bar{r}} A_{\bar{r}}''(\tau) e^{\hat{\kappa}_\beta \tau}. \\
&= a\sigma_r^2 I_r \left[A_r''(\tau) + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r} \kappa_r H_{\bar{r}}(\tau) e^{-\kappa_{\bar{r}}\tau} \right] e^{\hat{\kappa}_\beta \tau} \\
&= a\sigma_r^2 I_r \left[A_r''(\tau) + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r} \kappa_r \left(1 + \int_0^\tau A_r''(\tau') e^{\kappa_{\bar{r}}\tau'} d\tau' \right) e^{-\kappa_{\bar{r}}\tau} \right] e^{\hat{\kappa}_\beta \tau} \\
&= a\sigma_r^2 I_r \left[-\kappa_r e^{-\kappa_r \tau} + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r} \kappa_r \left(1 - \kappa_r \int_0^\tau e^{(\kappa_{\bar{r}} - \kappa_r)\tau'} d\tau' \right) e^{-\kappa_{\bar{r}}\tau} \right] e^{\hat{\kappa}_\beta \tau} \\
&= a\sigma_r^2 I_r \kappa_r \left[-1 + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r} \left(1 - \frac{\kappa_r \left(e^{(\kappa_{\bar{r}} - \kappa_r)\tau} - 1 \right)}{\kappa_{\bar{r}} - \kappa_r} \right) e^{(\kappa_r - \kappa_{\bar{r}})\tau} \right] e^{(\hat{\kappa}_\beta - \kappa_r)\tau} \\
&= a\sigma_r^2 I_r \kappa_r \left[-1 + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r} \left(e^{(\kappa_r - \kappa_{\bar{r}})\tau} - \frac{\kappa_r \left(1 - e^{(\kappa_r - \kappa_{\bar{r}})\tau} \right)}{\kappa_{\bar{r}} - \kappa_r} \right) \right] e^{(\hat{\kappa}_\beta - \kappa_r)\tau}.
\end{aligned}$$

The term in square brackets is an affine function of $e^{(\kappa_r - \kappa_{\bar{r}})\tau}$ and can hence change sign at most once. Since $A_r'(\tau)$ is decreasing and $A_{\bar{r}}'(\tau)$ is hump-shaped, $H_\beta'(\tau)$ is negative for large τ. Since it can change sign at most once, it is either negative or positive and then negative. Therefore, $H_\beta(\tau)$ is either decreasing or increasing and then decreasing. Since $H_\beta(\tau)$ is positive for $\tau = 0$, it is either positive or positive and then negative. The first pattern is ruled out because when $\hat{\kappa}_\beta > 0$, $A_\beta'(\tau)$ is equal to zero for both $\tau = 0$ and $\tau \to \infty$.

The effect of an increase in $\bar{\beta}_t$ on the term structure of spot rates is described by the function $\frac{A_{\bar{\beta}}(\tau)}{\tau}$ and that on the term structure of instantaneous forward rates by the function $A_{\bar{\beta}}'(\tau)$. We will show that these functions have the following properties:

- $\frac{A_{\bar{\beta}}(\tau)}{\tau} > 0$ and $A_{\bar{\beta}}'(\tau) > 0$ for $\tau > 0$.
- $\lim_{\tau \to 0} \frac{A_{\bar{\beta}}(\tau)}{\tau} = 0$ and $A_{\bar{\beta}}'(0) = 0$.
- For $\hat{\kappa}_\beta > 0$, $\lim_{\tau \to \infty} \frac{A_{\bar{\beta}}(\tau)}{\tau} = 0$ and $\lim_{\tau \to 0} A_{\bar{\beta}}'(\tau) = 0$.
- For $\hat{\kappa}_\beta > 0$, $\frac{A_{\bar{\beta}}(\tau)}{\tau}$ and $A_{\bar{\beta}}'(\tau)$ are hump-shaped in τ. For $\hat{\kappa}_\beta < 0$, $\frac{A_{\bar{\beta}}(\tau)}{\tau}$ and $A_{\bar{\beta}}'(\tau)$ are increasing in τ.

The above properties can be derived from those of $A_\beta(\tau)$ in the same way that the properties of $\frac{A_{\bar{r}}(\tau)}{\tau}$ and $A'_{\bar{r}}(\tau)$ are derived from those of $A_r(\tau)$ in the proof of Proposition 2. In particular, because $A_\beta(\tau)$ is positive, increasing, equal to zero for $\tau = 0$, and converging to a finite limit for $\tau \to \infty$ when $\hat{\kappa}_\beta > 0$, we can show that $\frac{A_{\bar{\beta}}(\tau)}{\tau}$ and $A'_{\bar{\beta}}(\tau)$ are positive, equal to zero for $\tau = 0$, and converging to zero for $\tau \to \infty$ when $\hat{\kappa}_\beta > 0$. The function $A''_{\bar{\beta}}(\tau)$ has the same sign as

$$H_{\bar{\beta}}(\tau) \equiv \left[A'_\beta(\tau) - \kappa_{\bar{\beta}} A_\beta(\tau)\right] e^{\kappa_{\bar{\beta}}\tau} + \kappa_{\bar{\beta}}^2 \int_0^\tau A_\beta(\tau') e^{\kappa_{\bar{\beta}}\tau'} d\tau'.$$

The function $H_{\bar{\beta}}(\tau)$ is equal to $A'_\beta(0) = 0$ for $\tau = 0$, and its derivative is

$$H'_{\bar{\beta}}(\tau) = A''_\beta(\tau) e^{\kappa_{\bar{\beta}}\tau}.$$

When $\hat{\kappa}_\beta \leq 0$, $A''_\beta(\tau)$ is positive. Therefore, $A''_{\bar{\beta}}(\tau)$ is also positive and the functions $\frac{A_{\bar{\beta}}(\tau)}{\tau}$ and $A'_{\bar{\beta}}(\tau)$ are increasing. When $\hat{\kappa}_\beta > 0$, $A''_\beta(\tau)$ is positive and then negative. Therefore, $H_{\bar{\beta}}(\tau)$ is increasing and then decreasing. Since $H_{\bar{\beta}}(\tau)$ is equal to zero for $\tau = 0$, it is either positive or positive and then negative. The first pattern is ruled out when $\hat{\kappa}_\beta > 0$ because $A'_{\bar{\beta}}(\tau)$ is equal to zero for both $\tau = 0$ and $\tau \to \infty$.

The final step in the proof is to show that $\hat{\kappa}_\beta$ is a monotone function of κ_β. This will ensure that $\hat{\kappa}_\beta > 0$ corresponds to larger values of κ_β than $\hat{\kappa}_\beta \leq 0$ does. Since the function $A_\beta(\tau)$ is positive and increasing, Lemma A.1 implies that I_β is positive. Since the function

$$G(\hat{\kappa}_\beta) \equiv \hat{\kappa}_\beta - \kappa_\beta + a\sigma_\beta^2 I_\beta$$

is positive for $\hat{\kappa}_\beta \geq \kappa_\beta$, any solution $\hat{\kappa}_\beta$ to $G(\hat{\kappa}_\beta) = 0$ satisfies $\hat{\kappa}_\beta < \kappa_\beta$. Moreover, at the largest solution, which corresponds to our equilibrium selection, the function $G(\hat{\kappa}_\beta)$ crosses the x-axis from below. Since $G(\hat{\kappa}_\beta)$ is decreasing in κ_β, the largest solution is increasing in κ_β. ∎

Proof of Proposition 5: The humps on the instantaneous-forward-rate term structure associated to shocks to $\bar{r}_t$ and $\bar{\beta}_t$ are located at the solutions to

$$H_{\bar{r}}(\tau) = 1 + \int_0^\tau A''_r(\tau') e^{\kappa_{\bar{r}}\tau'} d\tau' = 0, \tag{A.29}$$

$$H_{\bar{\beta}}(\tau) = \int_0^\tau A''_\beta(\tau') e^{\kappa_{\bar{\beta}}\tau'} d\tau' = 0, \tag{A.30}$$

respectively. We denote these solutions by $(\tau_{\bar{r}}, \tau_{\bar{\beta}})$. Since

$$
\begin{aligned}
A''_\beta(\tau) &= H_\beta(\tau)e^{-\hat{\kappa}_\beta\tau} \\
&= \left\{a\sigma_r^2 I_r + \int_0^\tau \left[a\sigma_r^2 I_r A''_r(\tau')e^{\hat{\kappa}_\beta\tau'} + a\sigma_{\bar{r}}^2 I_{\bar{r}} A''_{\bar{r}}(\tau')e^{\hat{\kappa}_\beta\tau'}\right] d\tau'\right\} e^{-\hat{\kappa}_\beta\tau},
\end{aligned}
$$

we can write (A.30) as

$$
\begin{aligned}
&\int_0^\tau A''_\beta(\tau')e^{\kappa_{\bar{\beta}}\tau'}d\tau' = 0 \\
&\Leftrightarrow \int_0^\tau \left\{a\sigma_r^2 I_r + \int_0^{\tau'} \left[a\sigma_r^2 I_r A''_r(\tau'')e^{\hat{\kappa}_\beta\tau''} + a\sigma_{\bar{r}}^2 I_{\bar{r}} A''_{\bar{r}}(\tau'')e^{\hat{\kappa}_\beta\tau''}\right] d\tau''\right\} e^{-\hat{\kappa}_\beta\tau'}e^{\kappa_{\bar{\beta}}\tau'}d\tau' = 0 \\
&\Leftrightarrow \int_0^\tau e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau'}d\tau' + \int_0^\tau \left(\int_{\tau''}^\tau e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau'}d\tau'\right)\left(A''_r(\tau'') + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r}A''_{\bar{r}}(\tau'')\right)e^{\hat{\kappa}_\beta\tau''}d\tau'' = 0 \\
&\Leftrightarrow 1 + \int_0^\tau \left(A''_r(\tau') + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r}A''_{\bar{r}}(\tau')\right)\frac{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau+\hat{\kappa}_\beta\tau'} - e^{\kappa_{\bar{\beta}}\tau'}}{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau} - 1}d\tau' = 0. \qquad \text{(A.31)}
\end{aligned}
$$

A sufficient condition for $\tau_{\bar{\beta}} > \tau_{\bar{r}}$ is that

$$
\frac{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}+\hat{\kappa}_\beta\tau'} - e^{\kappa_{\bar{\beta}}\tau'}}{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}} - 1} < e^{\kappa_{\bar{r}}\tau'} \quad \text{for} \quad 0 < \tau' < \tau_{\bar{r}}. \qquad \text{(A.32)}
$$

This is because

$$
\begin{aligned}
&H_{\bar{r}}(\tau_{\bar{r}}) = 1 + \int_0^{\tau_{\bar{r}}} A''_r(\tau')e^{\kappa_{\bar{r}}\tau'}d\tau' = 0 \\
&\Rightarrow 1 + \int_0^{\tau_{\bar{r}}} A''_r(\tau')\frac{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}+\hat{\kappa}_\beta\tau'} - e^{\kappa_{\bar{\beta}}\tau'}}{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}} - 1} > 0 \\
&\Rightarrow 1 + \int_0^{\tau_{\bar{r}}} \left(A''_r(\tau') + \frac{\sigma_{\bar{r}}^2 I_{\bar{r}}}{\sigma_r^2 I_r}A''_{\bar{r}}(\tau')\right)\frac{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}+\hat{\kappa}_\beta\tau'} - e^{\kappa_{\bar{\beta}}\tau'}}{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}} - 1} > 0 \\
&\Rightarrow H_{\bar{\beta}}(\tau_{\bar{r}}) > 0, \qquad \text{(A.33)}
\end{aligned}
$$

where the second step follows from (A.32) and $A''_r(\tau') < 0$, and the third step follows because $A''_{\bar{r}}(\tau') > 0$ for $\tau' < \tau_{\bar{r}}$. Since $H_{\bar{\beta}}(\tau)$ has the same sign of $A_{\bar{\beta}}(\tau)$, and the latter is positive if and only if $\tau < \tau_{\bar{\beta}}$, (A.33) implies that $\tau_{\bar{r}} < \tau_{\bar{\beta}}$.

Equation (A.32) is equivalent to

$$h(\tau') \equiv \frac{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}+(\hat{\kappa}_\beta-\kappa_{\bar{r}})\tau'} - e^{(\kappa_{\bar{\beta}}-\kappa_{\bar{r}})\tau'}}{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}} - 1} < 1 \quad \text{for} \quad 0 < \tau' < \tau_{\bar{r}}.$$

The function $h(\tau')$ is equal to one for $\tau' = 0$ and to zero for $\tau' = \tau_{\bar{r}}$. Its derivative is

$$h'(\tau') = \frac{(\hat{\kappa}_\beta - \kappa_{\bar{r}})e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}+(\hat{\kappa}_\beta-\kappa_{\bar{r}})\tau'} - (\kappa_{\bar{\beta}} - \kappa_{\bar{r}})e^{(\kappa_{\bar{\beta}}-\kappa_{\bar{r}})\tau'}}{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)\tau_{\bar{r}}} - 1}$$

and has the same sign as

$$h_1(\tau') \equiv \frac{(\hat{\kappa}_\beta - \kappa_{\bar{r}})e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)(\tau_{\bar{r}}-\tau')} - (\kappa_{\bar{\beta}} - \kappa_{\bar{r}})}{\kappa_{\bar{\beta}} - \hat{\kappa}_\beta}.$$

If $\kappa_\beta \leq \kappa_{\bar{r}}$, then $\hat{\kappa}_\beta < \kappa_{\bar{r}}$. The function $h_1(\tau')$ is negative, as can be seen by writing it as

$$h_1(\tau') = (\hat{\kappa}_\beta - \kappa_{\bar{r}})\frac{e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)(\tau_{\bar{r}}-\tau')} - 1}{\kappa_{\bar{\beta}} - \hat{\kappa}_\beta} - 1.$$

Suppose next that $\kappa_{\bar{\beta}} \leq \kappa_{\bar{r}}$. If $\hat{\kappa}_\beta \leq \kappa_{\bar{r}}$, then $h_1(\tau')$ is negative because of the previous argument. If $\hat{\kappa}_\beta > \kappa_{\bar{r}}$, then $h_1(\tau')$ is negative, as can be seen by writing it as

$$h_1(\tau') = \frac{\kappa_{\bar{\beta}} - \kappa_{\bar{r}} - (\hat{\kappa}_\beta - \kappa_{\bar{r}})e^{(\kappa_{\bar{\beta}}-\hat{\kappa}_\beta)(\tau_{\bar{r}}-\tau')}}{\hat{\kappa}_\beta - \kappa_{\bar{\beta}}}.$$

Since $h_1(\tau')$ is negative, $h(\tau') < 1$ for $0 < \tau' < \tau_{\bar{r}}$, and hence (A.32) is satisfied. ∎

Proof of Proposition 6: Equation (19) implies that the function $A'_{\bar{r}}(\tau)$ decays at rate $e^{-\min\{\kappa_r,\kappa_{\bar{r}}\}\tau}$ for large τ. Equations (20) and (21) imply that the function $A'_{\bar{\beta}}(\tau)$ decays at rate $e^{-\min\{\kappa_r,\kappa_{\bar{r}},\gamma_1,\gamma_2\}\tau}$ for large τ. Therefore, the effect of $\bar{\beta}_t$-shocks on the instantaneous-forward-rate term structure decays with maturity at a slower rate than the effect of $\bar{r}_t$-shocks if

$$\min\{\gamma_1, \gamma_2\} \leq \min\{\kappa_r, \kappa_{\bar{r}}\}, \tag{A.34}$$

and at a strictly slower rate if (A.34) is strict. For $\sigma_{\bar{\beta}} = 0$, (27) implies that $(\gamma_1, \gamma_2) = (\hat{\kappa}_\beta, \kappa_{\bar{\beta}})$. The proposition follows from this observation, (A.34), and $\hat{\kappa}_\beta < \kappa_\beta$. ∎

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Table 2: Reaction of U.S. Treasury yields and forwards to major QE announcements. This table shows the 2-day change in zero coupon Treasury yields and 1-year forward rates surrounding policy announcements about the Fed's Large Scale Asset Purchase Programs. Yields and forwards are based on the fitted nominal Treasury curve estimated by Gürkaynak, Sack, and Wright (2007). All variables are all measured in percentage points. We classify events based on whether the announcement indicated that the Fed would be expanding or contracting its asset purchases. We compute the totals across all expansionary and contractionary events.

			Individual announcement events																	
			Change in zero coupon yields (%)									Change in forward rates (%)								
Date	Type	Event Description	1-yr	2-yr	3-yr	4-yr	5-yr	7-yr	10-yr	15-yr	20-yr	1-yr	2-yr	3-yr	4-yr	5-yr	7-yr	10-yr	15-yr	20-yr
11/25/08	Expand	**QE1**: Initial announcement: $100B of GSE and $500B MBS	-0.12	-0.17	-0.22	-0.26	-0.29	-0.32	-0.31	-0.26	-0.22	-0.12	-0.23	-0.33	-0.38	-0.41	-0.38	-0.25	-0.11	-0.10
12/1/08	Expand	**QE1**: Bernanke QE1 speech mentions extending to USTs	-0.06	-0.12	-0.18	-0.22	-0.25	-0.27	-0.25	-0.22	-0.23	-0.06	-0.19	-0.29	-0.34	-0.36	-0.30	-0.18	-0.16	-0.32
12/16/08	Expand	**QE1**: FOMC alludes to possibility of UST purchases	-0.01	0.00	-0.04	-0.10	-0.17	-0.29	-0.39	-0.37	-0.29	-0.01	0.01	-0.12	-0.30	-0.45	-0.62	-0.56	-0.21	0.01
3/18/09	Expand	**QE1**: Increased size of QE1 from $600B to $1.75T.	-0.12	-0.20	-0.28	-0.35	-0.40	-0.48	-0.51	-0.43	-0.30	-0.12	-0.28	-0.43	-0.55	-0.63	-0.67	-0.53	-0.11	0.21
8/10/10	Expand	**QE2**: Will reinvest coupon/principal from QE1.	-0.01	-0.03	-0.05	-0.08	-0.10	-0.14	-0.15	-0.13	-0.10	-0.01	-0.05	-0.11	-0.16	-0.20	-0.21	-0.16	-0.05	-0.01
9/21/10	Expand	**QE2**: Will continue to reinvest coupon/principal	-0.01	-0.03	-0.06	-0.09	-0.11	-0.15	-0.17	-0.16	-0.14	-0.01	-0.05	-0.12	-0.18	-0.22	-0.24	-0.19	-0.10	-0.07
11/3/10	Expand	**QE2**: Will purchase additional $600B of USTs.	-0.01	-0.02	-0.06	-0.10	-0.13	-0.16	-0.12	0.01	0.12	-0.01	-0.03	-0.13	-0.22	-0.26	-0.21	0.06	0.40	0.42
9/21/11	Expand	**MEP**: $400 billion Maturity Extension Program	0.02	0.05	0.03	-0.01	-0.07	-0.17	-0.26	-0.33	-0.39	0.02	0.09	-0.01	-0.15	-0.28	-0.45	-0.49	-0.47	-0.62
6/20/12	Expand	**MEP**: Continuing MEP through 2012	0.02	0.02	0.02	0.02	0.02	0.01	-0.01	-0.03	-0.05	0.02	0.02	0.02	0.01	0.00	-0.02	-0.05	-0.08	-0.11
8/22/12	Expand	**QE3**: "Additional accommodation may be required fairly soon."	-0.02	-0.04	-0.06	-0.08	-0.10	-0.13	-0.14	-0.13	-0.11	-0.02	-0.05	-0.10	-0.14	-0.17	-0.20	-0.16	-0.07	-0.07
9/13/12	Expand	**QE3**: Purchase $40B of MBS per month	0.00	0.01	0.02	0.03	0.04	0.07	0.12	0.18	0.20	0.00	0.01	0.03	0.06	0.09	0.16	0.25	0.31	0.23
12/12/12	Expand	**QE3**: Purchase $40B of MBS and $45B of USTs per month	0.00	0.01	0.03	0.04	0.06	0.08	0.09	0.08	0.07	0.00	0.03	0.06	0.09	0.11	0.13	0.11	0.05	0.04
11/4/09	Contract	**QE1**: Slowing pace of purchases to complete in 2010Q1	-0.03	-0.04	-0.03	-0.02	0.00	0.03	0.07	0.09	0.08	-0.03	-0.05	-0.02	0.02	0.07	0.13	0.16	0.10	0.05
5/22/13	Contract	**QE3**: Taper tantrum, may slow pace	-0.01	0.02	0.04	0.06	0.07	0.09	0.10	0.07	0.05	-0.01	0.04	0.08	0.11	0.13	0.14	0.08	0.00	0.00
6/19/13	Contract	**QE3**: Taper tantrum, may slow pace	0.00	0.07	0.14	0.20	0.24	0.25	0.23	0.19	0.18	0.00	0.13	0.30	0.38	0.38	0.26	0.12	0.14	0.16
12/18/13	Contract	**QE3**: Announces tapering of QE3	-0.01	0.02	0.05	0.09	0.11	0.12	0.09	0.04	0.02	-0.01	0.04	0.13	0.18	0.19	0.12	-0.01	-0.06	-0.02

	Total across all announcements																	
	Change in zero coupon yields (%)									Change in forward rates (%)								
	1-yr	2-yr	3-yr	4-yr	5-yr	7-yr	10-yr	15-yr	20-yr	1-yr	2-yr	3-yr	4-yr	5-yr	7-yr	10-yr	15-yr	20-yr
Expand	-0.33	-0.53	-0.86	-1.21	-1.53	-1.95	-2.11	-1.79	-1.44	-0.33	-0.73	-1.52	-2.27	-2.78	-3.01	-2.14	-0.62	-0.40
Contract	-0.05	0.06	0.20	0.33	0.42	0.50	0.48	0.40	0.34	-0.05	0.17	0.49	0.70	0.77	0.65	0.35	0.19	0.19