Long-Horizon Investing in a Non-CAPM World

Christopher Polk
London School of Economics
and CEPR

Dimitri Vayanos
London School of Economics
CEPR and NBER

Paul Woolley
London School of Economics

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Abstract

We study dynamic portfolio choice in a calibrated equilibrium model where value and momentum anomalies arise because capital slowly moves from under- to over-performing market segments. Over short horizons, momentum's Sharpe ratio exceeds value's, the value-momentum correlation is negative, and the conditional value-momentum correlation positively predicts Sharpe ratios of value and momentum. In contrast, over long horizons, value's Sharpe ratio can exceed momentum's, the value-momentum correlation turns positive, and the value spread becomes a better predictor of Sharpe ratios. Momentum's optimal portfolio weight relative to value's declines significantly as horizon increases. We provide novel empirical evidence supporting our model's predictions.

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1 Introduction

How should long-horizon investors choose their portfolio of financial assets? According to the CAPM, all investors should hold the market portfolio, which weighs assets according to their market capitalization. Many models of dynamic portfolio choice use the CAPM's basic insight and simplify portfolio choice between stocks and cash to one between a single risky asset and cash. They examine how the optimal investment in the risky asset depends on investor horizon and on variables that predict the asset return.[1]

The CAPM fails to describe asset prices and portfolio allocations well. A vast literature, surveyed in, e.g., Fama (1991) and Schwert (2003), documents that CAPM beta is a weak predictor of asset returns and that other variables such as value, momentum and size are stronger predictors. Trading strategies based on the latter variables are widely used in practice throughout asset management, and funds belonging to styles such as value and growth, momentum, and large- or small-cap are quite popular.

Despite the extensive evidence against the CAPM, academic guidance on long-horizon investing in a non-CAPM world is scarce. What is the optimal portfolio of CAPM anomalies that investors should hold? Should long-horizon investors hold a different portfolio of anomalies than short-horizon investors? Are anomaly returns predictable, and are the predictors different for long- and short-horizon investors? Since long-horizon investors such as pension funds and sovereign-wealth funds control a large fraction of social savings, guidance on these questions can yield large benefits to households as well as improvements in market efficiency.

To study dynamic portfolio choice in a non-CAPM world, an equilibrium approach, grounded on a model in which the CAPM fails to hold, is useful. Indeed, optimal portfolios for long-horizon investors depend on the dynamic evolution of asset prices, and specifying that evolution in the presence of CAPM anomalies involves many degrees of freedom. These include: how each anomaly is reflected in the cross-section of assets; which variables predict each anomaly's return; how anomaly returns correlate with each other; and how they correlate across time. Deriving the corresponding moments from an equilibrium model can provide a tight and internally consistent specification.[2]

In this paper, we study dynamic portfolio choice in a non-CAPM world using the equilibrium approach. We assume that asset prices are determined as in the model of Vayanos and Woolley (2013, VW), in which capital moves from under- to over-performing market segments and does so slowly. The model yields the value and momentum anomalies. Deepening VW's analysis, we provide answers to questions that are key to dynamic portfolio choice and that the theoretical

[1] References are in the literature review section at the end of the Introduction.

[2] Related arguments in favor of an equilibrium approach in dynamic portfolio choice are in Cochrane (2022).

literature has not addressed. We determine, in particular, how the returns of value and momentum depend on predictor variables; how they correlate with each other; and how they correlate across time. We show that that the profitability of anomalies and the main variables that predict them change significantly with investment horizon, and in ways that differ across value and momentum. We explore the implications of our results for dynamic portfolio choice and show that optimal portfolios vary significantly with investment horizon.

We describe our model in Section 2 and solve it in Section 3. The momentum and value anomalies arise from performance-driven flows across investment funds. Suppose that a negative shock hits the fundamental value of some assets. Investment funds holding those assets realize low returns, triggering outflows by investors who infer that fund managers' ability is likely to be low. Because of the outflows, funds sell assets they own, and these sales further depress the prices of the assets hit by the original shock. The momentum anomaly arises because the outflows are assumed to be gradual and because, despite their predictability, they lower expected returns. The value anomaly arises because outflows push prices below fundamental values, so expected returns eventually rise. Key to both anomalies is that capital moves from under- to over-performing market segments and does so slowly. Momentum and value strategies derive their profitability by exploiting these capital flows.

Section 4 defines value and momentum strategies, as well as performance measures for general strategies, and calibrates the model. An asset's value weight is assumed linear in the difference between the present value of the asset's expected dividends discounted at the riskless rate, and the asset's price. An asset's momentum weight is assumed linear in the asset's cumulative return over a given lookback window. Weights change continuously, implying continuous rebalancing of the strategies. We measure a strategy's performance by its annualized market-adjusted Sharpe ratio over a given horizon. A strategy maximizing the utility of an investor with mean-variance preferences over wealth over that horizon maximizes Sharpe ratio. The linear structure of our model makes it possible to compute Sharpe ratios in closed form, even over long horizons and even for strategies that rebalance continuously. We calibrate the model using empirical estimates of the size, price impact, and performance sensitivity of fund flows, available in the literature.

Using the equilibrium prices generated by the calibrated model, we compute the performance of trading strategies and show our main results. Section 5 evaluates strategies over an infinitesimal horizon, and Section 6 considers all horizons longer than infinitesimal.

Our first result concerns the performance of value and momentum strategies in isolation. Over short horizons of up to two years, the strategies' Sharpe ratios decrease with horizon. This reflects the short-horizon positive autocorrelation of strategies' returns, driven by asset-level momentum. Because of that autocorrelation, the annualized variance of returns increases with horizon, and

Sharpe ratios decrease. Over longer horizons, the Sharpe ratio of momentum becomes approximately independent of horizon, while that of value increases significantly. In our main calibration, value overtakes momentum for horizons longer than thirteen years in the case of unconditional Sharpe ratios and five years in the case of conditional ones. Intuitively, momentum has short memory because it weighs assets based only on recent performance. As a consequence, its returns are approximately independent over time when evaluated over longer horizons, and its annualized variance is constant. By contrast, value has long memory because it loads up on assets that have underperformed over a long period. If the assets held by value experience a further long period of underperformance, then their expected returns increase and so does the weight given to them by value. This boosts value's expected return, resulting in strong negative long-horizon autocorrelation of value returns.

Our second result concerns the diversification gains of combining value and momentum. Over short horizons, the strategies are negatively correlated, as has been documented empirically (Asness, Moskowitz, and Pedersen (2013)). This is because value loads up on assets that have underperformed over a long period, while momentum tends to short those assets as they have been trending down in the recent past. In contrast, over horizons longer than one year, the correlation turns positive. This change is mainly because of a positive lead-lag effect from value to momentum.

Our third result concerns the weights of value and momentum in their optimal combination. The optimal combination tilts away from momentum and towards value as horizon increases. Momentum's weight is almost twice that of value for horizons of up to two years. It then decreases with horizon, becoming one-half of value's weight at forty years. Value and momentum exhaust almost all the available gains in our model: the Sharpe ratio of their optimal combination is above 90% of the fully optimal strategy's.

Our fourth result concerns the performance of value and momentum conditional on predictor variables. This result can be understood in terms of the "flow cycle," which describes how capital moves across funds. Following a negative shock to the fundamentals of some assets, capital moves slowly out of funds holding those assets. Since those assets are expected to continue underperforming in the near term, and momentum goes short in them, it has high conditional short-horizon Sharpe ratio at the cycle's early stage. By contrast, value has negative Sharpe ratio because it goes long. Value's Sharpe ratio rises at the cycle's intermediate stage, when most capital has moved out: the assets are then severely undervalued, with high expected returns. It remains high at the cycle's late stage, when the undervalued assets begin to accumulate a history of good performance. At the late stage, momentum's Sharpe ratio is also high because it goes long in the undervalued assets. It is instead low at the intermediate stage, when the return history of the undervalued assets has not yet caught up with their high expected returns.

The variation of conditional short-horizon Sharpe ratios over the flow cycle is reflected into their relationship with two predictors implied by our model: the value spread, whose predictive power for value returns has been shown in Cohen, Polk, and Vuolteenaho (2003, CPV), and the short-horizon value-momentum correlation. Value and momentum are negatively correlated at the cycle's early stage, since value longs the assets that momentum shorts, and are positively correlated at the late stage, since they long the same assets. Therefore, their correlation is strongly positively related to value's Sharpe ratio. A positive relationship exists with momentum's Sharpe ratio as well. The value spread is positively related to value's Sharpe ratio, but the relationship is weaker than that involving the correlation. This weak link arises because at the cycle's early stage, the value spread is wide, but value's Sharpe ratio is negative. The value spread becomes strongly positively related to the strategies' long-horizon Sharpe ratios, while the correlation becomes weakly related.

Section 7 examines whether the theoretical patterns appear in the data. We use a monthly vector auto-regression (VAR) of value and momentum returns together with three predictors: the value spread, the value-momentum correlation, and the panic variable shown in Daniel and Moskowitz (2016, DM) to predict momentum returns. Consistent with our theory, we find that the value-momentum correlation positively predicts value and momentum returns and that including that variable improves the ability of the value spread to predict value returns. We next use the VAR to compute Sharpe ratios and correlations of value and momentum over general horizons and show that they depend on horizon in a way consistent with our theory. The autocorrelation and lead-lag patterns are also consistent with our theory.

Our paper is related to the dynamic portfolio choice literature. Merton (1969, 1971), Samuelson (1969) and Cox and Huang (1989, 1991) develop general methodologies and use them to derive closed-form solutions for return distributions that are constant over time. Kim and Omberg (1996), Brennan and Xia (2002), Wachter (2002) and Liu (2007) derive closed-form solutions for time-varying return distributions.

Brennan, Schwartz, and Lagnado (1997) and Barberis (2000) incorporate the empirically documented positive relationship between the aggregate stock market's expected return and dividend yield—a value effect for the aggregate market—in their numerical analysis of portfolio choice between a risky asset and cash. Campbell and Viceira (1999, 2002) and Chacko and Viceira (2005) analyze portfolio choice between a risky asset and cash using log-linear approximations. When the risky asset's expected return is positively related to the asset's dividend yield, the asset's Sharpe ratio increases with horizon, and long-horizon investors allocate a larger fraction of their wealth to the asset than short-horizon investors. These papers focus on the allocation between stocks and cash in a CAPM world, while we focus on portfolio choice over the cross-section of stocks (or other assets) in a non-CAPM world. Moreover, while these papers emphasize time-variation in the market's expected return, that return is constant in our model and is thus not driving the variation in value and momentum Sharpe ratios across investment horizon.

Jurek and Viceira (2011) study portfolio choice between a value index, a growth index and cash and show that long-horizon investors invest less in value than short-horizon investors. They estimate moments of value and growth returns using VARs. Because they do not include the value spread as a predictor, they do not pick up the negative long-horizon autocorrelation of value returns that we uncover that makes value attractive for long-horizon investors.

Our theory of value and momentum is based on Vayanos and Woolley (2013, VW) and relates to Barberis and Shleifer (2003, BS).[3] Investors in BS can trade multiple assets and move from under- to over-performing investment styles. Because investors do not anticipate future flows, momentum in BS is more profitable than in VW and our model, in which future flows are rationally anticipated. Both BS and VW compute Sharpe ratios of value and momentum, but only unconditionally and over short investment horizons—one period in BS and infinitesimal in VW.

Our paper is also related to the empirical literature showing that fund flows impact asset returns, e.g., Harris and Gurel (1986), Shleifer (1986), Coval and Stafford (2007), Greenwood and Thesmar (2011), Lou (2012), Anton and Polk (2014), Koijen and Yogo (2019) and Gabaix and Koijen (2020). Estimates of the size, price impact, and performance sensitivity of fund flows from that literature inform our calibration exercise. Our calibration shows that such estimates can be mapped to estimates of profitability of CAPM anomalies, such as conditional and unconditional Sharpe ratios of value and momentum.

Our results on horizon effects align with recent empirical findings. Laarits (2021) finds that the variance ratio of value rises above one over short horizons and declines to well below one over longer horizons, while that of momentum remains close to one. These results are shown non-parametrically, for horizons of up to ten years. Chernov, Lochstoer, and Lundeby (2021, CLL) find that linear factor models fail to price their own long-horizon factor returns. One explanation is that means and variances of factor returns are time-varying, causing the optimal combination of factors that prices assets to vary. In our model, assets are priced by a fund flow factor, whose covariance with value and momentum is time-varying in equilibrium. CLL do not find significant long-horizon pricing errors for the value factor, indicating that their method does not pick up the strong mean-reversion of value returns that we uncover. Our flow-based pricing model relates to that in Dou, Kogan, and Wu (2021), in which the premium of the flow factor changes with the amount of delegation.

[3] Other behavioral theories of value and momentum include Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998) and Hong and Stein (1999). Other rational theories of value and momentum include Berk, Green, and Naik (1999), Dasgupta, Prat, and Verardo (2011), Albuquerque and Miao (2014) and Ottaviani and Sorensen (2015).

2 Model

Time t is continuous and goes from zero to infinity. There are $N + 1$ assets. Asset zero is riskless and has an exogenous, continuously compounded return r. Assets $n = 1, .., N$ are risky and their prices are determined endogenously in equilibrium. We interpret the risky assets as stocks or as industry-sector portfolios. We denote by D_{nt} the cumulative dividend per share of asset $n = 1, .., N$, by S_{nt} the asset's price, by $dR_{nt} \equiv dD_{nt} + dS_{nt} - rS_{nt}dt$ the asset's return per share in excess of the riskless asset, and by η_n the asset's supply in terms of number of shares. We refer to dR_{nt} simply as return. We set $dR_t \equiv (dR_{1t}, .., dR_{Nt})'$, where v' denotes the transpose of the vector v.

There are three agents: a representative investor, a representative fund manager, and a representative hedger. The investor can invest in the riskless asset. She can also invest in the risky assets through a passive fund that tracks mechanically an index and through an active fund. The index includes η_n shares of risky asset n and is thus capitalization-weighted.

The index is not an optimal portfolio, and can thus be dominated by the active fund, because the hedger holds a portfolio other than the index. We denote by $\eta_n - \theta_n$ the number of shares of risky asset n held by the hedger. The number of shares held by the other agents is thus θ_n. We refer to θ_n as asset n's residual supply. Our assumption that the hedger does not hold the index amounts to the vectors $\eta = (\eta_1, .., \eta_N)$ and $\theta \equiv (\theta_1, .., \theta_N)$ being linearly independent. We set

$$\Delta \equiv \theta\Sigma\theta'\eta\Sigma\eta' - (\eta\Sigma\theta')^2 > 0.$$

The investor determines how to allocate her wealth between the riskless asset, the index fund, and the active fund. She maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-\mathbb{E}\int_0^\infty \exp(-\alpha c_t - \beta t)dt, \tag{2.1}$$

where α is the coefficient of absolute risk aversion, c_t is consumption, and β is the discount rate. The investor's control variables are consumption c_t and the number of shares x_t and y_t of the index and active fund, respectively.

The fund manager runs the active fund and can invest his personal wealth in it. He thus determines the active portfolio and the allocation of his wealth between the riskless asset and the fund. He maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-\mathbb{E}\int_0^\infty \exp(-\bar{\alpha}\bar{c}_t - \bar{\beta}t)dt, \tag{2.2}$$

where $\bar{\alpha}$ is the coefficient of absolute risk aversion, $\bar{c}_t$ is consumption, and $\bar{\beta}$ is the discount rate. The manager's control variables are consumption $\bar{c}_t$, the number of shares $\bar{y}_t$ of the active fund, and the active portfolio $z_t \equiv (z_{1t},..,z_{Nt})$, where z_{nt} denotes the number of shares of asset n included in one share of the active fund.

The assumption that the manager can invest his personal wealth in the active fund generates an objective that the fund maximizes: the manager chooses the fund's portfolio to maximize the utility that he derives from his stake in the fund. The same assumption generates a counterparty to the investor's flows, ensuring that markets can clear: when the investor reduces her stake in the active fund, prices change so the manager is induced to increase his stake. The manager can be interpreted as the aggregate of all smart-money agents absorbing the investor's flows.

The investor holds the index fund in addition to the active fund because she incurs a cost from investing in the active fund. The cost drives a wedge between the investor's net return from the active fund, and the gross return made of the dividends and capital gains of the stocks held by the fund. We interpret the cost as a perk that the manager extracts from the investor or as a reduced form for managerial ability (with high cost corresponding to low ability). In line with these interpretations, we assume that the cost is time-varying. It is because of this time-variation that the investor moves across funds and assets' expected returns are time-varying. The index fund entails no cost, so its gross and net returns coincide.

Formally, the investor receives net return $y_t(z_t dR_t - C_t dt)$ from the number of shares y_t of the active fund that she holds. This is the gross return $y_t z_t dR_t$ minus the cost $y_t C_t dt$. We assume that C_t follows the process

$$dC_t = \kappa(\bar{C} - C_t)dt + s dB_t^C, \tag{2.3}$$

where κ is a mean-reversion parameter, $\bar{C}$ is a long-run mean, s is a positive scalar, and B_t^C is a Brownian motion.

We normalize one share of the active fund so that its market value equals the equilibrium market value of the entire fund. Under this normalization, the number of fund shares held by the investor and the manager in equilibrium sum to one, i.e.,

$$y_t + \bar{y}_t = 1. \tag{2.4}$$

We normalize one share of the index fund to coincide with the market index η.

We denote the vector of the risky assets' cumulative dividends by $D_t \equiv (D_{1t}, .., D_{Nt})'$ and the vector of the risky assets' prices by $S_t \equiv (S_{1t}, .., S_{Nt})'$. We assume that D_t follows the process

$$dD_t = F_t dt + \sigma dB_t^D, \tag{2.5}$$

where $F_t \equiv (F_{1t}, .., F_{Nt})'$ is a time-varying drift equal to the expected dividend rate, σ is a constant matrix of diffusion coefficients, and B_t^D is a d-dimensional Brownian motion independent of B_t^C. We model time-variation in F_t through the process

$$dF_t = \kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F, \tag{2.6}$$

where the mean-reversion parameter κ is the same as for C_t for simplicity, $\bar{F}$ is a long-run mean, ϕ is a positive scalar, and B_t^F is a d-dimensional Brownian motion independent of B_t^C and B_t^D. The diffusion matrices for D_t and F_t are proportional for simplicity. We set $\Sigma \equiv \sigma\sigma'$.

We assume that the investor can adjust her active-fund holdings y_t only gradually. Gradual adjustment can result from limited attention or institutional decision lags. We model these frictions as a flow transaction cost $\frac{1}{2}\psi\left(\frac{dy_t}{dt}\right)^2$ that the investor must incur when changing y_t.

The manager observes all the variables in the model. The investor observes the returns and share prices of the index and active funds, but not the same variables for individual stocks. She does not observe C_t and F_t.

3 Equilibrium

The equilibrium, derived in Appendix A, is characterized by four state variables: the expected dividend rate F_t, the cost C_t of investing in the active fund, the investor's expectation $\hat{C}_t$ of that cost, and the investor's active-fund holdings y_t. The dynamics of y_t are described by

$$v_t \equiv \frac{dy_t}{dt} = b_0 - b_1\hat{C}_t - b_2 y_t, \tag{3.1}$$

where (b_0, b_1, b_2) are constants and (b_1, b_2) are positive. The investor's active-fund holdings y_t evolve towards the time-varying target $\frac{b_0 - b_1 \hat{C}_t}{b_2}$. When the investor becomes more pessimistic about the active fund ($\hat{C}_t$ rises), the target drops ($\frac{b_1}{b_2} > 0$) and the investor gradually flows from the active into the index fund. The long-run mean of y_t is $\bar{y} \equiv \frac{b_0 - b_1 \bar{C}}{b_2}$.

The dynamics of $\hat{C}_t$ are described by

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left[p_f \left[dD_t - E_t(dD_t) \right] - (C_t - \hat{C}_t)dt \right] - \beta_2 p_f \left[dS_t - E_t(dS_t) \right], \tag{3.2}$$

where (β_1, β_2) are positive constants. The investor becomes more pessimistic about the active fund if its return is low relative to the index fund's. We conduct our analysis in the steady state derived when t goes to infinity, in which the investor's conditional variance of C_t is constant.

The prices S_t of the risky assets take the form

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r + \kappa} - \frac{\alpha \bar{\alpha} f}{\alpha + \bar{\alpha}} \frac{\eta \Sigma \theta'}{\eta \Sigma \eta'} \Sigma \eta' - (\gamma_0 + \gamma_1 \hat{C}_t + \gamma_2 C_t + \gamma_3 y_t) \Sigma p_f', \tag{3.3}$$

where $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ are constants, (γ_1, γ_2) are positive, γ_3 is negative, and

$$p_f \equiv \theta - \frac{\eta \Sigma \theta'}{\eta \Sigma \eta'} \eta \tag{3.4}$$

is a "flow portfolio" describing the flows that the investor generates when moving across funds. The first two terms in (3.3) are the present value of expected dividends discounted at the riskless rate r. The third term is a risk discount proportional to the covariance $\Sigma \eta'$ with the index. This discount is constant over time and reflects an adjustment for index risk. The fourth term is a risk discount proportional to the covariance $\Sigma p_f'$ with the flow portfolio. This discount is time-varying and reflects the price impact of flows.

The flow portfolio p_f is equal to the residual-supply portfolio θ plus a short position in the index η such that the overall position has zero covariance with the index ($\eta \Sigma p_f' = 0$). Long positions in p_f are in risky assets that the active fund overweights relative to the index, and short positions in p_f are in assets that the active fund underweights.

When the investor becomes more pessimistic about the active fund, she gradually moves from the active into the index fund, selling a slice of p_f. As a consequence, assets covarying positively with p_f experience a price decline ($\gamma_1 > 0$), while assets covarying negatively experience a price rise.

Expected returns are described by the two-factor model

$$\mathbb{E}_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\mathbb{C}\text{ov}_t(dR_t, \eta dR_t) + \Lambda_t\mathbb{C}\text{ov}_t(dR_t, p_f dR_t), \tag{3.5}$$

with the factors being the index η and the flow portfolio p_f. The risk premium associated to η is constant over time. The risk premium Λ_t associated to p_f is time-varying and equal to

$$\Lambda_t = r\bar{\alpha} + \frac{1}{f+\frac{k\Delta}{\eta\Sigma\eta'}}\left(\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - k_1\bar{q}_1 - k_2\bar{q}_2\right), \tag{3.6}$$

where $f \equiv 1+\frac{\phi^2}{(r+\kappa)^2}$, $(\gamma_1^R, \gamma_2^R, \gamma_3^R, k, k_1, k_2, \bar{q}_1, \bar{q}_2)$ are constants, γ_1^R and $\gamma_1^R+\gamma_2^R$ are negative, and γ_3^R and k are positive. Equations (3.5) and (3.6) imply that expected returns follow a cycle with a cross-sectional and a time-series dimension. In the cross-section, assets are divided into two segments according to the sign of their covariance with p_f. That covariance reflects the pattern of fund holdings: assets overweighted by the active fund belong to one segment, and assets underweighted belong to the other. In the time-series, the expected returns of assets in each segment exhibit common variation depending on fund flows. When the investor begins to reallocate from one fund to the other, assets in the losing segment are expected to earn low returns. After some flows occur, the expected returns of assets in the losing segment become high. The initial decline in expected returns gives rise to short-run momentum, while the subsequent increase gives rise to long-run reversal.

To illustrate the patterns, consider an increase in $\hat{C}_t$, which triggers gradual outflows from the active into the index fund. Assets covarying positively with p_f experience a price decline ($\gamma_1 > 0$) and a decline in their expected returns ($\gamma_1^R < 0$). Over time, as the outflows from the active fund materialize, y_t drops. Assets covarying positively with p_f experience an increase in their expected returns ($\gamma_3^R > 0$), and that effect eventually dominates.

The initial decline in expected returns, which gives rise to short-run momentum, is surprising. Indeed, as the investor flows out of the active fund, the manager increases his holdings in the fund, absorbing the investor's flows. Why should the manager buy assets that the fund overweights, knowing that these assets' expected returns have declined? Why shouldn't instead those assets drop immediately to a level from which they are expected to earn higher future returns? The answer lies in the manager's intertemporal hedging demand, whose effect in this setting VW term *bird-in-the-hand effect*. The anticipation of outflows from the active fund causes assets covarying positively with p_f to be underpriced and to earn an attractive return over a long horizon (one bird in the hand). The manager could earn an even more attractive return on average (two birds in the bush) by buying these assets after the outflows occur. This exposes him, however, to the risk that the outflows might not occur, in which case the assets would cease to be underpriced and future investment opportunities would become unprofitable.

The bird-in-the-hand effect can be illustrated using a simple three-period example. An asset is expected to pay off 100 in Period 2. The asset's price is 92 in Period 0, and 80 or 100 in Period 1 with equal probabilities. Buying the asset in Period 0 earns an investor a two-period expected capital gain of 8. Buying in Period 1 earns an expected capital gain of 20 if the price is 80 and 0 if the price is 100. A risk-averse agent might prefer earning 8 rather than 20 or 0 with equal probabilities, even though the expected capital gain between Periods 0 and 1 is negative.

4 Trading Strategies and Performance Measures

4.1 Value and Momentum

We define a trading strategy by a vector of weights $w_t \equiv (w_{1t}, .., w_{Nt})$, where w_{nt} is the number of shares invested in risky asset n at time t. We include in the strategy a position $-\sum_{n=1}^{N} w_{nt} S_{nt}$ in the riskless asset, so that the value of the combined position is zero. The strategy rebalances continuously if the weights w_t change continuously over time. Any gains are paid out and any losses are covered continuously so that the value of the combined position remains zero.

We consider the value strategy

$$w_t^V \equiv \left(\frac{\bar{F}}{r} + \frac{\epsilon(F_t - \bar{F})}{r + \kappa} - S_t \right)', \tag{4.1}$$

where $\epsilon \in \{0, 1\}$. A risky asset's value weight increases linearly in the difference between the asset's fundamental value and price. We measure fundamental value by the present value of expected dividends discounted at r, and use two measures of expected dividends: the optimal forecast, which depends on the expected dividend rate F_t and corresponds to $\epsilon = 1$ in (4.1), and the crude forecast, which sets expected dividends equal to their unconditional mean $\bar{F}$ and corresponds to $\epsilon = 0$.

We consider the momentum strategy

$$w_t^M \equiv \left(\int_{t-\tau}^{t} dR_u\right)'. \tag{4.2}$$

A risky asset's momentum weight increases linearly in the asset's cumulative past return over the interval $[t-\tau, t]$. We refer to the length $\tau > 0$ of that interval as the lookback window.

4.2 Performance Measures

We measure the performance of a trading strategy w_t by the Sharpe ratio of its index-adjusted version

$$\hat{w}_t \equiv w_t - \frac{\mathbb{C}\mathrm{ov}_t(w_t dR_t, \eta dR_t)}{\mathbb{V}\mathrm{ar}_t(\eta dR_t)}\eta. \tag{4.3}$$

The index-adjusted strategy $\hat{w}_t$ is constructed by combining w_t with a position in the index such that the covariance between the overall position and the index is zero. The Sharpe ratio of the index-adjusted strategy represents compensation for risk orthogonal to the index.

The Sharpe ratio over an infinitesimal horizon dt is

$$SR_{w,t} \equiv \frac{\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t dR_t)}{\sqrt{\mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(\hat{w}_t dR_t) dt}}. \tag{4.4}$$

It is derived by dividing the expected excess return of $\hat{w}_t$ by the return's standard deviation, and expressing the ratio in annualized terms by dividing by $\sqrt{dt}$. The return moments are conditional on an information set $\mathcal{I}_t$ that depends on t. We use the subscript $\mathcal{I}_t$ for moments conditional on $\mathcal{I}_t$, the subscript t for moments conditional on all information available at time t, and no subscript for unconditional moments. We likewise omit the subscript t from the unconditional Sharpe ratio. We refer to $SR_{w,t}$ interchangeably as the Sharpe ratio of w_t or of $\hat{w}_t$.

Our use of $SR_{w,t}$ to measure performance measure can be motivated based on portfolio optimization. Suppose that an investor with horizon dt has mean-variance preferences, and can invest in the riskless asset, the index η and the strategy w_t. In Appendix C (Lemma C.1), we show that the investor's maximum utility is proportional to the sum of the squared Sharpe ratio of the index η and of w_t. In particular, it depends on w_t only through $SR_{w,t}$.

We extend our use of the Sharpe ratio over a general finite horizon T. The Sharpe ratio, expressed in annualized terms, is

$$SR_{w,t,T} \equiv \frac{\mathbb{E}_{\mathcal{I}_t}\left(\int_t^{t+T} \hat{w}_u dR_u\right)}{\sqrt{\mathbb{V}\mathrm{ar}_{\mathcal{I}_t}\left(\int_t^{t+T} \hat{w}_u dR_u\right) T}}, \tag{4.5}$$

and can be motivated based on portfolio optimization, as in the case of an infinitesimal horizon dt. The relationship between maximum utility and Sharpe ratio carries through provided that ηdR_u is uncorrelated with $\hat{w}_{u'} dR_{u'}$ conditionally on $\mathcal{I}_t$ for $t < u < u'$. That property holds under a condition on the strategy weights w_u, which is met for the strategies that we examine in the rest of this paper. The derivations are in Appendix C (Lemma C.2).

We define and calculate Sharpe ratios using returns per share rather than per dollar invested. This is because our CARA-normal model is better suited for calculating per share returns and their moments: the calculations are simplified by the properties that prices are linear in the state variables and that state variables are normally distributed. Since Sharpe ratios are unit-free, they have a similar economic interpretation for returns per share and returns per dollar.

4.3 Calibration

We next calibrate our model. The parameter values are summarized in Table I. The model-implied moments are calculated in Appendix C. A sensitivity analysis to different parameter values is in Appendix F. Our calibration differs from VW because we adopt a different set of target moments and mapping between parameters and moments.

We set some parameters to one using appropriate normalizations. By redefining the units of the consumption good, we set the investor's risk-aversion coefficient α to one. By redefining the dividend per share of each asset n, we set the asset's supply η_n to a value that is common across assets and such that the average residual supply $\bar{\theta} \equiv \frac{\sum_{n=1}^N \theta_n}{N}$ is equal to one. Since assets are supplied in the same number of shares, the index includes the same number of shares $\eta_n = \bar{\eta}$ of each asset n. By rescaling the index, we set $\bar{\eta}$ to one.

We interpret assets as industry-sector portfolios and set their number N to ten. We assume that all sector portfolios have the same expected dividends, the same standard deviation of dividends and the same pairwise correlation. We denote the vector of expected dividends per share by $\bar{F} = \mathcal{F}\mathbf{1}$ and the covariance matrix of dividends per share by $\Sigma = \hat{\sigma}^2(I + \omega\mathbf{1}\mathbf{1}')$, where $\mathbf{1}$ is the $N \times 1$ vector of ones, I is the $N \times N$ identity matrix, and $(\mathcal{F}, \hat{\sigma}, \omega)$ are scalars. We choose $\mathcal{F}$ so that the index's expected return per dollar in excess of the riskless rate is 4%. We choose $\Sigma_{nn} = \hat{\sigma}^2(1 + \omega)$ so that

Table I: Calibration of model parameters.

Parameter	**Symbol**	**Value**	**Target**
Investor's risk-aversion coefficient	α	1	Normalization
Manager's risk-aversion coefficient	$\bar{\alpha}$	29	Fraction of return variance generated by flows
Number of assets	N	10	Industry-sector portfolios
Number of shares of each asset in the index	η_n	1	Normalization
Average residual supply across assets	$\bar{\theta} \equiv \frac{\sum_{n=1}^{N} \theta_n}{N}$	1	Normalization
Standard deviation of residual supply across assets	$\sqrt{\frac{\sum_{n=1}^{N}(\theta_n - \bar{\theta})^2}{N}}$	0.2	Industry-sector level active share of aggregate portfolio of mutual funds
Expected dividends per share of each asset	$\bar{F}_n$	0.33	Expected excess return of index
Variance of dividends per share of each asset	Σ_{nn}	0.47	Sharpe ratio of index
Covariance of dividends per share of each asset pair	$\Sigma_{nn'}$	0.41	Correlation between average industry-sector portfolio and index
Shocks to expected dividends F_t relative to dividends D_t	ϕ	0.05	
Mean-reversion coefficient of C_t and F_t	κ	0.3	Mean-reversion of return gap
Standard deviation of shocks to C_t	s	1.2	Volume generated by fund flows
Long-run mean of C_t	$\bar{C}$	-0.22	Investor's share in active fund
Transaction cost	ψ	0.65	Horizon over which fund flows respond to performance
Riskless rate	r	0.04	

the annualized Sharpe ratio of the index is 30%. (The index's Sharpe ratio is horizon-independent in our model.) We choose $\Sigma_{nn'} = \hat{\sigma}^2\omega$ for $n' \neq n$ so that the return correlation between industry-sector portfolios and the index is 87%. This is the average correlation between sector portfolios and the index in Ang and Chen (2002). The remaining parameter describing dividends is ϕ. It is the size of shocks to the process F_t that drives expected dividends relative to shocks to the process D_t that drives dividends. Shocks to expected dividends render prices not fully revealing about C_t, and induce a causal link from fund performance to fund flows as the investor uses performance to learn about C_t. We set ϕ to 0.05, a value that maximizes the investor's uncertainty about C_t. Even under that value, uncertainty is small: the investor's conditional standard deviation of C_t is 18%

of the unconditional standard deviation. Changing ϕ has a small effect on Sharpe ratios.

With a symmetric covariance matrix of dividends, the only characteristic of residual supply θ_n that affects Sharpe ratios, beyond the average $\bar{\theta} = 1$ across assets, is the standard deviation $\sigma(\theta) \equiv \sqrt{\frac{\sum_{n=1}^{N}(\theta_n - \bar{\theta})^2}{N}}$. We choose $\sigma(\theta)$ based on the active share of the residual-supply portfolio (Cremers and Petajisto (2009)). Buffa, Vayanos, and Woolley (2022) find that the active share of the aggregate portfolio of all active equity mutual funds, computed at the industry-sector level, is 10.81%. Defining the residual supply portfolio to also include index funds, and taking index fund assets to be 10% of total fund assets (active and index), the active share of the residual supply portfolio is 9.73% (=90%× 10.81%). We set $\sigma(\theta) = 0.2$, which implies an active share of 10% under the assumption that θ_n is equal to $\bar{\theta} + \sigma(\theta) = 1.2$ for half of the assets and to $\bar{\theta} - \sigma(\theta) = 0.8$ for the other half.

We choose the manager's coefficient of absolute risk aversion $\bar{\alpha}$ based on the fraction of asset return variance generated by fund flows. Intuitively, when the manager is more risk-averse, the investor's flows have larger price impact and account for a larger fraction of price movements. Greenwood and Thesmar (2011) find that fund flows explain 8% of the variance of individual stocks. Gabaix and Koijen (2020) find that flows explain up to 50% of the volatility of the aggregate market. This amounts to 25% of the variance if flows are independent of fundamentals. We assume that the effect for industry-sector portfolios lies in-between, and use 15% as our target. The corresponding value of $\bar{\alpha}$ is 29, i.e., the manager is 29 times more risk-averse than the investor. In our sensitivity analysis, we allow for a lower target variance.

We choose the mean-reversion coefficient κ of the cost C_t by identifying C_t with the return gap in Kacperczyk, Sialm, and Zheng (2008, KSZ). KSZ define the return gap as the difference between a mutual fund's return over a given quarter and the return of a hypothetical portfolio invested in the stocks that the fund holds at the beginning of the quarter. This aligns with the definition of C_t in our model. We set κ to 0.3 to match KSZ's finding that shocks to the return gap shrink to about one-third of their size within four years.

We choose the diffusion coefficient s of C_t based on the volume generated by fund flows. Lou (2012) computes flow-induced trading (FIT) for each stock by aggregating the trades that all mutual funds perform on that stock in response to inflows or outflows they experience, and dividing by the funds' aggregate holdings of the stock. The spread in quarterly FIT between top and bottom stock deciles sorted based on FIT is 22.27% (= 16.76% − (−5.51%)). We set $s = 1.2$ to match that spread, thus assuming that the spread is the same for stocks as for industry-sector portfolios (i.e., flows take place at the level of sector portfolios). In our sensitivity analysis, we allow for a lower target spread.

We choose the long-run mean $\bar{C}$ of C_t based on the long-run mean $\bar{y}$ of the investor's share y_t in the active fund. The share y_t can be interpreted as the extent to which non-expert investors participate in trades that require financial expertise, which in our model consists in exploiting hedger-induced mispricing. Perfect risk-sharing, derived for C_t constant over time and equal to $\bar{C} = 0$, implies a share y_t that is constant over time and equal to $\bar{y} = \frac{\bar{\alpha}}{\alpha+\bar{\alpha}} = 96.8\%$. We use $\bar{y} = 90\%$ as our target. The corresponding value of $\bar{C}$ is -0.22. Allowing for a lower target $\bar{y} > 0$ for which $\bar{C}$ becomes positive, has a small effect on Sharpe ratios.

We choose the transaction cost ψ based on the horizon over which fund flows respond to performance. Coval and Stafford (2007) find that flows into a mutual fund during quarter t increase in the fund's return during quarters $t-1$ to $t-4$, and are roughly independent of the return during quarters $t-5$ to $t-8$. We set ψ to 0.65 in line with that finding: following a positive (negative) shock to the active fund's return, the fund experiences inflows (outflows) for 22 months, with 90% of the effect occurring within the first 13 months.

The FIT estimates in Lou (2012) concern the volume generated by all mutual fund flows. Since our calibration matches all mutual fund flows but our model assumes only flows between an aggregate of active funds and of index funds, the latter flows are unrealistically large in our calibration. The flows in our calibration could be interpreted as including flows between active funds. Since differences between active funds can be larger than between an aggregate of active funds and of index funds, we allow for a larger target active share in our sensitivity analysis.

Our sensitivity analysis reveals that the results in Sections 5 and 6 are relatively insensitive to the target active share and FIT spread. They are sensitive to the target variance: when the fraction of return variance generated by flows is smaller, the Sharpe ratio of momentum drops significantly, and the Sharpe ratio of value becomes less volatile and more correlated with the value spread.

5 Performance over an Infinitesimal Horizon

5.1 Optimal Strategy

In Appendix D (Lemma D.1) we show that the Sharpe ratio of a strategy w_t over an infinitesimal horizon dt is

$$SR_{w,t} = \frac{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}\left(\Lambda_t w_t \Sigma p_f'\right)}{\sqrt{f\left[\mathbb{E}_{\mathcal{I}_t}(w_t\Sigma w_t') - \frac{\mathbb{E}_{\mathcal{I}_t}[(w_t\Sigma\eta')^2]}{\eta\Sigma\eta'}\right] + k\mathbb{E}_{\mathcal{I}_t}[(w_t\Sigma p_f')^2]}}, \tag{5.1}$$

and is maximized for $w_t = \Lambda_t p_f$. The intuition why the strategy $w_t = \Lambda_t p_f$ is optimal comes from the two-factor model (3.5) for expected returns. The two factors are the index η, with a constant risk premium, and the flow portfolio p_f, with a time-varying risk premium Λ_t. Since the Sharpe ratio $SR_{w,t}$ concerns the index-adjusted version $\hat{w}_t$ of w_t, it reflects compensation for the risk corresponding to the second factor p_f only. Therefore, it is maximized for a strategy that invests only in p_f: risk that is uncorrelated with p_f (and η) is not compensated. The size of the investment in p_f is proportional to that factor's risk premium Λ_t.

The Sharpe ratio of the optimal strategy $w_t = \Lambda_t p_f$ is (Proposition D.1):

$$SR^*_{w,t} \equiv \sqrt{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}\mathbb{E}_{\mathcal{I}_t}(\Lambda_t^2)}. \tag{5.2}$$

The unconditional Sharpe ratio is proportional to $\sqrt{\mathbb{E}(\Lambda_t^2)}$. The conditional Sharpe ratio is proportional to the absolute value $|\Lambda_t|$ when the conditioning set $\mathcal{I}_t$ includes the time-t values of the state variables $(\hat{C}_t, C_t, y_t)$. Indeed, since Λ_t is a function of $(\hat{C}_t, C_t, y_t)$ only, $\mathbb{E}_{\mathcal{I}_t}(\Lambda_t^2) = \Lambda_t^2$. Since Λ_t is affine in $(\hat{C}_t, C_t, y_t)$, the conditional Sharpe ratio is high when $\hat{C}_t$, C_t or y_t are large in absolute value.

In our calibrated example, the unconditional Sharpe ratio of the optimal strategy is 70.21%. It is thus 2.34 times higher than the index's Sharpe ratio, which is 30%. The optimal strategy achieves its high Sharpe ratio while also representing risk orthogonal to the index.

The optimal strategy's conditional Sharpe ratio on $(\hat{C}_t, C_t, y_t)$ has mean 56.02% and standard deviation 42.32%. It thus varies significantly over time. It is lower on average than the unconditional Sharpe ratio because it tends to be high when the optimal strategy has high conditional standard deviation.

5.2 Value

In Appendix D (Proposition D.2) we derive a closed-form solution for the unconditional Sharpe ratio of the value strategy (4.1). In our calibrated example, the unconditional Sharpe ratio is 26.88% when the present value of expected dividends is computed using the optimal forecast ($\epsilon = 1$), and 27.05% when the crude forecast ($\epsilon = 0$) is used.

While the value strategy offers a Sharpe ratio comparable to that of the index, and with orthogonal risk, it achieves less than 40% of the optimal strategy's Sharpe ratio in relative terms ($\frac{27.05\%}{70.21\%} = 38.53\%$). This is because the value strategy fails to account for short-run momentum.

Consider an increase in $\hat{C}_t$, which triggers gradual outflows from the active fund. Assets covarying positively with the flow portfolio p_f experience an immediate price decline, and thus an increase in their value weight. Since these assets also experience a decline in their expected return, the value strategy earns a low expected return at this stage of the cycle. Over time, as the outflows from the active fund materialize and y_t drops, the value weight of these assets increases further. Their expected return switches to being high, and so does the expected return of the value strategy.

To characterize the performance of the value strategy at different stages of the cycle, we compute in Appendix D (Proposition D.3) the strategy's conditional Sharpe ratio. We condition on $(\hat{C}_t, y_t)$ only, because these are the key variables describing the cycle and are observed by the investor in our model (while C_t is not). Equations (3.3) and (4.1) imply that the expected weights of the value strategy conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$ and when $\epsilon = 1$ are

$$\mathbb{E}_{\mathcal{I}_t}\left(w_t^V\right) = \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta\Sigma + \left(\gamma_0 + (\gamma_1+\gamma_2)\hat{C}_t + \gamma_3 y_t\right)p_f\Sigma \tag{5.3}$$

Likewise, (3.6) implies that the conditional expectation of Λ_t is

$$\mathbb{E}_{\mathcal{I}_t}\left(\Lambda_t\right) = r\bar{\alpha} + \frac{1}{f+\frac{k\Delta}{\eta\Sigma\eta'}}\left((\gamma_1^R+\gamma_2^R)\hat{C}_t + \gamma_3^R y_t - k_1\bar{q}_1 - k_2\bar{q}_2\right). \tag{5.4}$$

An increase in $\hat{C}_t$ raises value weights of assets covarying positively with p_f because (γ_1, γ_2) are positive. It lowers Λ_t, thus lowering the expected returns of those assets, because $\gamma_1^R + \gamma_2^R$ is negative. By contrast, a decrease in y_t raises those assets' value weights and expected returns because (γ_3, γ_3^R) are negative. In line with these observations, the conditional Sharpe ratio of the value strategy is negative and large when $\hat{C}_t$ becomes large in absolute value (early stage of the cycle). It switches to being positive and large when y_t adjusts to the change in $\hat{C}_t$ by becoming large with opposite sign to $\hat{C}_t$ (late stage).

The unconditional mean and standard deviation of the value strategy's conditional Sharpe ratio are 20.60% and 65.03%, respectively, when $\epsilon = 1$, and 21.46% and 63.25%, respectively, when $\epsilon = 0$. The Sharpe ratio of the value strategy varies more than that of the optimal strategy, reflecting the sharply different performance of value at different stages in the cycle.

We next examine how the conditional Sharpe ratio of the value strategy correlates with the value spread. We define the value spread as the standard deviation of the market-to-book ratio in the cross-section of assets. We assume that all assets have the same book value, which we take to be the average price in the cross-section of assets and over time. We compute the value spread conditional on $(\hat{C}_t, y_t)$ in Appendix D (Proposition D.4). The value spread is high when $\hat{C}_t$ is large in absolute value because the price discrepancies between assets that covary positively and assets that covary negatively with the flow portfolio p_f are large. The value spread is even higher when y_t adjusts to the change in $\hat{C}_t$ because the price discrepancies become even larger.

The value spread correlates positively but imperfectly with the conditional Sharpe ratio of the value strategy. This is because $\hat{C}_t$ moves the two variables in opposite directions, while y_t moves them in the same direction and has a dominant effect. When $\hat{C}_t$ becomes large in absolute value (early stage of the cycle), the value spread is large and the conditional Sharpe ratio of value is negative. When y_t adjusts to the change in $\hat{C}_t$ by becoming large with opposite sign to $\hat{C}_t$ (late stage), the value spread is even larger and the conditional Sharpe ratio of value is positive. The unconditional correlation between the value spread and the conditional Sharpe ratio of the value strategy is 26.31% when $\epsilon = 1$ and 26.00% when $\epsilon = 0$. For the remainder of our analysis, we focus on the value strategy with $\epsilon = 0$.

5.3 Momentum

In Appendix D (Proposition D.5) we derive a closed-form solution for the unconditional Sharpe ratio of the momentum strategy (4.2). Figure 1 plots the unconditional Sharpe ratio in our calibrated example as function of the lookback window τ over which past returns are calculated. The unconditional Sharpe ratio reaches its maximum value 53.66% for a window of seven months, and exceeds 50% for windows ranging from four to eleven months. When the window goes to zero, the Sharpe ratio also goes to zero because performance over a very short interval is almost uninformative about future flows. Conversely, when the window becomes large, the Sharpe ratio becomes negative because momentum turns into reversal.

The momentum strategy with the 4-12 month lookback window performs significantly better than the value strategy because it is better aligned with movements in expected returns. Consider an increase in $\hat{C}_t$, which triggers gradual outflows from the active fund. Assets covarying positively with the flow portfolio p_f experience an immediate price decline, and thus a decrease in their momentum weight. Since these assets also experience a decline in their expected return, the momentum strategy earns a high expected return at the early stage of the cycle. It also earns a moderately high expected return at the late stage. Indeed, after the outflows materialize, the expected return of assets covarying positively with p_f is high. As a consequence, these assets' return history improves, and their momentum weight rises. Momentum's underperformance occurs at intermediate stages

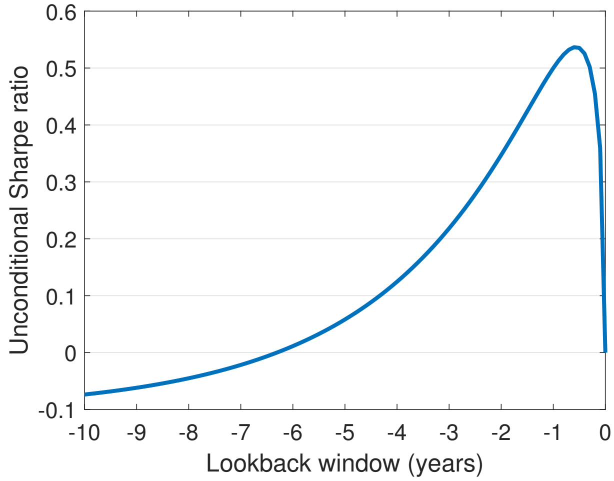

Figure 1: Unconditional Sharpe ratio of momentum as function of the lookback window τ over which past returns are calculated.

of the cycle. Indeed, the expected return of assets covarying positively with p_f has increased but their return history has not caught up with that increase.

To characterize the performance of the momentum strategy at different stages of the cycle, we compute in Appendix D (Proposition D.6) the strategy's conditional Sharpe ratio. As with the value strategy, we condition on $(\hat{C}_t, y_t)$ only. Using (4.2) and (A.4), we show in the Appendix that the expected weights of the momentum strategy conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$ are

$$\mathbb{E}_{\mathcal{I}_t}\left(w_t^M\right) = \frac{r\alpha\bar{\alpha}f\tau}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta\Sigma + \left(\delta_0^M + \delta_{12}^M\hat{C}_t + \delta_3^M y_t\right)p_f\Sigma, \tag{5.5}$$

where $(\delta_0^M, \delta_{12}^M, \delta_3^M)$ are constants. For the remainder of our analysis, we focus on the optimal momentum strategy with the seven month lookback window, for which $(\delta_{12}^M, \delta_3^M)$ are positive and the ratio $\frac{\delta_3^M}{\delta_{12}^M}$ is smaller than $\frac{\gamma_3^R}{\gamma_1^R+\gamma_2^R}$. An increase in $\hat{C}_t$ lowers momentum weights of assets covarying positively with p_f because δ_{12}^M is negative. Because it also lowers Λ_t, the conditional Sharpe ratio of the momentum strategy is positive and large when $\hat{C}_t$ is large in absolute value (early stage of the cycle). A decrease in y_t raises the momentum weights of assets covarying positively with p_f because δ_3^M is negative. Because it also raises Λ_t, the conditional Sharpe ratio of the momentum strategy is positive when y_t adjusts to the change in $\hat{C}_t$ by becoming large with opposite sign to $\hat{C}_t$ (late stage). The conditional Sharpe ratio is instead negative for a range of intermediate values of y_t. Indeed, because $\frac{\delta_3^M}{\delta_{12}^M} < \frac{\gamma_3^R}{\gamma_1^R+\gamma_2^R}$, Λ_t changes sign before momentum weights do during the process of y_t's adjustment.

The unconditional mean and standard deviation of the conditional Sharpe ratio of the momentum strategy are 40.74% and 46.58%, respectively. The Sharpe ratio of the momentum strategy varies less than that of the value strategy, reflecting the more limited variation in momentum's performance over the cycle.

Since momentum performs well at the early stage of the cycle, moderately well at the late stage, and poorly at intermediate stages, it is weakly correlated with the value spread. The unconditional correlation between the value spread and the conditional Sharpe ratio of momentum is -8.13%. The correlation between the conditional Sharpe ratios of momentum and value is 28.22%.

Figure 2 plots the dynamics following a shock that moves the state variables $(\hat{C}_t, y_t)$ away from their long-run means $(\bar{C}, \bar{y})$. The shock is a decline to the flow portfolio's return at time zero, equal to one standard deviation of the portfolio's annual return. The left panel plots the shock's effect on $(\hat{C}_t, y_t)$, as function of time t. Following the shock, the investor's expectation $\hat{C}_t$ of the active fund's cost jumps up and declines gradually to $\bar{C}$. The investor's share y_t in the active fund declines gradually for 22 months after the shock. After that time, it increases gradually to $\bar{y}$.

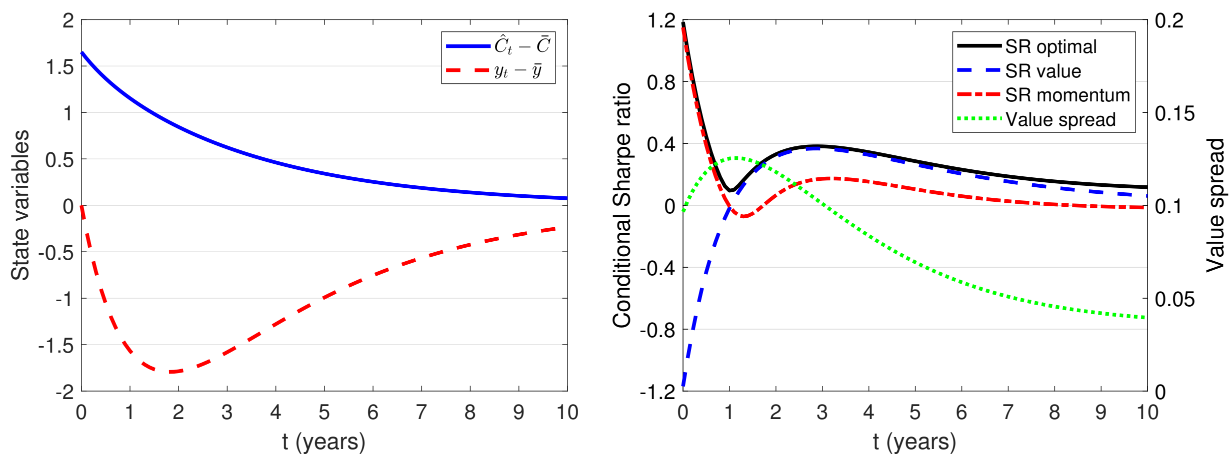

Figure 2: State variables (left panel) and conditional Sharpe ratios and the value spread (right panel) following an one standard deviation drop in the return of the flow portfolio p_f at time zero.

The right panel plots the Sharpe ratios conditional on $(\hat{C}_t, y_t)$ for the optimal strategy (black solid line), the value strategy (blue dashed line) and the momentum strategy (red dashed-dotted line), as function of time t. The value spread is also plotted (green dotted line). The units for the Sharpe ratios are shown in the left y-axis, and the units for the value spread are shown in the right y-axis.

When $(\hat{C}_t, y_t)$ are equal to their long-run means $(\bar{C}, \bar{y})$, the Sharpe ratios of momentum and value are close to zero: -2.48% for momentum and 0.38% for value. Following the shock, the Sharpe ratios experience large movements in opposite directions. The Sharpe ratio of momentum jumps up to 114.99%. It then declines rapidly, becomes negative twelve months after the shock, becomes positive again twenty-two months after the shock, reaches a maximum of 17.36% three years and two months after the shock, and finally declines to its value for $(\hat{C}_t, y_t) = (\bar{C}, \bar{y})$. The Sharpe ratio of value jumps down to -116.96%. It then rises rapidly, becomes positive thirteen months after the shock, reaches a maximum of 36.71% three years after the shock, and finally declines to its value for $(\hat{C}_t, y_t) = (\bar{C}, \bar{y})$. The value spread jumps up following the shock, and keeps increasing as the mispricing worsens. It reaches a maximum thirteen months after the shock, and then declines to its value for $(\hat{C}_t, y_t) = (\bar{C}, \bar{y})$.

The Sharpe ratio of the optimal strategy is remarkably close to that of value or of momentum. The difference between the optimal strategy's Sharpe ratio and the larger of the value and the momentum Sharpe ratios is smaller than 6% for a long period after the shock, which is composed of two sub-periods. During the first sub-period which lasts for the first five months after the shock, momentum is approximately optimal and its Sharpe ratio lies within 6% of the optimal strategy's. During the second sub-period which lasts from fourteen months to 10.5 years after the shock, value is instead approximately optimal and its Sharpe ratio lies within 6% of the optimal strategy's. Hence, within the context of our model, value and momentum span well the set of trading strategies, with each of them being approximately optimal at a different stage of the cycle.

The shock in Figure 2 is a decline to the flow portfolio's return. Under the opposite shock, $(\hat{C}_t - \bar{C}, y_t - \bar{y})$ would change sign, and the right panel would flip around the x-axis. The left panel would remain approximately the same, however. This is because the Sharpe ratios depend almost exclusively on the stage of the flow cycle and not on whether the flows occur from the active to the index fund or vice-versa. The conditional correlation between value and momentum, plotted in Figure 3, would also remain approximately the same.

5.4 Combining Value and Momentum

We next compute the gains from combining value and momentum. This requires computing the correlation between the two strategies' returns, an exercise of independent interest since it reveals how the strategies relate to each other. For two general strategies (w_t^A, w_t^B), the Sharpe ratio of their optimal (mean-variance maximizing) combination is (Appendix D, Lemma D.5)

$$SR_{w^{AB},t} \equiv \sqrt{\frac{SR^2_{w^A,t} + SR^2_{w^B,t} - 2SR_{w^A,t}SR_{w^B,t}\mathbb{C}\mathrm{orr}_{\mathcal{I}_t}(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t)}{1 - \mathbb{C}\mathrm{orr}_{\mathcal{I}_t}(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t)^2}}. \tag{5.6}$$

The Sharpe ratio of the optimal combination depends only on the two strategies' Sharpe ratios and on the correlation between the strategies' returns. The calculations in (5.6) are conditional on an information set $\mathcal{I}_t$, in the sense that strategy weights, Sharpe ratios and the correlation can depend on $\mathcal{I}_t$. The correlation in (5.6) is between the strategies' index-adjusted versions, but as with the Sharpe ratios we refer to it as pertaining to the strategies as well.

In Appendix D (Proposition D.7) we derive a closed-form solution for the unconditional correlation between value and momentum, and for the Sharpe ratio of the optimal unconditional combination of the two strategies ($\mathcal{I}_t = \emptyset$). In our calibrated example, this Sharpe ratio is 63.45%. It is 10% larger than the unconditional Sharpe ratio of momentum (53.66%) and 36% larger than that of value (27.05%). Thus, combining value and momentum improves significantly over using one or the other strategy and yields a Sharpe ratio close to that of the optimal strategy (70.21%). The improvement is due to the low unconditional correlation between value and momentum, which is -12.20%. The correlation is negative because value loads up on assets that have underperformed over a long period, while momentum tends to short those assets as they have been trending down in the recent past.

The low unconditional correlation between value and momentum masks large variation in the conditional correlation. We compute the value-momentum correlation conditionally on $(\hat{C}_t, y_t)$ in Appendix D (Proposition D.7). The conditional correlation has unconditional mean -8.13% and standard deviation 73.02%. Figure 3 illustrates the large variation in the conditional correlation by plotting its dynamics following the same shock as in Figure 2.

When $(\hat{C}_t, y_t)$ are equal to their long-run means $(\bar{C}, \bar{y})$, momentum and value are approximately independent, with a correlation of -4.53%. Following the shock, the correlation jumps down to -97.06%. The two strategies thus become approximately perfectly negatively correlated. The correlation remains below -80% for the ten months after the shock. It then rises gradually, becomes positive twenty months after the shock, reaches a maximum of 52.30% three years and four months after the shock, and finally declines to its value for $(\hat{C}_t, y_t) = (\bar{C}, \bar{y})$.

During the period of high negative correlation, the momentum strategy shorts assets that covary positively with the flow portfolio p_f because the shock drives down these assets' returns. By contrast, the value strategy longs these assets because their price is low. During the period of high

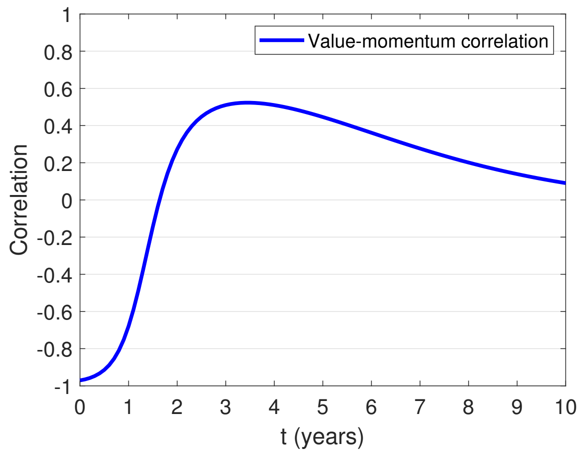

Figure 3: Conditional correlation between value and momentum following an one standard deviation drop in the return of the flow portfolio p_f at time zero.

positive correlation, value continues longing these assets, and momentum switches to longing them as well.

Combining value and momentum conditionally on $(\hat{C}_t, y_t)$ yields small gains over using one of the two strategies only. In the dynamics shown in Figure 2, the Sharpe ratio of the optimal value-momentum combination never exceeds the larger of the individual Sharpe ratios by more than 6%. (This can be anticipated from the closeness between the Sharpe ratio of the optimal strategy and the larger of the individual Sharpe ratios.) The improvement from combining value and momentum conditionally is smaller than unconditionally because of the variation in the relative performance of the two strategies. Momentum is the much better strategy for a set of values of $(\hat{C}_t, y_t)$ and value is for another set. Within either set, combining the strategies yields small gains relative to using the better strategy. When, however, information on $(\hat{C}_t, y_t)$ is not used, the identity of the better strategy is unknown. Combining the strategies then yields larger gains because the weight on the worse strategy is reduced and so is the scope for under-performance.

The conditional correlation between value and momentum is informative about the Sharpe ratio of each strategy. That information is particularly precise for value: the unconditional correlation between the conditional value-momentum correlation and the conditional Sharpe ratio of value is 86.02%. This can be anticipated from Figures 2 and 3, as the conditional correlation and the conditional Sharpe ratio of value respond similarly to the shock. While Figures 2 and 3 suggest a negative unconditional correlation between the conditional value-momentum correlation and the conditional Sharpe ratio of momentum, that correlation is positive and equal to 41.34%. Thus, a positive value-momentum correlation indicates high conditional Sharpe ratio of both value and momentum.

6 Performance over a General Finite Horizon

6.1 Optimal Strategy

We next allow the investment horizon to take any finite value. To determine how horizon influences the choice of strategy, we begin with an optimization exercise. Consider an investor who has horizon T and maximizes the unconditional Sharpe ratio $SR_{w,T}$. Suppose that the investor must follow a strategy of the form $w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ and can optimize over the coefficients $(\delta_0, \delta_1, \delta_2, \delta_3)$. The optimal coefficients depend on the horizon T, and we denote them by $(\delta^*_{0,T}, \delta^*_{1,T}, \delta^*_{2,T}, \delta^*_{3,T})$. We determine in Appendix E the Sharpe ratio $SR^*_{w,T}$ of the optimal strategy (Proposition E.1) and the strategy's correlation with value and momentum (Propositions E.5 and E.6).

The optimization is not over the full set of strategies: the strategies are assumed to invest only in the flow portfolio p_f; the investment in p_f is assumed linear in the state variables $(\hat{C}_t, C_t, y_t)$; and the coefficients in the linear function are assumed constant over time. The first and second assumptions are shown as results for the short-horizon optimal strategy $w_t = \Lambda_t p_f$ (Section 5.1), and we conjecture that these results extend to the long-horizon optimization. In particular, since the Sharpe ratio $SR_{w,T}$ reflects compensation only for risk corresponding to p_f, it should be maximized for a strategy that invests only in p_f. The third assumption is restrictive. Indeed, since the optimal coefficients $(\delta^*_{0,T}, \delta^*_{1,T}, \delta^*_{2,T}, \delta^*_{3,T})$ depend on the horizon T, the investor may want to change them as time passes and the end of the horizon approaches. Restricting the coefficients to be time-independent simplifies the calculation of the (constrained) optimal strategy and of its closeness to value and momentum (both of which are defined to be time-independent).

Figure 4 illustrates properties of the optimal strategy. The left panel plots the unconditional return correlation of the optimal strategy with the value strategy (blue dashed line) and with the momentum strategy (red dashed-dotted line), as function of the investment horizon. The correlation concerns returns computed over the horizon corresponding to the optimal strategy (e.g., one-year returns for the one-year optimal strategy, and ten-year returns for the ten-year optimal strategy). When the horizon is short, the optimal strategy correlates more highly with momentum than with value. The higher correlation with momentum is consistent with the finding in Section 5 that momentum has a higher Sharpe ratio than value over an infinitesimal horizon.

The main new observation from the figure concerns the variation of the correlation with the investment horizon. As horizon increases, the correlation of the optimal strategy with momentum decreases, while that with value increases and overtakes momentum's for horizons longer than thir-

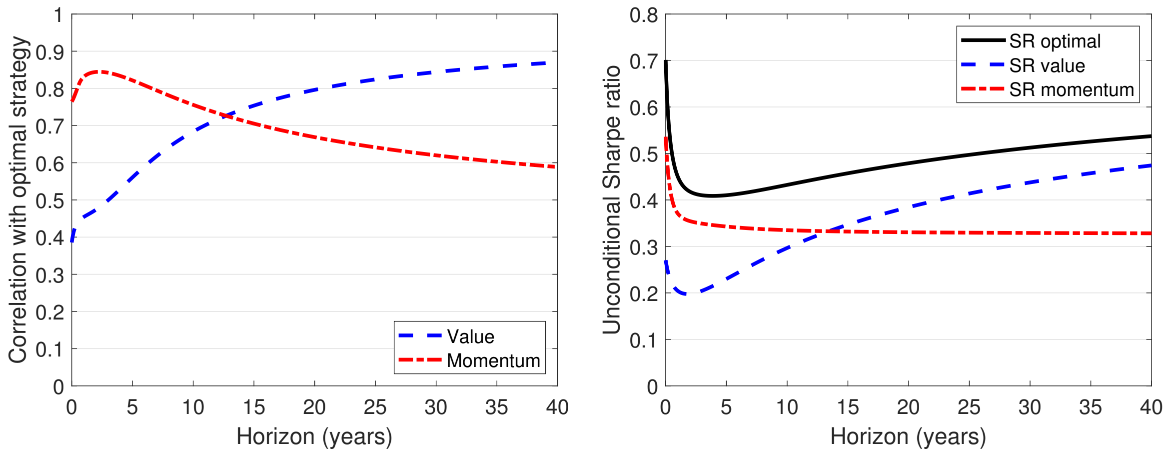

Figure 4: Unconditional correlation of the optimal strategy with value and momentum (left panel) and unconditional Sharpe ratios (right panel), as function of the investment horizon.

teen years. Similar conclusions follow when measuring closeness by weights in a tracking portfolio. When value and momentum are combined into a portfolio whose return is the closest to the optimal strategy's, as measured by unconditional variance, their weights have a similar dependence on horizon as the correlations. We defer a fuller discussion of optimal weights to Section 6.4.

The effects of investment horizon on the correlation that the optimal strategy has with value and momentum are related to the effects of horizon on the strategies' Sharpe ratios. The right panel of Figure 4 plots the unconditional Sharpe ratios of the optimal strategy (black line), value (blue dashed line) and momentum (red dashed-dotted line), as function of horizon. We compute the unconditional Sharpe ratios of value and momentum in Appendix E (Propositions E.2 and E.3, respectively). The Sharpe ratio of the optimal strategy is closer to that of momentum for short horizons and to that of value for long horizons. This is consistent with the results on the correlations. Consistent with those results is also that value's Sharpe ratio overtakes momentum's for horizons longer than thirteen years. The main new observation from the figure concerns the variation of Sharpe ratios with horizon. The Sharpe ratio of the optimal strategy is an inverse hump-shaped function of horizon, as is the Sharpe ratio of value. The Sharpe ratio of momentum initially decreases with horizon and then stays essentially flat.

The effects of horizon on Sharpe ratios are driven by the autocorrelation of strategy returns. In Appendix E (Lemma E.2) we show that the unconditional Sharpe ratio $SR_{w,T}$ of a strategy w_t over horizon T can be expressed in terms of the unconditional Sharpe ratio SR_w over an infinitesimal horizon and the unconditional autocovariance of returns over all horizons from zero to T:

$$SR_{w,T} = \frac{SR_w}{\sqrt{1 + 2\int_t^{t+T} \left(1 - \frac{u-t}{T}\right) \frac{\mathbb{C}\mathrm{ov}(\hat{w}_t dR_t, \hat{w}_u dR_u)}{\mathbb{V}\mathrm{ar}(\hat{w}_t dR_t)}}}. \tag{6.1}$$

The Sharpe ratio is independent of horizon when strategy returns are serially uncorrelated. This is because expected returns are horizon-independent when expressed in annualized terms (dividing by T), and lack of serial correlation implies that the same is true for standard deviation (dividing by $\sqrt{T}$). When instead strategy returns are positively autocorrelated, annualized variance increases with horizon, and $SR_{w,T}$ is smaller than SR_w. The converse is true when returns are negatively autocorrelated. Thus, the effects of horizon shown in the right panel of Figure 4 reflect variation in return autocorrelation. We examine that variation next, in the context of value and momentum.

6.2 Value

We compute unconditional autocorrelations of value and momentum returns in Appendix E (Proposition E.8), and plot them in Figure 5. The left panel plots correlations between value (blue dashed line) or momentum (red dashed-dotted line) returns over a lookback window τ ending at time t with the same strategy's returns over the year starting at time t. The right panel replaces returns one year ahead by returns ten years ahead. Value returns are positively autocorrelated over short horizons and lookback windows, reflecting asset-level momentum. They become negatively autocorrelated over long horizons or lookback windows, reflecting partly asset-level reversal and mostly the nature of the value strategy. Value loads up on assets that have performed poorly, and has low turnover because it is based on slow-moving signals. Suppose that the poorly performing assets held by value experience a further long period of underperformance, lowering value returns. The expected returns of those assets increase, and so does the weight given to them by the value strategy. This boosts value's expected return, resulting in negative autocorrelation of value returns over long lookback windows. Lengthening the horizon (moving from the left to the right panel) renders autocorrelations more negative, and negative for all lookback windows. This is because the effect of momentum is small over long horizons.

The autocorrelation pattern of value returns is reflected into the inverse hump-shaped pattern of value's Sharpe ratio. For short horizons, the relevant autocorrelations are those over short lookback windows in the left panel of Figure 5. Since these are positive, annualized variance increases with horizon and the Sharpe ratio decreases. For long horizons instead, the autocorrelations over long lookback windows become relevant, lowering the annualized variance and raising the Sharpe ratio. The long-window autocorrelations are larger in absolute terms than the short-horizon ones, and

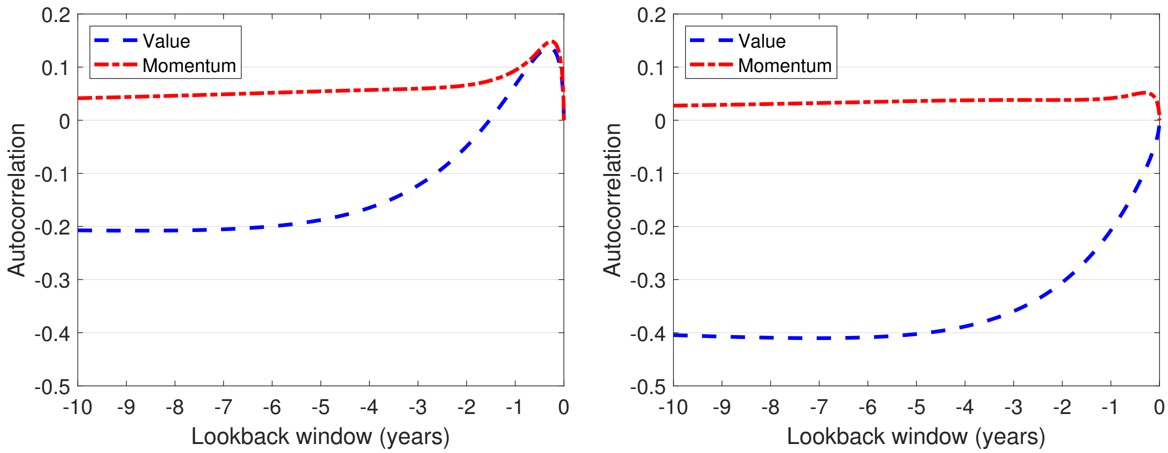

Figure 5: Unconditional autocorrelations of value and momentum, between returns over a lookback window τ ending at time t and returns over the year (left panel) and the ten years (right panel) starting at time t. Autocorrelations are plotted as function of τ.

die off to zero slowly when the window increases. Consequently, their effect on the Sharpe ratio is quantitatively important. While the unconditional Sharpe ratio of value is 27.05% over an infinitesimal horizon (Section 5) and drops to 19.78% for a two-year horizon, it rises to 29.66% for ten years, 38.43% for twenty years, and 47.42% for forty years.

The effect of horizon is even more pronounced on value's conditional Sharpe ratio. The left panel of Figure 6 plots the unconditional mean of the conditional Sharpe ratios of value (blue dashed line) and momentum (red dashed-dotted line), as function of horizon. We compute conditional Sharpe ratios in Appendix E (Proposition E.4). The unconditional mean of value's conditional Sharpe ratio drops from 21.46% for an infinitesimal horizon to 15.77% for a two-year horizon, and rises to 44.58% for ten years, 53.97% for twenty years, and 60.79% for forty years. Value's Sharpe ratio overtakes momentum's for horizons longer than five years (rather than thirteen years in the case of unconditional Sharpe ratios).

We next turn to the predictability of value's conditional Sharpe ratio. The conditional Sharpe ratio over a given finite horizon varies less than the infinitesimal-horizon Sharpe ratio. This is because it can be viewed as an average of current and future expected infinitesimal-horizon Sharpe ratios, adjusted for autocorrelation. Its variation in the case of value remains significant nonetheless. The unconditional standard deviation of the conditional Sharpe ratio of value drops from 63.25% for an infinitesimal horizon to 39.39% for an one-year horizon, 40.88% for five years, 31.33% for ten years, and 18.16% for twenty years.

Lengthening the horizon changes drastically the predictors of value's conditional Sharpe ratio. The value spread becomes a better predictor. Its unconditional correlation with the conditional

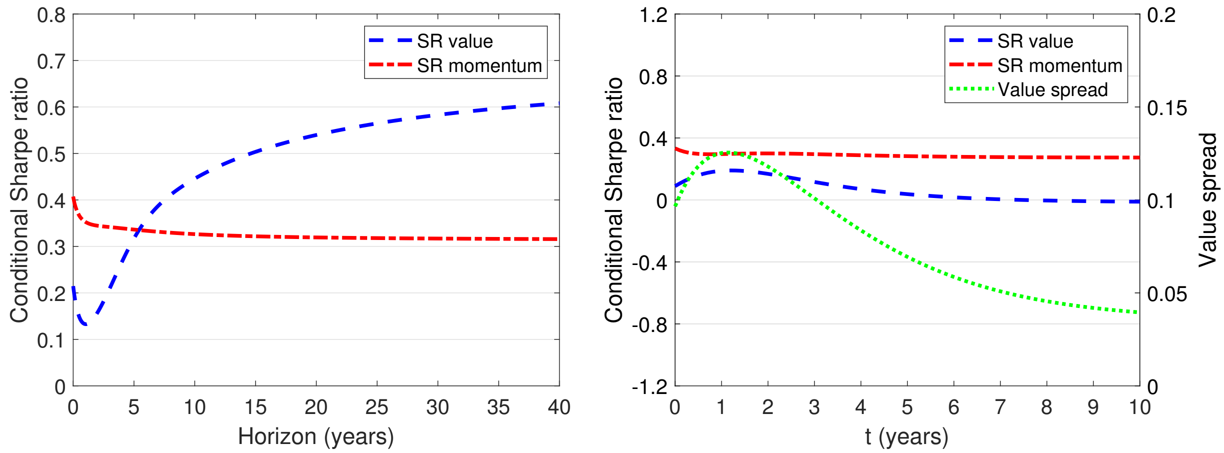

Figure 6: Unconditional mean of conditional Sharpe ratios as function of the investment horizon (left panel), and conditional five-year Sharpe ratios and the value spread following an one standard deviation drop in the return of the flow portfolio p_f at time zero (right panel).

Sharpe ratio of value rises from 26.00% for an infinitesimal horizon to 77.24% for an one-year horizon, 97.81% for five years, 96.92% for ten years, and 96.45% for twenty years. Conversely, the (instantaneous) conditional correlation becomes a worse predictor. Its unconditional correlation with the conditional Sharpe ratio of value drops from 86.02% for an infinitesimal horizon (Section 5) to 47.63% for an one-year horizon, -3.56% for five years, -7.72% for ten years, and -6.92% for twenty years.

The effect of horizon on the predictive relationships can be understood by plotting the response of the conditional Sharpe ratios to an one standard deviation drop in the flow portfolio's return. This exercise is performed for infinitesimal Sharpe ratios in Figure 2, and we repeat it for five-year Sharpe ratios in the right panel of Figure 6. While the infinitesimal Sharpe ratio of value drops substantially in response to the shock, its five-year counterpart rises. This is because the five-year Sharpe ratio incorporates future expected infinitesimal-horizon Sharpe ratios, which rise in response to the shock. As a consequence, the five-year Sharpe ratio moves more in synch with the value spread, resulting in a higher correlation. It also moves less in synch with the value-momentum correlation, resulting in a lower correlation.

6.3 Momentum

Similar to value returns, momentum returns are positively autocorrelated over short horizons and lookback windows, reflecting asset-level momentum. In contrast to value returns, the autocorrelation does not become negative for long horizons or lookback windows but instead drops to zero.

The autocorrelation vanishes because the momentum strategy has high turnover, holding assets based only on their recent performance.

The positive autocorrelation of momentum returns is reflected into momentum's Sharpe ratio, which decreases with investment horizon. The unconditional Sharpe ratio of momentum is 53.66% over an infinitesimal horizon (Section 5), and drops to 37.06% for an one-year horizon. It then stays essentially flat, equal to 34.28% for five years, 33.49% for ten years, 33.04% for twenty years and 32.82% for forty years.

We next turn to the predictability of momentum's conditional Sharpe ratio. Momentum's Sharpe ratio varies significantly less than value's, especially as horizon increases. The unconditional standard deviation of momentum's conditional Sharpe ratio drops from 46.58% for an infinitesimal horizon to 11.93% for an one-year horizon, 6.52% for five years, 4.52% for ten years, and 2.35% for twenty years.

The predictability of momentum's conditional Sharpe ratio becomes similar to value's as horizon increases. Momentum's conditional Sharpe ratio is well predicted by the value spread, with the unconditional correlation rising from -8.13% for an infinitesimal horizon to 27.13% for an one-year horizon, 87.99% for five years, 91.75% for ten years, and 91.53% for twenty years. Conversely, it is not well predicted by the conditional correlation between value and momentum. Its unconditional correlation with that variable rises from 41.34% for an infinitesimal horizon (Section 5) to 53.91% for an one-year horizon, but subsequently drops to 18.47% for five years, 8.01% for ten years, and 6.64% for twenty years.

6.4 Combining Value and Momentum

We compute the unconditional correlation between value and momentum returns over a general investment horizon in Appendix E (Proposition E.7). Figure 7 plots this correlation as function of horizon. The correlation is negative for returns computed over horizons up to seven months and turns positive for longer horizons. It is relatively small in absolute value, rising from -12.20% for an infinitesimal horizon (Section 6.4) to 19.92% for a forty-year horizon. As a consequence, combining value and momentum improves significantly over using one or the other strategy and yields a Sharpe ratio close to that of the optimal strategy. The difference between the Sharpe ratio of the optimal strategy and of the optimal value-momentum combination drops from 6.76% for an infinitesimal horizon to 3.20% for an one-year horizon, 1.76% for five years, 1.30% for ten years, 0.94% for twenty years, and 0.64% for forty years.

The change in sign of the value-momentum correlation from negative to positive as horizon increases reflects the cross-autocorrelations between the strategies (lead-lag effects). We compute

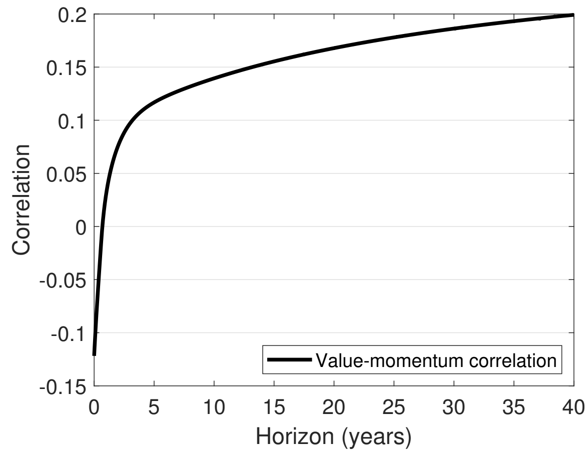

Figure 7: Unconditional correlation between value and momentum as function of the investment horizon.

unconditional cross-autocorrelations for value and momentum returns in Appendix E (Proposition E.8), and plot them in Figure 8. The left panel plots correlations between value (blue dashed line) or momentum (red dashed-dotted line) returns over a lookback window τ ending at time t with the other strategy's returns over the year starting at time t. The right panel replaces returns one year ahead by returns ten years ahead.

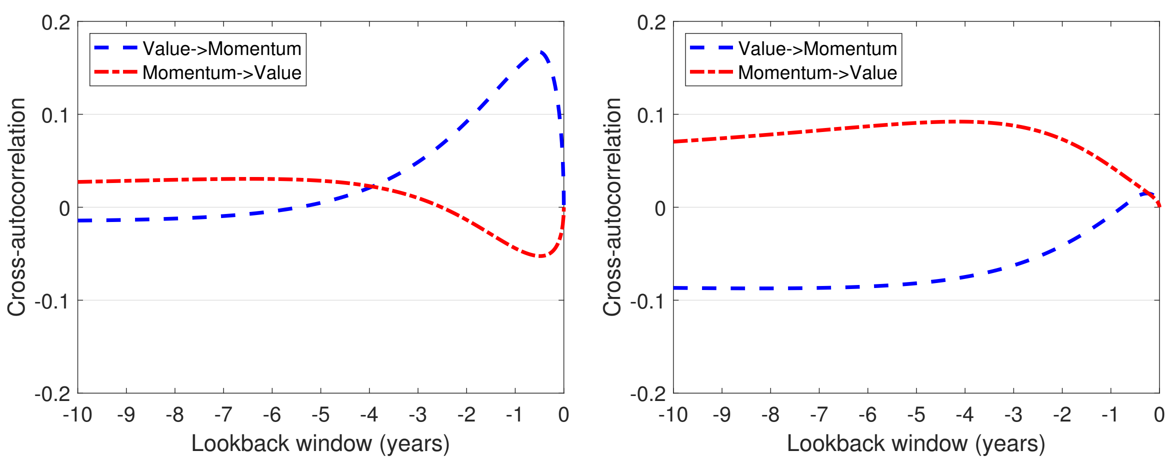

Figure 8: Unconditional cross-autocorrelations (lead-lag effects) of value and momentum, between returns over a lookback window τ ending at time t and returns over the year (left panel) and the ten years (right panel) starting at time t. Cross-autocorrelations are plotted as function of τ.

Lead-lag effects differ for short- and long-horizon returns. Over short horizons (left panel), they are mainly driven by the joint variation of the strategies' expected returns during the flow cycle, and they are present only from value to momentum and only over short lookback windows.[4]

[4]In Appendix E (Equations (E.6) and (E.7)), we show that the correlation $\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u^j dR_u, \hat{w}_{u'}^k dR_{u'}\right)$ between

High returns on value predict high returns on momentum (positive lead-lag effect). This is because value earns high expected returns towards the end of the flow cycle, and as these returns cumulate, momentum starts earning high expected returns as well. By contrast, momentum returns do not predict short-horizon returns on value. This is because momentum earns high expected returns at the beginning or at the end of the flow cycle, and these are followed by low expected returns of value in the former case and by high expected returns in the latter case.

Lead-lag effects over long horizons (right panel) are mainly driven by the response of the strategies' expected returns to shocks, and they are present from both value to momentum and from momentum to value. A long period of underperformance by the assets held by value indicates that those assets will earn high future expected returns. Hence, those assets will be included in momentum portfolios, which will perform well on average (negative lead-lag effect). A long period of overperformance by momentum indicates that flows out of funds holding poorly performing assets and into funds holding well performing ones are larger than expected. This indicates high mispricing and thus high future expected returns by value (positive lead-lag effect).

The cross-autocorrelation from value to momentum changes sign to positive as horizon increases because the positive lead-lag effects dominate the negative ones. Therefore, the drivers of the switch in sign of the value-momentum correlation are the short-horizon lead-lag effect from value to momentum, and the long-horizon one from momentum to value.

Using the unconditional Sharpe ratios and correlation of value and momentum, we compute in Appendix E (Proposition E.9) the investment in these strategies in their optimal (mean-variance maximizing) combination. We express the investment in normalized terms by rescaling strategy weights in the assets so that strategy standard deviation times investor risk aversion is equal to one. We refer to the normalized investment as the weight given by the investor in a strategy. The weights for value and momentum are plotted as function of horizon in the left panel of Figure 9. Momentum's weight is 169% that of value for an infinitesimal horizon, and rises to 199% for a two-year horizon. It then decreases with horizon, becoming equal to value's weight for thirteen years, and to 57% of value's weight for forty years. Momentum's weight declines with horizon relative to value's weight because value's Sharpe ratio increases while momentum's stays essentially flat. Momentum's weight declines with horizon in absolute terms because value's weight rises and because the value-momentum correlation turns positive.

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instantaneous returns of strategy j at time u and strategy k at time $u' > u$ can be written as

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\mathbb{E}_u(\hat{w}^j_u dR_u), \mathbb{E}_{u'}(\hat{w}^k_{u'} dR_{u'})\right] + \mathbb{E}_{\mathcal{I}_t}\left[\hat{w}^j_u \mathbb{C}\text{ov}_u(dR_u, \hat{w}^k_{u'}\mathbb{E}_{u'}(dR_{u'}))\right].$$

The first term is the correlation between expected returns at u and u', and drives lead-lag effects over short horizons in our calibration. The second term describes the response of expected returns at u' to shocks at u, and drives lead-lag effects over long horizons.

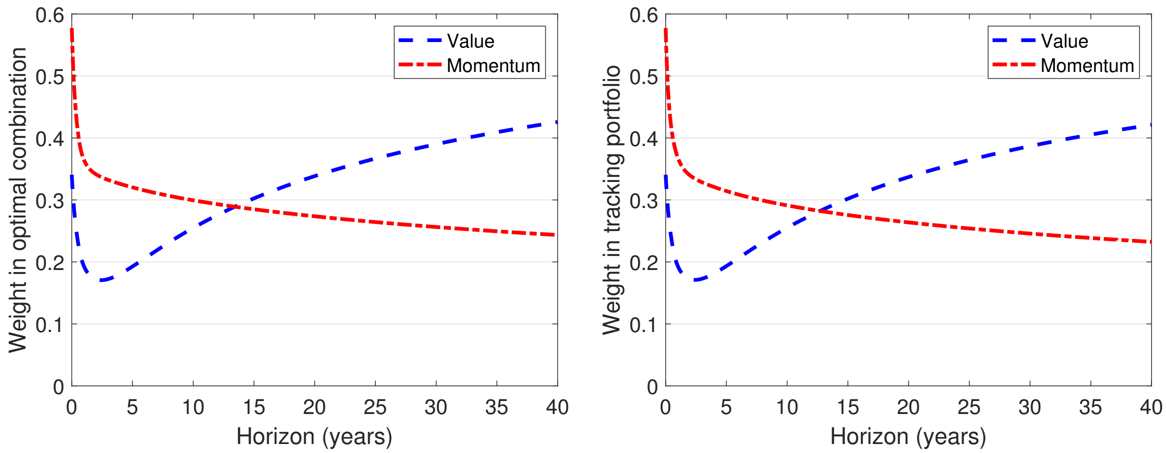

Figure 9: Weights of value and momentum in the combination of the two strategies that maximizes the unconditional Sharpe ratio (left panel) and in the combination that minimizes the unconditional variance of the difference in returns between that combination and the optimal strategy (right panel), as function of the investment horizon.

The right panel of Figure 9 plots value and momentum weights in the combination that best approximates the optimal strategy derived in Section 6.1. We construct that combination by minimizing the unconditional variance of the difference in returns between that combination and the optimal strategy. The weights of value and momentum in that combination are computed in Appendix E (Proposition E.10). Figure 9 shows that they are nearly identical to those in the mean-variance maximizing value-momentum combination.

7 Empirical Analysis

7.1 Data

We next examine whether the theoretical patterns shown in Section 6 appear in the data. We use a monthly vector auto-regression (VAR) of value and momentum returns together with the value spread, the panic variable of DM, and the value-momentum correlation. Our sample runs from January 1940 to December 2021.[5]

Value and momentum returns are those of the HML and UMD factors, respectively, sourced from Ken French's website. We compute these returns over a monthly horizon, and express them in logarithms.

To construct the value spread, we first create an annual series following CPV. Our $\frac{BE}{ME}$ values

[5]We start the sample in 1940 following CPV who exclude the pre-1938 data because the poor disclosure regulation at that time (documented in the accounting literature) could have resulted in unreliable book equity data. Our main findings are robust to this choice.

come from Ken French's website; we use the ratio based on market equity ME at the end of June of each year, and book equity BE measured at the end of December of the previous year but adjusted for net stock issuance up to the end of June. The value spread at the end of June is the difference in the logarithm of $\frac{BE}{ME}$ between the high and the low $\frac{BE}{ME}$ portfolios. We next create a monthly series following Campbell and Vuolteenaho (2004) and Campbell, Giglio, Polk, and Turley (2018). Specifically, the value spread at the end of each month from July to May is constructed by adding to the end-of-June value spread the cumulative log return on the low $\frac{BE}{ME}$ portfolio from the end of June, and subtracting the cumulative log return on the high $\frac{BE}{ME}$ portfolio.

To construct DM's panic variable at the end of each month, we multiply an indicator variable equal to one if the market return in the previous two-year period is negative by the realized variance of the market return during the previous six-month period computed using daily observations. We take the value-weighted CRSP portfolio as our market proxy and source return data from Ken French's website. To construct the correlation between HML and UMD at the end of each month, we compute the realized correlation using daily observations during that month.

A difference between our empirical analysis and our calibration is that the former is done at the stock level while the latter is done at the level of industry sectors. We assume industry sectors in our calibration for parsimony: going at the stock level requires additional assumptions on how large the sector-specific component of returns is relative to the stock-specific component, what the inter- and intra-sector distributions of fund holdings are, etc. On the other hand, constructing empirical value strategies at the sector level is challenging because book value may not be comparable across sectors. Stock-level value strategies are less affected by this problem because they also exploit intra-sector variation. Indeed, CPV find that the profitability of stock-level value strategies derives primarily from intra- rather that inter-sector variation. Our theory suggests that it should be possible to construct profitable sector-level value strategies in the data using measures of fundamental value that are comparable across sectors.

7.2 Results

Figure 10 plots the value spread (red), the panic variable (black) and the value-momentum correlation (blue) over our sample period. Each variable is demeaned and standardized by its in-sample standard deviation.

The value-momentum correlation varies significantly over time and is persistent. Its average in our sample is -6%, and its standard deviation is 52%. It is negative over some extended periods, e.g., the years around the dot-com boom and those around the global financial crisis. It is positive over some extended periods as well, e.g., between those two episodes. The value spread is more

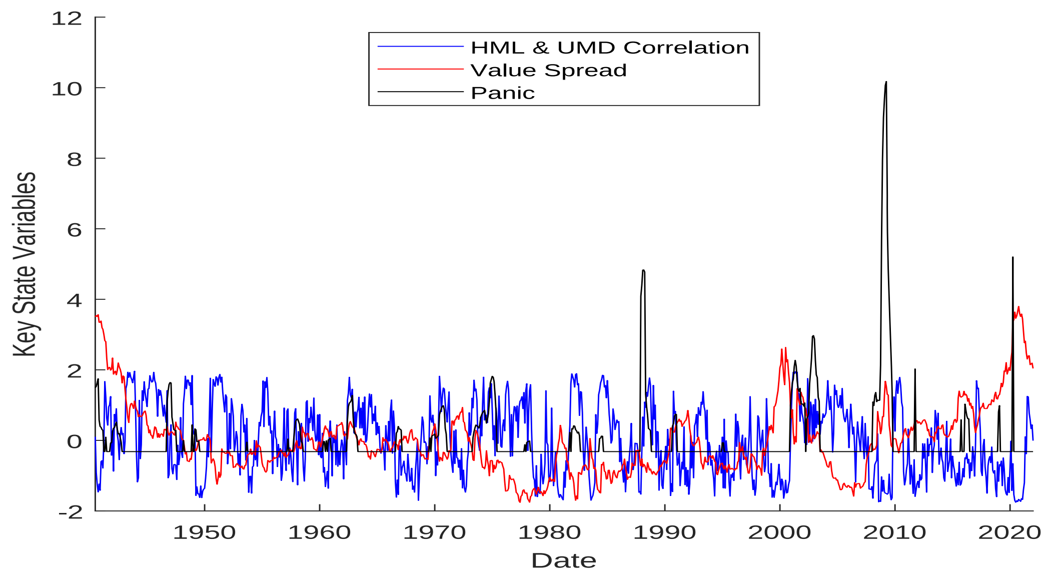

Figure 10: Predictors of value and momentum returns.

persistent. Its three most recent peaks are during the dot-com boom, the global financial crisis, and 2020. The panic variable is equal to zero much of the time and has occasional spikes, the largest of which occurred in 2009, towards the end of the global financial crisis.

Table II shows the VAR results. Each pair of rows reports results for the regression predicting the corresponding variable in the first column. The regression coefficients are in the upper row and the t-statistics are in the lower row. Boldface indicates statistical significance at the 5% level.

Table II: VAR results.

	Intercept	HML	UMD	Corr	VS	Panic	R2
HML	**-0.01415**	**0.15334**	-0.00832	**0.00418**	**0.01166**	-11.38827	2.9%
	-2.05	**4.74**	-0.34	**2.39**	**2.58**	-1.49	
UMD	0.01604	-0.05716	-0.01689	0.00404	-0.00419	**-57.42914**	3.4%
	1.74	-1.32	-0.52	1.73	-0.70	**-5.65**	
Corr	**0.22005**	**3.48653**	-0.00340	**0.73941**	**-0.16341**	73.76268	61.3%
	2.68	**9.07**	-0.01	**35.48**	**-3.04**	0.81	
VS	**0.03865**	**-0.16348**	0.01683	-0.00248	**0.97449**	7.66643	96.7%
	4.11	**-3.72**	0.51	-1.04	**158.59**	0.74	
Panic	0.00001	**0.00014**	0.00008	-0.00001	0.00000	**0.91011**	82.1%
	0.44	**2.45**	1.85	-1.59	-0.30	**66.00**	

The return on HML is positively predicted by its lagged value. Thus, HML has short-run momentum, consistent with Ehsani and Linnainmaa (2019). The return on HML is also positively predicted by the value spread, as in CPV, and by the value-momentum correlation. All three predictive links are consistent with our theory, and we are the first to document that the value-momentum correlation predicts value returns. Indeed, in unreported results, we find that the coefficient on the value spread is much weaker when we exclude the lagged HML return and the value-momentum correlation from the HML regression. This finding is consistent with the importance of taking our theory's flow cycle view of value predictability into account when using the value spread to forecast value returns.

The return on UMD is predicted negatively by the panic variable, as in DM. It is positively predicted by the value-momentum correlation, with statistical significance at the 10% level.[6] The latter finding is consistent with our theory.

Following high HML returns, the value-momentum correlation increases, and the value spread decreases. Intuitively, high HML returns render value stocks more likely to be selected into momentum portfolios, raising the the value-momentum correlation. They also lower the value stocks' book-to-market ratio, lowering the value spread.

Using the VAR coefficients in Table II, together with the covariance matrix of VAR residuals, we compute Sharpe ratios and correlations of value and momentum returns over general horizons and compare to our results in Section 6. The calculations are in Appendix G.

The left panel of Figure 11 plots the unconditional Sharpe ratio of value (blue dashed line) and momentum (red dashed-dotted line) as function of horizon. The right panel plots the unconditional correlation between value (blue dashed line) or momentum (red dashed-dotted line) returns over a lookback window τ ending at time t with the same strategy's returns over the year starting at time t. In this and the next figure, the thin lines are one-standard-deviation bounds, produced by bootstrapping the VAR.

Figure 11 provides some support to our theory. While the standard-deviation bounds are wide due to the noise in returns, the general patterns of the Sharpe ratios and autocorrelations resemble those in Section 6. Over short horizons, the Sharpe ratios of value and momentum decrease with horizon. Over long horizons, momentum's Sharpe ratio becomes approximately flat, while value's Sharpe ratio increases. Reflecting these patterns, the autocorrelations are positive over short lookback windows and become approximately equal to zero for momentum and negative for value over long lookback windows. All these properties are consistent with our theory. A discrepancy arises in the magnitude of the effects, which are more modest than in Section 6. While value's Sharpe ratio increases non-trivially with horizon (it is 36% larger in relative terms over

[6] If we exclude the other forecasting variables from the UMD regression, the coefficient on the value-momentum correlation increases by roughly 19% and has a t-statistic of 2.11.

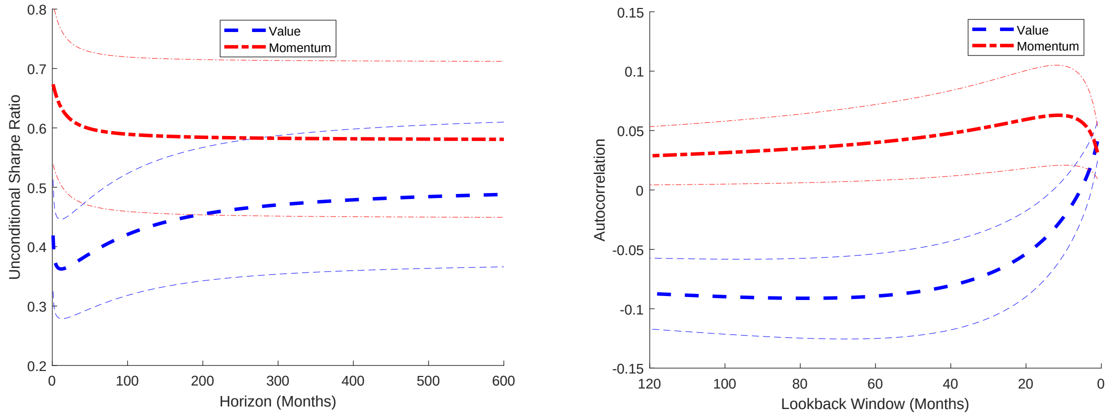

Figure 11: Sharpe ratios of value and momentum as function of horizon (left panel) and autocorrelations of value and momentum, between returns over a lookback window τ ending at time t and returns over the year starting at time t as function of τ (right panel). Moments are unconditional and computed using the empirical estimates from the VAR.

a forty-year horizon than over an one-year horizon), it does not overtake momentum's over long horizons (right panel of Figure 4). Moreover, value's return autocorrelations are not as negative as in Section 6 over long lookback windows (left panel of Figure 5).

Figure 12 plots correlations and lead-lag effects. The left panel plots the correlation between value and momentum returns as function of horizon. The right panel plots the correlation between value (blue dashed line) or momentum (red dashed-dotted line) returns over a lookback window τ ending at time t with the other strategy's returns over the year starting at time t.

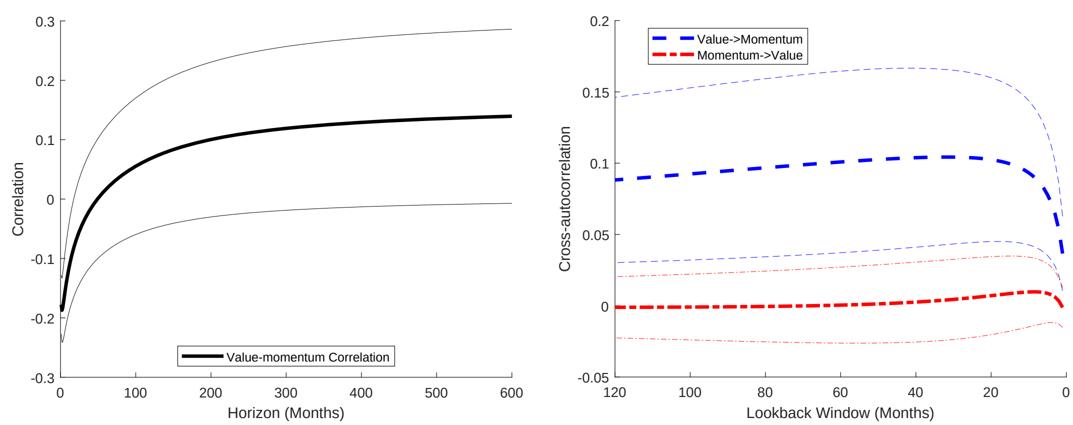

Figure 12: Correlation between value and momentum returns as function of horizon (left panel) and cross-autocorrelations between returns over a lookback window τ ending at time t and returns over the year starting at time t as function of τ (right panel). Moments are unconditional and computed using the empirical estimates from the VAR.

The correlation between value and momentum depends on horizon in a way consistent with our theory: it is negative over short horizons, and turns positive over longer horizons. Also consistent with our theory, the dominant lead-lag effect is from value to momentum and is positive. Figure 12 differs from its lead-lag counterpart in Section 6 (right panel of Figure 8) in that the lead-lag effect from value to momentum does not decay to zero as fast as horizon increases.

8 Conclusion

We study dynamic portfolio choice in a calibrated equilibrium model where value and momentum anomalies arise because capital moves from under- to over-performing market segments and does so slowly. Our model provides answers to questions that are key to dynamic portfolio choice and that the theoretical literature has not addressed so far. We determine, in particular, how the Sharpe ratios of value and momentum depend on investment horizon, how value and momentum returns correlate with each other, and how these returns and Sharpe ratios depend on predictor variables. We provide novel empirical evidence supporting our model's predictions.

A common thread running through our results is that Sharpe ratios, correlations, and predictor variables depend significantly on horizon. Over short horizons, momentum's Sharpe ratio exceeds value's, the value-momentum correlation is negative, and the conditional value-momentum correlation positively predicts the Sharpe ratios of value and momentum. In contrast, over long horizons, value's Sharpe ratio can exceed momentum's, the value-momentum correlation turns positive, and the value spread becomes a better predictor of Sharpe ratios.

Our results imply that performance metrics computed using returns over short horizons, e.g., monthly or annual, can give poor guidance to investors such as pension funds and sovereign-wealth funds, whose horizons span decades. Constraints on portfolio deviations from benchmarks, which are pervasive in practice, can be at odds with long-horizon objectives for similar reasons. For example, tracking-error constraints evaluate risk over a short horizon and can overestimate it over a long horizon in the same way that the value strategy's short-horizon risk overestimates its long-horizon risk. Our analysis suggests that performance metrics and constraints of long-horizon investors should be re-examined on the basis of the investment behavior and horizon that they induce.

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Appendix

A Proofs of Results in Section 3

Equation (3.3) follows from combining VW equation (28) with (29) and (B34). VW equation (B34) implies

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \gamma_0\Sigma p_f'$$

with

$$\gamma_0 = \frac{\kappa(\gamma_1+\gamma_2)\bar{C} + b_0\gamma_3 - k_1\bar{q}_1 - k_2\bar{q}_2}{r} + \bar{\alpha}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right). \tag{A.1}$$

Equation (3.1) is VW equation (30). Equation (3.2) is VW equation (31), shown in Proposition 4. Equations (3.5) and (3.6) are VW equations (22) and (38), respectively, shown in Corollary 9.

Additional equations from VW that we use in subsequent proofs are those describing the covariance matrix of returns (VW equation (37) shown in Corollary 8)

$$\mathbb{C}\mathrm{ov}_t(dR_t, dR_t') = \left(f\Sigma + k\Sigma p_f' p_f\Sigma\right)dt, \tag{A.2}$$

the properties of the flow portfolio (stated between VW equations (A28) and (A29))

$$\eta\Sigma p_f' = 0,$$

$$\theta\Sigma p_f' = p_f\Sigma p_f' = \frac{\Delta}{\eta\Sigma\eta'},$$

the investor's stock holdings (stated just before VW equation (A63))

$$x_t\eta + y_t z_t = y_t p_f + \frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta, \tag{A.3}$$

stock returns (VW equation (B7))

$$dR_t = \left[\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left(r\gamma_0 + \gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right)\Sigma p_f'\right]dt$$

$$+\left(\sigma+\beta_1\gamma_1\Sigma p_f' p_f\sigma\right)dB_t^D+\frac{\phi}{r+\kappa}\left(\sigma+\beta_2\gamma_1\Sigma p_f' p_f\sigma\right)dB_t^F-s\gamma_2\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\Sigma p_f' dB_t^C, \tag{A.4}$$

where

$$\gamma_1^R \equiv (r+\kappa+\rho)\gamma_1+b_1\gamma_3, \tag{A.5}$$

$$\gamma_2^R \equiv (r+\kappa)\gamma_2-\rho\gamma_1, \tag{A.6}$$

$$\gamma_3^R \equiv (r+b_2)\gamma_3, \tag{A.7}$$

$$\rho \equiv \beta_1\left(1-\frac{(r+\kappa)\gamma_2\Delta}{\eta\Sigma\eta'}\right), \tag{A.8}$$

and the dynamics of $\hat{C}_t$ (VW equation (B6) combined with (B8))

$$d\hat{C}_t=\kappa(\bar{C}-\hat{C}_t)dt+\rho(C_t-\hat{C}_t)dt-\beta_1 p_f\sigma dB_t^D-\beta_2\left(\frac{\phi p_f\sigma dB_t^F}{r+\kappa}-\frac{s\gamma_2\Delta dB_t^C}{\eta\Sigma\eta'}\right), \tag{A.9}$$

where

$$\beta_1 \equiv T\left[1-(r+k)\frac{\gamma_2\Delta}{\eta\Sigma\eta'}\right]\frac{\eta\Sigma\eta'}{\Delta}, \tag{A.10}$$

$$\beta_2 \equiv \frac{s^2\gamma_2}{\frac{\phi^2}{(r+\kappa)^2}+\frac{s^2\gamma_2^2\Delta}{\eta\Sigma\eta'}}, \tag{A.11}$$

and T, the investor's conditional variance of C_t, is the positive solution to

$$T^2\left(1-(r+\kappa)\frac{\gamma_2\Delta}{\eta\Sigma\eta'}\right)^2\frac{\eta\Sigma\eta'}{\Delta}+2\kappa T-\frac{\frac{s^2\phi^2}{(r+\kappa)^2}}{\frac{\phi^2}{(r+\kappa)^2}+\frac{s^2\gamma_2^2\Delta}{\eta\Sigma\eta'}}=0. \tag{A.12}$$

We focus on the steady state reached when t goes to infinity, where the coefficients (β_1, β_2, T) are time-independent, and thus so are all other coefficients describing the equilibrium.

Substituting γ_0 from (A.1) and using (3.6), we can write (A.4) as

$$dR_t=\left[\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta'+\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\Lambda_t\Sigma p_f'\right]dt$$

$$+ \left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f \sigma\right) dB_t^D + \frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right) dB_t^F - s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\Sigma p_f' dB_t^C. \tag{A.13}$$

The term in square brackets in (A.13) is $\mathbb{E}_t(dR_t)$ and maps to the two-factor model (3.5).

B Additional Background Notation and Results

Lemma B.1 determines the state variables $(F_t, \hat{C}_t, C_t, y_t)$ in steady state as function of all past Brownian shocks.

Lemma B.1. *The values of $(F_t, \hat{C}_t, C_t, y_t)$ in the steady state reached when $t \to \infty$ are*

$$F_t = \bar{F} + \int_{-\infty}^{t} e^{-\kappa(t-u)}\phi\sigma dB_u^F, \tag{B.1}$$

$$\hat{C}_t = \bar{C} + \int_{-\infty}^{t} e^{-\kappa(t-u)} s dB_u^C - \int_{-\infty}^{t} e^{-(\kappa+\rho)(t-u)}\left[\beta_1 p_f \sigma dB_u^D + \frac{\phi\beta_2 p_f \sigma dB_u^F}{r+\kappa} + s\left(1 - \frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right) dB_u^C\right], \tag{B.2}$$

$$C_t = \bar{C} + \int_{-\infty}^{t} e^{-\kappa(t-u)} s dB_u^C, \tag{B.3}$$

$$\begin{aligned} y_t = \bar{y} &+ \int_{-\infty}^{t} \frac{b_1}{\kappa - b_2}\left[e^{-\kappa(t-u)} - e^{-b_2(t-u)}\right] s dB_u^C \\ &- \int_{-\infty}^{t} \frac{b_1}{\kappa+\rho-b_2}\left[e^{-(\kappa+\rho)(t-u)} - e^{-b_2(t-u)}\right]\left[\beta_1 p_f \sigma dB_u^D + \frac{\phi\beta_2 p_f \sigma dB_u^F}{r+\kappa} + s\left(1 - \frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right) dB_u^C\right]. \end{aligned} \tag{B.4}$$

Proof: The dynamics of F_t are given by the stochastic differential equation (2.6). Integrating that equation with initial condition F_0, and letting $t \to \infty$, we find (B.1). The dynamics of $(\hat{C}_t, C_t, y_t)$ are given by the system of stochastic differential equations (A.9) and (2.3), and ordinary differential equation (3.1). Integrating that system with initial conditions $(\hat{C}_0, C_0, y_0)$, and letting $t \to \infty$, we find (B.2)-(B.4). ∎

We next introduce some notation, which we use together with Lemma B.1 to compute autocovariances of $(F_t, \hat{C}_t, C_t, y_t, dR_t)$ in Lemma B.3. For scalars $(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)$ and a function $\nu(\omega, \mathcal{T})$, we define the function $G(\psi_1, \psi_2, \psi_3, \mathcal{T}, \nu)$ by

$$G(\psi_1, \psi_2, \psi_3, \mathcal{T}, \nu) \equiv$$

$$- \left[\psi_1 \nu(\kappa+\rho, \mathcal{T}) + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(\nu(\kappa+\rho,\mathcal{T}) - \nu(b_2,\mathcal{T})\right)\right] \beta_1 \left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right)$$

$$- \left[(\psi_1+\psi_2)\nu(\kappa, \mathcal{T}) + \frac{\psi_3 b_1}{\kappa-b_2}\left(\nu(\kappa,\mathcal{T}) - \nu(b_2,\mathcal{T})\right)\right] s^2\gamma_2 \left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right),$$

the function $H(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \mathcal{T}, \nu)$ by

$$H(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \mathcal{T}, \nu) \equiv$$

$$\left[\frac{1}{2(\kappa+\rho)}\left(\hat{\psi}_1 + \frac{\hat{\psi}_3 b_1}{\kappa+\rho-b_2}\right)\left(\psi_1 - \frac{\psi_3 b_1}{\kappa+\rho+b_2}\right)\nu(\kappa+\rho,\mathcal{T})\right.$$

$$\left. - \frac{\hat{\psi}_3 b_1}{(\kappa+\rho+b_2)(\kappa+\rho-b_2)}\left(\psi_1 - \frac{\psi_3 b_1}{2b_2}\right)\nu(b_2,\mathcal{T})\right]\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\eta\Sigma\eta'}\right]\frac{\Delta}{\eta\Sigma\eta'}$$

$$+ \left[\frac{1}{2\kappa+\rho}\left(\hat{\psi}_1 + \hat{\psi}_2 + \frac{\hat{\psi}_3 b_1}{\kappa-b_2}\right)\left(\psi_1 - \frac{\psi_3 b_1}{\kappa+b_2}\right)\nu(\kappa,\mathcal{T})\right.$$

$$- \frac{1}{2\kappa+\rho}\left(\hat{\psi}_1 + \frac{\hat{\psi}_3 b_1}{\kappa+\rho-b_2}\right)\left(\frac{\psi_1\kappa}{\kappa+\rho} - \psi_2 - \frac{\psi_3 b_1\kappa}{(\kappa+\rho)(\kappa+\rho+b_2)}\right)\nu(\kappa+\rho,\mathcal{T})$$

$$\left. - \frac{\hat{\psi}_3 b_1}{(\kappa+b_2)(\kappa+\rho-b_2)}\left(\frac{2\psi_1\kappa\rho}{(\kappa-b_2)(\kappa+\rho+b_2)} + \psi_2 - \frac{\psi_3\kappa b_1\rho}{b_2(\kappa-b_2)(\kappa+\rho+b_2)}\right)\nu(b_2,\mathcal{T})\right]\frac{s^2\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}$$

$$+ \left[\frac{1}{2\kappa}\left(\hat{\psi}_1 + \hat{\psi}_2 + \frac{\hat{\psi}_3 b_1}{\kappa-b_2}\right)\left(\frac{\psi_1\rho}{2\kappa+\rho} + \psi_2 - \frac{\psi_3 b_1\rho}{(2\kappa+\rho)(\kappa+b_2)}\right)\nu(\kappa,\mathcal{T})\right.$$

$$- \frac{1}{2\kappa+\rho}\left(\hat{\psi}_1 + \frac{\hat{\psi}_3 b_1}{\kappa+\rho-b_2}\right)\left(\frac{\psi_1\rho}{2(\kappa+\rho)} + \psi_2 - \frac{\psi_3 b_1\rho}{2(\kappa+\rho)(\kappa+\rho+b_2)}\right)\nu(\kappa+\rho,\mathcal{T})$$

$$\left. - \frac{\hat{\psi}_3 b_1\rho}{(\kappa+b_2)(\kappa-b_2)(\kappa+\rho-b_2)}\left(\frac{\psi_1\rho}{\kappa+\rho+b_2} + \psi_2 - \frac{\psi_3 b_1\rho}{2b_2(\kappa+\rho+b_2)}\right)\nu(b_2,\mathcal{T})\right] s^2,$$

and the functions $K_1(\psi_1, \psi_3, \mathcal{T}, \nu)$ and $K_2(\psi_1, \psi_3, \mathcal{T}, \nu)$ by

$$K_1(\psi_1, \psi_3, \mathcal{T}, \nu) \equiv - \frac{1}{2\kappa+\rho}\left(\psi_1 - \frac{\psi_3 b_1}{\kappa+b_2}\right)\nu(\kappa,\mathcal{T})\frac{\phi^2\beta_2}{r+\kappa},$$

$$K_2(\psi_1, \psi_3, \mathcal{T}, \nu) \equiv - \left[\frac{1}{2\kappa+\rho}\left(\psi_1 + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\right)\nu(\kappa+\rho,\mathcal{T}) - \frac{\psi_3 b_1}{(\kappa+b_2)(\kappa+\rho-b_2)}\nu(b_2,\mathcal{T})\right]\frac{\phi^2\beta_2}{r+\kappa}.$$

We define the functions $\nu_0(t)$ and $\{\nu_i(\omega,\mathcal{T})\}_{i=1,..,4}$ for $\mathcal{T}=(t,\Delta t)$ and $\Delta t>0$ by

$$\nu_0(\omega,t) \equiv e^{-\omega t},$$

$$\nu_1(\omega,\mathcal{T}) \equiv \int_{t-\Delta t}^{t} \nu_0(\omega,|u|)1_{\{u\geq 0\}}du,$$

$$\nu_2(\omega,\mathcal{T}) \equiv \int_{t-\Delta t}^{t} \nu_0(\omega,|u|)1_{\{u\leq 0\}}du,$$

$$\nu_3(\omega,\mathcal{T}) \equiv \int_{u'=t-\Delta t}^{t}\int_{u=-\Delta t}^{0} \nu_0(\omega,|u'-u|)1_{\{u\leq u'\}}dudu',$$

$$\nu_4(\omega,\mathcal{T}) \equiv \int_{u'=t-\Delta t}^{t}\int_{u=-\Delta t}^{0} \nu_0(\omega,|u'-u|)1_{\{u\geq u'\}}dudu'.$$

We define the scalars (L_1, L_2) by

$$L_1 \equiv \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}, \tag{B.5}$$

$$L_2 \equiv r\bar{\alpha}\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)+(\gamma_1^R+\gamma_2^R)\bar{C}+\gamma_3^R\bar{y}-k_1\bar{q}_1-k_2\bar{q}_2=\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}(\Lambda_t), \tag{B.6}$$

and the scalars $(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ by

$$\Delta_1 \equiv f\left(\eta\Sigma^3\eta'-\frac{(\eta\Sigma^2\eta')^2}{\eta\Sigma\eta'}\right)+k\left(\eta\Sigma^2p_f'\right)^2,$$

$$\Delta_2 \equiv f\left(\eta\Sigma^3p_f'-\frac{\eta\Sigma^2\eta'\eta\Sigma^2p_f'}{\eta\Sigma\eta'}\right)+k\eta\Sigma^2p_f'p_f\Sigma^2p_f',$$

$$\Delta_3 \equiv f\left(p_f\Sigma^3p_f'-\frac{(\eta\Sigma^2p_f')^2}{\eta\Sigma\eta'}\right)+k\left(p_f\Sigma^2p_f'\right)^2,$$

$$\Delta_4 \equiv f\left(Tr(\Sigma^2)-\frac{\eta\Sigma^3\eta'}{\eta\Sigma\eta'}\right)+kp_f\Sigma^3p_f',$$

where $Tr(M)$ denotes the trace of the matrix M.

Lemma B.2 derives closed-form solutions for the functions $\{\nu_i(\omega,\mathcal{T})\}_{i=1,2,3,4}$.

Lemma B.2. *The functions* $\{\nu_i(\omega,\mathcal{T})\}_{i=1,2,3,4}$ *are equal to*

$$\nu_1(\omega,\mathcal{T}) = \frac{e^{-\omega\max\{t-\Delta t,0\}}-e^{-\omega\max\{t,0\}}}{\omega},$$

$$\nu_2(\omega,\mathcal{T}) = \nu_1(\omega,(-t+\Delta t,\Delta t)) = \frac{e^{\omega\min\{t,0\}} - e^{\omega\min\{t-\Delta t,0\}}}{\omega},$$

$$\nu_3(\omega,\mathcal{T}) = \frac{e^{-\omega\max\{t+\Delta t,0\}} + e^{-\omega\max\{t-\Delta t,0\}} - 2e^{-\omega\max\{t,0\}}}{\omega^2}$$

$$+ \frac{\min\{\max\{t,-\Delta t\},0\} - \min\{\max\{t-\Delta t,-\Delta t\},0\}}{\omega},$$

$$\nu_4(\omega,\mathcal{T}) = \nu_3(\omega,(-t,\Delta t)) = \frac{e^{\omega\min\{t+\Delta t,0\}} + e^{\omega\min\{t-\Delta t,0\}} - 2e^{\omega\min\{t,0\}}}{\omega^2}$$

$$+ \frac{\max\{\min\{t,0\},-\Delta t\} - \max\{\min\{t-\Delta t,0\},-\Delta t\}}{\omega}.$$

Proof: We first compute $(\nu_1(\omega,\mathcal{T}),\nu_2(\omega,\mathcal{T}))$. Since the variable u in $\nu_1(\omega,\mathcal{T})$ is non-negative, we can drop the indicator function $1_{\{u\geq 0\}}$ and change the integration bounds and the argument of $\nu_0(\omega,t)$ as follows:

$$\nu_1(\omega,\mathcal{T}) = \int_{\max\{t-\Delta t,0\}}^{\max\{t,0\}} e^{-\omega u}du = \frac{e^{-\omega\max\{t-\Delta t,0\}} - e^{-\omega\max\{t,0\}}}{\omega}.$$

This is the expression in the lemma. To compute $\nu_2(\omega,\mathcal{T})$, we make the change of variable from u to $-u$:

$$\begin{aligned}
\nu_2(\omega,\mathcal{T}) &= \int_{-t}^{-t+\Delta t} \nu_0(\omega,|u|)1_{\{u\geq 0\}}du \\
&= \int_{-(t-\Delta t)-\Delta t}^{-(t-\Delta t)} \nu_0(\omega,|u|)1_{\{u\geq 0\}}du \\
&= \nu_1(\omega,(-t+\Delta t,\Delta t)) \\
&= \frac{e^{-\omega\max\{-t+\Delta t-\Delta t,0\}} - e^{-\omega\max\{-t+\Delta t,0\}}}{\omega} \\
&= \frac{e^{-\omega\max\{-t,0\}} - e^{-\omega\max\{-(t-\Delta t),0\}}}{\omega} \\
&= \frac{e^{\omega\min\{t,0\}} - e^{\omega\min\{t-\Delta t,0\}}}{\omega},
\end{aligned}$$

which is the expression in the lemma.

We next compute $(\nu_3(\omega,\mathcal{T}),\nu_4(\omega,\mathcal{T}))$. Since the difference $u'-u$ in $\nu_3(\omega,\mathcal{T})$ is non-negative and $u\geq -\Delta t$, u' must also exceed $-\Delta t$. Proceeding as for $(\nu_1(\omega,\mathcal{T}),\nu_2(\omega,\mathcal{T}))$, we drop the indicator function $1_{\{u \leq u'\}}$ and change the integration bounds and the argument of $\nu_0(\omega, t)$ as follows:

$$\begin{aligned}
\nu_3(\omega, \mathcal{T}) &= \int_{u'=\max\{t-\Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \int_{u=-\Delta t}^{\min\{u', 0\}} e^{\omega(u-u')} du du' \\
&= \int_{\max\{t-\Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(\min\{u',0\}-u')} - e^{\omega(-\Delta t - u')}}{\omega} du'. \qquad \text{(B.7)}
\end{aligned}$$

Integrating the second term inside the integral yields

$$\begin{aligned}
\int_{\max\{t-\Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(-\Delta t - u')}}{\omega} du' &= \frac{e^{\omega(-\Delta t - \max\{t-\Delta t, -\Delta t\})} - e^{\omega(-\Delta t - \max\{t, -\Delta t\})}}{\omega^2} \\
&= \frac{e^{-\omega \max\{t, 0\}} - e^{-\omega \max\{t+\Delta t, 0\}}}{\omega^2} \qquad \text{(B.8)}
\end{aligned}$$

To integrate the first term inside the integral, we separate it into two using indicator functions, and then change the integration bounds:

$$\begin{aligned}
&\int_{\max\{t-\Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(\min\{u',0\}-u')}}{\omega} du' \\
&= \int_{\max\{t-\Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(\min\{u',0\}-u')} 1_{\{u' \geq 0\}}}{\omega} du' + \int_{\max\{t-\Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(\min\{u',0\}-u')} 1_{\{u' \leq 0\}}}{\omega} du' \\
&= \int_{\max\{\max\{t-\Delta t, -\Delta t\}, 0\}}^{\max\{\max\{t, -\Delta t\}, 0\}} \frac{e^{-\omega u'}}{\omega} du' + \int_{\min\{\max\{t-\Delta t, -\Delta t\}, 0\}}^{\min\{\max\{t, -\Delta t\}, 0\}} \frac{1}{\omega} du' \\
&= \int_{\max\{t-\Delta t, 0\}}^{\max\{t, 0\}} \frac{e^{-\omega u'}}{\omega} du' + \int_{\min\{\max\{t-\Delta t, -\Delta t\}, 0\}}^{\min\{\max\{t, -\Delta t\}, 0\}} \frac{1}{\omega} du' \\
&= \frac{e^{-\omega \max\{t-\Delta t, 0\}} - e^{-\omega \max\{t, 0\}}}{\omega^2} + \frac{\min\{\max\{t, -\Delta t\}, 0\} - \min\{\max\{t-\Delta t, -\Delta t\}, 0\}}{\omega} \qquad \text{(B.9)}
\end{aligned}$$

Substituting (B.8) and (B.9) into (B.7) yields

$$\begin{aligned}
\nu_3(\omega, \mathcal{T}) &= \frac{e^{-\omega \max\{t-\Delta t, 0\}} - e^{-\omega \max\{t, 0\}}}{\omega^2} - \frac{e^{-\omega \max\{t, 0\}} - e^{-\omega \max\{t+\Delta t, 0\}}}{\omega^2} \\
&\quad + \frac{\min\{\max\{t, -\Delta t\}, 0\} - \min\{\max\{t-\Delta t, -\Delta t\}, 0\}}{\omega},
\end{aligned}$$

which is the expression in the lemma. To compute $\nu_4(\omega, \mathcal{T})$, we revert the order of integration, and add $-t$ to both integrands:

$$\begin{aligned}
\nu_4(\omega, \mathcal{T}) &= \int_{u=-\Delta t}^{0} \int_{u'=t-\Delta t}^{t} \nu_0(\omega, |u'-u|) 1_{\{u \geq u'\}} du' du \\
&= \int_{u'=-\Delta t}^{0} \int_{u=t-\Delta t}^{t} \nu_0(\omega, |u'-u|) 1_{\{u \leq u'\}} du du' \\
&= \int_{u'=-t-\Delta t}^{-t} \int_{u=-\Delta t}^{0} \nu_0(\omega, |u'-u|) 1_{\{u \leq u'\}} du du' \\
&= \nu_3(\omega, (-t, Dt)) \\
&= \frac{e^{-\omega \max\{-t+\Delta t, 0\}} + e^{-\omega \max\{-t-\Delta t, 0\}} - 2e^{-\omega \max\{-t, 0\}}}{\omega^2} \\
&\quad + \frac{\min\{\max\{-t, -\Delta t\}, 0\} - \min\{\max\{-t-\Delta t, -\Delta t\}, 0\}}{\omega} \\
&= \frac{e^{-\omega \max\{-(t-\Delta t), 0\}} + e^{-\omega \max\{-(t+\Delta t), 0\}} - 2e^{-\omega \max\{-t, 0\}}}{\omega^2} \\
&\quad + \frac{\min\{-\min\{t, \Delta t\}, 0\} - \min\{-\min\{t+\Delta t, \Delta t\}, 0\}}{\omega} \\
&= \frac{e^{\omega \min\{t-\Delta t, 0\}} + e^{\omega \min\{t+\Delta t, 0\}} - 2e^{\omega \min\{t, 0\}}}{\omega^2} \\
&\quad + \frac{-\max\{\min\{t, \Delta t\}, 0\} + \max\{\min\{t+\Delta t, \Delta t\}, 0\}}{\omega}.
\end{aligned}$$

Adding $-\Delta t$ to both terms inside each maximum in the last line, we can write it as

$$\begin{aligned}
&\frac{-\max\{\min\{t-\Delta t, \Delta t - \Delta t\}, -\Delta t\} + \max\{\min\{t+\Delta t - \Delta t, \Delta t - \Delta t\}, -\Delta t\}}{\omega} \\
&= \frac{-\max\{\min\{t-\Delta t, 0\}, -\Delta t\} + \max\{\min\{t, 0\}, -\Delta t\}}{\omega},
\end{aligned}$$

and thus obtain the expression in the lemma. ∎

Lemma B.2 derives covariances between the state variables and between them and returns.

Lemma B.3. *For $t' > t$,*

$$\mathbb{C}\text{ov}_t(dR_t, \psi_1 \hat{C}_{t'} + \psi_2 C_{t'} + \psi_3 y_{t'}) = G(\psi_1, \psi_2, \psi_3, t'-t, \nu_0) \Sigma p_f' dt, \tag{B.10}$$

$$\mathbb{C}\text{ov}_t(dR_t, F_{t'}') = \frac{\phi^2}{r+\kappa} \left(\Sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \Sigma\right) \nu_0(\kappa, t'-t) dt, \tag{B.11}$$

$$\mathbb{C}\text{ov}_t(dR_t, dR'_{t'}) = G(\gamma_1^R, \gamma_2^R, \gamma_3^R, t'-t, \nu_0)\Sigma p'_f p_f \Sigma dt dt', \tag{B.12}$$

and for $t' \geq t$,

$$\mathbb{C}\text{ov}\left(\psi_1 \hat{C}_t + \psi_2 C_t + \psi_3 y_t, \hat{\psi}_1 \hat{C}_{t'} + \hat{\psi}_2 C_{t'} + \hat{\psi}_3 y_{t'}\right) = H(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, t'-t, \nu_0), \tag{B.13}$$

$$\mathbb{C}\text{ov}\left(\psi_1 \hat{C}_t + \psi_2 C_t + \psi_3 y_t, F_{t'}\right) = K_1(\psi_1, \psi_3, t'-t, \nu_0)\Sigma p'_f, \tag{B.14}$$

$$\mathbb{C}\text{ov}\left(F_t, \psi_1 \hat{C}_{t'} + \psi_2 C_{t'} + \psi_3 y_{t'}\right) = K_2(\psi_1, \psi_3, t'-t, \nu_0)\Sigma p'_f, \tag{B.15}$$

$$\mathbb{C}\text{ov}(F_t, F'_{t'}) = \frac{\phi^2 \Sigma}{2\kappa}\nu_0(\kappa, t'-t). \tag{B.16}$$

Proof: We first show (B.10). Since the covariance is conditional as of time t, it involves only the Brownian terms in dR_t and not the drift terms. Using (A.4) and (B.2)-(B.4), and noting that the only non-zero covariances are between Brownian increments of the same process as of time t, we find

$$\begin{aligned}
&\mathbb{C}\text{ov}_t(dR_t, \psi_1 \hat{C}_{t'} + \psi_2 C_{t'} + \psi_3 y_{t'}) \\
=& -\left(\sigma + \beta_1\gamma_1\Sigma p'_f p_f \sigma\right)\left[\psi_1 e^{-(\kappa+\rho)(t'-t)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right)\right]\beta_1\sigma' p'_f dt \\
&- \frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p'_f p_f \sigma\right)\left[\psi_1 e^{-(\kappa+\rho)(t'-t)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right)\right]\frac{\phi\beta_2}{r+\kappa}\sigma' p'_f dt \\
&- s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\Sigma p'_f\left[(\psi_1+\psi_2)e^{-\kappa(t'-t)} + \frac{\psi_3 b_1}{\kappa-b_2}\left(e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right)\right. \\
&\left. - \left[\psi_1 e^{-(\kappa+\rho)(t'-t)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right)\right]\left(1 - \frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right)\right] s dt \\
=& \left\{-\left[\psi_1 e^{-(\kappa+\rho)(t'-t)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right)\right]\right. \\
&\times\left[\beta_1\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right) + \left(\frac{\phi^2\beta_2}{(r+\kappa)^2} - s^2\gamma_2\left(1 - \frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right)\right)\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\right] \\
&\left. - \left[(\psi_1+\psi_2)e^{-\kappa(t'-t)} + \frac{\psi_3 b_1}{\kappa-b_2}\left(e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right)\right] s^2\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\right\}\Sigma p'_f dt \\
=& \left\{-\left[\psi_1 e^{-(\kappa+\rho)(t'-t)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right)\right]\beta_1\left(1 + \frac{\beta_1\gamma_1\Delta}{\eta\Sigma\eta'}\right)\right.
\end{aligned}$$

$$-\left[(\psi_1+\psi_2)e^{-\kappa(t'-t)}+\frac{\psi_3 b_1}{\kappa-b_2}\left(e^{-\kappa(t'-t)}-e^{-b_2(t'-t)}\right)\right]s^2\gamma_2\left(1+\frac{\beta_2\gamma_1\Delta}{\eta\Sigma\eta'}\right)\Bigg\}\Sigma p_f' dt, \quad \text{(B.17)}$$

where the third step follows from (A.11). Equation (B.17) yields (B.10).

We next show (B.11). Using (A.4) and (B.1), and noting that the conditional covariance involves only the Brownian terms in dR_t, and that the only non-zero covariances are between the Brownian increments of the process F_t as of time t, we find

$$\mathbb{C}\text{ov}_t(dR_t, F_{t'}') = \frac{\phi}{r+\kappa}\left(\sigma+\beta_2\gamma_1\Sigma p_f' p_f\sigma\right)\phi\sigma' e^{-\kappa(t'-t)}dt,$$

which yields (B.11).

We next show (B.12). Using (A.4), and noting that the conditional covariance involves only the Brownian terms in dR_t and only the drift terms in $dR_{t'}$, we find

$$\begin{aligned}\mathbb{C}\text{ov}_t(dR_t, dR_{t'}') =& \mathbb{C}\text{ov}_t(dR_t, E_{t'}(dR_{t'}'))\\ =& \mathbb{C}\text{ov}_t(dR_t, \gamma_1^R\hat{C}_{t'}+\gamma_2^R C_{t'}+\gamma_3^R y_{t'})p_f\Sigma dt', \qquad \text{(B.18)}\end{aligned}$$

where the second step follows from (A.4). Combining (B.18) with (B.10) yields (B.12).

We next show (B.13). Using (B.2)-(B.4) and noting that the only non-zero covariances are between Brownian increments of the same process as of the same time $u \in (-\infty, t]$, we find

$$\mathbb{C}\text{ov}\left(\psi_1\hat{C}_t+\psi_2 C_t+\psi_3 y_t, \hat{\psi}_1\hat{C}_{t'}+\hat{\psi}_2 C_{t'}+\hat{\psi}_3 y_{t'}\right) \qquad \text{(B.19)}$$

$$\begin{aligned}=&\int_{-\infty}^{t}\left[\psi_1 e^{-(\kappa+\rho)(t-u)}+\frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t-u)}-e^{-b_2(t-u)}\right)\right]\\ &\times\left[\hat{\psi}_1 e^{-(\kappa+\rho)(t'-u)}+\frac{\hat{\psi}_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-u)}-e^{-b_2(t'-u)}\right)\right]\left(\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right)\frac{\Delta}{\eta\Sigma\eta'}du\\ &+\int_{-\infty}^{t}\left[(\psi_1+\psi_2)e^{-\kappa(t-u)}+\frac{\psi_3 b_1}{\kappa-b_2}\left(e^{-\kappa(t-u)}-e^{-b_2(t-u)}\right)\right.\\ &-\left.\left[\psi_1 e^{-(\kappa+\rho)(t-u)}+\frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t-u)}-e^{-b_2(t-u)}\right)\right]\left(1-\frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right)\right]\\ &\times\left[(\hat{\psi}_1+\hat{\psi}_2)e^{-\kappa(t'-u)}+\frac{\hat{\psi}_3 b_1}{\kappa-b_2}\left(e^{-\kappa(t'-u)}-e^{-b_2(t'-u)}\right)\right.\end{aligned}$$

$$-\left[\hat{\psi}_1 e^{-(\kappa+\rho)(t'-u)} + \frac{\hat{\psi}_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-u)} - e^{-b_2(t'-u)}\right)\right]\left(1-\frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right)\Bigg] s^2 du. \quad \text{(B.20)}$$

Integrating all products of exponentials in (B.20) and summing, yields (B.13). To perform the algebra, we separate the terms in $\left(1-\frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}\right)^2$ into quadratic terms in $\frac{\beta_2\gamma_2\Delta}{\eta\Sigma\eta'}$, linear terms and constant terms.

We next show (B.14) and (B.15). Using (B.1)-(B.4) and noting that the only non-zero covariances are between Brownian increments of the same process as of the same time $u \in (-\infty, t]$, we find

$$\begin{aligned}
&\mathbb{C}\text{ov}\left(\psi_1\hat{C}_t + \psi_2 C_t + \psi_3 y_t, F_{t'}\right)\\
&= -\int_{-\infty}^{t}\left[\psi_1 e^{-(\kappa+\rho)(t-u)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t-u)} - e^{-b_2(t-u)}\right)\right] e^{-\kappa(t'-u)}\frac{\phi^2\beta_2\Sigma p_f'}{r+\kappa}du
\end{aligned} \quad \text{(B.21)}$$

and

$$\begin{aligned}
&\mathbb{C}\text{ov}\left(F_t, \psi_1\hat{C}_{t'} + \psi_2 C_{t'} + \psi_3 y_{t'}\right)\\
&= -\int_{-\infty}^{t} e^{-\kappa(t-u)}\left[\psi_1 e^{-(\kappa+\rho)(t'-u)} + \frac{\psi_3 b_1}{\kappa+\rho-b_2}\left(e^{-(\kappa+\rho)(t'-u)} - e^{-b_2(t'-u)}\right)\right]\frac{\phi^2\beta_2\Sigma p_f'}{r+\kappa}du.
\end{aligned} \quad \text{(B.22)}$$

Integrating all products of exponentials in (B.21) and (B.22) and summing, yields (B.14) and (B.15), respectively.

We finally show (B.16). Using (B.1) and noting that the only non-zero covariances are between Brownian increments of the same process as of the same time $u \in (-\infty, t]$, we find

$$\mathbb{C}\text{ov}(F_t, F_{t'}') = \int_{-\infty}^{t}\phi^2\Sigma e^{-\kappa(t-u)}e^{-\kappa(t'-u)}du \quad \text{(B.23)}$$

Integrating (B.23), we find (B.16). ∎

C Proofs of Results in Section 4

The portfolio optimization problem corresponding to $SR_{w,t}$ is as follows. Consider an investor at time t with infinitesimal horizon dt, who can invest in the riskless asset, the index η and the strategy w_t. The investor has mean-variance preferences

$$\mathbb{E}_{\mathcal{I}_t}(dW_t) - \frac{a}{2}\mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(dW_t). \tag{C.1}$$

She chooses an overall exposure $\hat{\hat{x}}_t$ to the index and a position $\hat{y}_t$ in the strategy. These positions can depend on information in $\mathcal{I}_t$. The investor's overall exposure to the index at time t is

$$\hat{\hat{x}}_t = \hat{x}_t + \hat{y}_t \frac{\mathbb{C}\mathrm{ov}_t(w_t dR_t, \eta dR_t)}{\mathbb{V}\mathrm{ar}_t(\eta dR_t)},$$

the sum of a position $\hat{x}_t$ in the index and an exposure resulting from the strategy. The investor's budget constraint is

$$\begin{aligned} dW_t &= rW_t dt + \hat{x}_t \eta dR_t + \hat{y}_t w_t dR_t \\ &= rW_t dt + \left(\hat{x}_t + \hat{y}_t \frac{\mathbb{C}\mathrm{ov}_t(w_t dR_t, \eta dR_t)}{\mathbb{V}\mathrm{ar}_t(\eta dR_t)}\right) \eta dR_t + \hat{y}_t \hat{w}_t dR_t \\ &= rW_t dt + \hat{\hat{x}}_t \eta dR_t + \hat{y}_t \hat{w}_t dR_t. \end{aligned} \tag{C.2}$$

Lemma C.1. *The investor's maximum utility is*

$$\frac{1}{2a}\left(SR_\eta^2 + SR_{w,t}^2\right) dt. \tag{C.3}$$

Proof: Substituting dW_t from (C.2), and noting that $(\eta dR_t, \hat{w}_t dR_t)$ are uncorrelated, we can write (C.1) as

$$\hat{\hat{x}}_t \mathbb{E}_{\mathcal{I}_t}(\eta dR_t) + \hat{y}_t \mathbb{E}_{\mathcal{I}_t}(\hat{w}_t dR_t) - \frac{a}{2}\left(\hat{\hat{x}}_t^2 \mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(\eta dR_t) + \hat{y}_t^2 \mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(\hat{w}_t dR_t)\right). \tag{C.4}$$

Maximizing (C.4) over $(\hat{\hat{x}}_t, \hat{y}_t)$ yields the utility

$$\frac{1}{2a}\left(SR_{\eta,t}^2 + SR_{w,t}^2\right) dt. \tag{C.5}$$

Since $\eta\Sigma p_f' = 0$ and (A.4) imply

$$\eta dR_t = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\eta\Sigma\theta' dt + \eta\sigma\left(dB_t^D + \frac{\phi dB_t^F}{r+\kappa}\right), \tag{C.6}$$

$\mathbb{E}_{\mathcal{I}_t}(\eta dR_t)$ and $\mathbb{V}\text{ar}_{\mathcal{I}_t}(\eta dR_t)$ coincide with their unconditional values. Therefore, $SR_{\eta,t} = SR_\eta$, and (C.5) coincides with (C.3). ∎

The portfolio optimization problem corresponding to $SR_{w,t,T}$ is as follows. Consider an investor at time t with horizon T, who can invest in the riskless asset, the index η and the strategy w_t. The investor has mean-variance preferences

$$\mathbb{E}_{\mathcal{I}_t}(\Delta W_{t+T}) - \frac{a}{2}\mathbb{V}\text{ar}_{\mathcal{I}_t}(\Delta W_{t+T}) \tag{C.7}$$

over the increment $\Delta W_{t+T} \equiv W_{t+T}e^{-rT} - W_t$ in discounted wealth at the riskless rate r. She chooses an overall exposure $\hat{\hat{x}}_t$ to the index and a position $\hat{y}_t$ in the strategy at time t. These positions can depend on information in $\mathcal{I}_t$. The investor is assumed to scale up these positions over time at the riskless rate r, to $\hat{\hat{x}}_u = \hat{\hat{x}}_t e^{r(u-t)}$ and $\hat{y}_u = \hat{y}_t e^{r(u-t)}$, respectively, at time u. The investor's overall exposure to the index at time u is

$$\hat{\hat{x}}_u = \hat{x}_u + \hat{y}_u \frac{\mathbb{C}\text{ov}_u(w_u dR_u, \eta dR_u)}{\mathbb{V}\text{ar}_u(\eta dR_u)},$$

the sum of a position $\hat{x}_u$ in the index and an exposure resulting from the strategy. The investor's budget constraint is

$$\begin{aligned} dW_u &= rW_u dt + \hat{x}_u \eta dR_u + \hat{y}_u w_u dR_u \\ &= rW_u dt + \left(\hat{x}_u + \hat{y}_u \frac{\mathbb{C}\text{ov}_u(w_u dR_u, \eta dR_u)}{\mathbb{V}\text{ar}_u(\eta dR_u)}\right)\eta dR_u + \hat{y}_u \hat{w}_u dR_u \qquad \text{(C.8)} \\ &= rW_u dt + \hat{\hat{x}}_t e^{r(u-t)}\eta dR_u + \hat{y}_t e^{r(u-t)}\hat{w}_u dR_u, \qquad \text{(C.9)} \end{aligned}$$

and integrates to

$$\Delta W_{t+T} = \hat{\hat{x}}_t \int_t^{t+T} \eta dR_u + \hat{y}_t \int_t^{t+T} \hat{w}_u dR_u \tag{C.10}$$

from time t to $t+T$.

Lemma C.2. *Suppose* $\mathbb{C}\text{ov}_{\mathcal{I}_t}(\eta dR_u, \hat{w}_{u'}dR_{u'}) = 0$ *for* $t < u < u'$*. The investor's maximum utility is*

$$\frac{1}{2a}\left(SR_\eta^2 + SR_{w,t,T}^2\right)T. \tag{C.11}$$

Proof: Substituting ΔW_{t+T} from (C.10), we can write (C.7) as

$$\begin{aligned}&\hat{\hat{x}}_t\mathbb{E}_{\mathcal{I}_t}\left(\int_t^{t+T}\eta dR_u\right) + \hat{y}_t\mathbb{E}_{\mathcal{I}_t}\left(\int_t^{t+T}\hat{w}_u dR_u\right)\\ &-\frac{a}{2}\left[\hat{\hat{x}}_t^2\mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\int_t^{t+T}\eta dR_u\right) + \hat{y}_t^2\mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\int_t^{t+T}\hat{w}_u dR_u\right) + 2\hat{\hat{x}}_t\hat{y}_t\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\int_t^{t+T}\eta dR_u, \int_t^{t+T}\hat{w}_u dR_u\right)\right].\end{aligned} \tag{C.12}$$

To compute the covariance in (C.12), we write it as

$$\int_t^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\eta dR_u, \hat{w}_u dR_u\right) + \int_{u=t}^{t+T}\int_{u'=t}^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\eta dR_u, \hat{w}_{u'}dR_{u'}\right), \tag{C.13}$$

where the first term in (C.13) is the covariance between contemporaneous returns and the second term is the covariance between lagged returns. The first term is zero because of the definition (4.3) of $\hat{w}_t$. Since (C.6) implies that the covariance $\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\eta dR_u, \hat{w}_{u'}dR_{u'}\right)$ for $u > u'$ is zero, the second term is

$$\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\eta dR_u, \hat{w}_{u'}dR_{u'}\right),$$

and is zero because of the assumption $\mathbb{C}\text{ov}_{\mathcal{I}_t}(\eta dR_u, \hat{w}_{u'}dR_{u'}) = 0$ for $t < u < u'$. In the proof of Proposition E.10 we show that this assumption is satisfied for the strategies that we examine in this paper. With a zero covariance in (C.12), maximization over $(\hat{\hat{x}}_t, \hat{y}_t)$ yields the maximum utility

$$\frac{1}{2a}\left(SR_{\eta,t,T}^2 + SR_{w,t,T}^2\right)T. \tag{C.14}$$

Since (C.6) implies that the conditional moments $\mathbb{E}_{\mathcal{I}_t}\left(\int_t^{t+T}\eta dR_u\right)$ and $\mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\int_t^{t+T}\eta dR_u\right)$ coincide with their unconditional values, $SR_{\eta,t,T} = SR_{\eta,T}$. Since, in addition, $\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\eta dR_u, \eta dR_{u'}\right)$ $= \mathbb{C}\text{ov}\left(\eta dR_u, \eta dR_{u'}\right) = 0$ for $u \neq u'$, Lemma E.2 implies $SR_{\eta,T} = SR_\eta$. Therefore, (C.14) coincides with (C.11). ∎

We next move to the calibration. We compute model-implied moments for general asset payoffs, and specialize them to symmetric assets, with $\eta = \mathbf{1}'$, $\bar{F} = \mathcal{F}\mathbf{1}$ and $\Sigma = \hat{\sigma}^2(I + \omega\mathbf{1}\mathbf{1}')$, in Lemma C.5. The calculations of Sharpe ratios and correlations in Appendices D and E also concern general asset payoffs, except when symmetry is explicitly mentioned. Lemma C.3 computes the Sharpe ratio of the index, the correlation between an asset and the index, and the fraction of an asset's variance that is generated by fund flows.

Lemma C.3. *The Sharpe ratio of the index η is*

$$SR_\eta = SR_{\eta,T} = \frac{r\alpha\bar{\alpha}\sqrt{f}}{\alpha + \bar{\alpha}} \frac{\eta\Sigma\theta'}{\sqrt{\eta\Sigma\eta'}}. \tag{C.15}$$

The correlation between asset n and the index is

$$\mathbb{C}\mathrm{orr}(dR_{nt}, \eta dR_t) = \frac{\sqrt{f}(\eta\Sigma)_n}{\sqrt{\eta\Sigma\eta' \left[f\Sigma_{nn} + k[(p_f\Sigma)_n]^2\right]}}. \tag{C.16}$$

The fraction of asset n's variance that is generated by fund flows is

$$\frac{k[(p_f\Sigma)_n]^2}{f\Sigma_{nn} + k[(p_f\Sigma)_n]^2}. \tag{C.17}$$

Proof: Equation (C.6) implies

$$\mathbb{E}(\eta dR_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}}\eta\Sigma\theta' dt, \tag{C.18}$$

$$\mathbb{V}\mathrm{ar}(\eta dR_t) = f\eta\Sigma\eta' dt. \tag{C.19}$$

Substituting (C.18) and (C.19) into (4.4), and noting that $SR_{\eta,T} = SR_\eta$, we find (C.15). The correlation between asset n and the index is

$$\mathbb{C}\mathrm{orr}(dR_{nt}, \eta dR_t) = \frac{\mathbb{C}\mathrm{ov}(dR_{nt}, \eta dR_t)}{\sqrt{\mathbb{V}\mathrm{ar}(dR_{nt})\mathbb{V}\mathrm{ar}(\eta dR_t)}} = \frac{(f\eta\Sigma)_n}{\sqrt{f\eta\Sigma\eta'(f\Sigma + k\Sigma p_f' p_f\Sigma)_{nn}}}, \tag{C.20}$$

where the second step follows from (A.2), (C.19) and $\eta\Sigma p_f' = 0$. Equation (C.20) implies (C.16). Equation (C.17) follows from (A.2). ∎

We approximate the calculation of the index's expected return per dollar by dividing the index's expected return per share by the index's expected price

$$\frac{\mathbb{E}(\eta dR_t)}{\mathbb{E}(\eta S_t)} = \frac{\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\eta\Sigma\theta' dt}{\eta\frac{\bar{F}}{r} - \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta\Sigma\eta'}. \tag{C.21}$$

The active-share calculation is as follows. The active share of the residual supply portfolio is

$$AS_\theta = \frac{1}{2}\sum_{n=1}^{N}\left|\frac{\theta_n S_n}{\sum_{m=1}^{N}\theta_m S_m} - \frac{\eta_n S_n}{\sum_{m=1}^{N}\eta_m S_m}\right|. \tag{C.22}$$

Since asset prices vary over time, active share does too. We use expected active share, and approximate its calculation by replacing prices S_n in the numerator and denominator of (C.22) by their expectations. Our approximation for expected active share thus is

$$\overline{AS}_\theta = \frac{1}{2}\sum_{n=1}^{N}\left|\frac{\theta_n \mathbb{E}(S_n)}{\sum_{m=1}^{N}\theta_m \mathbb{E}(S_m)} - \frac{\eta_n \mathbb{E}(S_n)}{\sum_{m=1}^{N}\eta_m \mathbb{E}(S_m)}\right|. \tag{C.23}$$

Using (3.3) and Lemma B.1, we find that expected prices are

$$\mathbb{E}(S_t) = \frac{\bar{F}}{r} - \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' - \left(\gamma_0 + (\gamma_1+\gamma_2)\bar{C} + g_3\bar{y}\right)\Sigma p_f'.$$

Lemma C.4 computes the standard deviation of flow-induced trading for asset n.

Lemma C.4. *The standard deviation of the change in the investor's holdings of asset n between t and $t+\Delta t$ is*

$$\sqrt{2\left[H(0,0,1,0,0,1,0,\nu_0) - H(0,0,1,0,0,1,\Delta t,\nu_0)\right]}\,|(p_f)_n|\,. \tag{C.24}$$

Proof: Equation (A.3) implies that the change in the investor's holdings of asset n between t and $t+\tau$ is

$$(x_{t+\tau}\eta + y_{t+\Delta t}z_{t+\Delta t})_n - (x_t\eta + y_t z_t)_n = (y_{t+\Delta t} - y_t)(p_f)_n.$$

The standard deviation of that change is

$$\sqrt{\mathbb{V}\text{ar}(y_{t+\Delta t} - y_t)}\,|(p_f)_n|\,. \tag{C.25}$$

Since

$$\begin{aligned}\mathbb{V}\text{ar}(y_{t+\Delta t} - y_t) =& \mathbb{V}\text{ar}(y_{t+\Delta t}) + \mathbb{V}\text{ar}(y_t) - 2\mathbb{C}\text{ov}(y_t, y_{t+\Delta t})\\ =& 2\mathbb{V}\text{ar}(y_t) - 2\mathbb{C}\text{ov}(y_t, y_{t+\Delta t}),\end{aligned}$$

where the second step follows in steady state, (B.13) and (C.25) imply (C.24). ∎

The standard deviation in Lemma C.4 is computed for a given asset n over time and is expressed per share of the asset. When assets are symmetric and θ_n is equal to $\bar{\theta} + \sigma(\theta)$ for half of the assets and to $\bar{\theta} - \sigma(\theta)$ for the other half, $|(p_f)_n|$ is the same for all n, and the standard deviation (C.24) of the change in asset holdings is the same across assets. Hence, changes in asset holdings are drawn from the same distribution for all assets, and the standard deviation (C.24) describes both the cross-section and the time-series. Lou (2012) computes a spread in changes in asset holdings between top and bottom deciles of 22.27%. This translates to a standard deviation of 6.55% (=22.27%/3.4). The counterpart quantity in our model is the standard deviation (C.24) divided by the number of shares held by the active and the index funds. That number is $\bar{\theta} = 1$ for the average asset.

The ratio of the investor's conditional standard deviation of C_t to the unconditional standard deviation is $\sqrt{\frac{T}{\frac{s^2}{2\kappa}}}$, where T is given from (A.12). Equation (B.10) implies that the response of the investor's share y_t in the active fund at time t' to a shock dR_t at time t is proportional to $G(0, 0, 1, t' - t, \nu_0)$.

Lemma C.5 derives formulas for symmetric assets. We use these formulas to simplify the model-implied moments computed in Appendix C, and the Sharpe ratios and correlations computed in Appendices D and E.

Lemma C.5. *Suppose $\eta = \boldsymbol{1}'$ and $\Sigma = \hat{\sigma}^2(I + \omega \boldsymbol{1}\boldsymbol{1}')$. For all $i \in \mathbb{N}$,*

$$\eta\Sigma^i\eta' = \hat{\sigma}^{2i}(1 + \omega N)^i N, \tag{C.26}$$

$$\eta\Sigma^i p_f' = 0, \tag{C.27}$$

$$p_f\Sigma^i p_f' = \hat{\sigma}^{2i}\sigma(\theta)^2 N, \tag{C.28}$$

$$Tr(\Sigma^i) = \hat{\sigma}^{2i}\left[(1 + \omega N)^i + N - 1\right], \tag{C.29}$$

$$\Sigma^i p_f' = \hat{\sigma}^{2i}(\theta' - \bar{\theta}\boldsymbol{1}). \tag{C.30}$$

Proof: Using the binomial formula and $\eta = \mathbf{1}'$, we find

$$\Sigma^i = \hat{\sigma}^{2i}\left(\sum_{i'=0}^{i} C(i,i')\omega^{i'}(\mathbf{1}\mathbf{1}')^{i'}\right) = \hat{\sigma}^{2i}\left(I + \sum_{i'=1}^{i} C(i,i')\omega^{i'}N^{i'-1}\mathbf{1}\mathbf{1}'\right). \tag{C.31}$$

Post-multiplying (C.31) by η' and θ' yields

$$\Sigma^i\eta' = \hat{\sigma}^{2i}\left(1 + \sum_{i'=1}^{i} C(i,i')\omega^{i'}N^{i'}\right)\mathbf{1} = \hat{\sigma}^{2i}(1+\omega N)^i\mathbf{1}, \tag{C.32}$$

$$\Sigma^i\theta' = \hat{\sigma}^{2i}\left[\theta' + \left(\sum_{i'=1}^{i} C(i,i')\omega^{i'}N^{i'}\right)\bar{\theta}\mathbf{1}\right] = \hat{\sigma}^{2i}\left[\theta' - \bar{\theta}\mathbf{1} + (1+\omega N)^i\bar{\theta}\mathbf{1}\right], \tag{C.33}$$

respectively. Pre-multiplying (C.32) and (C.33) by η, and setting $i = 1$, yields

$$\eta\Sigma\theta' = \bar{\theta}\eta\Sigma\eta', \tag{C.34}$$

which in turn implies

$$p_f = \theta - \bar{\theta}\eta = \theta - \bar{\theta}\mathbf{1}'. \tag{C.35}$$

Pre-multiplying (C.32) by η yields (C.26). Post-multiplying Σ^i by p_f' and using (C.32)-(C.35) yields (C.30). Pre-multiplying (C.30) by η yields (C.27). Pre-multiplying (C.30) by p_f and using (C.30) yields (C.28). Summing the diagonal terms in (C.31) yields (C.29). ∎

The model-implied moments computed in this section depend on θ through the aggregate quantities in Lemma C.5 and the components of the vector $p_f = \theta - \bar{\theta}\mathbf{1}' = \theta - \mathbf{1}'$. The aggregate quantities depend on θ only through $\bar{\theta} = 1$ and $\sigma(\theta)$. To compute the components of p_f, we assume that θ_n is equal to $\bar{\theta} + \sigma(\theta)$ for half of the assets and to $\bar{\theta} - \sigma(\theta)$ for the other half.

D Proofs of Results in Section 5

Lemma D.1 computes the Sharpe ratio of a general strategy w_t over an infinitesimal horizon dt. It also characterizes the optimal strategy and its Sharpe ratio.

Lemma D.1. *The Sharpe ratio of a strategy w_t over horizon dt is $SR_{w,t} = \frac{\mathcal{N}_{w,t}}{\sqrt{\mathcal{D}_{w,t}}}$, where*

$$\mathcal{N}_{w,t} \equiv \frac{1}{dt}\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t dR_t) = \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}\left(\Lambda_t w_t \Sigma p_f'\right), \tag{D.1}$$

$$\mathcal{D}_{w,t} \equiv \frac{1}{dt}\mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(\hat{w}_t dR_t) = f\left[\mathbb{E}_{\mathcal{I}_t}(w_t\Sigma w_t') - \frac{\mathbb{E}_{\mathcal{I}_t}\left[(w_t\Sigma\eta')^2\right]}{\eta\Sigma\eta'}\right] + k\mathbb{E}_{\mathcal{I}_t}[(w_t\Sigma p_f')^2]. \tag{D.2}$$

It is maximized for the strategy $w_t = \Lambda_t p_f$. The Sharpe ratio of the optimal strategy is given by (5.2).

Proof: Lemma D.1 coincides with VW Proposition 8 in the case of the unconditional Sharpe ratio. The arguments in that proposition extend to the case of the conditional Sharpe ratio. ∎

Proposition D.1 computes the optimal strategy's unconditional Sharpe ratio, as well as its Sharpe ratios conditional on different information sets.

Proposition D.1. *The unconditional Sharpe ratio of the optimal strategy is*

$$SR_w^* = \sqrt{\frac{\Delta}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\eta\Sigma\eta'}\left[L_2^2 + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, 0, \nu_0)\right]}. \tag{D.3}$$

When $\mathcal{I}_t$ includes $(\hat{C}_t, C_t, y_t)$, the Sharpe ratio of the optimal strategy is

$$SR_{w,t}^* = \sqrt{\frac{\Delta}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\eta\Sigma\eta'}}\left|L_2 + \gamma_1^R(\hat{C}_t - \bar{C}) + \gamma_2^R(C_t - \bar{C}) + \gamma_3^R(y_t - \bar{y})\right| \tag{D.4}$$

and has unconditional expectation

$$\begin{aligned}\mathbb{E}\left(SR_{w,t}^*\right) = &\sqrt{\frac{\Delta}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\eta\Sigma\eta'}H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, 0, \nu_0)} \\ &\times\left[\sqrt{\frac{2}{\pi}}e^{-\frac{R(\Lambda_t)^2}{2}} + R(\Lambda_t)\left[1 - 2N\left(-R(\Lambda_t)\right)\right]\right]\end{aligned} \tag{D.5}$$

and unconditional variance

$$\mathbb{V}\mathrm{ar}\left(SR_{w,t}^*\right) = \frac{\Delta}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\eta\Sigma\eta'}H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, 0, \nu_0)\left[R(\Lambda_t)^2 + 1\right] - \mathbb{E}\left(SR_{w,t}^*\right)^2, \tag{D.6}$$

where $R(\Lambda_t) \equiv \frac{L_2}{\sqrt{H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,0,\nu_0)}}$. *When instead* $\mathcal{I}_t = (\hat{C}_t, y_t)$, *the Sharpe ratio of the optimal strategy is*

$$SR^*_{w,t} = \sqrt{\frac{\Delta}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\eta\Sigma\eta'}\left[\left(L_2 + (\gamma_1^R + \gamma_2^R)(\hat{C}_t - \bar{C}) + \gamma_3^R(y_t - \bar{y})\right)^2 + (\gamma_2^R)^2 T\right]}. \tag{D.7}$$

Proof: To derive (D.3), we set $\mathcal{I}_t = \emptyset$ in (5.2) and note that

$$\begin{aligned}\mathbb{E}(\Lambda_t^2) &= \mathbb{E}(\Lambda_t)^2 + \mathbb{V}\text{ar}(\Lambda_t)\\ &= \frac{1}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2}\left[L_2^2 + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,0,\nu_0)\right],\end{aligned}$$

where the second step follows because the definition (B.6) of L_2 implies

$$\mathbb{E}(\Lambda_t) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} L_2. \tag{D.8}$$

and because (3.6) and (B.13) imply

$$\mathbb{V}\text{ar}(\Lambda_t) = \frac{1}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2} H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,0,\nu_0). \tag{D.9}$$

To derive (D.4), we note that when $\mathcal{I}_t$ includes $(\hat{C}_t, C_t, y_t)$, $\mathbb{E}(\Lambda_t^2) = \Lambda_t^2$. Substituting into (5.2), we find

$$SR^*_{w,t} = \sqrt{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\frac{\Delta}{\eta\Sigma\eta'}}\,|\Lambda_t|\,. \tag{D.10}$$

Equation (D.4) follows from (D.10) and because (B.6) and (B.3)-(B.4) imply

$$\Lambda_t = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}}\left[L_2 + \gamma_1^R(\hat{C}_t - \bar{C}) + \gamma_2^R(C_t - \bar{C}) + \gamma_3^R(y_t - \bar{y})\right]. \tag{D.11}$$

Since Λ_t is normally distributed,

$$\mathbb{E}\left(|\Lambda_t|\right) = \sqrt{\mathbb{V}\text{ar}(\Lambda_t)}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{R(\Lambda_t)^2}{2}} + R(\Lambda_t)\left[1 - 2N\left(-R(\Lambda_t)\right)\right]\right] \tag{D.12}$$

and unconditional variance

$$\mathbb{V}\text{ar}\left(|\Lambda_t|\right) = \mathbb{V}\text{ar}(\Lambda_t)\left[R(\Lambda_t)^2 + 1\right] - \mathbb{E}\left(SR^*_{w,t}\right)^2, \tag{D.13}$$

where $R(\Lambda_t) \equiv \frac{\mathbb{E}(\Lambda_t)}{\sqrt{\mathbb{V}\text{ar}(\Lambda_t)}}$ and $N(.)$ is the cumulative distribution function of the standard normal. Equations (D.5) and (D.6) follow from (D.8)-(D.10), (D.12) and (D.13). To derive (D.7), we set $\mathcal{I}_t = (\hat{C}_t, y_t)$ in (5.2) and note that

$$\begin{aligned}
\mathbb{E}_{\mathcal{I}_t}(\Lambda_t^2) &= \mathbb{E}_{\mathcal{I}_t}(\Lambda_t)^2 + \mathbb{V}\text{ar}_{\mathcal{I}_t}(\Lambda_t) \\
&= \frac{1}{\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2}\left[\left(L_2 + (\gamma_1^R + \gamma_2^R)(\hat{C}_t - \bar{C}) + \gamma_3^R(y_t - \bar{y})\right)^2 + (\gamma_2^R)^2 T\right],
\end{aligned}$$

where the second step follows from (D.11) and because conditionally on $(\hat{C}_t, y_t)$, C_t is normal with mean $\hat{C}_t$ and variance T. ∎

Lemma D.2 specializes the Sharpe ratio formula that Lemma D.1 derives for a general strategy to a class of strategies whose moments have a specific form. Value and momentum strategies belong to that class.

Lemma D.2. *Suppose that for a strategy w_t and information set $\mathcal{I}_t$,*

$$\mathbb{E}_{\mathcal{I}_t}(w_t) = \Phi_{1t}\eta\Sigma + \Phi_{2t}p_f\Sigma, \tag{D.14}$$

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_t, w_t) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}}\Phi_t^{\Lambda}p_f\Sigma, \tag{D.15}$$

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}(w_t', w_t) = \hat{\Phi}_t^{\Sigma}\Sigma + \hat{\Phi}_t\Sigma p_f' p_f\Sigma. \tag{D.16}$$

Then, the Sharpe ratio of w_t conditional on $\mathcal{I}_t$ is $\frac{\mathcal{N}_{w,t}}{\sqrt{\mathcal{D}_{w,t}}}$, with

$$\mathcal{N}_{w,t} = \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}(\Lambda_t)\left(\Phi_{1t}\eta\Sigma^2 p_f' + \Phi_{2t}p_f\Sigma^2 p_f'\right) + \Phi_t^{\Lambda}p_f\Sigma^2 p_f',$$

$$\mathcal{D}_{w,t} = \Phi_{1t}^2 \Delta_1 + 2\Phi_{1t}\Phi_{2t}\Delta_2 + \left(\Phi_{2t}^2 + \hat{\Phi}_t\right)\Delta_3 + \hat{\Phi}_t^{\Sigma}\Delta_4.$$

Proof: We can write the numerator in (5.1) as

$$\begin{aligned}
&\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\mathbb{E}_{\mathcal{I}_t}(\Lambda_t)\mathbb{E}_{\mathcal{I}_t}(w_t) + \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(\Lambda_t, w_t)\right]\Sigma p_f' \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}(\Lambda_t)\left(\Phi_{1t}\eta\Sigma^2 p_f' + \Phi_{2t} p_f \Sigma^2 p_f'\right) + \Phi_t^{\Lambda} p_f \Sigma^2 p_f',
\end{aligned} \tag{D.17}$$

where the second step follows by substituting $\mathbb{E}_{\mathcal{I}_t}(w_t)$ and $\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(\Lambda_t, w_t)$ from (D.14) and (D.15), respectively. We can write the term inside the square root in the denominator in (5.1) as

$$\begin{aligned}
&f\left[\mathbb{E}_{\mathcal{I}_t}(w_t)\Sigma\mathbb{E}_{\mathcal{I}_t}(w_t') - \frac{\left[\mathbb{E}_{\mathcal{I}_t}(w_t)\Sigma\eta'\right]^2}{\eta\Sigma\eta'}\right] + k\left[\mathbb{E}_{\mathcal{I}_t}(w_t)\Sigma p_f'\right]^2 \\
&\quad + f\left[\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(w_t, \Sigma w_t') - \frac{\mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(w_t\Sigma\eta')}{\eta\Sigma\eta'}\right] + k\mathbb{V}\mathrm{ar}_{\mathcal{I}_t}(w_t\Sigma p_f') \\
&= f\left[\mathbb{E}_{\mathcal{I}_t}(w_t)\Sigma\mathbb{E}_{\mathcal{I}_t}(w_t') - \frac{\left[\mathbb{E}_{\mathcal{I}_t}(w_t)\Sigma\eta'\right]^2}{\eta\Sigma\eta'}\right] + k\left[\mathbb{E}_{\mathcal{I}_t}(w_t)\Sigma p_f'\right]^2 \\
&\quad + f\left[Tr\left(\Sigma\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(w_t', w_t)\right) - \frac{\eta\Sigma\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(w_t, w_t')\Sigma\eta'}{\eta\Sigma\eta'}\right] + k p_f\Sigma\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(w_t, w_t')\Sigma p_f' \\
&= \Phi_{1t}^2\left[f\left(\eta\Sigma^3\eta' - \frac{(\eta\Sigma^2\eta')^2}{\eta\Sigma\eta'}\right) + k\left(\eta\Sigma^2 p_f'\right)^2\right] \\
&\quad + 2\Phi_{1t}\Phi_{2t}\left[f\left(\eta\Sigma^3 p_f' - \frac{\eta\Sigma^2\eta'\eta\Sigma^2 p_f'}{\eta\Sigma\eta'}\right) + k\eta\Sigma^2 p_f' p_f \Sigma^2 p_f'\right] \\
&\quad + \left(\Phi_{2t}^2 + \hat{\Phi}_t\right)\left[f\left(p_f\Sigma^3 p_f' - \frac{(\eta\Sigma^2 p_f')^2}{\eta\Sigma\eta'}\right) + k\left(p_f\Sigma^2 p_f'\right)^2\right] \\
&\quad + \hat{\Phi}_t^{\Sigma}\left[f\left(Tr(\Sigma^2) - \frac{\eta\Sigma^3\eta'}{\eta\Sigma\eta'}\right) + k p_f\Sigma^3 p_f'\right] \\
&= \Phi_{1t}^2\Delta_1 + 2\Phi_1\Phi_{2t}\Delta_2 + \left(\Phi_{2t}^2 + \hat{\Phi}_t\right)\Delta_3 + \hat{\Phi}_t^{\Sigma}\Delta_4,
\end{aligned} \tag{D.18}$$

where the third step follows by substituting $\mathbb{E}_{\mathcal{I}_t}(w_t)$ and $\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}(w_t', w_t)$ from (D.14) and (D.16), respectively. The lemma follows from (D.17) and (D.18). ∎

Proposition D.2 computes the unconditional Sharpe ratio of the value strategy.

Proposition D.2. *The unconditional Sharpe ratio of the value strategy (4.1) is* $SR_{w^V} = \frac{\mathcal{N}_{w^V}}{\sqrt{\mathcal{D}_{w^V}}}$ *where*

$$\mathcal{N}_{w^V} = \frac{L_1 L_2}{r}\eta\Sigma^2 p_f' + \left(\frac{L_2^2}{r} - \frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R, \gamma_3^R, 0, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0)\right) p_f \Sigma^2 p_f',$$

$$\begin{aligned}\mathcal{D}_{w^V} &= \frac{L_1^2}{r^2}\Delta_1 + \frac{2L_1L_2}{r^2}\Delta_2 \\ &+ \left(\frac{L_2^2}{r^2} - \frac{2(1-\epsilon)}{r+\kappa}K_1(\gamma_1, \gamma_3, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0)\right)\Delta_3 + \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2\kappa}\Delta_4.\end{aligned}$$

Proof: Substituting (3.3) into (4.1), we can write the value weights as

$$w_t^V = -\frac{(1-\epsilon)(F_t - \bar{F})'}{r+\kappa} + \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta\Sigma + (\gamma_0 + \gamma_1\hat{C}_t + \gamma_2 C_t + \gamma_3 y_t)p_f\Sigma. \tag{D.19}$$

Taking unconditional expectations in (D.19), we find

$$\begin{aligned}\mathbb{E}\left(w_t^V\right) &= \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\eta\Sigma + \left(\gamma_0 + (\gamma_1+\gamma_2)\bar{C} + \gamma_3\bar{y}\right)p_f\Sigma \\ &= \frac{L_1}{r}\eta\Sigma + \frac{L_2}{r}p_f\Sigma,\end{aligned} \tag{D.20}$$

where the second step follows from (B.5) and because (A.1), (A.5)-(A.7) and (B.6) imply

$$\frac{L_2}{r} = \gamma_0 + (\gamma_1+\gamma_2)\bar{C} + \gamma_3\bar{y}. \tag{D.21}$$

Taking the unconditional covariance of (D.19) with (3.6), and using (B.13) and (B.15), we find

$$\mathbb{C}\mathrm{ov}\left(\Lambda_t, w_t^V\right) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}}\left(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R, \gamma_3^R, 0, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0)\right)p_f\Sigma. \tag{D.22}$$

Taking the unconditional covariance of (D.19) with the transpose of (D.19), and using (B.13), (B.15) and (B.16), we find

$$\mathbb{C}\mathrm{ov}\left(\left(w_t^V\right)', w_t^V\right) = \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2\kappa}\Sigma + \left(-\frac{2(1-\epsilon)}{r+\kappa}K_1(\gamma_1, \gamma_3, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0)\right)\Sigma p_f' p_f \Sigma. \tag{D.23}$$

Equations (D.20), (D.22) and (D.23) imply that the unconditional Sharpe ratio of the value strategy can be deduced from Lemma D.2 by setting $\mathcal{I}_t = \emptyset$,

$$\Phi_{1t} = \frac{L_1}{r},$$

$$\Phi_{2t} = \frac{L_2}{r},$$

$$\Phi_t^{\Lambda} = -\frac{1-\epsilon}{r+\kappa} K_1(\gamma_1^R, \gamma_3^R, 0, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0),$$

$$\hat{\Phi}_t^{\Sigma} = \frac{(1-\epsilon)^2 \phi^2}{2(r+\kappa)^2 \kappa},$$

$$\hat{\Phi}_t = -\frac{2(1-\epsilon)}{r+\kappa} K_1(\gamma_1, \gamma_3, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0).$$

The proposition follows from this observation and (D.8). ∎

Lemma D.3 computes moments of value weights conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$. We denote the covariance matrix of $(\hat{C}_t, y_t)$ by

$$\Sigma^{\hat{C}y} = \begin{pmatrix} \mathbb{V}\mathrm{ar}(\hat{C}_t) & \mathbb{C}\mathrm{ov}(\hat{C}_t, y_t) \\ \mathbb{C}\mathrm{ov}(\hat{C}_t, y_t) & \mathbb{V}\mathrm{ar}(y_t) \end{pmatrix}$$

$$= \begin{pmatrix} H(1,0,0,1,0,0,0,\nu_0) & H(1,0,0,0,0,1,0,\nu_0) \\ H(1,0,0,0,0,1,0,\nu_0) & H(0,0,1,0,0,1,0,\nu_0) \end{pmatrix}.$$

Lemma D.3. *For $t'' \geq t' \geq t$,*

$$E_{\mathcal{I}_t}\left(w_{t'}^V\right) = \frac{L_1}{r}\eta\Sigma + \left(\frac{L_2}{r} + \delta_{12,t'-t}^V(\hat{C}_t - \bar{C}) + \delta_{3,t'-t}^V(y_t - \bar{y})\right) p_f \Sigma, \tag{D.24}$$

$$E_{\mathcal{I}_t}(\Lambda_{t'}) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}}\left(L_2 + \delta_{12,t'-t}^{\Lambda}(\hat{C}_t - \bar{C}) + \delta_{3,t'-t}^{\Lambda}(y_t - \bar{y})\right), \tag{D.25}$$

$$\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(w_{t'}^V, \Lambda_{t''}\right) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \mathcal{C}_{t'-t,t''-t}^{V\Lambda} p_f \Sigma, \tag{D.26}$$

$$\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\Lambda_{t'}, w_{t''}^V\right) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \mathcal{C}_{t'-t,t''-t}^{\Lambda V} p_f \Sigma, \tag{D.27}$$

$$\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\left(w_{t'}^V\right)', w_{t''}^V\right) = \mathcal{C}_{t''-t'}^{V\Sigma}\Sigma + \mathcal{C}_{t'-t,t''-t}^{V}\Sigma p_f' p_f \Sigma, \tag{D.28}$$

$$\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\Lambda_{t'}, \Lambda_{t''}\right) = \mathcal{C}_{t'-t,t''-t}^{\Lambda}, \tag{D.29}$$

where

$$\begin{pmatrix} \delta^V_{12,t'-t} \\ \delta^V_{3,t'-t} \end{pmatrix} \equiv \left(\Sigma^{\hat{C}y}\right)^{-1} \begin{pmatrix} -\frac{(1-\epsilon)K_1(1,0,t'-t,\nu_0)}{r+\kappa} + H(1,0,0,\gamma_1,\gamma_2,\gamma_3,t'-t,\nu_0) \\ -\frac{(1-\epsilon)K_1(0,1,t'-t,\nu_0)}{r+\kappa} + H(0,0,1,\gamma_1,\gamma_2,\gamma_3,t'-t,\nu_0) \end{pmatrix}, \tag{D.30}$$

$$\begin{pmatrix} \delta^\Lambda_{12,t'-t} \\ \delta^\Lambda_{3,t'-t} \end{pmatrix} \equiv \left(\Sigma^{\hat{C}y}\right)^{-1} \begin{pmatrix} H(1,0,0,\gamma^R_1,\gamma^R_2,\gamma^R_3,t'-t,\nu_0) \\ H(0,0,1,\gamma^R_1,\gamma^R_2,\gamma^R_3,t'-t,\nu_0) \end{pmatrix}, \tag{D.31}$$

$$\mathcal{C}^{V\Lambda}_{t'-t,t''-t} \equiv -\frac{(1-\epsilon)K_2(\gamma^R_1,\gamma^R_3,t''-t',\nu_0)}{r+\kappa} + H(\gamma_1,\gamma_2,\gamma_3,\gamma^R_1,\gamma^R_2,\gamma^R_3,t''-t',\nu_0)$$

$$- (\delta^V_{12,t'-t},\delta^V_{3,t'-t})\Sigma^{\hat{C}y} \begin{pmatrix} \delta^\Lambda_{12,t''-t} \\ \delta^\Lambda_{3,t''-t} \end{pmatrix},$$

$$\mathcal{C}^{\Lambda V}_{t'-t,t''-t} \equiv -\frac{(1-\epsilon)K_1(\gamma^R_1,\gamma^R_3,0,\nu_0)}{r+\kappa} + H(\gamma^R_1,\gamma^R_2,\gamma^R_3,\gamma_1,\gamma_2,\gamma_3,0,\nu_0)$$

$$- (\delta^\Lambda_{12,t'-t},\delta^\Lambda_{3,t'-t})\Sigma^{\hat{C}y} \begin{pmatrix} \delta^V_{12,t''-t} \\ \delta^V_{3,t''-t} \end{pmatrix},$$

$$\mathcal{C}^{V\Sigma}_{t''-t'} \equiv \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2\kappa}\nu_0(\kappa,t''-t')$$

$$\mathcal{C}^{V}_{t'-t,t''-t} \equiv -\frac{1-\epsilon}{r+\kappa}\left[K_1(\gamma_1,\gamma_3,t''-t',\nu_0) + K_2(\gamma_1,\gamma_3,t''-t',\nu_0)\right]$$

$$+ H(\gamma_1,\gamma_2,\gamma_3,\gamma_1,\gamma_2,\gamma_3,t''-t',\nu_0) - (\delta^V_{12,t'-t},\delta^V_{3,t'-t})\Sigma^{\hat{C}y} \begin{pmatrix} \delta^V_{12,t''-t} \\ \delta^V_{3,t''-t} \end{pmatrix},$$

$$\mathcal{C}^{\Lambda}_{t'-t,t''-t} \equiv H(\gamma^R_1,\gamma^R_2,\gamma^R_3,\gamma^R_1,\gamma^R_2,\gamma^R_3,t''-t',\nu_0) - (\delta^\Lambda_{12,t'-t},\delta^\Lambda_{3,t'-t})\Sigma^{\hat{C}y} \begin{pmatrix} \delta^\Lambda_{12,t''-t} \\ \delta^\Lambda_{3,t''-t} \end{pmatrix}.$$

Proof: Using the joint normality of $\left(\left(w^V_{t'}\right)', \Lambda_{t'}, \hat{C}_t, y_t\right)$, (D.8) and (D.20), we can set

$$w^V_{t'} - \left(\frac{L_1}{r}\eta\Sigma + \frac{L_2}{r}p_f\Sigma\right) = \Delta^V_{12,t'-t}(\hat{C}_t - \bar{C}) + \Delta^V_{3,t'-t}(y_t - \bar{y}) + \zeta^V_{t'}, \tag{D.32}$$

$$\Lambda_{t'} - \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}}L_2 = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}}\left[\delta^\Lambda_{12,t'-t}(\hat{C}_t - \bar{C}) + \delta^\Lambda_{3,t'-t}(y_t - \bar{y}) + \zeta^\Lambda_{t'}\right], \tag{D.33}$$

where the error terms $\left(\zeta^V_{t'}, \zeta^\Lambda_{t'}\right)$ have mean zero and are independent of $(\hat{C}_t, y_t)$.

Taking covariances of both sides of (D.32) with $\hat{C}_t$ and y_t, and using (B.13), (B.14), (D.19) and the independence of $\zeta_{t'}^V$ from $(\hat{C}_t, y_t)$, we find

$$\left(-\frac{(1-\epsilon)K_1(1,0,t'-t,\nu_0)}{r+\kappa} + H(1,0,0,\gamma_1,\gamma_2,\gamma_3,t'-t,\nu_0)\right) p_f\Sigma = \Delta^V_{12,t'-t}\Sigma^{\hat{C}y}_{11} + \Delta^V_{3,t'-t}\Sigma^{\hat{C}y}_{12}, \tag{D.34}$$

$$\left(-\frac{(1-\epsilon)K_1(0,1,t'-t,\nu_0)}{r+\kappa} + H(0,0,1,\gamma_1,\gamma_2,\gamma_3,t'-t,\nu_0)\right) p_f\Sigma = \Delta^V_{12,t'-t}\Sigma^{\hat{C}y}_{21} + \Delta^V_{3,t'-t}\Sigma^{\hat{C}y}_{22}, \tag{D.35}$$

respectively. Equations (D.34) and (D.35) imply $\Delta^V_{12,t'-t} = \delta^V_{12,t'-t}p_f\Sigma$ and $\Delta^V_{3,t'-t} = \delta^V_{3,t'-t}p_f\Sigma$ for two scalars $(\delta^V_{12,t'-t}, \delta^V_{3,t'-t})$. Writing (D.34) and (D.35) in terms of $(\delta^V_{12,t'-t}, \delta^V_{3,t'-t})$, and solving for $(\delta^V_{12,t'-t}, \delta^V_{3,t'-t})$, we find (D.30). Equation (D.24) follows from (D.32) because $\zeta^V_{t'}$ has mean zero and is independent of $(\hat{C}_t, y_t)$.

Taking covariances of both sides of (D.33) with $\hat{C}_t$ and y_t, and using (3.6), (B.13), (B.14) and the independence of $\zeta^\Lambda_{t'}$ from $(\hat{C}_t, y_t)$, we find

$$H(1,0,0,\gamma_1^R,\gamma_2^R,\gamma_3^R,t'-t,\nu_0) = \delta^\Lambda_{12,t'-t}\Sigma^{\hat{C}y}_{11} + \delta^\Lambda_{3,t'-t}\Sigma^{\hat{C}y}_{12}, \tag{D.36}$$

$$H(0,0,1,\gamma_1^R,\gamma_2^R,\gamma_3^R,t'-t,\nu_0) = \delta^\Lambda_{12,t'-t}\Sigma^{\hat{C}y}_{21} + \delta^\Lambda_{3,t'-t}\Sigma^{\hat{C}y}_{22}, \tag{D.37}$$

respectively. Solving (D.36) and (D.37) for $(\delta^\Lambda_{12,t'-t}, \delta^\Lambda_{3,t'-t})$, we find (D.31). Equation (D.25) follows from (D.33) because $\zeta^\Lambda_{t'}$ has mean zero and is independent of $(\hat{C}_t, y_t)$.

Writing that the covariance between the left-hand side of (D.32) evaluated at t' and the left-hand side of (D.33) evaluated at t'' is equal to the covariance between the corresponding right-hand sides, and using (3.6), (B.13), (B.14), (D.19), $\Delta^V_{12,t'-t} = \delta^V_{12,t'-t}p_f\Sigma$, $\Delta^V_{3,t'-t} = \delta^V_{3,t'-t}p_f\Sigma$ and the independence of $(\zeta^V_{t'}, \zeta^\Lambda_{t''})$ from $(\hat{C}_t, y_t)$, we find

$$\begin{aligned}&\left(-\frac{(1-\epsilon)K_2(\gamma_1^R,\gamma_3^R,t''-t',\nu_0)}{r+\kappa} + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,t''-t',\nu_0)\right) p_f\Sigma \\ &= (\delta^V_{12,t'-t}, \delta^V_{3,t'-t}) \begin{pmatrix} \Sigma^{\hat{C}y}_{11} \\ \Sigma^{\hat{C}y}_{21} \end{pmatrix} \begin{pmatrix} \delta^\Lambda_{12,t''-t} \\ \delta^\Lambda_{3,t''-t} \end{pmatrix} p_f\Sigma + \mathbb{C}\mathrm{ov}(\zeta^V_{t'}, \zeta^\Lambda_{t''}).\end{aligned} \tag{D.38}$$

Equation (D.26) follows from (D.38) by noting that $\frac{1}{f+\frac{k\Delta}{\eta\Sigma\eta'}}\mathbb{C}\mathrm{ov}(\zeta^V_{t'}, \zeta^\Lambda_{t''}) = \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(w^V_{t'}, \Lambda_{t''}\right)$.

Writing that the covariance between the left-hand side of (D.33) evaluated at t' and the left-hand side of (D.32) evaluated at t'' is equal to the covariance between the corresponding right-hand sides, and using (3.6), (B.13), (B.14), (D.19), $\Delta^V_{12,t'-t} = \delta^V_{12,t'-t} p_f \Sigma$, $\Delta^V_{3,t'-t} = \delta^V_{3,t'-t} p_f \Sigma$ and the independence of $(\zeta^\Lambda_{t'}, \zeta^V_{t''})$ from $(\hat{C}_t, y_t)$, we find

$$\left(-\frac{(1-\epsilon)K_1(\gamma_1^R, \gamma_3^R, t''-t', \nu_0)}{r+\kappa} + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, t''-t', \nu_0)\right) p_f \Sigma$$
$$= (\delta^\Lambda_{12,t'-t}, \delta^\Lambda_{3,t'-t}) \begin{pmatrix} \Sigma^{\hat{C}y}_{11} \\ \Sigma^{\hat{C}y}_{21} \end{pmatrix} \begin{pmatrix} \delta^V_{12,t''-t} \\ \delta^V_{3,t''-t} \end{pmatrix} p_f \Sigma + \mathbb{C}\mathrm{ov}(\zeta^\Lambda_{t'}, \zeta^V_{t''}). \tag{D.39}$$

Equation (D.27) follows from (D.39) by noting that $\frac{1}{f+\frac{k\Delta}{\eta\Sigma\eta'}}\mathbb{C}\mathrm{ov}(\zeta^\Lambda_{t'}, \zeta^V_{t''}) = \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\Lambda_{t'}, w^V_{t''}\right)$.

Writing that the covariance between the left-hand side of (D.32) evaluated at t'' and the transpose of the left-hand side of (D.32) evaluated at t' is equal to the covariance between the corresponding right-hand sides, and using (B.13)-(B.16), (D.19), $\Delta^V_{12,t'-t} = \delta^V_{12,t'-t} p_f \Sigma$, $\Delta^V_{3,t'-t} = \delta^V_{3,t'-t} p_f \Sigma$ and the independence of $(\zeta^V_{t'}, \zeta^V_{t''})$ from $(\hat{C}_t, y_t)$, we find

$$\frac{(1-\epsilon)^2\phi^2}{2\kappa}\nu_0(\kappa, t''-t')\Sigma + \left(-\frac{1-\epsilon}{r+\kappa}\left[K_1(\gamma_1, \gamma_3, t''-t', \nu_0) + K_2(\gamma_1, \gamma_3, t''-t', \nu_0)\right]\right.$$
$$\left. + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, t''-t', \nu_0)\right) \Sigma p_f' p_f \Sigma$$
$$= (\delta^V_{12,t'-t}, \delta^V_{3,t'-t}) \begin{pmatrix} \Sigma^{\hat{C}y}_{11} \\ \Sigma^{\hat{C}y}_{21} \end{pmatrix} \begin{pmatrix} \delta^V_{12,t''-t} \\ \delta^V_{3,t''-t} \end{pmatrix} \Sigma p_f' p_f \Sigma + \mathbb{C}\mathrm{ov}\left(\left(\zeta^V_{t'}\right)', \zeta^V_{t''}\right). \tag{D.40}$$

Equation (D.26) follows from (D.38) by noting that $\mathbb{C}\mathrm{ov}\left(\left(\zeta^V_{t'}\right)', \zeta^V_{t''}\right) = \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\left(w^V_{t'}\right)', w^V_{t''}\right)$.

Writing that the covariance of the left-hand side of (D.33) evaluated at t'' and the left-hand side of (D.33) evaluated at t' is equal to the covariance between the corresponding right-hand sides, and using (3.6), (B.13) and the independence of $(\zeta^\Lambda_{t'}, \zeta^\Lambda_{t''})$ from $(\hat{C}_t, y_t)$, we find

$$H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, t''-t', \nu_0) = (\delta^\Lambda_{12,t'-t}, \delta^\Lambda_{3,t'-t}) \begin{pmatrix} \Sigma^{\hat{C}y}_{11} \\ \Sigma^{\hat{C}y}_{21} \end{pmatrix} \begin{pmatrix} \delta^\Lambda_{12,t''-t} \\ \delta^\Lambda_{3,t''-t} \end{pmatrix} + \mathbb{C}\mathrm{ov}(\zeta^\Lambda_{t'}, \zeta^\Lambda_{t''}). \tag{D.41}$$

Equation (D.29) follows from (D.41) by noting that $\mathbb{C}\text{ov}(\zeta^{\Lambda}_{t'}, \zeta^{\Lambda}_{t''}) = \mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_{t'}, \Lambda_{t''})$. ∎

Proposition D.3 computes the Sharpe ratio of the value strategy conditional on $(\hat{C}_t, y_t)$.

Proposition D.3. *The Sharpe ratio of the value strategy (4.1) conditional on* $(\hat{C}_t, y_t)$ *is* $SR_{w^V,t} = \frac{\mathcal{N}_{w^V,t}}{\sqrt{\mathcal{D}_{w^V,t}}}$*, where*

$$\mathcal{N}_{w^V,t} = \left[L_2 + \delta^{\Lambda}_{12,0}(\hat{C}_t - \bar{C}) + \delta^{\Lambda}_{3,0}(y_t - \bar{y})\right]$$

$$\times \left[\frac{L_1}{r}\eta\Sigma^2 p'_f + \left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right) p_f\Sigma^2 p'_f\right] + \mathcal{C}^{VL}_{0,0} p_f \Sigma^2 p'_f,$$

$$\mathcal{D}_{w^V,t} = \frac{L_1^2}{r^2}\Delta_1 + 2\frac{L_1}{r}\left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)\Delta_2$$

$$+ \left[\left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)^2 + \mathcal{C}^V_{0,0}\right]\Delta_3 + \mathcal{C}^{V\Sigma}_0\Delta_4.$$

Proof: Lemma D.3 implies that the Sharpe ratio of the value strategy conditional on $(\hat{C}_t, y_t)$ can be deduced from Lemma D.2 by setting $\mathcal{I}_t = \{\hat{C}_t, y_t\}$,

$$\Phi_{1t} = \frac{L_1}{r},$$

$$\Phi_{2t} = \frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y}),$$

$$\Phi^{\Lambda}_t = \mathcal{C}^{V\Lambda}_{0,0},$$

$$\hat{\Phi}^{\Sigma}_t = \mathcal{C}^{V\Sigma}_0,$$

$$\hat{\Phi}_t = \mathcal{C}^V_{0,0}.$$

The proposition follows from this observation and (D.25). ∎

We next compute the value spread. We define the value spread as the standard deviation of the market-to-book ratio in the cross-section of assets,

$$VS_t = \sqrt{\frac{\sum_{n=1}^N \left(\frac{S_{nt}}{B_{nt}} - \frac{\sum_{n'=1}^N \frac{S_{n't}}{B_{n't}}}{N}\right)^2}{N}}, \tag{D.42}$$

and assume that all assets have the same book value, which we take to be the average price in the cross-section of assets and over time,

$$B_{nt} = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}(S_{nt}) \equiv B. \tag{D.43}$$

Proposition D.4 computes the variance of the market-to-book ratio conditional on $(\hat{C}_t, y_t)$ in the cross-section of symmetric assets. We take the square root of that quantity as our measure of the value spread conditional on $(\hat{C}_t, y_t)$.

Proposition D.4. *Suppose $\eta = \mathbf{1}'$, $\bar{F} = \mathcal{F}\mathbf{1}$ and $\Sigma = \hat{\sigma}^2(I + \omega\mathbf{1}\mathbf{1}')$. The value spread conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$ is*

$$\sqrt{\mathbb{E}_{\mathcal{I}_t} VS_t^2} = \frac{\sqrt{\left[\left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)^2 + \mathcal{C}^V_{0,0}\right]\hat{\sigma}^4\sigma(\theta)^2 + \mathcal{C}_0^{V\Sigma}\frac{(N-1)\hat{\sigma}^2}{N}}}{\frac{\mathcal{F}}{r} - \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\bar{\theta}\hat{\sigma}^2(1+\omega N)}, \tag{D.44}$$

where $(\delta^V_{12,0}, \delta^V_{3,0}, \mathcal{C}^V_{0,0}, \mathcal{C}_0^{V\Sigma})$ are derived in Lemma D.3 for $\epsilon = 0$.

Proof: Using $B_{nt} = B$, we can write (D.42) as

$$VS_t = \frac{1}{B}\sqrt{\frac{\sum_{n=1}^N\left(S_{nt} - \frac{\sum_{n'=1}^N S_{n't}}{N}\right)^2}{N}} = \frac{1}{B}\sqrt{\frac{\sum_{n=1}^N\left(w^V_{nt} - \frac{\sum_{n'=1}^N w^V_{n't}}{N}\right)^2}{N}}, \tag{D.45}$$

where the second step follows by using (4.1) and setting $\epsilon = 0$ and $\bar{F} = \mathcal{F}\mathbf{1}$. Equation (D.32) implies

$$\begin{aligned}
w^V_{nt} - \frac{\sum_{n'=1}^N w^V_{n't}}{N} &= \frac{L_1}{r}\left((\eta\Sigma)_n - \frac{\sum_{n'=1}^N(\eta\Sigma)_{n'}}{N}\right) \\
&+ \left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)\left((p_f\Sigma)_n - \frac{\sum_{n'=1}^N(p_f\Sigma)_{n'}}{N}\right) + \left(\zeta^V_t\right)_n - \frac{\sum_{n'=1}^N\left(\zeta^V_t\right)_{n'}}{N} \\
&= \left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)(p_f\Sigma)_n + \left(\zeta^V_t\right)_n - \frac{\sum_{n'=1}^N\left(\zeta^V_t\right)_{n'}}{N},
\end{aligned} \tag{D.46}$$

where the second step follows from $\eta = \mathbf{1}'$ and $\Sigma = \hat{\sigma}^2(I + \omega\mathbf{1}\mathbf{1}')$. Squaring both sides of (D.46), taking expectations conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$, and denoting by e_n the $N \times 1$ vector with n'th element equal to one and all other elements equal to zero, we find

$$\begin{aligned}
&\mathbb{E}_{\mathcal{I}_t}\left(w^V_{nt} - \frac{\sum_{n'=1}^N w^V_{n't}}{N}\right)^2 \\
&= \left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)^2 \left((p_f\Sigma)_n\right)^2 + \mathbb{E}_{\mathcal{I}_t}\left(\left(\zeta^V_t\right)_n - \frac{\sum_{n'=1}^N \left(\zeta^V_t\right)_{n'}}{N}\right)^2 \\
&= \left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y})\right)^2 \left((p_f\Sigma)_n\right)^2 \\
&\quad + \left(e_n - \frac{1}{N}\mathbf{1}\right)' \left(\mathcal{C}^{V\Sigma}_0\Sigma + \mathcal{C}^V_{0,0}\Sigma p_f' p_f \Sigma\right)\left(e_n - \frac{1}{N}\mathbf{1}\right), \\
&= \left[\left(\frac{L_2}{r} + \delta^V_{12,t'-t}(\hat{C}_t - \bar{C}) + \delta^V_{3,t'-t}(y_t - \bar{y})\right)^2 \left((p_f\Sigma)_n\right)^2 + \mathcal{C}^V_{0,0}\right]\left((p_f\Sigma)_n\right)^2 + \mathcal{C}^{V\Sigma}_0 \frac{(N-1)\hat{\sigma}^2}{N},
\end{aligned} \tag{D.47}$$

where the second step follows from (D.28) and the third step follows from

$$\left(e_n - \frac{1}{N}\mathbf{1}\right)' \Sigma p_f' p_f \Sigma \left(e_n - \frac{1}{N}\mathbf{1}\right) = \left(p_f\Sigma\left(e_n - \frac{1}{N}\eta'\right)\right)^2 = \left(p_f\Sigma e_n\right)^2 = \left((p_f\Sigma)_n\right)^2$$

and

$$\begin{aligned}
\left(e_n - \frac{1}{N}\mathbf{1}\right)' \Sigma \left(e_n - \frac{1}{N}\mathbf{1}\right) &= \left(e_n - \frac{1}{N}\mathbf{1}\right)' \hat{\sigma}^2(I + \omega\mathbf{1}\mathbf{1}')\left(e_n - \frac{1}{N}\mathbf{1}\right) \\
&= \left(e_n - \frac{1}{N}\mathbf{1}\right)' \hat{\sigma}^2 I \left(e_n - \frac{1}{N}\mathbf{1}\right)' \\
&= \hat{\sigma}^2\left[\left(\frac{N-1}{N}\right)^2 + (N-1)\left(\frac{1}{N}\right)^2\right] = \hat{\sigma}^2\frac{N-1}{N}.
\end{aligned}$$

Summing (D.47) across assets and using (C.30), we find the term inside the square root in (D.44). To compute B, we note that since $\eta = \mathbf{1}'$, $B = \frac{\eta\mathbb{E}(S_t)}{N} = \frac{\mathbb{E}(\eta S_t)}{N}$. Using the expression for $\mathbb{E}(\eta S_t)$ in the denominator of (C.21), together with $\eta = \mathbf{1}'$, (C.26) and (C.34), we find the denominator of (D.44). ∎

Proposition D.5 computes the unconditional Sharpe ratio of the momentum strategy.

Proposition D.5. *The unconditional Sharpe ratio of the momentum strategy (4.2) is* $SR_{w^M} =$

$\frac{\mathcal{N}_{w^M}}{\sqrt{\mathcal{D}_{w^M}}}$ *where*

$$\mathcal{N}_{w^M} = L_1 L_2 \tau \eta \Sigma^2 p_f' + \left[L_2^2 \tau + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_2)\right] p_f \Sigma^2 p_f',$$

$$\mathcal{D}_{w^M} = L_1^2 \tau^2 \Delta_1 + 2 L_1 L_2 \tau^2 \Delta_2$$
$$+ \left[L_2^2 \tau^2 + 2H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_4) + 2G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_4) + k\tau\right] \Delta_3 + f\tau\Delta_4,$$

and $\mathcal{T} = (0, \tau)$.

Proof: Substituting (3.3) into (4.1), we can write the momentum weights as

$$w_t^M = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}} \frac{\eta\Sigma\theta'}{\eta\Sigma\eta'} \tau\eta\Sigma + \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \left(\int_{t-\tau}^{t} \Lambda_u du\right) p_f \Sigma + \int_{t-\tau}^{t} [dR_u - E_u(dR_u)]' \tag{D.48}$$

Taking unconditional expectations in (D.48), we find

$$\mathbb{E}\left(w_t^M\right) = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}} \frac{\eta\Sigma\theta'}{\eta\Sigma\eta'} \tau\eta\Sigma + \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \left(\int_{t-\tau}^{t} \mathbb{E}\left(\Lambda_u\right) du\right) p_f \Sigma$$
$$= L_1 \tau \eta \Sigma + L_2 \tau p_f \Sigma, \tag{D.49}$$

where the second step follows from (B.5) and (D.8). Taking the unconditional covariance of (D.48) with (3.6), we find

$$\mathbb{C}\text{ov}\left(\Lambda_t, w_t^M\right) = \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \left(\int_{t-\tau}^{t} \mathbb{C}\text{ov}\left(\Lambda_t, \Lambda_u\right) du\right) p_f \Sigma + \int_{t-\tau}^{t} \mathbb{C}\text{ov}\left(\Lambda_t, [dR_u - E_u(dR_u)]'\right). \tag{D.50}$$

To compute the second term in (D.50), we note that for a random variable X_t that depends on information up to time t

$$\mathbb{C}\text{ov}\left(X_t, dR_u - E_u(dR_u)\right) = \mathbb{E}\left(X_t[dR_u - E_u(dR_u)]\right)$$
$$= \mathbb{E}\left[\mathbb{E}_u\left(X_t[dR_u - E_u(dR_u)]\right)\right]$$
$$= \mathbb{E}\left[\mathbb{C}\text{ov}_u\left(X_t, dR_u\right)\right]. \tag{D.51}$$

Using (3.6), (B.10), (B.13) and (D.51), we can write (D.50) as

$$\mathbb{C}\text{ov}\left(\Lambda_t, w_t^M\right) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \left[H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_2)\right] p_f \Sigma. \quad \text{(D.52)}$$

Taking the unconditional covariance of (D.48) with the transpose of (D.48), we find

$$\begin{aligned}
\mathbb{C}\text{ov}\left(\left(w_t^M\right)', w_t^M\right) &= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \left(\int_{u'=t-\tau}^{t} \int_{u=t-\tau}^{t} \mathbb{C}\text{ov}\left(\Lambda_u, \Lambda_{u'}\right) du du'\right) \Sigma p_f' p_f \Sigma \\
&+ \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \Sigma p_f' \left(\int_{u'=t-\tau}^{t} \int_{u=t-\tau}^{t} \mathbb{C}\text{ov}\left(\Lambda_u, [dR_{u'} - E_{u'}(dR_{u'})]'\right) du\right) \\
&+ \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \left(\int_{u'=t-\tau}^{t} \int_{u=t-\tau}^{t} \mathbb{C}\text{ov}\left(\Lambda_{u'}, dR_u - E_u(dR_u)\right) du'\right) p_f \Sigma \\
&+ \int_{t-\tau}^{t} \mathbb{C}\text{ov}\left(dR_u - E_u(dR_u), [dR_u - E_u(dR_u)]'\right). \qquad \text{(D.53)}
\end{aligned}$$

To compute the second and third terms in (D.53), we note that the covariance in (D.51) is zero for $t < u$. To compute the fourth term in (D.53), we note that it is equal to $\mathbb{E}[\mathbb{C}\text{ov}_u(dR_u, dR_u')]$. We can thus write (D.53) as

$$\begin{aligned}
\mathbb{C}\text{ov}\left(\left(w_t^M\right)', w_t^M\right) &= 2\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \left(\int_{u'=t-\tau}^{t} \int_{u=u'-\tau}^{t} \mathbb{C}\text{ov}\left(\Lambda_u, \Lambda_{u'}\right) du du'\right) \Sigma p_f' p_f \Sigma \\
&+ \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \Sigma p_f' \left(\int_{u'=t-\tau}^{t} \int_{u=u'}^{t} \mathbb{C}\text{ov}\left(\Lambda_u, [dR_{u'} - E_{u'}(dR_{u'})]'\right) du'\right) \\
&+ \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \left(\int_{u=t-\tau}^{t} \int_{u'=u}^{t} \mathbb{C}\text{ov}\left(\Lambda_{u'}, dR_u - E_u(dR_u)\right) du\right) p_f \Sigma \\
&+ \int_{t-\tau}^{t} \mathbb{E}[\mathbb{C}\text{ov}_u(dR_u, dR_u')]. \qquad \text{(D.54)}
\end{aligned}$$

Using (A.2), (3.6), (B.10), (B.13) and (D.54), we find

$$\begin{aligned}
\mathbb{C}\text{ov}\left(\left(w_t^M\right)', w_t^M\right) &= \left[2H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_4) + 2G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_4)\right] \Sigma p_f' p_f \Sigma \\
&+ \tau\left(f\Sigma + k\Sigma p_f' p_f \Sigma\right). \qquad \text{(D.55)}
\end{aligned}$$

Equations (D.49), (D.52) and (D.55) imply that the unconditional Sharpe ratio of the momentum strategy can be deduced from Lemma D.2 by setting $\mathcal{I}_t = \emptyset$,

$$\Phi_{1t} = L_1\tau,$$

$$\Phi_{2t} = L_2\tau,$$

$$\Phi_t^{\Lambda} = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_2),$$

$$\hat{\Phi}_t^{\Sigma} = f\tau,$$

$$\hat{\Phi}_t = 2H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_4) + 2G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_4) + k\tau.$$

The proposition follows from this observation and (D.8). ∎

Lemma D.4 computes moments of momentum weights conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$.

Lemma D.4. *For $t'' \geq t' \geq t$,*

$$E_{\mathcal{I}_t}\left(w_{t'}^M\right) = L_1\tau\eta\Sigma + \left(L_2\tau + \delta_{12,t'-t}^M(\hat{C}_t - \bar{C}) + \delta_{3,t'-t}^M(y_t - \bar{y})\right) p_f\Sigma, \tag{D.56}$$

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(w_{t'}^M, \Lambda_{t''}\right) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \mathcal{C}_{t'-t,t''-t}^{M\Lambda} p_f\Sigma, \tag{D.57}$$

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\Lambda_{t'}, w_{t''}^M\right) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \mathcal{C}_{t'-t,t''-t}^{\Lambda M} p_f\Sigma, \tag{D.58}$$

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\left(w_{t'}^M\right)', w_{t''}^M\right) = \mathcal{C}_{t''-t'}^{M\Sigma}\Sigma + \mathcal{C}_{t'-t,t''-t}^{M}\Sigma p_f' p_f\Sigma, \tag{D.59}$$

where

$$\begin{pmatrix} \delta_{12,t'-t}^M \\ \delta_{3,t'-t}^M \end{pmatrix} \equiv \left(\Sigma^{\hat{C}y}\right)^{-1} \begin{pmatrix} H(1,0,0,\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_1) \\ +H(\gamma_1^R, \gamma_2^R, \gamma_3^R, 1, 0, 0, \mathcal{T}, \nu_2) + G(1,0,0,\mathcal{T}, \nu_2) \\ H(0,0,1,\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_1) \\ +H(\gamma_1^R, \gamma_2^R, \gamma_3^R, 0,0,1, \mathcal{T}, \nu_2) + G(0,0,1,\mathcal{T}, \nu_2) \end{pmatrix}, \tag{D.60}$$

$$\begin{aligned} \mathcal{C}_{t'-t,t''-t}^{M\Lambda} \equiv{} & H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^{-}, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^{-}, \nu_2) \\ & - (\delta_{12,t'-t}^M, \delta_{3,t'-t}^M)\Sigma^{\hat{C}y} \begin{pmatrix} \delta_{12,t''-t}^{\Lambda} \\ \delta_{3,t''-t}^{\Lambda} \end{pmatrix}, \end{aligned}$$

$$\mathcal{C}_{t'-t,t''-t}^{\Lambda M} \equiv H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2)$$

$$+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2) - (\delta^{\Lambda}_{12,t'-t}, \delta^{\Lambda}_{3,t'-t}) \Sigma^{\hat{C}y} \begin{pmatrix} \delta^M_{12,t''-t} \\ \delta^M_{3,t''-t} \end{pmatrix},$$

$$\mathcal{C}^{M\Sigma}_{t''-t'} \equiv f \max\{\tau + t' - t'', 0\}, \tag{D.61}$$

$$\mathcal{C}^M_{t'-t,t''-t} \equiv H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4)$$

$$+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4)$$

$$+ k \max\{\tau + t' - t'', 0\} - (\delta^M_{12,t'-t}, \delta^M_{3,t'-t}) \Sigma^{\hat{C}y} \begin{pmatrix} \delta^M_{12,t''-t} \\ \delta^M_{3,t''-t} \end{pmatrix},$$

$\mathcal{T} \equiv (t' - t, \tau)$, $\mathcal{T}' \equiv (t'' - t', \tau)$ *and* $\mathcal{T}'^{-} \equiv (t' - t'', \tau)$.

Proof: Using the joint normality of $\left((w^M_{t'})', \hat{C}_t, y_t \right)$ and (D.49), we can set

$$w^M_{t'} - (L_1 \tau \eta \Sigma + L_2 \tau p_f \Sigma) = \Delta^M_{12,t'-t}(\hat{C}_t - \bar{C}) + \Delta^M_{3,t'-t}(y_t - \bar{y}) + \zeta^M_{t'}, \tag{D.62}$$

where the error term $\zeta^M_{t'}$ has mean zero and is independent of $(\hat{C}_t, y_t)$.

Taking covariances of both sides of (D.62) with $\hat{C}_t$ and y_t, and using (D.48) and the independence of $\zeta^M_{t'}$ from $(\hat{C}_t, y_t)$, we find

$$\left(f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left[\int_{t'-\tau}^{t'} \mathbb{C}\text{ov} \left(\hat{C}_t, \Lambda_u \right) du \right] p_f \Sigma + \int_{t'-\tau}^{t'} \mathbb{C}\text{ov} \left(\hat{C}_t, [dR_u - E_u(dR_u)]' \right)$$

$$= \Delta^M_{12,t'-t} \Sigma^{\hat{C}y}_{11} + \Delta^M_{3,t'-t} \Sigma^{\hat{C}y}_{12}, \tag{D.63}$$

$$\left(f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left[\int_{t'-\tau}^{t'} \mathbb{C}\text{ov} \left(y_t, \Lambda_u \right) du \right] p_f \Sigma + \int_{t'-\tau}^{t'} \mathbb{C}\text{ov} \left(y_t, [dR_u - E_u(dR_u)]' \right)$$

$$= \Delta^M_{12,t'-t} \Sigma^{\hat{C}y}_{21} + \Delta^M_{3,t'-t} \Sigma^{\hat{C}y}_{22}. \tag{D.64}$$

Noting that the covariances in the second term of (D.63) and (D.64) are zero for $t < u$, and using (3.6), (B.10), (B.13) and (D.51), we can write (D.63) and (D.64) as

$$\left[H(1,0,0,\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}, \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, 1,0,0, \mathcal{T}, \nu_2) + G(1,0,0, \mathcal{T}, \nu_2) \right] p_f \Sigma$$

$$= \Delta^M_{12,t'-t} \Sigma^{\hat{C}y}_{11} + \Delta^M_{3,t'-t} \Sigma^{\hat{C}y}_{12}, \tag{D.65}$$

$$\left[H(0,0,1,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T},\nu_1)+H(\gamma_1^R,\gamma_2^R,\gamma_3^R,0,0,1,\mathcal{T},\nu_2)+G(0,0,1,\mathcal{T},\nu_2)\right]p_f\Sigma$$
$$=\Delta^M_{12,t'-t}\Sigma^{\hat{C}y}_{21}+\Delta^M_{3,t'-t}\Sigma^{\hat{C}y}_{22}, \tag{D.66}$$

respectively. Equations (D.65) and (D.66) imply $\Delta^M_{12,t'-t}=\delta^M_{12,t'-t}p_f\Sigma$ and $\Delta^M_{3,t'-t}=\delta^M_{3,t'-t}p_f\Sigma$ for two scalars $(\delta^M_{12,t'-t},\delta^M_{3,t'-t})$. Writing (D.65) and (D.66) in terms of $(\delta^M_{12,t'-t},\delta^M_{3,t'-t})$, and solving for $(\delta^M_{12,t'-t},\delta^M_{3,t'-t})$, we find (D.60). Equation (D.56) follows from (D.62) because $\zeta^M_{t'}$ has mean zero and is independent of $(\hat{C}_t,y_t)$.

Writing that the covariance between the left-hand side of (D.62) evaluated at t' and the left-hand side of (D.33) evaluated at t'' is equal to the covariance between the corresponding right-hand sides, and using

$$\mathbb{C}\text{ov}\left(w^M_{t'},\Lambda_{t''}\right)=\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\left(\int_{t'-\tau}^{t'}\mathbb{C}\text{ov}\left(\Lambda_u,\Lambda_{t''}\right)du\right)p_f\Sigma+\int_{t'-\tau}^{t'}\mathbb{C}\text{ov}\left([dR_u-E_u(dR_u)]',\Lambda_{t''}\right) \tag{D.67}$$

which generalizes (D.50) from (Λ_t,w^M_t) to $(\Lambda_{t''},w^M_{t'})$, together with (3.6), (B.10), (B.13), (D.51), $\Delta^M_{12,t'-t}=\delta^M_{12,t'-t}p_f\Sigma$, $\Delta^M_{3,t'-t}=\delta^M_{3,t'-t}p_f\Sigma$ and the independence of $(\zeta^M_{t'},\zeta^\Lambda_{t''})$ from $(\hat{C}_t,y_t)$, we find

$$\left[H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)+G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)\right]p_f\Sigma$$
$$=(\delta^M_{12,t'-t},\delta^M_{3,t'-t})\begin{pmatrix}\Sigma^{\hat{C}y}_{11}\\ \Sigma^{\hat{C}y}_{21}\end{pmatrix}\begin{pmatrix}\delta^\Lambda_{12,t''-t}\\ \delta^\Lambda_{3,t''-t}\end{pmatrix}p_f\Sigma+\mathbb{C}\text{ov}(\zeta^M_{t'},\zeta^\Lambda_{t''}). \tag{D.68}$$

Equation (D.57) follows from (D.68) by noting that $\frac{1}{f+\frac{k\Delta}{\eta\Sigma\eta'}}\mathbb{C}\text{ov}(\zeta^M_{t'},\zeta^\Lambda_{t''})=\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(w^M_{t'},\Lambda_{t''}\right)$.

Writing that the covariance between the left-hand side of (D.33) evaluated at t' and the left-hand side of (D.62) evaluated at t'' is equal to the covariance between the corresponding right-hand sides, and using

$$\mathbb{C}\text{ov}\left(\Lambda_{t'},w^M_{t''}\right)=\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\left(\int_{t''-\tau}^{t''}\mathbb{C}\text{ov}\left(\Lambda_{t'},\Lambda_u\right)du\right)p_f\Sigma+\int_{t''-\tau}^{t''}\mathbb{C}\text{ov}\left(\Lambda_{t'},[dR_u-E_u(dR_u)]'\right) \tag{D.69}$$

which generalizes (D.50) from (Λ_t, w_t^M) to $(\Lambda_{t'}, w_{t''}^M)$, together with (3.6), (B.10), (B.13), (D.51), $\Delta^M_{12,t'-t} = \delta^M_{12,t'-t} p_f \Sigma$, $\Delta^M_{3,t'-t} = \delta^M_{3,t'-t} p_f \Sigma$ and the independence of $(\zeta^\Lambda_{t'}, \zeta^M_{t''})$ from $(\hat{C}_t, y_t)$, we find

$$\left[H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_2) + G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_2)\right] p_f \Sigma$$
$$= (\delta^M_{12,t'-t}, \delta^M_{3,t'-t}) \begin{pmatrix} \Sigma_{11}^{\hat{C}y} \\ \Sigma_{21}^{\hat{C}y} \end{pmatrix} \begin{pmatrix} \delta^\Lambda_{12,t''-t} \\ \delta^\Lambda_{3,t''-t} \end{pmatrix} p_f \Sigma + \mathbb{C}\mathrm{ov}(\zeta^\Lambda_{t'}, \zeta^M_{t''}). \tag{D.70}$$

Equation (D.26) follows from (D.38) by noting that $\frac{1}{f+\frac{k\Delta}{\eta\Sigma\eta'}} \mathbb{C}\mathrm{ov}(\zeta^\Lambda_{t'}, \zeta^M_{t''}) = \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\Lambda_{t'}, w^M_{t''}\right)$.

Writing that the covariance between the left-hand side of (D.62) evaluated at t'' and the transpose of the left-hand side of (D.62) evaluated at t' is equal to the covariance between the corresponding right-hand sides, and using

$$\begin{aligned}
\mathbb{C}\mathrm{ov}\left(\left(w^M_{t'}\right)', w^M_{t''}\right) = {} & \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \left(\int_{u''=t''-\tau}^{t''} \int_{u'=t'-\tau}^{t'} \mathbb{C}\mathrm{ov}\left(\Lambda_{u'}, \Lambda_{u''}\right) du' du''\right) \Sigma p_f' p_f \Sigma \\
& + \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \Sigma p_f' \left(\int_{u''=t''-\tau}^{t''} \int_{u'=t'-\tau}^{t'} \mathbb{C}\mathrm{ov}\left(\Lambda_{u'}, [dR_{u''} - E_{u''}(dR_{u''})]'\right) du'\right) \\
& + \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right) \left(\int_{u''=t''-\tau}^{t''} \int_{u'=t'-\tau}^{t'} \mathbb{C}\mathrm{ov}\left(dR_{u'} - E_{u'}(dR_{u'}), \Lambda_{u''}\right) du''\right) p_f \Sigma \\
& + 1_{\{\tau+t'-t''>0\}} \int_{t''-\tau}^{t'} \mathbb{C}\mathrm{ov}\left(dR_u - E_u(dR_u), [dR_u - E_u(dR_u)]'\right),
\end{aligned} \tag{D.71}$$

which generalizes (D.53) from (w_t^M, w_t^M) to $(w_{t'}^M, w_{t''}^M)$, together with (A.2), (3.6), (B.10), (B.13), (D.51), $\Delta^M_{12,t'-t} = \delta^M_{12,t'-t} p_f \Sigma$, $\Delta^M_{3,t'-t} = \delta^M_{3,t'-t} p_f \Sigma$ and the independence of $\zeta^M_{t'}$ from $(\hat{C}_t, y_t)$, we find

$$\begin{aligned}
& \left[H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_3) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_4)\right. \\
& \left.+ G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_3) + G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_4)\right] \Sigma p_f' p_f \Sigma \\
& + \max\{\tau + t' - t'', 0\} \left(f\Sigma + k\Sigma p_f' p_f \Sigma\right) \\
& = (\delta^M_{12,t'-t}, \delta^M_{3,t'-t}) \begin{pmatrix} \Sigma_{11}^{\hat{C}y} \\ \Sigma_{21}^{\hat{C}y} \end{pmatrix} \begin{pmatrix} \delta^M_{12,t''-t} \\ \delta^M_{3,t''-t} \end{pmatrix} \Sigma p_f' p_f \Sigma + \mathbb{C}\mathrm{ov}\left(\left(\zeta^M_{t'}\right)', \zeta^M_{t''}\right).
\end{aligned} \tag{D.72}$$

Equation (D.59) follows from (D.72) by noting that $\mathbb{C}\text{ov}\left(\left(\zeta_{t'}^M\right)', \zeta_{t''}^M\right) = \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\left(w_{t'}^M\right)', w_{t''}^M\right)$. ∎

Proposition D.6 computes the Sharpe ratio of the momentum strategy conditional on $(\hat{C}_t, y_t)$.

Proposition D.6. *The Sharpe ratio of the momentum strategy (4.2) conditional on* $(\hat{C}_t, y_t)$ *is* $SR_{w^M,t} = \frac{\mathcal{N}_{w^M,t}}{\sqrt{\mathcal{D}_{w^M,t}}}$, *where*

$$\mathcal{N}_{w^M,t} = \left[L_2 + \delta_{12,0}^{\Lambda}(\hat{C}_t - \bar{C}) + \delta_{3,0}^{\Lambda}(y_t - \bar{y})\right]$$

$$\times \left[L_1\tau\eta\Sigma^2 p_f' + \left(L_2\tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y})\right) p_f \Sigma^2 p_f'\right] + \mathcal{C}_{0,0}^{M\Lambda} p_f \Sigma^2 p_f',$$

$$\mathcal{D}_{w^M,t} = L_1^2\tau^2\Delta_1 + 2L_1\tau\left(L_2\tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y})\right)\Delta_2$$

$$+ \left[\left(L_2\tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y})\right)^2 + \mathcal{C}_{0,0}^M\right]\Delta_3 + \mathcal{C}_0^{M\Sigma}\Delta_4.$$

Proof: Lemma D.4 implies that the Sharpe ratio of the momentum strategy conditional on $(\hat{C}_t, y_t)$ can be deduced from Lemma D.2 by setting $\mathcal{I}_t = \{\hat{C}_t, y_t\}$,

$$\Phi_{1t} = L_1\tau,$$

$$\Phi_{2t} = L_2\tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y}),$$

$$\Phi_t^{\Lambda} = \mathcal{C}_{0,0}^{M\Lambda},$$

$$\hat{\Phi}_t^{\Sigma} = \mathcal{C}_0^{M\Sigma},$$

$$\hat{\Phi}_t = \mathcal{C}_{0,0}^M.$$

The proposition follows from this observation and (D.25). ∎

Lemma D.5 computes the Sharpe ratio of the optimal (mean-variance maximizing) combination of two strategies (w_t^A, w_t^B).

Lemma D.5. *The maximum Sharpe ratio of a combination of* (w_t^A, w_t^B) *is given by (5.6).*

Proof: Consider an investor at time t with infinitesimal horizon dt, who can invest in the riskless asset, the index η and the strategies (w_t^A, w_t^B). The investor's optimization problem is as in Lemma

C.1, except that the budget constraint (C.2) is replaced by

$$dW_t = rW_t dt + \hat{\hat{x}}_t \eta dR_t + \hat{y}_t^A \hat{w}_t^A dR_t^A + \hat{y}^M \hat{w}_t^B dR_t^B. \tag{D.73}$$

Substituting dW_t from (D.73), and noting that ηdR_t is uncorrelated with $(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t)$, we can write the investor's objective (C.1) as

$$\hat{\hat{x}}_t \mathbb{E}_{\mathcal{I}_t}(\eta dR_t) + \hat{y}_t^A \mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^A dR_t) + \hat{y}^M \mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^B dR_t) - \frac{a}{2}\Big(\hat{\hat{x}}_t^2 \mathbb{V}\text{ar}_{\mathcal{I}_t}(\eta dR_t)$$
$$+ \left(\hat{y}_t^A\right)^2 \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right) + \left(\hat{y}^M\right)^2 \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^B dR_t\right) + 2\hat{y}_t^A \hat{y}^M \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right)\Big). \tag{D.74}$$

Maximizing (D.74) over $(\hat{\hat{x}}_t, \hat{y}_t^A, \hat{y}^M)$ yields the utility

$$\frac{1}{2a}\left(SR_{\eta,t}^2 dt + \left(\mathbb{E}_{\mathcal{I}_t}^{AB}\right)' \left(\mathbb{C}\text{ov}_{\mathcal{I}_t}^{AB}\right)^{-1} \mathbb{E}_{\mathcal{I}_t}^{AB}\right), \tag{D.75}$$

where $\mathbb{E}_{\mathcal{I}_t}^{AB} \equiv \left(\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^A dR_t), \mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^B dR_t)\right)'$ and

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}^{AB} \equiv \begin{pmatrix} \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right) & \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right) \\ \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right) & \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^B dR_t\right) \end{pmatrix}.$$

Comparison of (C.5) and (D.75) yields

$$SR_{w^{AB},t} = \sqrt{\frac{1}{dt}\left(\mathbb{E}_{\mathcal{I}_t}^{AB}\right)' \left(\mathbb{C}\text{ov}_{\mathcal{I}_t}^{AB}\right)^{-1} \mathbb{E}_{\mathcal{I}_t}^{AB}}. \tag{D.76}$$

The term inside the square root in (D.76) has numerator

$$\left(\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^A dR_t)\right)^2 \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^B dR_t\right) + \left(\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^B dR_t)\right)^2 \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right)$$
$$- 2\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^A dR_t)\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t^B dR_t)\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right)$$

and denominator

$$\left[\mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right) \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^B dR_t\right) - \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right)^2\right] dt.$$

Dividing numerator and denominator by $\mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right) \mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\hat{w}_t^B dR_t\right) dt$, we can write the term inside the square root in (D.76) as the term inside the square root in (5.6). ∎

Lemma D.5 computes the covariance between the returns of (the index-adjusted versions of) two strategies (w_t^A, w_t^B).

Lemma D.6. *The covariance between the returns of (w_t^A, w_t^B) conditional on $\mathcal{I}_t$ is given by*

$$\begin{aligned}\mathcal{G}_{w^A,w^B,t} &\equiv \frac{1}{dt}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right)\\ &= f\left[\mathbb{E}_{\mathcal{I}_t}\left(w_t^A\Sigma\left(w_t^B\right)'\right) - \frac{\mathbb{E}_{\mathcal{I}_t}\left(w_t^A\Sigma\eta' w_t^B\Sigma\eta'\right)}{\eta\Sigma\eta'}\right] + k\mathbb{E}_{\mathcal{I}_t}\left(w_t^A\Sigma p_f' w_t^B\Sigma p_f'\right). \qquad \text{(D.77)}\end{aligned}$$

Suppose that for $i = A, B$

$$\mathbb{E}_{\mathcal{I}_t}\left(w_t^i\right) = \Phi_{1t}^i\eta\Sigma + \Phi_{2t}^i p_f\Sigma \qquad \text{(D.78)}$$

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\left(w_t^A\right)', w_t^B\right) = \hat{\Phi}_t^{AB\Sigma}\Sigma + \hat{\Phi}_t^{AB}\Sigma p_f' p_f\Sigma. \qquad \text{(D.79)}$$

Then, the covariance between the returns of (w_t^A, w_t^B) conditional on $\mathcal{I}_t$ is given by

$$\mathcal{G}_{w^A,w^B,t} = \Phi_{1t}^A\Phi_{1t}^B\Delta_1 + \left(\Phi_{1t}^A\Phi_{2t}^B + \Phi_{1t}^B\Phi_{2t}^A\right)\Delta_2 + \left(\Phi_{2t}^A\Phi_{2t}^B + \hat{\Phi}_t^{AB}\right)\Delta_3 + \hat{\Phi}_t^{AB\Sigma}\Delta_4. \qquad \text{(D.80)}$$

Proof: The covariance between the returns of (w_t^A, w_t^B) conditional on $\mathcal{I}_t$ is

$$\begin{aligned}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right) &= \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t \hat{w}_t^B dR_t\right) - \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right)\mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t\right)\\ &= \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A dR_t \hat{w}_t^B dR_t\right)\\ &= \mathbb{E}_{\mathcal{I}_t}\mathbb{E}_t\left(\hat{w}_t^A dR_t \hat{w}_t^B dR_t\right)\\ &= \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A \mathbb{E}_t(dR_t dR_t')\left(\hat{w}_t^B\right)'\right)\\ &= \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A\left(\mathbb{C}\text{ov}_t(dR_t dR_t') + E_t(dR_t)E_t(dR_t)'\right)\left(\hat{w}_t^B\right)'\right)\\ &= \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A \mathbb{C}\text{ov}_t(dR_t dR_t')\left(\hat{w}_t^B\right)'\right)\\ &= \mathbb{E}_{\mathcal{I}_t}\left(\hat{w}_t^A\left(f\Sigma + k\Sigma p_f' p_f\Sigma\right)\left(\hat{w}_t^B\right)'\right)dt, \qquad \text{(D.81)}\end{aligned}$$

where the second and sixth steps follow because the term that is dropped is of order $(dt)^2$ while the term that is kept is of order dt, and the last step follows from (A.2).

Using (4.3), we can write (D.81) divided by dt as

$$\mathcal{G}_{w^A,w^B,t} = \mathbb{E}_{\mathcal{I}_t}\left[\left(w_t^A - \frac{\mathbb{C}\mathrm{ov}_t(w_t^A dR_t, \eta dR_t)}{\mathbb{V}\mathrm{ar}_t(\eta dR_t)}\eta\right)(f\Sigma + k\Sigma p_f' p_f \Sigma)\left(w_t^B - \frac{\mathbb{C}\mathrm{ov}_t(w_t^B dR_t, \eta dR_t)}{\mathbb{V}\mathrm{ar}_t(\eta dR_t)}\eta\right)'\right]$$

$$\mathbb{E}_{\mathcal{I}_t}\left[\left(w_t^A - \frac{w_t^A \Sigma\eta'}{\eta\Sigma\eta'}\eta\right)(f\Sigma + k\Sigma p_f' p_f \Sigma)\left(w_t^B - \frac{w_t^B \Sigma\eta'}{\eta\Sigma\eta'}\eta\right)'\right], \tag{D.82}$$

where the second step follows from (A.2) and $\eta\Sigma p_f' = 0$. Expanding the products in (D.82) and using $\eta\Sigma p_f' = 0$ yields (D.77).

We can write the right-hand side of (D.77) as

$$f\left[\mathbb{E}_{\mathcal{I}_t}\left(w_t^A\right)\Sigma\mathbb{E}_{\mathcal{I}_t}\left(\left(w_t^B\right)'\right) - \frac{\mathbb{E}_{\mathcal{I}_t}\left(w_t^A\right)\Sigma\eta'\mathbb{E}_{\mathcal{I}_t}\left(w_t^B\right)\Sigma\eta'}{\eta\Sigma\eta'}\right] + k\mathbb{E}_{\mathcal{I}_t}\left(w_t^A\right)\Sigma p_f'\mathbb{E}_{\mathcal{I}_t}\left(w_t^B\right)\Sigma p_f'$$

$$+ f\left[\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(w_t^A, \Sigma\left(w_t^B\right)'\right) - \frac{\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(w_t^A\Sigma\eta', w_t^B\Sigma\eta'\right)}{\eta\Sigma\eta'}\right] + k\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(w_t^A\Sigma p_f', w_t^B\Sigma p_f'\right). \tag{D.83}$$

Using (D.78), (D.79) and (D.83), and proceeding as in the derivation of (D.18), we can derive (D.80). ∎

Lemma D.7 computes covariances between the weights of value and momentum strategies conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$.

Lemma D.7. *For $t'' \geq t' \geq t$,*

$$\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\left(w_{t'}^V\right)', w_{t''}^M\right) = \mathcal{C}_{t''-t'}^{VM\Sigma}\Sigma + \mathcal{C}_{t'-t,t''-t}^{VM}\Sigma p_f' p_f \Sigma, \tag{D.84}$$

$$\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\left(w_{t'}^M\right)', w_{t''}^V\right) = \mathcal{C}_{t''-t'}^{MV\Sigma}\Sigma + \mathcal{C}_{t'-t,t''-t}^{MV}\Sigma p_f' p_f \Sigma, \tag{D.85}$$

where

$$\mathcal{C}_{t'-t,t''-t}^{VM} \equiv -\frac{1-\epsilon}{r+\kappa}[K_2(\gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_1) + K_1(\gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_2)] - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa, \mathcal{T}')$$

$$+ H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, \mathcal{T}', \nu_2)$$

$$+ G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}', \nu_2) - (\delta_{12,t'-t}^V, \delta_{3,t'-t}^V)\Sigma^{\hat{C}y}\begin{pmatrix}\delta_{12,t''-t}^M \\ \delta_{3,t''-t}^M\end{pmatrix},$$

$$\mathcal{C}^{VM\Sigma}_{t''-t'} \equiv -\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}'),$$

$$\mathcal{C}^{MV}_{t'-t,t''-t} \equiv -\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,\mathcal{T}'^{-},\nu_2) - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}'^{-})$$

$$+ H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^{-},\nu_2) + G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^{-},\nu_2)$$

$$-(\delta^M_{12,t'-t},\delta^M_{3,t'-t})\Sigma^{\hat{C}y}\begin{pmatrix}\delta^V_{12,t''-t}\\ \delta^V_{3,t''-t}\end{pmatrix},$$

$$\mathcal{C}^{MV\Sigma}_{t'-t,t''-t} \equiv -\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}'^{-}),$$

$\mathcal{T}' \equiv (t''-t',\tau)$, $\mathcal{T}'^{-} \equiv (t'-t'',\tau)$, and $(\delta^V_{12,u},\delta^V_{3,u})$ and $(\delta^M_{12,u},\delta^M_{3,u})$ are defined in Lemmas D.3 and D.4, respectively.

Proof: Writing that the covariance between the left-hand side of (D.62) evaluated at t'' and the transpose of the left-hand side of (D.32) evaluated at t' is equal to the covariance between the corresponding right-hand sides, and using (D.19), (D.48), $\Delta^V_{12,t''-t} = \delta^V_{12,t''-t}p_f\Sigma$, $\Delta^V_{3,t''-t} = \delta^V_{3,t''-t}p_f\Sigma$, $\Delta^M_{12,t'-t} = \delta^M_{12,t'-t}p_f\Sigma$, $\Delta^M_{3,t'-t} = \delta^M_{3,t'-t}p_f\Sigma$ and the independence of $(\zeta^V_{t'},\zeta^M_{t''})$ from $(\hat{C}_t,y_t)$, we find

$$\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\left(-\frac{1-\epsilon}{r+\kappa}\int_{u''=t''-\tau}^{t''}\mathbb{C}\mathrm{ov}\left(\Lambda_{u''},F_{t'}\right)du''p_f\Sigma \right. \tag{D.86}$$

$$\left. +\int_{u''=t''-\tau}^{t''}\mathbb{C}\mathrm{ov}\left(\Lambda_{u''},\gamma_1\hat{C}_{t'}+\gamma_2C_{t'}+\gamma_3y_{t'}\right)du''\Sigma p_f'p_f\Sigma\right)$$

$$+\left(-\frac{1-\epsilon}{r+\kappa}\int_{u''=t''-\tau}^{t''}\mathbb{C}\mathrm{ov}\left(F_{t'},[dR_{u''}-E_{u''}(dR_{u''})]'\right)du'\right.$$

$$\left.+\Sigma p_f'\int_{u''=t''-\tau}^{t''}\mathbb{C}\mathrm{ov}\left(\gamma_1\hat{C}_{t'}+\gamma_2C_{t'}+\gamma_3y_{t'},[dR_{u''}-E_{u''}(dR_{u''})]'\right)du''\right)$$

$$=(\delta^V_{12,t'-t},\delta^V_{3,t'-t})\begin{pmatrix}\Sigma^{\hat{C}y}_{11}\\ \Sigma^{\hat{C}y}_{21}\end{pmatrix}\begin{pmatrix}\delta^M_{12,t''-t}\\ \delta^M_{3,t''-t}\end{pmatrix}\Sigma p_f'p_f\Sigma+\left(\left(\zeta^V_{t'}\right)',\zeta^M_{t''}\right). \tag{D.87}$$

Using (3.6), (B.10), (B.11), (B.13), (B.14), and (D.51), we can write (D.87) as

$$
\begin{aligned}
&-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}')\Sigma+\Bigg[-\frac{1-\epsilon}{r+\kappa}[K_2(\gamma_1^R,\gamma_3^R,\mathcal{T}',\nu_1)+K_1(\gamma_1^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)]\\
&+H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1)+H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}',\nu_2)\\
&-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}')+G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}',\nu_2)\Bigg]\Sigma p_f'p_f\Sigma\\
&=(\delta^V_{12,t'-t},\delta^V_{3,t'-t})\begin{pmatrix}\Sigma_{11}^{\hat{C}y}\\ \Sigma_{21}^{\hat{C}y}\end{pmatrix}\begin{pmatrix}\delta^M_{12,t''-t}\\ \delta^M_{3,t''-t}\end{pmatrix}\Sigma p_f'p_f\Sigma+\mathbb{C}\mathrm{ov}\left(\left(\zeta^V_{t'}\right)',\zeta^M_{t''}\right).
\end{aligned}
\tag{D.88}
$$

Equation (D.84) follows from (D.88) by noting that $\mathbb{C}\mathrm{ov}\left(\left(\zeta^V_{t'}\right)',\zeta^M_{t''}\right)=\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\left(w^V_{t'}\right)',w^M_{t''}\right)$.

Writing that the covariance between the left-hand side of (D.32) evaluated at t'' and the transpose of the left-hand side of (D.62) evaluated at t' is equal to the covariance between the corresponding right-hand sides, we likewise find

$$
\begin{aligned}
&\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\Bigg(-\frac{1-\epsilon}{r+\kappa}\Sigma p_f'\int_{u'=t'-\tau}^{t'}\mathbb{C}\mathrm{ov}\left(\Lambda_{u'},F'_{t''}\right)du' \qquad \text{(D.89)}\\
&+\int_{u'=t'-\tau}^{t'}\mathbb{C}\mathrm{ov}\left(\Lambda_{u'},\gamma_1\hat{C}_{t''}+\gamma_2C_{t''}+\gamma_3y_{t''}\right)du'\Sigma p_f'p_f\Sigma\Bigg)\\
&+\Bigg(-\frac{1-\epsilon}{r+\kappa}\int_{u'=t'-\tau}^{t'}\mathbb{C}\mathrm{ov}\left(dR_{u'}-E_{u'}(dR_{u'}),F'_{t''}\right)du'\\
&+\int_{u'=t'-\tau}^{t'}\mathbb{C}\mathrm{ov}\left(dR_{u'}-E_{u'}(dR_{u'}),\gamma_1\hat{C}_{t''}+\gamma_2C_{t''}+\gamma_3y_{t''}\right)du'p_f\Sigma\Bigg)\\
&=(\delta^M_{12,t'-t},\delta^M_{3,t'-t})\begin{pmatrix}\Sigma_{11}^{\hat{C}y}\\ \Sigma_{21}^{\hat{C}y}\end{pmatrix}\begin{pmatrix}\delta^M_{12,t''-t}\\ \delta^M_{3,t''-t}\end{pmatrix}\Sigma p_f'p_f\Sigma+\mathbb{C}\mathrm{ov}\left(\left(\zeta^M_{t'}\right)',\zeta^V_{t''}\right). \qquad \text{(D.90)}
\end{aligned}
$$

Using (3.6), (B.10), (B.11), (B.13), (B.14), and (D.51), we can write (D.90) as

$$
\begin{aligned}
&-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}'^-)\Sigma+\Bigg[-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)+H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^-,\nu_2)\\
&-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}'^-)+G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^-,\nu_2)\Bigg]\Sigma p_f'p_f\Sigma
\end{aligned}
$$

$$= (\delta^M_{12,t'-t}, \delta^M_{3,t'-t}) \begin{pmatrix} \Sigma^{\hat{C}y}_{11} \\ \Sigma^{\hat{C}y}_{21} \end{pmatrix} \begin{pmatrix} \delta^V_{12,t''-t} \\ \delta^V_{3,t''-t} \end{pmatrix} \Sigma p_f' p_f \Sigma + \mathbb{C}\text{ov}\left(\left(\zeta^M_{t'}\right)', \zeta^V_{t''}\right). \tag{D.91}$$

Equation (D.85) follows from (D.91) by noting that $\mathbb{C}\text{ov}\left(\left(\zeta^M_{t'}\right)', \zeta^V_{t''}\right) = \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\left(w^M_{t'}\right)', w^V_{t''}\right)$. ∎

Proposition D.7 computes the correlation between the returns of value and momentum strategies, both unconditionally and conditionally on $(\hat{C}_t, y_t)$.

Proposition D.7. *The unconditional correlation between the returns of the value strategy (4.1) and the momentum strategy (4.2) is* $\mathbb{C}\text{orr}(\hat{w}^V_t dR_t, \hat{w}^M_t dR_t) = \frac{\mathcal{G}_{w^V,w^M}}{\sqrt{\mathcal{D}_{w^V}\mathcal{D}_{w^M}}}$*, where*

$$\mathcal{G}_{w^V,w^M} = \frac{L_1^2}{r}\tau\Delta_1 + 2\frac{L_1 L_2}{r}\tau\Delta_2 + \left(\frac{L_2^2}{r}\tau - \frac{1-\epsilon}{r+\kappa} K_1(\gamma_1^R, \gamma_3^R, \mathcal{T}, \nu_2) - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa, \mathcal{T}) \right.$$
$$\left. + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, \mathcal{T}, \nu_2) + G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}, \nu_2) \right) \Delta_3 - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa, \mathcal{T})\Delta_4,$$

$\mathcal{T} = (0, \tau)$*, and* $\mathcal{D}_{w^V}$ *and* $\mathcal{D}_{w^M}$ *are defined in Propositions D.2 and D.5, respectively. The correlation between the returns of the value and momentum strategies conditional on* $\mathcal{I}_t = (\hat{C}_t, y_t)$ *is* $\mathbb{C}\text{orr}_{\mathcal{I}_t}(\hat{w}^V_t dR_t, \hat{w}^M_t dR_t) = \frac{\mathcal{G}_{w^V,w^M,t}}{\sqrt{\mathcal{D}_{w^V,t}\mathcal{D}_{w^M,t}}}$*, where*

$$\mathcal{G}_{w^V,w^M,t} = \frac{L_1^2}{r}\tau\Delta_1 + \frac{L_1}{r}\left(L_2\tau + \delta^M_{12,0}(\hat{C}_t - \bar{C}) + \delta^M_{3,0}(y_t - \bar{y}) \right)\Delta_2 + L_1\tau\left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y}) \right)$$
$$+ \left[\left(\frac{L_2}{r} + \delta^V_{12,0}(\hat{C}_t - \bar{C}) + \delta^V_{3,0}(y_t - \bar{y}) \right)\left(L_2\tau + \delta^M_{12,0}(\hat{C}_t - \bar{C}) + \delta^M_{3,0}(y_t - \bar{y}) \right) + \mathcal{C}^{MV}_{0,0} \right]\Delta_3 + \mathcal{C}^{MV\Sigma}_0\Delta_4,$$

and $\mathcal{D}_{w^V,t}$ *and* $\mathcal{D}_{w^M,t}$ *are defined in Propositions D.3 and D.6, respectively.*

Proof: To show the equation for the unconditional correlation, we need to show that the unconditional covariance between the returns of the value and momentum strategies is $\mathcal{G}_{w^V,w^M}dt$. Since the unconditional expectation of value weights is given by (D.20) and of momentum weights is given by (D.49), the result follows from Lemma D.6 provided that

$$\mathbb{C}\text{ov}\left(\left(w^M_t\right)', w^V_t\right) = -\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa, \mathcal{T})\Sigma + \left(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R, \gamma_3^R, \mathcal{T}, \nu_2) - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa, \mathcal{T}) \right.$$
$$\left. + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, \mathcal{T}, \nu_2) + G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}, \nu_2) \right) \Sigma p_f' p_f \Sigma. \tag{D.92}$$

Equation (D.92) follows by noting that $\mathbb{C}\text{ov}\left(\left(w_t^M\right)', w_t^V\right)$ is equal to the left-hand side of (D.91) for $t'' = t' = t$.

To show the equation for the correlation conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$, we need to show that the conditional covariance between the returns of the value and momentum strategies is $\mathcal{G}_{w^V,w^M,t}dt$. Since the conditional expectation of value weights is given by (D.24) and of momentum weights is given by (D.56), the result follows from Lemma D.6 provided that

$$\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\left(w_t^M\right)', w_t^V\right) = \mathcal{C}_0^{MV\Sigma}\Sigma + \mathcal{C}_{0,0}^{MV}\Sigma p_f' p_f \Sigma. \tag{D.93}$$

Equation (D.93) follows from Lemma D.7, by setting $t'' = t' = t$ in (D.85). ∎

The unconditional expectations and standard deviations of functions of $(\hat{C}_t, y_t)$ are calculated using the unconditional distribution of $(\hat{C}_t, y_t)$, which is normal with mean $(\bar{C}, \bar{y})$ and covariance matrix $\Sigma^{\hat{C}y}$.

E Proofs of Results in Section 6

Lemma E.1 expresses the Sharpe ratio over investment horizon T of a general strategy w_t in terms of expectations, variances, and autocovariances of instantaneous returns.

Lemma E.1. *The Sharpe ratio of a strategy w_t over investment horizon T is $SR_{w,t,T} = \frac{\mathcal{N}_{w,t,T}}{\sqrt{\mathcal{D}_{w,t,T}+\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T}+\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t,T}}}$, where*

$$\mathcal{N}_{w,t,T} \equiv \frac{1}{T}\int_t^{t+T} \mathbb{E}_{\mathcal{I}_t}(\hat{w}_u dR_u), \tag{E.1}$$

$$\mathcal{D}_{w,t,T} \equiv \frac{1}{T}\int_t^{t+T} \mathbb{V}\text{ar}_{\mathcal{I}_t}(\hat{w}_u dR_u), \tag{E.2}$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T} \equiv \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T} \mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u \mathbb{E}_u(dR_u), \hat{w}_{u'}\mathbb{E}_{u'}(dR_{u'})\right], \tag{E.3}$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t,T} \equiv \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T} \mathbb{E}_{\mathcal{I}_t}\left\{\hat{w}_u \mathbb{C}\text{ov}_u\left[dR_u, \hat{w}_{u'}\mathbb{E}_{u'}(dR_{u'})\right]\right\}. \tag{E.4}$$

Proof: The Lemma will follow from the definition (4.5) of the Sharpe ratio provided that

$$\frac{1}{T}\mathbb{V}\text{ar}_{\mathcal{I}_t}\left(\int_t^{t+T}\hat{w}_u dR_u\right) = \mathcal{D}_{w,t,T} + \mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T} + \mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t,T}. \tag{E.5}$$

We can write the left-hand side of (E.5) as

$$\begin{aligned}
&\frac{1}{T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\int_t^{t+T}\hat{w}_u dR_u, \int_t^{t+T}\hat{w}_u dR_u\right)\\
&= \frac{1}{T}\int_t^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u dR_u, \hat{w}_u dR_u\right) + \frac{1}{T}\int_{u=t}^{t+T}\int_{u'=t}^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'}\right)\\
&= \mathcal{D}_{w,t,T} + \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'}\right),
\end{aligned} \tag{E.6}$$

where the second step follows by separating the covariance between contemporaneous returns and the covariance between lagged returns. We can write the second term in (E.6) as

$$\begin{aligned}
&\frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\left\{\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u dR_u, \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right] + \mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right]\right\}\\
&= \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u dR_u, \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right]\\
&= \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\left\{\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\mathbb{E}_u(\hat{w}_u dR_u), \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right] + \mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u dR_u - \mathbb{E}_u(\hat{w}_u dR_u), \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right]\right\}\\
&= \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\left\{\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\mathbb{E}_u(\hat{w}_u dR_u), \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right] + \mathbb{E}_{\mathcal{I}_t}\left[\mathbb{C}\text{ov}_u(\hat{w}_u dR_u, \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'}))\right]\right\}\\
&= \frac{2}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\left\{\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u\mathbb{E}_u(dR_u), \hat{w}_{u'}\mathbb{E}_{u'}(dR_{u'})\right] + \mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u\mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u'}\mathbb{E}_{u'}(dR_{u'}))\right]\right\}\\
&= \mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T} + \mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t,T},
\end{aligned} \tag{E.7}$$

where the second step follows from writing $\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right]$ as

$$\mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u dR_u(\hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'}))\right] - \mathbb{E}_{\mathcal{I}_t}(\hat{w}_u dR_u)\mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})\right]$$

and noting that each term is zero because of the Law of Iterative Expectations, and the fourth step follows from (D.51). Combining (E.6) and (E.7), we find (E.5). ∎

Lemma E.2 specializes Lemma E.1 to the unconditional Sharpe ratio ($\mathcal{I}_t = \emptyset$).

Lemma E.2. *The unconditional Sharpe ratio of a strategy* w_t *over investment horizon* T *is* $SR_{w,T} = \frac{\mathcal{N}_w}{\sqrt{\mathcal{D}_w + \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1} + \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2}}}$, *where*

$$\mathcal{N}_w \equiv \frac{1}{dt}\mathbb{E}(\hat{w}_t dR_t) = \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left(\Lambda_t w_t \Sigma p_f'\right),$$

$$\mathcal{D}_w \equiv \frac{1}{dt}\mathbb{V}\text{ar}(\hat{w}_t dR_t) = f\left[\mathbb{E}(w_t\Sigma w_t') - \frac{\mathbb{E}\left[(w_t\Sigma\eta')^2\right]}{\eta\Sigma\eta'}\right] + k\mathbb{E}[(w_t\Sigma p_f')^2],$$

are the unconditional versions of $(\mathcal{N}_{w,t}, \mathcal{D}_{w,t})$ *defined in Lemma D.1, and*

$$\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1} \equiv \frac{2}{dt}\int_t^{t+T}\left(1 - \frac{u-t}{T}\right)\mathbb{C}\text{ov}\left[\hat{w}_t\mathbb{E}_t(dR_t), \hat{w}_u\mathbb{E}_u(dR_u)\right], \tag{E.8}$$

$$\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2} \equiv \frac{2}{dt}\int_t^{t+T}\left(1 - \frac{u-t}{T}\right)\mathbb{E}\left\{\hat{w}_t\mathbb{C}\text{ov}_t\left[dR_t, \hat{w}_u\mathbb{E}_u(dR_u)\right]\right\}. \tag{E.9}$$

Proof: The lemma follows from Lemma E.1 by noting that when $\mathcal{I}_t = \emptyset$:

$$\mathcal{N}_{w,t,T} = \frac{1}{T}\int_t^{t+T}\mathbb{E}(\hat{w}_u dR_u) = \frac{1}{T}\int_t^{t+T}\frac{\mathbb{E}(\hat{w}_u dR_u)}{du}du = \frac{1}{T}\int_t^{t+T}\frac{\mathbb{E}(\hat{w}_t dR_t)}{dt}du = \mathcal{N}_w,$$

where the third step follows because the expectation is unconditional;

$$\mathcal{D}_{w,t,T} = \frac{1}{T}\int_t^{t+T}\mathbb{V}\text{ar}(\hat{w}_u dR_u) = \frac{1}{T}\int_t^{t+T}\frac{\mathbb{V}\text{ar}(\hat{w}_u dR_u)}{du}du = \frac{1}{T}\int_t^{t+T}\frac{\mathbb{V}\text{ar}(\hat{w}_t dR_t)}{dt}du = \mathcal{D}_w,$$

where the third step follows because the variance is unconditional;

$$\begin{aligned}
\mathcal{D}_{w,t,T}^{\mathbb{C}\text{ov}_1} &= \frac{2}{T}\int_{u=t}^{t+T}\int_{s=0}^{t+T-u}\mathbb{C}\text{ov}\left[\hat{w}_u\mathbb{E}_u(dR_u), \hat{w}_{u+s}\mathbb{E}_{u+s}(dR_{u+s})\right] \\
&= \frac{2}{T}\int_{s=0}^{T}\int_{u=t}^{t+T-s}\mathbb{C}\text{ov}\left[\hat{w}_u\mathbb{E}_u(dR_u), \hat{w}_{u+s}\mathbb{E}_{u+s}(dR_{u+s})\right] \\
&= \frac{2}{T}\int_{s=0}^{T}\int_{u=t}^{t+T-s}\frac{\mathbb{C}\text{ov}\left[\hat{w}_u\mathbb{E}_u(dR_u), \hat{w}_{u+s}\mathbb{E}_{u+s}(dR_{u+s})\right]}{du}du \\
&= \frac{2}{T}\int_{s=0}^{T}\int_{u=t}^{t+T-s}\frac{\mathbb{C}\text{ov}\left[\hat{w}_t\mathbb{E}_t(dR_t), \hat{w}_{t+s}\mathbb{E}_{t+s}(dR_{t+s})\right]}{dt}du \\
&= \frac{2}{T}\int_{s=0}^{T}(T-s)\frac{\mathbb{C}\text{ov}\left[\hat{w}_t\mathbb{E}_t(dR_t), \hat{w}_{t+s}\mathbb{E}_{t+s}(dR_{t+s})\right]}{dt}
\end{aligned}$$

$$= \frac{2}{T} \int_{u=t}^{t+T} [T - (u-t)] \frac{\mathbb{C}\text{ov}\left[\hat{w}_t \mathbb{E}_t(dR_t), \hat{w}_u \mathbb{E}_u(dR_u)\right]}{dt} = \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1},$$

where the first and sixth steps follow from the change of variable $s = u - t$, the second step follows by changing the order of the integrals, and the fourth step follows because the covariance is unconditional and depends only on s; and

$$\begin{aligned}
\mathcal{D}_{w,t,T}^{\mathbb{C}\text{ov}_2} &= \frac{2}{T} \int_{u=t}^{t+T} \int_{s=0}^{t+T-u} \mathbb{E}\left\{\hat{w}_u \mathbb{C}\text{ov}_u \left[dR_u, \hat{w}_{u+s} \mathbb{E}_{u+s}(dR_{u+s})\right]\right\} \\
&= \frac{2}{T} \int_{s=0}^{T} \int_{u=t}^{t+T-s} \mathbb{E}\left\{\hat{w}_u \mathbb{C}\text{ov}_u \left[dR_u, \hat{w}_{u+s} \mathbb{E}_{u+s}(dR_{u+s})\right]\right\} \\
&= \frac{2}{T} \int_{s=0}^{T} \int_{u=t}^{t+T-s} \frac{\mathbb{E}\left\{\hat{w}_u \mathbb{C}\text{ov}_u \left[dR_u, \hat{w}_{u+s} \mathbb{E}_{u+s}(dR_{u+s})\right]\right\}}{du} du \\
&= \frac{2}{T} \int_{s=0}^{T} \int_{u=t}^{t+T-s} \frac{\mathbb{E}\left\{\hat{w}_t \mathbb{C}\text{ov}_t \left[dR_t, \hat{w}_{t+s} \mathbb{E}_{t+s}(dR_{t+s})\right]\right\}}{dt} du \\
&= \frac{2}{T} \int_{s=0}^{T} (T-s) \frac{\mathbb{E}\left\{\hat{w}_t \mathbb{C}\text{ov}_t \left[dR_t, \hat{w}_{t+s} \mathbb{E}_{t+s}(dR_{t+s})\right]\right\}}{dt} \\
&= \frac{2}{T} \int_{u=t}^{t+T} [T-(u-t)] \frac{\mathbb{E}\left\{\hat{w}_t \mathbb{C}\text{ov}_t \left[dR_t, \hat{w}_{t+s} \mathbb{E}_{t+s}(dR_{t+s})\right]\right\}}{dt} = \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2},
\end{aligned}$$

where the first and sixth steps follow from the change of variable $s = u - t$, and the fourth step follows because the expectation is unconditional and depends only on s.

Since the same argument as in the derivation of (E.7) implies

$$\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1} + \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2} = \frac{2}{dt} \int_{t}^{t+T} \left(1 - \frac{u-t}{T}\right) \mathbb{C}\text{ov}\left(\hat{w}_t dR_t, \hat{w}_u dR_u\right),$$

we can write the Sharpe ratio $SR_{w,T}$ as

$$SR_{w,T} = \frac{\frac{1}{dt}\mathbb{E}(\hat{w}_t dR_t)}{\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}(\hat{w}_t dR_t) + \frac{2}{dt}\int_t^{t+T}\left(1 - \frac{u-t}{T}\right)\mathbb{C}\text{ov}\left(\hat{w}_t dR_t, \hat{w}_u dR_u\right)}}.$$

Dividing numerator and denominator by $\sqrt{\frac{1}{dt}\mathbb{V}\text{ar}(\hat{w}_t dR_t)}$ and using the definition (4.5) of the Sharpe ratio over an infinitesimal horizon, we find (6.1). ∎

The terms $\{\mathcal{D}_{w,t,T}^{\mathbb{C}\text{ov}_i}\}_{i=1,2}$ in Lemma E.1 and $\{\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_i}\}_{i=1,2}$ in Lemma E.2 involve covariances between products of random variables, such as $\hat{w}_u \mathbb{E}_u(dR_u)$ and $\hat{w}_{u'} \mathbb{E}_{u'}(dR_{u'})$. Lemma E.3 computes covariances between products of normal random variables.

Lemma E.3. *If the random variables $\{X_i\}_{i=1,2,3,4}$ are jointly normal, then*

$$\mathbb{C}\text{ov}(X_1 X_2, X_3) = \mathbb{E}(X_1)\mathbb{C}\text{ov}(X_2, X_3) + \mathbb{E}(X_2)\mathbb{C}\text{ov}(X_1, X_3) \tag{E.10}$$

$$\begin{aligned} \mathbb{C}\text{ov}(X_1 X_2, X_3 X_4) = & \mathbb{E}(X_1)\mathbb{E}(X_3)Cov(X_2, X_4) + \mathbb{E}(X_1)\mathbb{E}(X_4)\mathbb{C}\text{ov}(X_2, X_3) \\ & + \mathbb{E}(X_2)\mathbb{E}(X_3)\mathbb{C}\text{ov}(X_1, X_4) + \mathbb{E}(X_2)\mathbb{E}(X_4)\mathbb{C}\text{ov}(X_1, X_3) \\ & + \mathbb{C}\text{ov}(X_1, X_3)\mathbb{C}\text{ov}(X_2, X_4) + \mathbb{C}\text{ov}(X_1, X_4)\mathbb{C}\text{ov}(X_2, X_3). \end{aligned} \tag{E.11}$$

Proof: We first show (E.10) and (E.11) in the special case where $\{X_i\}_{i=1,2,3,4}$ are mean zero. Since these variables are normal, we can set

$$X_i = \frac{\mathbb{C}\text{ov}(X_i, X_3)}{\mathbb{V}\text{ar}(X_3)} X_3 + \epsilon_i, \tag{E.12}$$

for $i = 1, 2, 4$, where ϵ_i is normal, mean zero and independent of X_3.

Using (E.12), we can write the left-hand side of (E.10) as

$$\begin{aligned} & \frac{\mathbb{C}\text{ov}(X_1, X_3)\mathbb{C}\text{ov}(X_2, X_3)}{\mathbb{V}\text{ar}(X_3)^2}\mathbb{C}\text{ov}(X_3^2, X_3) + \frac{\mathbb{C}\text{ov}(X_1, X_3)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_3\epsilon_2, X_3) \\ & + \frac{\mathbb{C}\text{ov}(X_2, X_3)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_3\epsilon_1, X_3) + \mathbb{C}\text{ov}(\epsilon_1\epsilon_2, X_3). \end{aligned} \tag{E.13}$$

The first term in (E.13) is zero because

$$\mathbb{C}\text{ov}(X_3^2, X_3) = \mathbb{E}(X_3^3) - \mathbb{E}(X_3^2)\mathbb{E}(X_3) = 0,$$

where the second step follows because the normality and mean zero properties of X_3 imply $\mathbb{E}(X_3^3) = \mathbb{E}(X_3) = 0$. The second and third terms in (E.13) are zero because

$$\mathbb{C}\text{ov}(X_3\epsilon_i, X_3) = \mathbb{E}(X_3^2\epsilon_i) - \mathbb{E}(X_3\epsilon_i)\mathbb{E}(X_3) = \left[\mathbb{E}(X_3^2) - \mathbb{E}(X_3)^2\right]\mathbb{E}(\epsilon_i) = 0,$$

for $i = 1, 2$, where the second step follows because ϵ_i is independent of X_3, and the third step follows because ϵ_i is mean zero. The fourth term in (E.13) is zero because (ϵ_1, ϵ_2) are independent of X_3. Therefore, (E.13) is equal to zero, which implies (E.10).

Using (E.12), we can write the left-hand side of (E.11) as

$$
\begin{aligned}
&\frac{\mathbb{C}\text{ov}(X_3,X_4)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_1X_2,X_3^2)+\mathbb{C}\text{ov}(X_1X_2,X_3\epsilon_4)\\
&=\frac{\mathbb{C}\text{ov}(X_3,X_4)}{\mathbb{V}\text{ar}(X_3)}\left[\frac{\mathbb{C}\text{ov}(X_1,X_3)\mathbb{C}\text{ov}(X_2,X_3)}{\mathbb{V}\text{ar}(X_3)^2}\mathbb{C}\text{ov}(X_3^2,X_3^2)+\frac{\mathbb{C}\text{ov}(X_1,X_3)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_3\epsilon_2,X_3^2)\right.\\
&\left.+\frac{\mathbb{C}\text{ov}(X_2,X_3)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_3\epsilon_1,X_3^2)+\mathbb{C}\text{ov}(\epsilon_1\epsilon_2,X_3^2)\right]+\frac{\mathbb{C}\text{ov}(X_1,X_3)\mathbb{C}\text{ov}(X_2,X_3)}{\mathbb{V}\text{ar}(X_3)^2}\mathbb{C}\text{ov}(X_3^2,X_3\epsilon_4)\\
&+\frac{\mathbb{C}\text{ov}(X_1,X_3)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_3\epsilon_2,X_3\epsilon_4)+\frac{\mathbb{C}\text{ov}(X_2,X_3)}{\mathbb{V}\text{ar}(X_3)}\mathbb{C}\text{ov}(X_3\epsilon_1,X_3\epsilon_4)+\mathbb{C}\text{ov}(\epsilon_1\epsilon_2,X_3\epsilon_4). \qquad \text{(E.14)}
\end{aligned}
$$

To compute the first term in (E.14), we note that

$$\mathbb{C}\text{ov}(X_3^2,X_3^2)=\mathbb{E}(X_3^4)-\mathbb{E}(X_3^2)^2=2\mathbb{E}(X_3^2)^2=2\mathbb{V}\text{ar}(X_3)^2, \qquad \text{(E.15)}$$

where the second step follows because the mean-zero property of X_3 implies that $\frac{\mathbb{E}(X_3^4)}{\mathbb{E}(X_3^2)^2}$ is the kurtosis of X_3 and because the normality of X_3 implies that X_3 has a kurtosis of three. The second and third terms in (E.14) are zero because

$$\mathbb{C}\text{ov}(X_3\epsilon_i,X_3^2)=\mathbb{E}(X_3^3\epsilon_i)-\mathbb{E}(X_3\epsilon_i)\mathbb{E}(X_3^2)=\left[\mathbb{E}(X_3^3)-\mathbb{E}(X_3)\mathbb{E}(X_3^2)\right]\mathbb{E}(\epsilon_i)=0,$$

for $i=1,2$, where the second step follows because ϵ_i is independent of X_3, and the third step follows because ϵ_i is mean zero. The fourth term in (E.14) is zero because (ϵ_1,ϵ_2) are independent of X_3. The fifth term in (E.14) is zero because

$$\mathbb{C}\text{ov}(X_3^2,X_3\epsilon_4)=\mathbb{E}(X_3^3\epsilon_4)-\mathbb{E}(X_3^2)\mathbb{E}(X_3\epsilon_4)=\left[\mathbb{E}(X_3^3)-\mathbb{E}(X_3^2)\mathbb{E}(X_3)\right]\mathbb{E}(\epsilon_4)=0,$$

where the second step follows because ϵ_4 is independent of X_3, and the third step follows because ϵ_4 is mean zero. To compute the sixth and seventh terms in (E.14), we note that

$$
\begin{aligned}
\mathbb{C}\text{ov}(X_3\epsilon_i,X_3\epsilon_4)&=\mathbb{E}(X_3^2\epsilon_i\epsilon_4)-\mathbb{E}(X_3\epsilon_i)\mathbb{E}(X_3\epsilon_4)\\
&=\mathbb{E}(X_3^2)\mathbb{E}(\epsilon_i\epsilon_4)-\mathbb{E}(X_3)^2\mathbb{E}(\epsilon_i)\mathbb{E}(\epsilon_4)\\
&=\mathbb{E}(X_3^2)\mathbb{E}(\epsilon_i\epsilon_4)\\
&=\mathbb{V}\text{ar}(X_3^2)\mathbb{C}\text{ov}(\epsilon_i,\epsilon_4)
\end{aligned}
$$

$$= \mathbb{V}\text{ar}(X_3^2)\mathbb{C}\text{ov}(X_i, \epsilon_4), \tag{E.16}$$

for $i = 1, 2$, where the second step follows because (ϵ_i, ϵ_4) are independent of X_3, the third and fourth step follow because $(X_3, \epsilon_i, \epsilon_4)$ are mean zero, and the fifth step follows from (E.12) and the independence of (X_3, ϵ_4). The eighth term in (E.14) is zero because

$$\mathbb{C}\text{ov}(\epsilon_1\epsilon_2, X_3\epsilon_4) = \mathbb{E}(\epsilon_1\epsilon_2 X_3\epsilon_4) - \mathbb{E}(\epsilon_1\epsilon_2)\mathbb{E}(X_3\epsilon_4) = \mathbb{E}(X_3)\left[\mathbb{E}(\epsilon_1\epsilon_2\epsilon_4) - \mathbb{E}(\epsilon_1\epsilon_2)\mathbb{E}(\epsilon_4)\right] = 0,$$

where the second step follows because $(\epsilon_1, \epsilon_2, \epsilon_4)$ are independent of X_3, and the third step follows because X_3 is mean zero. Suppressing all zero terms and using (E.15) and (E.16), we can write (E.14) as

$$\begin{aligned}
&\frac{2\mathbb{C}\text{ov}(X_1, X_3)\mathbb{C}\text{ov}(X_2, X_3)\mathbb{C}\text{ov}(X_3, X_4)}{\mathbb{V}\text{ar}(X_3)} + \mathbb{C}\text{ov}(X_1, X_3)\mathbb{C}\text{ov}(X_2, \epsilon_4) + \mathbb{C}\text{ov}(X_2, X_3)\mathbb{C}\text{ov}(X_1, \epsilon_4) \\
&= \mathbb{C}\text{ov}(X_1, X_3)\mathbb{C}\text{ov}\left[X_2, \frac{\mathbb{C}\text{ov}(X_3, X_4)}{\mathbb{V}\text{ar}(X_3)}X_3 + \epsilon_4\right] + \mathbb{C}\text{ov}(X_2, X_3)\mathbb{C}\text{ov}\left[X_1, \frac{\mathbb{C}\text{ov}(X_3, X_4)}{\mathbb{V}\text{ar}(X_3)}X_3 + \epsilon_4\right] \\
&= \mathbb{C}\text{ov}(X_1, X_3)\mathbb{C}\text{ov}(X_2, X_4) + \mathbb{C}\text{ov}(X_2, X_3)\mathbb{C}\text{ov}(X_1, X_4),
\end{aligned}$$

which implies (E.11).

We next show (E.10) and (E.11) when $\{X_i\}_{i=1,2,3,4}$ can have a non-zero mean. We set $\hat{X}_i \equiv X_i - \mathbb{E}(X_i)$ for $i = 1, 2, 3, 4$.

We can write the left-hand side of (E.10) as

$$\begin{aligned}
&\mathbb{C}\text{ov}\left[(\mathbb{E}(X_1) + \hat{X}_1)(\mathbb{E}(X_2) + \hat{X}_2), X_3\right] \\
&= \mathbb{E}(X_1)\mathbb{C}\text{ov}(\hat{X}_2, X_3) + \mathbb{E}(X_2)\mathbb{C}\text{ov}(\hat{X}_1, X_3) + \mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, X_3) \\
&= \mathbb{E}(X_1)\mathbb{C}\text{ov}(X_2, X_3) + \mathbb{E}(X_2)\mathbb{C}\text{ov}(X_1, X_3) + \mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, \hat{X}_3).
\end{aligned} \tag{E.17}$$

Combining (E.17) with (E.10) applied to $\{\hat{X}_i\}_{i=1,2,3}$, we find (E.10) applied to $\{X_i\}_{i=1,2,3}$.

We can write the left-hand side of (E.10) as

$$\begin{aligned}
&\mathbb{C}\text{ov}\left[(\mathbb{E}(X_1) + \hat{X}_1)(\mathbb{E}(X_2) + \hat{X}_2), X_3X_4\right] \\
&= \mathbb{E}(X_1)\mathbb{C}\text{ov}(\hat{X}_2, X_3X_4) + \mathbb{E}(X_2)\mathbb{C}\text{ov}(\hat{X}_1, X_3X_4) + \mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, X_3X_4).
\end{aligned} \tag{E.18}$$

Equation (E.10) implies

$$\begin{aligned}\mathbb{C}\text{ov}(\hat{X}_i, X_3X_4) =& \mathbb{E}(X_3)\mathbb{C}\text{ov}(\hat{X}_i, X_4) + \mathbb{E}(X_4)\mathbb{C}\text{ov}(\hat{X}_i, X_3)\\ =& \mathbb{E}(X_3)\mathbb{C}\text{ov}(X_i, X_4) + \mathbb{E}(X_4)\mathbb{C}\text{ov}(X_i, X_3)\end{aligned} \tag{E.19}$$

for $i = 1, 2$. Moreover,

$$\begin{aligned}\mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, X_3X_4) =& \mathbb{C}\text{ov}\left[\hat{X}_1\hat{X}_2, (\mathbb{E}(X_3) + \hat{X}_3)(\mathbb{E}(X_4) + \hat{X}_4)\right]\\ =& \mathbb{E}(X_3)\mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, \hat{X}_4) + \mathbb{E}(X_4)\mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, \hat{X}_3) + \mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, \hat{X}_3\hat{X}_4)\\ =& \mathbb{C}\text{ov}(\hat{X}_1\hat{X}_2, \hat{X}_3\hat{X}_4),\end{aligned} \tag{E.20}$$

where the second step follows from (E.10) and because $\hat{X}_i$ is mean zero. Combining (E.18)-(E.20) with (E.11) applied to $\hat{X}_i$, and noting that

$$\mathbb{C}\text{ov}(\hat{X}_i, \hat{X}_{i'}) = \mathbb{C}\text{ov}(X_i, X_{i'})$$

for $i = 1, 2$ and $i' = 3, 4$, we find (E.11) applied to X_i. ∎

Lemmas E.4 and E.5 use Lemma E.3 to compute the terms $\{\mathcal{D}^{\mathbb{C}\text{ov}_i}_{w,t,T}\}_{i=1,2}$ in Lemma E.1 and $\{\mathcal{D}^{\mathbb{C}\text{ov}_i}_{w,T}\}_{i=1,2}$ in Lemma E.2. To ensure that the normality assumption in Lemma E.3 is met, we restrict trading strategies to be linear, in the sense that strategy weights must be integrals of the Brownian shocks with constant coefficients. The value strategy (4.1), the momentum strategy (4.2), and all strategies of the form $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ are linear.

Lemma E.4. *For linear trading strategies,* $\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T} = \frac{2}{T}\sum_{i=1}^{6}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathcal{D}^{\mathbb{C}\text{ov}_1,i}_{w,t}(u,u')du du'$*, where*

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,1}_{w,t}(u,u') \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}_{\mathcal{I}_t}(\Lambda_u)\mathbb{E}_{\mathcal{I}_t}(\Lambda_{u'})\mathbb{C}\text{ov}_{\mathcal{I}_t}(w_u\Sigma p'_f, w_{u'}\Sigma p'_f),$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,2}_{w,t}(u,u') \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}_{\mathcal{I}_t}(\Lambda_u)\mathbb{E}_{\mathcal{I}_t}(w_{u'}\Sigma p'_f)\mathbb{C}\text{ov}_{\mathcal{I}_t}(w_u\Sigma p'_f, \Lambda_{u'}),$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,3}_{w,t}(u,u') \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}_{\mathcal{I}_t}(w_u\Sigma p'_f)\mathbb{E}_{\mathcal{I}_t}(\Lambda_{u'})\mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_u, w_{u'}\Sigma p'_f),$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,4}_{w,t}(u,u') \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}_{\mathcal{I}_t}(w_u\Sigma p'_f)\mathbb{E}_{\mathcal{I}_t}(w_{u'}\Sigma p'_f)\mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_u, \Lambda_{u'}),$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,5}_{w,t}(u,u') \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_u, \Lambda_{u'})\mathbb{C}\text{ov}_{\mathcal{I}_t}(w_u\Sigma p'_f, w_{u'}\Sigma p'_f),$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,6}_{w,t}(u,u') \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_u, w_{u'}\Sigma p'_f)\mathbb{C}\text{ov}_{\mathcal{I}_t}(w_u\Sigma p'_f, \Lambda_{u'}),$$

and $\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t} = \frac{2}{T}\sum_{i=1}^{2}\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathcal{D}^{\mathbb{C}\text{ov}_2,i}_{w,t}(u,u')dudu'$, *where*

$$\mathcal{D}^{\mathbb{C}\text{ov}_2,1}_{w,t}(u,u') \equiv \frac{1}{du}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u\Lambda_{u'}\mathbb{C}\text{ov}_u(dR_u, w_{u'}\Sigma p'_f)\right],$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_2,2}_{w,t}(u,u') \equiv \frac{1}{du}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u w_{u'}\Sigma p'_f\mathbb{C}\text{ov}_u(dR_u, \Lambda_{u'})\right].$$

Proof: Equations (4.3) and (A.13) imply

$$\begin{aligned}
\hat{w}_t\mathbb{E}_u(dR_t) &= \left(w_t - \frac{\mathbb{C}\text{ov}_t(w_t dR_t, \eta dR_t)}{\mathbb{V}\text{ar}_t(\eta dR_t)}\eta\right)\left[\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\Lambda_t\Sigma p'_f\right]dt \\
&= \left(w_t - \frac{w_t\Sigma\eta'}{\eta\Sigma\eta'}\eta\right)\left[\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\eta\Sigma\theta'}{\eta\Sigma\eta'}\Sigma\eta' + \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\Lambda_t\Sigma p'_f\right]dt \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\Lambda_t w_t\Sigma p'_f, \qquad \text{(E.21)}
\end{aligned}$$

where the second step follows from (A.2), (C.19) and $\eta\Sigma p'_f = 0$, and the third step follows from $\eta\Sigma p'_f = 0$.

Using (E.21), we can write (E.3) as

$$\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T} = \frac{2}{T}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathbb{C}\text{ov}_{\mathcal{I}_t}(\Lambda_u w_u\Sigma p'_f, \Lambda_{u'}w_{u'}\Sigma p'_f)dudu'. \qquad \text{(E.22)}$$

The equation for $\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w,t,T}$ in the lemma follows from (E.22) by using (E.11) and setting $X_1 = \Lambda_u$, $X_2 = w_u\Sigma p'_f$, $X_3 = \Lambda_{u'}$ and $X_4 = w_{u'}\Sigma p'_f$.

Using (E.21), we can write (E.4) as

$$\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t,T} = \frac{2}{T}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\int_{u=t}^{t+T}\int_{u'=u}^{t+T}\mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u\mathbb{C}\text{ov}_u(dR_u, \Lambda_{u'}w_{u'}\Sigma p'_f)\right]du'. \qquad \text{(E.23)}$$

The equation for $\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w,t,T}$ in the lemma follows from (E.23) by using (E.10) and the Law of Iterative

Expectations and setting $X_1 = dR_u$, $X_2 = \Lambda_{u'}$ and $X_3 = w_{u'}\Sigma p_f'$. ∎

Lemma E.5. *For linear trading strategies,* $\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1} = 2\sum_{i=1}^{6}\int_t^{t+T}\left(1-\frac{u-t}{T}\right)\mathcal{D}_w^{\mathbb{C}\text{ov}_1,i}(u)du$*, where*

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_1,1}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 E(\Lambda_t)^2 \mathbb{C}\text{ov}(w_t\Sigma p_f', w_u\Sigma p_f'),$$

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_1,2}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}(\Lambda_t)\mathbb{E}(w_t\Sigma p_f')\mathbb{C}\text{ov}(w_t\Sigma p_f', \Lambda_u),$$

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_1,3}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}(\Lambda_t)\mathbb{E}(w_t\Sigma p_f')\mathbb{C}\text{ov}(\Lambda_t, w_u\Sigma p_f'),$$

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_1,4}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}(w_t\Sigma p_f')^2\mathbb{C}\text{ov}(\Lambda_t, \Lambda_u),$$

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_1,5}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{C}\text{ov}(\Lambda_t, \Lambda_u)\mathbb{C}\text{ov}(w_t\Sigma p_f', w_u\Sigma p_f'),$$

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_1,6}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{C}\text{ov}(\Lambda_t, w_u\Sigma p_f')\mathbb{C}\text{ov}(w_t\Sigma p_f', \Lambda_u),$$

and $\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2} = 2\sum_{i=1}^{2}\int_t^{t+T}\left(1-\frac{u-t}{T}\right)\mathcal{D}_w^{\mathbb{C}\text{ov}_2,i}(u)du$*, where*

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_2,1}(u) \equiv \frac{1}{dt}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left[\hat{w}_t\Lambda_u\mathbb{C}\text{ov}_t(dR_t, w_u\Sigma p_f')\right],$$

$$\mathcal{D}_w^{\mathbb{C}\text{ov}_2,2}(u) \equiv \frac{1}{dt}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left[\hat{w}_t w_u\Sigma p_f'\mathbb{C}\text{ov}_t(dR_t, \Lambda_u)\right].$$

Proof: Using (E.21), we can write (E.8) as

$$\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1} = 2\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \int_t^{t+T}\left(1 - \frac{u-t}{T}\right)\mathbb{C}\text{ov}(\Lambda_t w_t\Sigma p_f', \Lambda_u w_u\Sigma p_f')du. \tag{E.24}$$

The equation for $\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1}$ in the lemma follows from (E.24) by using (E.11) and setting $X_1 = \Lambda_t$, $X_2 = w_t\Sigma p_f'$, $X_3 = \Lambda_u$ and $X_4 = w_u\Sigma p_f'$, and by noting that when expectations are unconditional, $\mathbb{E}(\Lambda_t) = \mathbb{E}(\Lambda_u)$ and $\mathbb{E}(w_t\Sigma p_f') = \mathbb{E}(w_u\Sigma p_f')$.

Using (E.21), we can write (E.9) as

$$\mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2} = \frac{2}{dt}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\int_t^{t+T}\left(1 - \frac{u-t}{T}\right)\mathbb{E}\left\{\hat{w}_t\mathbb{C}\text{ov}_t\left[dR_t, \Lambda_u w_u\Sigma p_f'\right]\right\}du. \tag{E.25}$$

The equation for $\mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_2}$ in the lemma follows from (E.25) by using (E.10) and the Law of Iterative Expectations and setting $X_1 = dR_t$, $X_2 = \Lambda_u$ and $X_3 = w_u \Sigma p_f'$. ∎

Proposition E.1 computes the unconditional Sharpe ratio over investment horizon T of a general strategy of the form $w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f$. The formula for the Sharpe ratio is expressed in terms of integrals. While the integrals can be computed in closed form, we do not present the closed-form solutions because they require introducing additional notation.

Proposition E.1. *The unconditional Sharpe ratio of a strategy $w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f$ over investment horizon T is $SR_{w,T} = \frac{\mathcal{N}_w}{\sqrt{\mathcal{D}_w + \mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_1} + \mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_2}}}$, where*

$$\mathcal{N}_w = \left[L_2 \left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, 0, \nu_0)\right] \frac{\Delta}{\eta \Sigma \eta'},$$

$$\mathcal{D}_w = \left(f + \frac{k\Delta}{\eta \Sigma \eta'}\right) \left[\left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right)^2 + H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, 0, \nu_0)\right] \frac{\Delta}{\eta \Sigma \eta'},$$

and $(\mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_1}, \mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_2})$ are derived in Lemma E.5, with

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_1,1}(u) = L_2^2 H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_1,2}(u) = L_2 \left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right) H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_1,3}(u) = L_2 \left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right) H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_1,4}(u) = \left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right)^2 H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_1,5}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_1,6}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, u - t, \nu_0) H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_2,1}(u) = \left[L_2 \left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right) + H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0)\right]$$
$$\times G(\delta_1, \delta_2, \delta_3, u - t, \nu_0) \left(\frac{\Delta}{\eta \Sigma \eta'}\right)^2,$$

$$\mathcal{D}_w^{\mathbb{C}\mathrm{ov}_2,2}(u) = \left[\left(\delta_0 + (\delta_1 + \delta_2)\bar{C} + \delta_3 \bar{y}\right)^2 + H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0)\right]$$

$$\times G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0)\left(\frac{\Delta}{\eta\Sigma\eta'}\right)^2.$$

Proof: The Sharpe ratio has the form in Lemma E.2. To compute $\mathcal{N}_w$, we note from Lemma E.2 that

$$\begin{aligned}
\mathcal{N}_w &= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left[\Lambda_t(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)\right] p_f\Sigma p_f' \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\mathbb{E}(\Lambda_t)\mathbb{E}(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t) + \mathbb{C}\text{ov}(\Lambda_t, \delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)\right]\frac{\Delta}{\eta\Sigma\eta'} \\
&= \left[L_2\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, 0, \nu_0)\right]\frac{\Delta}{\eta\Sigma\eta'},
\end{aligned}$$

where the first step follows from $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$, the second step follows from $p_f\Sigma p_f' = \frac{\Delta}{\eta\Sigma\eta'}$, and the third step follows from (3.6), (B.13) and (D.8). To compute $\mathcal{D}_w$, we note from Lemma E.2 that

$$\begin{aligned}
\mathcal{D}_w &= f\mathbb{E}\left[(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2\right] p_f\Sigma p_f' + k\mathbb{E}\left[(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2\right]\left(p_f\Sigma p_f'\right)^2 \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left[(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2\right]\frac{\Delta}{\eta\Sigma\eta'} \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\mathbb{E}(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2 + \mathbb{V}\text{ar}(\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)\right]\frac{\Delta}{\eta\Sigma\eta'} \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)^2 + H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, 0, \nu_0)\right]\frac{\Delta}{\eta\Sigma\eta'},
\end{aligned}$$

where the first step follows from $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ and $\eta\Sigma p_f' = 0$, the second step follows from $p_f\Sigma p_f' = \frac{\Delta}{\eta\Sigma\eta'}$, and the fourth step follows from (B.13). To compute $\{\mathcal{D}_w^{\mathbb{C}\text{ov}_1,i}(u)\}_{i=1,..,6}$, we use their definitions in Lemma E.5 together with $w_t = (\delta_0+\delta_1\hat{C}_t+\delta_2 C_t+\delta_3 y_t)p_f$, $p_f\Sigma p_f' = \frac{\Delta}{\eta\Sigma\eta'}$, (3.6), (B.13) and (D.8). To compute $\{\mathcal{D}_w^{\mathbb{C}\text{ov}_2,i}(u)\}_{i=1,2}$, we use their definitions in Lemma E.5 together with $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$, $\eta\Sigma p_f' = 0$, $p_f\Sigma p_f' = \frac{\Delta}{\eta\Sigma\eta'}$, (3.6), (B.10), (B.13) and (D.8). ∎

To determine the optimal strategy for a given investment horizon T, we maximize numerically the Sharpe ratio in Proposition E.1 over $(\delta_0, \delta_1, \delta_2, \delta_3)$. Since the Sharpe ratio is the same for $(\delta_0, \delta_1, \delta_2, \delta_3)$ and $(\lambda\delta_0, \lambda\delta_1, \lambda\delta_2, \lambda\delta_3)$ for any $\lambda > 0$, we can fix the value of one of the four arguments $(\delta_0, \delta_1, \delta_2, \delta_3)$ to one if the argument is positive at the optimum and to minus one if it is negative.

We do that for δ_3, which we set to minus one because it is negative at the optimum. Proposition E.2 computes the unconditional Sharpe ratio over investment horizon T of the value strategy.

Proposition E.2. *The unconditional Sharpe ratio of the value strategy (4.1) over investment horizon T is $SR_{w^V,T} = \frac{\mathcal{N}_{w^V}}{\sqrt{\mathcal{D}_{w^V}+\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^V,T}+\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^V,T}}}$, where $(\mathcal{N}_{w^V},\mathcal{D}_{w^V})$ are derived in Proposition D.2 and $(\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^V,T},\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^V,T})$ are defined in Lemma E.5, with*

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,1}_{w^V}(u) = L_2^2\Bigg[\Bigg(-\frac{1-\epsilon}{r+\kappa}\left[K_1(\gamma_1,\gamma_3,u-t,\nu_0)+K_2(\gamma_1,\gamma_3,u-t,\nu_0)\right] + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\Bigg)\left(p_f\Sigma^2p_f'\right)^2 + \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2\kappa}\nu_0(\kappa,u-t)p_f\Sigma^3p_f'\Bigg],$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,2}_{w^V}(u) = L_2\left(\frac{L_1}{r}\eta\Sigma^2p_f' + \frac{L_2}{r}p_f\Sigma^2p_f'\right)\Bigg(-\frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\Bigg)p_f\Sigma^2p_f',$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,3}_{w^V}(u) = L_2\left(\frac{L_1}{r}\eta\Sigma^2p_f' + \frac{L_2}{r}p_f\Sigma^2p_f'\right)\Bigg(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\Bigg)p_f\Sigma^2p_f',$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,4}_{w^V}(u) = \left(\frac{L_1}{r}\eta\Sigma^2p_f' + \frac{L_2}{r}p_f\Sigma^2p_f'\right)^2 H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0),$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,5}_{w^V}(u) = H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\Bigg[\Bigg(-\frac{1-\epsilon}{r+\kappa}\left[K_1(\gamma_1,\gamma_3,u-t,\nu_0)+K_2(\gamma_1,\gamma_3,u-t,\nu_0)\right] + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\Bigg)\left(p_f\Sigma^2p_f'\right)^2 + \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2\kappa}\nu_0(\kappa,u-t)p_f\Sigma^3p_f'\Bigg],$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_1,6}_{w^V}(u) = \left(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right) \times\left(-\frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right)\left(p_f\Sigma^2p_f'\right)^2$$

$$\mathcal{D}^{\mathbb{C}\text{ov}_2,1}_{w^V}(u) = \Bigg[\frac{L_1L_2}{r}\eta\Sigma^2p_f' + \Bigg(\frac{L_2^2}{r} - \frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\Bigg)p_f\Sigma^2p_f'\Bigg] \times\left(-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_0(\kappa,u-t) + G(\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)p_f\Sigma^2p_f'$$

$$- \left[\frac{L_1 L_2}{r} \left(\eta\Sigma^3 p_f' - \frac{\eta\Sigma^2\eta'\eta\Sigma^2 p_f'}{\eta\Sigma\eta'} \right) + \left(\frac{L_2^2}{r} - \frac{1-\epsilon}{r+\kappa} K_2(\gamma_1^R, \gamma_3^R, u-t, \nu_0) \right.\right.$$

$$\left.\left. + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \right) \left(p_f \Sigma^3 p_f' - \frac{(\eta\Sigma^2 p_f')^2}{\eta\Sigma\eta'} \right) \right] \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \nu_0(\kappa, u-t),$$

$$\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_2,2}(u) = \left[\left(\frac{L_1}{r} \eta\Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right)^2 + \left(-\frac{1-\epsilon}{r+\kappa} \left[K_1(\gamma_1, \gamma_3, u-t, \nu_0) + K_2(\gamma_1, \gamma_3, u-t, \nu_0) \right] \right.\right.$$

$$\left.\left. + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u-t, \nu_0) \right) (p_f \Sigma^2 p_f')^2 + \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2\kappa} \nu_0(\kappa, u-t) p_f \Sigma^3 p_f' \right]$$

$$\times G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0).$$

Proof: The Sharpe ratio has the form in Lemma E.2. To compute $\{\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_1,i}(u)\}_{i=1,..,6}$, we use their definitions in Lemma E.5, together with (3.6), (B.13)-(B.16), (D.8), (D.19), (D.20), and the derivations in the proof of Lemma D.3. To compute $\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_2,1}(u)$, we use its definition in Lemma E.5 and note that (B.10), (B.11) and (D.19) imply

$$\mathbb{C}\mathrm{ov}_t(dR_t, w_u^V \Sigma p_f') = -\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \left(\Sigma^2 p_f' + \beta_2 \gamma_1 p_f \Sigma^2 p_f' \Sigma p_f' \right) \nu_0(\kappa, u-t) dt$$
$$+ G(\gamma_1, \gamma_2, \gamma_3, u-t, \nu_0) p_f \Sigma^2 p_f' \Sigma p_f' dt. \tag{E.26}$$

Combining (E.26) with (4.3) and $\eta\Sigma p_f' = 0$, we find

$$\mathbb{E}\left[\hat{w}_t^V \Lambda_u \mathbb{C}\mathrm{ov}_t(dR_t, w_u^V \Sigma p_f') \right]$$
$$= \mathbb{E}(w_t^V \Sigma p_f' \Lambda_u) \left(-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \beta_2 \gamma_1 \nu_0(\kappa, u-t) + G(\gamma_1, \gamma_2, \gamma_3, u-t, \nu_0) \right) p_f \Sigma^2 p_f' dt$$
$$- \mathbb{E}\left(w_t^V \left(\Sigma^2 p_f' - \frac{\Sigma\eta'}{\eta\Sigma\eta'} \eta\Sigma^2 p_f' \right) \Lambda_u \right) \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \nu_0(\kappa, u-t) dt. \tag{E.27}$$

Combining (E.27) with (3.6), (B.13), (B.15), (D.8), (D.19) and (D.20), we find that $\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_2,1}(u)$ is as in the proposition. To compute $\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_2,2}(u)$, we use its definition in Lemma E.5 and note that (B.10) and (3.6) imply

$$\mathbb{C}\mathrm{ov}_t(dR_t, \Lambda_u) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \Sigma p_f' dt. \tag{E.28}$$

Combining (E.28) with (4.3) and $\eta\Sigma p_f' = 0$, we find

$$\mathbb{E}\left[\hat{w}_t^V w_u^V \Sigma p_f' \mathbb{C}\mathrm{ov}_t(dR_t, \Lambda_u)\right] = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \mathbb{E}(w_t^V \Sigma p_f' w_u^V \Sigma p_f') G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0)dt. \tag{E.29}$$

Combining (E.29) with (D.20) and $\mathbb{C}\mathrm{ov}(w_t^V \Sigma p_f', w_u^V \Sigma p_f')$ from the derivation of $\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_1,1}(u)$, we find that $\mathcal{D}_{w^V}^{\mathbb{C}\mathrm{ov}_2,2}(u)$ is as in the proposition. ∎

Proposition E.3 computes the unconditional Sharpe ratio over investment horizon T of the momentum strategy.

Proposition E.3. *The unconditional Sharpe ratio of the momentum strategy (4.2) over investment horizon T is $SR_{w^M,T} = \frac{\mathcal{N}_{w^M}}{\sqrt{\mathcal{D}_{w^M} + \mathcal{D}^{\mathbb{C}\mathrm{ov}_1}_{w^M,T} + \mathcal{D}^{\mathbb{C}\mathrm{ov}_2}_{w^M,T}}}$, where $(\mathcal{N}_{w^M}, \mathcal{D}_{w^M})$ are derived in Proposition D.5 and $(\mathcal{D}^{\mathbb{C}\mathrm{ov}_1}_{w^M,T}, \mathcal{D}^{\mathbb{C}\mathrm{ov}_2}_{w^M,T})$ are defined in Lemma E.5, with*

$$\begin{aligned}
\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,1}(u) = L_2^2 \big[\big(& H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4) \\
& + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4) + k\max\{\tau + t - u, 0\} \big) \left(p_f \Sigma^2 p_f'\right)^2 \\
& + f\max\{\tau + t - u, 0\} p_f \Sigma^3 p_f' \big],
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,2}(u) = L_2(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f')(& H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) \\
& + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2)) p_f \Sigma^2 p_f',
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,3}(u) = L_2(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f')(& H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) \\
& + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2)) p_f \Sigma^2 p_f',
\end{aligned}$$

$$\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,4}(u) = (L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f')^2 H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0),$$

$$\begin{aligned}
\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,5}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \big[\big(& H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) \\
& + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4) \\
& + k\max\{\tau + t - u, 0\} \big) \left(p_f \Sigma^2 p_f'\right)^2 + f\max\{\tau + t - u, 0\} p_f \Sigma^3 p_f' \Big],
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,6}(u) = (& H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2)) \\
& \times (H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2)
\end{aligned}$$

$$+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2)) \left(p_f \Sigma^2 p_f'\right)^2$$

$$\mathcal{D}_{w^M}^{\mathbb{C}\text{ov}_2,1}(u) = \left[L_1 L_2 \tau \eta \Sigma^2 p_f' + \left(L_2^2 \tau + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2)\right) p_f \Sigma^2 p_f'\right]$$

$$\times \left(G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + k 1_{\{\tau+t-u>0\}}\right) p_f \Sigma^2 p_f'$$

$$+ \left[L_1 L_2 \tau \left(\eta \Sigma^3 p_f' - \frac{\eta \Sigma^2 \eta' \eta \Sigma^2 p_f'}{\eta \Sigma \eta'}\right) + \left(L_2^2 \tau + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2)\right.\right.$$

$$\left.\left.+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2)\right) \left(p_f \Sigma^3 p_f' - \frac{(\eta \Sigma^2 p_f')^2}{\eta \Sigma \eta'}\right)\right] f 1_{\{\tau+t-u>0\}},$$

$$\mathcal{D}_{w^M}^{\mathbb{C}\text{ov}_2,2}(u) = \left[\left(L_1 \tau \eta \Sigma^2 p_f' + L_2 \tau p_f \Sigma^2 p_f'\right)^2 + \left(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3)\right.\right.$$

$$+ H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_4)$$

$$\left.\left.+ k \max\{\tau + t - u, 0\}\right) \left(p_f \Sigma^2 p_f'\right)^2 + f \max\{\tau + t - u, 0\} p_f \Sigma^3 p_f'\right] G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0),$$

where $\mathcal{T}' = (u - t, \tau)$ *and* $\mathcal{T}'^- = (t - u, \tau)$.

Proof: The Sharpe ratio has the form in Lemma E.2. To compute $\{\mathcal{D}_{w^M}^{\mathbb{C}\text{ov}_1,i}(u)\}_{i=1,..6}$, we use their definitions in Lemma E.5 together with (A.2), (3.6), (B.10), (B.13), (D.8), (D.48), (D.49), (D.51) and the derivations in the proof of Lemma D.4. To compute $\mathcal{D}_{w^M}^{\mathbb{C}\text{ov}_2,1}(u)$, we use its definition in Lemma E.5 and note that (D.48) implies

$$\begin{aligned}
\mathbb{C}\text{ov}_t(dR_t, w_u^M \Sigma p_f') &= \left(f + \frac{k\Delta}{\eta \Sigma \eta'}\right) \left(\int_{u-\tau}^{u} \mathbb{C}\text{ov}_t\left(dR_t, \Lambda_{t'}\right) dt'\right) p_f \Sigma^2 p_f' + \mathbb{C}\text{ov}_t(dR_t, dR_t') 1_{\{\tau+t-u>0\}} \Sigma p_f' \\
&= G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) p_f \Sigma^2 p_f' \Sigma p_f' + (f \Sigma^2 p_f' + k p_f \Sigma^2 p_f' \Sigma p_f') 1_{\{\tau+t-u>0\}},
\end{aligned} \tag{E.30}$$

where the second step follows from (A.2), (3.6) and (B.10). Combining (E.30) with (4.3) and $\eta \Sigma p_f' = 0$, we find

$$\begin{aligned}
&\mathbb{E}\left[\hat{w}_t^M \Lambda_u \mathbb{C}\text{ov}_t(dR_t, w_u^M \Sigma p_f')\right] \\
&= \mathbb{E}(w_t^M \Sigma p_f' \Lambda_u) \left(G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + k 1_{\{\tau+t-u>0\}}\right) p_f \Sigma^2 p_f' dt \\
&\quad - \mathbb{E}\left(w_t^M \left(\Sigma^2 p_f' - \frac{\Sigma \eta'}{\eta \Sigma \eta'} \eta \Sigma^2 p_f'\right) \Lambda_u\right) f 1_{\{\tau+t-u>0\}} dt.
\end{aligned} \tag{E.31}$$

Combining (E.31) with (D.8), (D.49) and $\mathbb{C}\mathrm{ov}(w_t^M \Sigma p_f', \Lambda_u)$ from the derivation of $\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,2}(u)$, we find that $\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_2,1}(u)$ is as in the proposition. To compute $\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_2,2}(u)$, we use its definition in Lemma E.5 and combine

$$\mathbb{E}\left[\hat{w}_t^M w_u^M \Sigma p_f' \mathbb{C}\mathrm{ov}_t(dR_t, \Lambda_u)\right] = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \mathbb{E}(w_t^M \Sigma p_f' w_u^M \Sigma p_f') G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) dt,$$

which is the counterpart of (E.29) for momentum, with (D.49) and $\mathbb{C}\mathrm{ov}(w_t^M \Sigma p_f', w_u^M \Sigma p_f')$ from the derivation of $\mathcal{D}_{w^M}^{\mathbb{C}\mathrm{ov}_1,1}(u)$. ∎

Proposition E.4 computes the Sharpe ratios of the value strategy and the momentum strategy conditional on $(\hat{C}_t, y_t)$ and over investment horizon T.

Proposition E.4. *The Sharpe ratios of the value strategy (4.1) and the momentum strategy (4.2) conditional on $(\hat{C}_t, y_t)$ and over investment horizon T are $SR_{w^j,t,T} = \frac{\mathcal{N}_{w^j,t,T}}{\sqrt{\mathcal{D}_{w^j,t,T} + \mathcal{D}_{w^j,t,T}^{\mathbb{C}\mathrm{ov}_1} + \mathcal{D}_{w^j,t,T}^{\mathbb{C}\mathrm{ov}_2}}}$, where $j = V$ for value and $j = M$ for momentum,*

$$\mathcal{N}_{w^j,t,T} = \frac{1}{T}\int_t^{t+T} \left[\left(L_2 + \delta_{12,u-t}^{\Lambda}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{\Lambda}(y_t - \bar{y})\right)\right.$$
$$\left. \times \left(L_1 z^j \eta \Sigma^2 p_f' + \left(L_2 z^j + \delta_{12,u-t}^{j}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{j}(y_t - \bar{y})\right) p_f \Sigma^2 p_f'\right) + \mathcal{C}_{u-t,u-t}^{j\Lambda} p_f \Sigma^2 p_f'\right] du,$$

$$\mathcal{D}_{w^j,t,T} = \frac{1}{T}\int_t^{t+T} \left[L_1^2 (z^j)^2 \Delta_1 + 2L_1 z^j \left(L_2 z^j + \delta_{12,u-t}^{j}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{j}(y_t - \bar{y})\right)\Delta_2\right.$$
$$\left. \times \left(\left(L_2 z^j + \delta_{12,u-t}^{j}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{j}(y_t - \bar{y})\right)^2 + \mathcal{C}_{u-t,u-t}^{j}\right)\Delta_3 + \mathcal{C}_{u-t}^{j\Sigma}\Delta_4\right] du,$$

and $(\mathcal{D}_{w^j,t,T}^{\mathbb{C}\mathrm{ov}_1}, \mathcal{D}_{w^j,t,T}^{\mathbb{C}\mathrm{ov}_2})$ are defined in Lemma E.4, with

$$\mathcal{D}_{w^j,t}^{\mathbb{C}\mathrm{ov}_1,1}(u,u') = \left(L_2 + \delta_{12,u-t}^{\Lambda}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{\Lambda}(y_t - \bar{y})\right)$$
$$\times \left(L_2 + \delta_{12,u'-t}^{\Lambda}(\hat{C}_t - \bar{C}) + \delta_{3,u'-t}^{\Lambda}(y_t - \bar{y})\right)\left(\mathcal{C}_{u-t,u'-t}^{j}(p_f \Sigma^2 p_f')^2 + \mathcal{C}_{u'-u}^{j\Sigma} p_f \Sigma^3 p_f'\right),$$

$$\mathcal{D}_{w^j,t}^{\mathbb{C}\mathrm{ov}_1,2}(u,u') = \left(L_2 + \delta_{12,u-t}^{\Lambda}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{\Lambda}(y_t - \bar{y})\right)$$
$$\times \left(L_1 z^j \eta \Sigma^2 p_f' + \left(L_2 z^j + \delta_{12,u'-t}^{j}(\hat{C}_t - \bar{C}) + \delta_{3,u'-t}^{j}(y_t - \bar{y})\right) p_f \Sigma^2 p_f'\right)\mathcal{C}_{u-t,u'-t}^{j\Lambda} p_f \Sigma^2 p_f',$$

$$\mathcal{D}_{w^j,t}^{\mathbb{C}\mathrm{ov}_1,3}(u,u') = \left(L_1 z^j \eta \Sigma^2 p_f' + \left(L_2 z^j + \delta_{12,u-t}^{j}(\hat{C}_t - \bar{C}) + \delta_{3,u-t}^{j}(y_t - \bar{y})\right) p_f \Sigma^2 p_f'\right)$$

$$\times\left(L_2+\delta^{\Lambda}_{12,u'-t}(\hat{C}_t-\bar{C})+\delta^{\Lambda}_{3,u'-t}(y_t-\bar{y})\right)\mathcal{C}^{\Lambda j}_{u-t,u'-t}p_f\Sigma^2p_f',$$

$$\mathcal{D}^{\mathbb{C}\mathrm{ov}_1,4}_{w^j,t}(u,u')=\left(L_1z^j\eta\Sigma^2p_f'+\left(L_2z^j+\delta^j_{12,u-t}(\hat{C}_t-\bar{C})+\delta^j_{3,u-t}(y_t-\bar{y})\right)p_f\Sigma^2p_f'\right)$$

$$\times\left(L_1z^j\eta\Sigma^2p_f'+\left(L_2z^j+\delta^j_{12,u'-t}(\hat{C}_t-\bar{C})+\delta^j_{3,u'-t}(y_t-\bar{y})\right)p_f\Sigma^2p_f'\right)\mathcal{C}^{\Lambda}_{u-t,u'-t},$$

$$\mathcal{D}^{\mathbb{C}\mathrm{ov}_1,5}_{w^M,t}(u,u')=\mathcal{C}^{\Lambda}_{u-t,u'-t}\left(\mathcal{C}^{j}_{u-t,u'-t}(p_f\Sigma^2p_f')^2+\mathcal{C}^{j\Sigma}_{u'-u}p_f\Sigma^3p_f'\right),$$

$$\mathcal{D}^{\mathbb{C}\mathrm{ov}_1,6}_{w^M,t}(u,u')=\mathcal{C}^{j\Lambda}_{u-t,u'-t}\mathcal{C}^{\Lambda j}_{u-t,u'-t}(p_f\Sigma^2p_f')^2,$$

$$\mathcal{D}^{\mathbb{C}\mathrm{ov}_2,1}_{w^M,t}(u,u')=\Bigg[\left(L_1z^j\eta\Sigma^2p_f'+\left(L_2z^j+\delta^j_{12,u-t}(\hat{C}_t-\bar{C})+\delta^j_{3,u-t}(y_t-\bar{y})\right)p_f\Sigma^2p_f'\right)$$

$$\times\left(L_2+\delta^{\Lambda}_{12,u'-t}(\hat{C}_t-\bar{C})+\delta^{\Lambda}_{3,u'-t}(y_t-\bar{y})\right)+\mathcal{C}^{j\Lambda}_{u-t,u'-t}p_f\Sigma^2p_f'\Bigg]$$

$$\times\Bigg[1_{\{j=V\}}\left(-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_0(\kappa,u'-u)+G(\gamma_1,\gamma_2,\gamma_3,u'-u,\nu_0)\right)$$

$$+1_{\{j=M\}}\left(G(\gamma_1^R,\gamma_2^R,\gamma_3^R,(u'-u,\tau),\nu_1)+k1_{\{\tau+u-u'>0\}}\right)\Bigg]p_f\Sigma^2p_f'$$

$$+\Bigg[\left(L_2+\delta^{\Lambda}_{12,u-t}(\hat{C}_t-\bar{C})+\delta^{\Lambda}_{3,u-t}(y_t-\bar{y})\right)\left(L_1z^j\left(\eta\Sigma^3p_f'-\frac{\eta\Sigma^2\eta'\eta\Sigma^2p_f'}{\eta\Sigma\eta'}\right)\right.$$

$$\left.+\left(L_2z^j+\delta^j_{12,u'-t}(\hat{C}_t-\bar{C})+\delta^j_{3,u'-t}(y_t-\bar{y})\right)\left(p_f\Sigma^3p_f'-\frac{(\eta\Sigma^2p_f')^2}{\eta\Sigma\eta'}\right)\right)$$

$$+\mathcal{C}^{j\Lambda}_{u-t,u'-t}\left(p_f\Sigma^3p_f'-\frac{(\eta\Sigma^2p_f')^2}{\eta\Sigma\eta'}\right)\Bigg]\left[1_{\{j=M\}}f1_{\{\tau+u-u'>0\}}-1_{\{j=V\}}\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_0(\kappa,u'-u)\right],$$

$$\mathcal{D}^{\mathbb{C}\mathrm{ov}_2,2}_{w^M,t}(u,u')=\Bigg[\left(L_1z^j\eta\Sigma^2p_f'+\left(L_2z^j+\delta^j_{12,u-t}(\hat{C}_t-\bar{C})+\delta^j_{3,u-t}(y_t-\bar{y})\right)p_f\Sigma^2p_f'\right)$$

$$\times\left(L_1z^j\eta\Sigma^2p_f'+\left(L_2z^j+\delta^j_{12,u'-t}(\hat{C}_t-\bar{C})+\delta^j_{3,u'-t}(y_t-\bar{y})\right)p_f\Sigma^2p_f'\right)$$

$$+\mathcal{C}^{j}_{u-t,u'-t}(p_f\Sigma^2p_f')^2+\mathcal{C}^{j\Sigma}_{u'-u}p_f\Sigma^3p_f'\Bigg]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0),$$

and $(z^V,z^M)=\left(\frac{1}{r},\tau\right)$.

Proof: The Sharpe ratio has the form in Lemma E.1. To compute $\mathcal{N}_{w^j,t,T}$, we note that (D.1) and (E.1) imply

$$\mathcal{N}_{w^j,t,T}\equiv\frac{1}{T}\int_t^{t+T}\left(f+\frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}_{\mathcal{I}_t}\left(\Lambda_u w^j_u\Sigma p_f'\right)du,$$

and we compute the integrand using the decomposition in (D.17) together with (D.24)-(D.26), (D.56) and (D.57). To compute $\mathcal{D}_{w^j,t,T}$, we note that (D.2) and (E.2) imply

$$\mathcal{D}_{w^j,t,T} \equiv \frac{1}{T}\int_t^{t+T}\left\{f\left[\mathbb{E}_{\mathcal{I}_t}\left(w_u^j\Sigma(w_u^j)'\right) - \frac{\mathbb{E}_{\mathcal{I}_t}\left[\left(w_u^j\Sigma\eta'\right)^2\right]}{\eta\Sigma\eta'}\right] + k\mathbb{E}_{\mathcal{I}_t}\left[\left(w_u^j\Sigma p_f'\right)^2\right]\right\}du,$$

and we compute the integrand using the decomposition in (D.18) together with (D.24), (D.28) (D.56) and (D.59). To compute $\{\mathcal{D}_{w^j,t}^{\mathbb{C}\text{ov}_1,i}(u)\}_{i=1,..,6}$, we use their definitions in Lemma E.4 together with (D.24)-(D.29) and (D.56)-(D.59). To compute $\{\mathcal{D}_{w^j,t}^{\mathbb{C}\text{ov}_2,i}(u)\}_{i=1,2}$, we use their definitions in Lemma E.4 and proceed as in the proofs of Propositions E.2 and E.3 replacing unconditional expectations $\mathbb{E}(w_t^j\Sigma p_f'\Lambda_u)$ and $\mathbb{E}(w_t^j\Sigma p_f' w_u^j\Sigma p_f')$ by conditional expectations $\mathbb{E}_{\mathcal{I}_t}(w_u^j\Sigma p_f'\Lambda_{u'})$ and $\mathbb{E}_{\mathcal{I}_t}(w_u^j\Sigma p_f' w_{u'}^j\Sigma p_f')$. ∎

Lemma E.6 computes the unconditional covariance between the returns of (the index-adjusted versions of) two strategies (w_t^A, w_t^B) over investment horizon T.

Lemma E.6. *The unconditional covariance between the returns of (w_t^A, w_t^B) over investment horizon T is given by*

$$\frac{1}{T}\mathbb{C}\text{ov}\left(\int_0^T \hat{w}_t^A dR_t, \int_0^T \hat{w}_t^B dR_t\right) = \mathcal{G}_{w^A,w^B} + \mathcal{G}_{w^A,w^B,T}^{\mathbb{C}\text{ov}_1} + \mathcal{G}_{w^A,w^B,T}^{\mathbb{C}\text{ov}_2}, \tag{E.32}$$

where

$$\begin{aligned}\mathcal{G}_{w^A,w^B} &\equiv \frac{1}{dt}\mathbb{C}\text{ov}\left(\hat{w}_t^A dR_t, \hat{w}_t^B dR_t\right)\\ &= f\left[\mathbb{E}\left(w_t^A\Sigma\left(w_t^B\right)'\right) - \frac{\mathbb{E}\left(w_t^A\Sigma\eta' w_t^B\Sigma\eta'\right)}{\eta\Sigma\eta'}\right] + k\mathbb{E}\left(w_t^A\Sigma p_f' w_t^B\Sigma p_f'\right),\end{aligned}$$

is the unconditional version of $\mathcal{G}_{w^A,w^B,t}$ defined in Lemma D.6, and

$$\mathcal{G}_{w^A,w^B,T}^{\mathbb{C}\text{ov}_1} \equiv \frac{1}{dt}\int_t^{t+T}\left(1-\frac{u-t}{T}\right)\left\{\mathbb{C}\text{ov}\left[\hat{w}_t^A\mathbb{E}_t(dR_t), \hat{w}_u^B\mathbb{E}_u(dR_u)\right] + \mathbb{C}\text{ov}\left[\hat{w}_t^B\mathbb{E}_t(dR_t), \hat{w}_u^A\mathbb{E}_u(dR_u)\right]\right\}, \tag{E.33}$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2}_{w^A,w^B,T} \equiv \frac{1}{dt}\int_t^{t+T}\left(1-\frac{u-t}{T}\right)\left(\mathbb{E}\left\{\hat{w}_t^A \mathbb{C}\mathrm{ov}_t\left[dR_t, \hat{w}_u^B \mathbb{E}_u(dR_u)\right]\right\} + \mathbb{E}\left\{\hat{w}_t^B \mathbb{C}\mathrm{ov}_t\left[dR_t, \hat{w}_u^A \mathbb{E}_u(dR_u)\right]\right\}\right). \tag{E.34}$$

When the strategies (w_t^A, w_t^B) *are linear,* $\mathcal{G}^{\mathbb{C}\mathrm{ov}_1}_{w^A,w^B,T} = \sum_{i=1}^6 \int_t^{t+T}\left(1-\frac{u-t}{T}\right)\left[\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,i}_{w^A,w^B}(u) + \mathcal{G}^{\mathbb{C}\mathrm{ov}_1,i}_{w^B,w^A}(u)\right] du$*, where*

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,1}_{w^j,w^k}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 E(\Lambda_t)^2 \mathbb{C}\mathrm{ov}(w_t^j \Sigma p_f', w_u^k \Sigma p_f'),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,2}_{w^j,w^k}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}(\Lambda_t)\mathbb{E}(w_t^k \Sigma p_f')\mathbb{C}\mathrm{ov}(w_t^j \Sigma p_f', \Lambda_u),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,3}_{w^j,w^k}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}(\Lambda_t)\mathbb{E}(w_t^j \Sigma p_f')\mathbb{C}\mathrm{ov}(\Lambda_t, w_u^k \Sigma p_f'),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w^j,w^k}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{E}(w_t^j \Sigma p_f')\mathbb{E}(w_t^k \Sigma p_f')\mathbb{C}\mathrm{ov}(\Lambda_t, \Lambda_u),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w^j,w^k}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{C}\mathrm{ov}(\Lambda_t, \Lambda_u)\mathbb{C}\mathrm{ov}(w_t^j \Sigma p_f', w_u^k \Sigma p_f'),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w^j,w^k}(u) \equiv \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)^2 \mathbb{C}\mathrm{ov}(w_t^j \Sigma p_f', \Lambda_u)\mathbb{C}\mathrm{ov}(\Lambda_t, w_u^k \Sigma p_f'),$$

and $\mathcal{G}^{\mathbb{C}\mathrm{ov}_2}_{w^A,w^B,T} = \sum_{i=1}^2 \int_t^{t+T}\left(1-\frac{u-t}{T}\right)\left[\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,i}_{w^A,w^B}(u) + \mathcal{G}^{\mathbb{C}\mathrm{ov}_2,i}_{w^B,w^A}(u)\right] du$*, where*

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w^j,w^k}(u) \equiv \frac{1}{dt}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left[\hat{w}_t^j \Lambda_u \mathbb{C}\mathrm{ov}_t(dR_t, w_u^k \Sigma p_f')\right],$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w^j,w^k}(u) \equiv \frac{1}{dt}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\mathbb{E}\left[\hat{w}_t^j w_u^k \Sigma p_f' \mathbb{C}\mathrm{ov}_t(dR_t, \Lambda_u)\right],$$

for $j, k \in \{A, B\}$ *and* $j \neq k$*.*

Proof: The covariance between the returns of (w_t^A, w_t^B) conditional on $\mathcal{I}_t$ and over investment horizon T is given by

$$\frac{1}{T}\mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\int_t^{t+T} \hat{w}_u^A dR_u, \int_t^{t+T} \hat{w}_u^B dR_u\right)$$
$$= \frac{1}{T}\int_t^{t+T} \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\hat{w}_u^A dR_u, \hat{w}_u^B dR_u\right) + \frac{1}{T}\int_{u=t}^{t+T}\int_{u'=t}^{t+T} \mathbb{C}\mathrm{ov}_{\mathcal{I}_t}\left(\hat{w}_u^A dR_u, \hat{w}_{u'}^B dR_{u'}\right)$$

$$= \frac{1}{T}\int_{t}^{t+T} \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u^A dR_u, \hat{w}_u^B dR_u\right) + \frac{1}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T} \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u^A dR_u, \hat{w}_{u'}^B dR_{u'}\right)$$

$$+ \frac{1}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T} \mathbb{C}\text{ov}_{\mathcal{I}_t}\left(\hat{w}_u^B dR_u, \hat{w}_{u'}^A dR_{u'}\right) \tag{E.35}$$

where the first step follows by separating the covariance between contemporaneous returns and the covariance between lagged returns. Proceeding as in the derivation of (E.7), we can write the second term in (E.35) as

$$\frac{1}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T} \left\{\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u^A \mathbb{E}_u(dR_u), \hat{w}_{u'}^B \mathbb{E}_{u'}(dR_{u'})\right] + \mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u^B \mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u'}^A \mathbb{E}_{u'}(dR_{u'}))\right]\right\} \tag{E.36}$$

and the third term as

$$\frac{1}{T}\int_{u=t}^{t+T}\int_{u'=u}^{t+T} \left\{\mathbb{C}\text{ov}_{\mathcal{I}_t}\left[\hat{w}_u^B \mathbb{E}_u(dR_u), \hat{w}_{u'}^A \mathbb{E}_{u'}(dR_{u'})\right] + \mathbb{E}_{\mathcal{I}_t}\left[\hat{w}_u^B \mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u'}^A \mathbb{E}_{u'}(dR_{u'}))\right]\right\}. \tag{E.37}$$

When the covariance is unconditional ($\mathcal{I}_t = \emptyset$), we can proceed as in the proof of Lemma E.2 to show that the first term in (E.35) becomes $\mathcal{G}_{w^A,w^B}$, and (E.36) and (E.37) become $\mathcal{G}^{\mathbb{C}\text{ov}_1}_{w^A,w^B,T}$ and $\mathcal{G}^{\mathbb{C}\text{ov}_2}_{w^A,w^B,T}$, respectively. When the strategies (w_t^A, w_t^B) are linear, we can proceed as in the proof of Lemma E.5 to show $\mathcal{G}^{\mathbb{C}\text{ov}_1}_{w^A,w^B,T} = \sum_{i=1}^{6}\int_t^{t+T}\left(1-\frac{u-t}{T}\right)\left[\mathcal{G}^{\mathbb{C}\text{ov}_1,i}_{w^A,w^B}(u) + \mathcal{G}^{\mathbb{C}\text{ov}_1,i}_{w^B,w^A}(u)\right] du$ and $\mathcal{G}^{\mathbb{C}\text{ov}_2}_{w^A,w^B,T} = \sum_{i=1}^{2}\int_t^{t+T}\left(1-\frac{u-t}{T}\right)\left[\mathcal{G}^{\mathbb{C}\text{ov}_2,i}_{w^A,w^B}(u) + \mathcal{G}^{\mathbb{C}\text{ov}_2,i}_{w^B,w^A}(u)\right] du$, with $\left\{\{\mathcal{G}^{\mathbb{C}\text{ov}_1,i}_{w^j,w^k}(u)\}_{i=1,..,6}, \{\mathcal{G}^{\mathbb{C}\text{ov}_2,i}_{w^j,w^k}(u)\}_{i=1,2}\right\}_{j,k\in\{A,B\},j\neq k}$ as in the proposition. ∎

Proposition E.5 computes the unconditional instantaneous correlation of a general strategy of the form $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ with value and momentum.

Proposition E.5. *The unconditional instantaneous correlation of a strategy $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ with value and momentum is $\mathbb{C}\text{orr}(\hat{w}_t^j dR_t, \hat{w}_t dR_t) = \frac{\mathcal{G}_{w^j,w}}{\sqrt{\mathcal{D}_{w^j}\mathcal{D}_w}}$, where $j = V$ for value and $j = M$ for momentum, $\mathcal{D}_w$ is derived in Proposition E.1, $\{\mathcal{D}_{w^j}\}_{j=V,M}$ are derived in Propositions D.2 and D.5,*

$$\mathcal{G}_{w^V,w} = \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\left(\frac{L_1}{r}\eta\Sigma^2 p_f' + \frac{L_2}{r}p_f\Sigma^2 p_f'\right)\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)\right.$$
$$\left. + \left(-\frac{1-\epsilon}{r+\kappa}K_1(\delta_1,\delta_3,0,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3,0,\nu_0)\right)p_f\Sigma^2 p_f'\right],$$

$$\mathcal{G}_{w^M,w} = \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\left(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f'\right)\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)\right.$$
$$\left. + \left(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, \mathcal{T}, \nu_2) + G(\delta_1, \delta_2, \delta_3, \mathcal{T}, \nu_2)\right) p_f\Sigma^2 p_f'\right],$$

and $\mathcal{T} = (0, \tau)$.

Proof: Since $\mathcal{D}_{w^V} = \mathbb{V}\text{ar}(\hat{w}_t^V dR_t)$, $\mathcal{D}_{w^M} = \mathbb{V}\text{ar}(\hat{w}_t^M dR_t)$ and $\mathcal{D}_w = \mathbb{V}\text{ar}(\hat{w}_t dR_t)$, the results on the unconditional instantaneous correlation will follow if we show $\mathbb{C}\text{ov}(\hat{w}_t^V dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^V,w}dt$ and $\mathbb{C}\text{ov}(\hat{w}_t^M dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^M,w}dt$. Using Lemma D.6 and noting that $\eta\Sigma p_f' = 0$ implies $w_t\Sigma\eta' = 0$, we find

$$\begin{aligned}
&\mathbb{C}\text{ov}\left(\hat{w}_t^j dR_t, \hat{w}_t dR_t\right) \\
&= f\mathbb{E}\left(w_t^j \Sigma w_t'\right) + k\mathbb{E}\left(w_t^j \Sigma p_f' w_t \Sigma p_f'\right) \\
&= f\mathbb{E}(w_t^j)\Sigma\mathbb{E}(w_t)' + k\mathbb{E}\left(w_t^j\right)\Sigma p_f'\mathbb{E}(w_t)\Sigma p_f' + f\mathbb{C}\text{ov}\left(w_t^j, \Sigma w_t'\right) + k\mathbb{C}\text{ov}\left(w_t^j\Sigma p_f', w_t\Sigma p_f'\right) \\
&= \left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left[\mathbb{E}\left(w_t^j\right)\Sigma p_f'\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right) + \mathbb{C}\text{ov}\left(w_t^j, \delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t\right)\Sigma p_f'\right]
\end{aligned} \tag{E.38}$$

for $j = V, M$, where the third step follows from $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$. Combining (E.38) with (D.20), and computing $\mathbb{C}\text{ov}\left(w_t^V, \delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t\right)$ as in the derivation of (D.22), we find $\mathbb{C}\text{ov}(\hat{w}_t^V dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^V,w}dt$. Combining (E.38) with (D.49), and computing $\mathbb{C}\text{ov}\left(w_t^M, \delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t\right)$ as in the derivation of (D.52), we find $\mathbb{C}\text{ov}(\hat{w}_t^M dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^M,w}dt$. ∎

Proposition E.6 computes the unconditional correlation of a general strategy of the form $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ with value and momentum over investment horizon T.

Proposition E.6. *The unconditional correlation of a strategy* $w_t = (\delta_0 + \delta_1\hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ *with value and momentum over investment horizon* T *is*

$$\mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^j dR_u, \int_t^{t+T}\hat{w}_u dR_u\right) = \frac{\mathcal{G}_{w^j,w} + \mathcal{G}_{w^j,w,T}^{\mathbb{C}\text{ov}_1} + \mathcal{G}_{w^j,w,T}^{\mathbb{C}\text{ov}_2}}{\sqrt{\left(\mathcal{D}_{w^j} + \mathcal{D}_{w^j,T}^{\mathbb{C}\text{ov}_1} + \mathcal{D}_{w^j,T}^{\mathbb{C}\text{ov}_2}\right)\left(\mathcal{D}_w + \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_1} + \mathcal{D}_{w,T}^{\mathbb{C}\text{ov}_2}\right)}}, \tag{E.39}$$

where $j = V$ for value and $j = M$ for momentum, $\left(\mathcal{D}_w, \mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_1}, \mathcal{D}_{w,T}^{\mathbb{C}\mathrm{ov}_2}\right)$ *are derived in Proposition E.1,* $\{\mathcal{D}_{w^j}\}_{j=V,M}$ *are derived in Propositions D.2 and D.5,* $\left\{\left(\mathcal{D}_{w^j,T}^{\mathbb{C}\mathrm{ov}_1}, \mathcal{D}_{w^j,T}^{\mathbb{C}\mathrm{ov}_2}\right)\right\}_{j=V,M}$ *are derived in Propositions E.2 and E.3,* $\{\mathcal{G}_{w^j,w}\}_{j=V,M}$ *are derived in Proposition E.5, and* $\left\{\left(\mathcal{G}_{w^j,w,T}^{\mathbb{C}\mathrm{ov}_1}, \mathcal{G}_{w^j,w,T}^{\mathbb{C}\mathrm{ov}_2}\right)\right\}_{j=V,M}$ *are defined in Lemma E.6, with*

$$\mathcal{G}_{w^V,w}^{\mathbb{C}\mathrm{ov}_1,1}(u) = L_2^2\left(-\frac{1-\epsilon}{r+\kappa}K_2(\delta_1,\delta_3,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}_{w,w^V}^{\mathbb{C}\mathrm{ov}_1,1}(u) = L_2^2\left(-\frac{1-\epsilon}{r+\kappa}K_1(\delta_1,\delta_3,u-t,\nu_0) + H(\delta_1,\delta_2,\delta_3,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}_{w^M,w}^{\mathbb{C}\mathrm{ov}_1,1}(u) = L_2^2\left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,\mathcal{T}'^-,\nu_2) + G(\delta_1,\delta_2,\delta_3,\mathcal{T}'^-,\nu_2)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\begin{aligned}\mathcal{G}_{w,w^M}^{\mathbb{C}\mathrm{ov}_1,1}(u) = L_2^2\big(&H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,\mathcal{T}',\nu_2)\\ &+G(\delta_1,\delta_2,\delta_3,\mathcal{T}',\nu_2)\big)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},\end{aligned}$$

$$\begin{aligned}\mathcal{G}_{w^V,w}^{\mathbb{C}\mathrm{ov}_1,2}(u) = L_2&\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)\\ &\times\left(-\frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},\end{aligned}$$

$$\mathcal{G}_{w,w^V}^{\mathbb{C}\mathrm{ov}_1,2}(u) = L_2\left(\frac{L_1}{r}\eta\Sigma^2 p_f' + \frac{L_2}{r}p_f\Sigma^2 p_f'\right)H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\begin{aligned}\mathcal{G}_{w^M,w}^{\mathbb{C}\mathrm{ov}_1,2}(u) = L_2&\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)\\ &\times\left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2) + G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},\end{aligned}$$

$$\mathcal{G}_{w,w^M}^{\mathbb{C}\mathrm{ov}_1,2}(u) = L_2\left(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f'\right)H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}_{w^V,w}^{\mathbb{C}\mathrm{ov}_1,3}(u) = L_2\left(\frac{L_1}{r}\eta\Sigma^2 p_f' + \frac{L_2}{r}p_f\Sigma^2 p_f'\right)H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\begin{aligned}\mathcal{G}_{w,w^V}^{\mathbb{C}\mathrm{ov}_1,3}(u) = L_2&\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)\\ &\times\left(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},\end{aligned}$$

$$\mathcal{G}_{w^M,w}^{\mathbb{C}\mathrm{ov}_1,3}(u) = L_2\left(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f'\right)H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,3}_{w^M,w}(u) = L_2\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)\left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1)\right.$$

$$\left. + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_2) + G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_2)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w^V,w}(u) = \mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w,w^V}(u) = \left(\frac{L_1}{r}\eta\Sigma^2 p_f' + \frac{L_2}{r}p_f\Sigma^2 p_f'\right)\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)$$

$$\times H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w^M,w}(u) = \mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w,w^M}(u) = \left(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f'\right)\left(\delta_0 + (\delta_1+\delta_2)\bar{C} + \delta_3\bar{y}\right)$$

$$\times H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w^V,w}(u) = H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)$$

$$\times\left(-\frac{1-\epsilon}{r+\kappa}K_2(\delta_1,\delta_3,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w,w^V}(u) = H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)$$

$$\times\left(-\frac{1-\epsilon}{r+\kappa}K_1(\delta_1,\delta_3,u-t,\nu_0) + H(\delta_1,\delta_2,\delta_3,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w^M,w}(u) = H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)$$

$$\times\left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,\mathcal{T}'^{-},\nu_2) + G(\delta_1,\delta_2,\delta_3,\mathcal{T}'^{-},\nu_2)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w^M,w}(u) = H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\left(H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1)\right.$$

$$\left. + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,\mathcal{T}',\nu_2) + G(\delta_1,\delta_2,\delta_3,\mathcal{T}',\nu_2)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w^V,w}(u) = \left(-\frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right)$$

$$\times H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w,w^V}(u) = H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)$$

$$\times\left(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w^M,w}(u) = \left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^{-},\nu_2) + G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^{-},\nu_2)\right)$$

$$\times H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w,w^M}(u) = H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1)\right.$$

$$\left.+H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_2)+G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_2)\right)\frac{p_f\Sigma^2 p_f'\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w^V,w}(u) = \left[\frac{L_1L_2}{r}\eta\Sigma^2 p_f' + \left(\frac{L_2^2}{r} - \frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right)\right.$$

$$\left.\times p_f\Sigma^2 p_f'\right]G(\delta_1,\delta_2,\delta_3,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w,w^V}(u) = \left[L_2\left(\delta_0+(\delta_1+\delta_2)\bar{C}+\delta_3\bar{y}\right)+H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right]p_f\Sigma^2 p_f'$$

$$\times\left[\left(-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_0(\kappa,u-t)+G(\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)\frac{\Delta}{\eta\Sigma\eta'} - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_0(\kappa,u-t)\right],$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w^M,w}(u) = \left[L_1L_2\tau\eta\Sigma^2 p_f' + \left(L_2^2\tau + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)+G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)\right)\right.$$

$$\left.\times p_f\Sigma^2 p_f'\right]G(\delta_1,\delta_2,\delta_3,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w,w^M}(u) = \left[L_2\left(\delta_0+(\delta_1+\delta_2)\bar{C}+\delta_3\bar{y}\right)+H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right]p_f\Sigma^2 p_f'$$

$$\times\left[\left(G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1)+k1_{\{\tau+t-u>0\}}\right)\frac{\Delta}{\eta\Sigma\eta'}+f1_{\{\tau+t-u>0\}}\right],$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w^V,w}(u) = \left[\left(\frac{L_1}{r}\eta\Sigma^2 p_f' + \frac{L_2}{r}p_f\Sigma^2 p_f'\right)\left(\delta_0+(\delta_1+\delta_2)\bar{C}+\delta_3\bar{y}\right)+\left(-\frac{1-\epsilon}{r+\kappa}K_2(\delta_1,\delta_3,u-t,\nu_0)\right.\right.$$

$$\left.\left.+H(\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3,u-t,\nu_0)\right)p_f\Sigma^2 p_f'\right]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w,w^V}(u) = \left[\left(\frac{L_1}{r}\eta\Sigma^2 p_f' + \frac{L_2}{r}p_f\Sigma^2 p_f'\right)\left(\delta_0+(\delta_1+\delta_2)\bar{C}+\delta_3\bar{y}\right)+\left(-\frac{1-\epsilon}{r+\kappa}K_1(\delta_1,\delta_3,u-t,\nu_0)\right.\right.$$

$$\left.\left.+H(\delta_1,\delta_2,\delta_3,\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)p_f\Sigma^2 p_f'\right]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w^M,w}(u) = \left[\left(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f'\right)\left(\delta_0+(\delta_1+\delta_2)\bar{C}+\delta_3\bar{y}\right)+\left(H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,\mathcal{T}'^-,\nu_2)\right.\right.$$

$$\left.\left.+G(\delta_1,\delta_2,\delta_3,\mathcal{T}'^-,\nu_2)\right)p_f\Sigma^2 p_f'\right]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w,w^M}(u) = \left[\left(L_1\tau\eta\Sigma^2 p_f' + L_2\tau p_f\Sigma^2 p_f'\right)\left(\delta_0+(\delta_1+\delta_2)\bar{C}+\delta_3\bar{y}\right)+\left(H(\delta_1,\delta_2,\delta_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1)\right.\right.$$

$$\left.\left.+H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\delta_1,\delta_2,\delta_3,\mathcal{T}',\nu_2)+G(\delta_1,\delta_2,\delta_3,\mathcal{T}',\nu_2)\right)p_f\Sigma^2 p_f'\right]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\frac{\Delta}{\eta\Sigma\eta'},$$

where $\mathcal{T}' = (u-t,\tau)$ *and* $\mathcal{T}'^{-} = (t-u,\tau)$.

Proof: To show the proposition, we need to show that the definitions of $\left\{\left\{\mathcal{G}^{\mathbb{C}\text{ov}_1,i}_{w^j,w,T}(u)\right\}_{i=1,..,6},\right.$ $\left.\left\{\mathcal{G}^{\mathbb{C}\text{ov}_2,i}_{w^j,w,T}(u)\right\}_{i=1,2}\right\}_{j=V,M}$ in Lemma E.6 yield the equations in the proposition. The equations follow from the derivations in Propositions E.2 and E.3. (These derivations determine the covariances $\left\{\mathbb{C}\text{ov}(w^j_t\Sigma p'_f,(\delta_0+\delta_1\hat{C}_u+\delta_2 C_u+\delta_3 y_u))\right\}_{j=V,M}$ and $\left\{\mathbb{C}\text{ov}((\delta_0+\delta_1\hat{C}_t+\delta_2 C_t+\delta_3 y_t),w^j_u\Sigma p'_f)\right\}_{j=V,M}$ by replacing $(\gamma^R_1,\gamma^R_2,\gamma^R_3)$ by $(\delta_1,\delta_2,\delta_3)$.) ∎

Proposition E.7 computes the unconditional correlation between the returns of value and momentum strategies over investment horizon T.

Proposition E.7. *The unconditional correlation between the returns of the value strategy (4.1) and the momentum strategy (4.2) over investment horizon T is*

$$\mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}^V_u dR_u,\int_t^{t+T}\hat{w}^M_u dR_u\right)=\frac{\mathcal{G}_{w^V,w^M}+\mathcal{G}^{\mathbb{C}\text{ov}_1}_{w^V,w^M,T}+\mathcal{G}^{\mathbb{C}\text{ov}_2}_{w^V,w^M,T}}{\sqrt{\left(\mathcal{D}_{w^V}+\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^V,T}+\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^V,T}\right)\left(\mathcal{D}_{w^M}+\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^M,T}+\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^M,T}\right)}},\tag{E.40}$$

where $\mathcal{D}_{w^V}$, $\mathcal{D}_{w^M}$ and $\mathcal{G}_{w^V,w^M}$ are derived in Propositions D.2, D.5 and D.7, respectively, $(\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^V,T},\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^V,T})$ and $(\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^M,T},\mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^M,T})$ are derived in Propositions E.2 and E.3, respectively, and $(\mathcal{G}^{\mathbb{C}\text{ov}_1}_{w^V,w^M,T},\mathcal{G}^{\mathbb{C}\text{ov}_2}_{w^V,w^M,T})$ are defined in Lemma E.6, with

$$\begin{aligned}\mathcal{G}^{\mathbb{C}\text{ov}_1,1}_{w^V,w^M}(u)=L_2^2\Bigg[\Bigg(&-\frac{1-\epsilon}{r+\kappa}\left[K_2(\gamma^R_1,\gamma^R_3,\mathcal{T}',\nu_1)+K_1(\gamma^R_1,\gamma^R_3,\mathcal{T}',\nu_2)\right]-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}')\\&+H(\gamma_1,\gamma_2,\gamma_3,\gamma^R_1,\gamma^R_2,\gamma^R_3,\mathcal{T}',\nu_1)+H(\gamma^R_1,\gamma^R_2,\gamma^R_3,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}',\nu_2)\\&+G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}',\nu_2)\Bigg)(p_f\Sigma p'_f)^2-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}')p_f\Sigma^3 p'_f\Bigg],\\ \mathcal{G}^{\mathbb{C}\text{ov}_1,1}_{w^M,w^V}(u)=L_2^2\Bigg[\Bigg(&-\frac{1-\epsilon}{r+\kappa}K_1(\gamma^R_1,\gamma^R_3,\mathcal{T}'^{-},\nu_2)-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}'^{-})+H(\gamma^R_1,\gamma^R_2,\gamma^R_3,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^{-},\nu_2)\\&+G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^{-},\nu_2)\Bigg)(p_f\Sigma p'_f)^2-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}'^{-})p_f\Sigma^3 p'_f\Bigg],\\ \mathcal{G}^{\mathbb{C}\text{ov}_1,2}_{w^V,w^M}(u)=L_2&\left(L_1\tau\eta\Sigma^2 p'_f+L_2\tau p_f\Sigma^2 p'_f\right)\end{aligned}$$

$$\times \left(-\frac{1-\epsilon}{r+\kappa} K_2(\gamma_1^R, \gamma_3^R, u-t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \right) p_f \Sigma^2 p_f',$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,2}_{w^M,w^V}(u) = L_2 \left(\frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right)$$

$$\times \left(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) \right) p_f \Sigma^2 p_f',$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,3}_{w^V,w^M}(u) = L_2 \left(\frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right) \left(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) \right.$$

$$\left. + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2) \right) p_f \Sigma^2 p_f',$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,3}_{w^M,w^V}(u) = L_2 \left(L_1 \tau \eta \Sigma^2 p_f' + L_2 \tau p_f \Sigma^2 p_f' \right)$$

$$\times \left(-\frac{1-\epsilon}{r+\kappa} K_1(\gamma_1^R, \gamma_3^R, u-t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, u-t, \nu_0) \right) p_f \Sigma^2 p_f',$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w^V,w^M}(u) = \mathcal{G}^{\mathbb{C}\mathrm{ov}_1,4}_{w^M,w^V}(u) = \left(L_1 \eta \Sigma^2 p_f' + L_2 p_f \Sigma^2 p_f' \right)^2 \frac{\tau}{r} H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w^V,w^M}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \left[\left(-\frac{1-\epsilon}{r+\kappa} \left[K_2(\gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_1) + K_1(\gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_2) \right] \right. \right.$$

$$- \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, \mathcal{T}') + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, \mathcal{T}', \nu_2)$$

$$\left. \left. + G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}', \nu_2) \right) (p_f \Sigma p_f')^2 - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \nu_2(\kappa, \mathcal{T}') p_f \Sigma^3 p_f' \right],$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,5}_{w^M,w^V}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \left[\left(-\frac{1-\epsilon}{r+\kappa} K_1(\gamma_1^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) \right. \right.$$

$$- \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, \mathcal{T}'^-) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, \mathcal{T}'^-, \nu_2)$$

$$\left. \left. + G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}'^-, \nu_2) \right) (p_f \Sigma p_f')^2 - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \nu_2(\kappa, \mathcal{T}'^-) p_f \Sigma^3 p_f' \right],$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w^V,w^M}(u) = \left(-\frac{1-\epsilon}{r+\kappa} K_2(\gamma_1^R, \gamma_3^R, u-t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u-t, \nu_0) \right)$$

$$\times \left(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2) \right.$$

$$\left. + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}', \nu_2) \right) \left(p_f \Sigma^2 p_f' \right)^2,$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,6}_{w^M,w^V}(u) = \left(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \mathcal{T}'^-, \nu_2) \right)$$

$$\times \left(-\frac{1-\epsilon}{r+\kappa} K_1(\gamma_1^R, \gamma_3^R, u-t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, u-t, \nu_0) \right) \left(p_f \Sigma^2 p_f' \right)^2,$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w^V,w^M}(u) = \Bigg[\frac{L_1L_2}{r}\eta\Sigma^2p_f' + \left(\frac{L_2^2}{r} - \frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0) + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\right)$$

$$\times p_f\Sigma^2p_f'\Bigg]\left(G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1) + k1_{\{\tau+t-u>0\}}\right)p_f\Sigma^2p_f'$$

$$+\Bigg[\frac{L_1L_2}{r}\left(\eta\Sigma^3p_f' - \frac{\eta\Sigma^2\eta'\eta\Sigma^2p_f'}{\eta\Sigma\eta'}\right) + \Bigg(\frac{L_2^2}{r} - \frac{1-\epsilon}{r+\kappa}K_2(\gamma_1^R,\gamma_3^R,u-t,\nu_0)$$

$$+ H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0)\Bigg)\left(p_f\Sigma^3p_f' - \frac{(\eta\Sigma^2p_f')^2}{\eta\Sigma\eta'}\right)\Bigg] f1_{\{\tau+t-u>0\}},$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,1}_{w^M,w^V}(u) = \big[L_1L_2\tau\eta\Sigma^2p_f' + \big(L_2^2\tau + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2) + G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)\big)$$

$$\times p_f\Sigma^2p_f'\big]\left(-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_0(\kappa,u-t) + G(\gamma_1,\gamma_2,\gamma_3,u-t,\nu_0)\right)p_f\Sigma^2p_f'$$

$$-\Bigg[L_1L_2\tau\left(\eta\Sigma^3p_f' - \frac{\eta\Sigma^2\eta'\eta\Sigma^2p_f'}{\eta\Sigma\eta'}\right) + \big(L_2^2\tau + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)$$

$$+G(\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}'^-,\nu_2)\big)\left(p_f\Sigma^3p_f' - \frac{(\eta\Sigma^2p_f')^2}{\eta\Sigma\eta'}\right)\Bigg]\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_0(\kappa,u-t),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w^V,w^M}(u) = \Bigg[\left(L_1\eta\Sigma^2p_f' + L_2p_f\Sigma^2p_f'\right)^2\frac{\tau}{r} + \Bigg(-\frac{1-\epsilon}{r+\kappa}\left[K_2(\gamma_1^R,\gamma_3^R,\mathcal{T}',\nu_1) + K_1(\gamma_1^R,\gamma_3^R,\mathcal{T}',\nu_2)\right]$$

$$-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}') + H(\gamma_1,\gamma_2,\gamma_3,\gamma_1^R,\gamma_2^R,\gamma_3^R,\mathcal{T}',\nu_1) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}',\nu_2)$$

$$+ G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}',\nu_2)\Bigg)(p_f\Sigma p_f')^2 - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}')p_f\Sigma^3p_f'\Bigg]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0),$$

$$\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,2}_{w^M,w^V}(u) = \Bigg[\left(L_1\eta\Sigma^2p_f' + L_2p_f\Sigma^2p_f'\right)^2\frac{\tau}{r} + \Bigg(-\frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R,\gamma_3^R,\mathcal{T}'^-,\nu_1)$$

$$-\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\beta_2\gamma_1\nu_2(\kappa,\mathcal{T}'^-) + H(\gamma_1^R,\gamma_2^R,\gamma_3^R,\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^-,\nu_2)$$

$$+ G(\gamma_1,\gamma_2,\gamma_3,\mathcal{T}'^-,\nu_2)\Bigg)(p_f\Sigma p_f')^2 - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2}\nu_2(\kappa,\mathcal{T}'^-)p_f\Sigma^3p_f'\Bigg]G(\gamma_1^R,\gamma_2^R,\gamma_3^R,u-t,\nu_0),$$

where $\mathcal{T}' = (u-t,\tau)$ *and* $\mathcal{T}'^- = (t-u,\tau)$.

Proof: To show the proposition, we need to show that the definitions of $\{\mathcal{G}^{\mathbb{C}\mathrm{ov}_1,i}_{w^V,w^M,T}(u)\}_{i=1,..6}$ and $\{\mathcal{G}^{\mathbb{C}\mathrm{ov}_2,i}_{w^V,w^M,T}(u)\}_{i=1,2}$ in Lemma E.6 yield the equations in the proposition. The equations follow from the derivations in Propositions E.2 and E.3, and the derivations of $\mathbb{C}\mathrm{ov}(w_t^V\Sigma p_f', w_u^M\Sigma p_f')$ and

$\mathbb{C}\text{ov}(w_t^M \Sigma p_f', w_u^V \Sigma p_f')$ in Lemma D.7. ∎

Proposition E.8 computes the unconditional correlation between the return of strategy w_t^j over an interval $[t, t+T]$ and the return of strategy w_t^k over a subsequent interval $[t', t'+T']$, with $j, k \in \{V, M\}$ and $t' \geq t+T$. The autocorrelation of value returns follows by setting $j = k = V$, the autocorrelation of momentum returns follows by setting $j = k = M$, and the cross-autocorrelations between the two strategies' returns follow by setting $(j,k) = (V, M)$ and $(j,k) = (M, V)$.

Proposition E.8. *Consider intervals $[t, t+T]$ and $[t', t'+T']$, with (T, T') positive and $t' \geq t+T$. The autocorrelation between the return of strategy w_t^j over the interval $[t, t+T]$ and the return of strategy w_t^k over the interval $[t', t'+T']$, with $j, k \in \{V, M\}$, is*

$$\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^j dR_u, \int_{t'}^{t'+T'} \hat{w}_u^k dR_u\right) = \frac{\mathcal{A}^{\mathbb{C}\text{ov}_1}_{w^j,w^k,T,t',T'} + \mathcal{A}^{\mathbb{C}\text{ov}_2}_{w^j,w^k,T,t'T'}}{\sqrt{\left(\mathcal{D}_{w^j} + \mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^j,T} + \mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^j,T}\right)\left(\mathcal{D}_{w^k} + \mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^k,T'} + \mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^k,T'}\right)}},$$

where $\{\mathcal{D}_{w^j}\}_{j=V,M}$ are derived in Propositions D.2 and D.5, $\left\{(\mathcal{D}^{\mathbb{C}\text{ov}_1}_{w^j,T}, \mathcal{D}^{\mathbb{C}\text{ov}_2}_{w^j,T})\right\}_{j=V,M}$ are derived in Propositions E.2 and E.3, and $(\mathcal{A}^{\mathbb{C}\text{ov}_1}_{w^j,w^k,T,t',T'}, \mathcal{A}^{\mathbb{C}\text{ov}_2}_{w^j,w^k,T,t',T'})$ are defined as

$$\mathcal{A}^{\mathbb{C}\text{ov}_1}_{w^j,w^k,T,t',T'} \equiv \frac{1}{\sqrt{TT'}} \sum_{i=1}^{6} \int_{t'-T}^{t'+T'} F(u) \mathcal{D}^{\mathbb{C}\text{ov}_1,i}_{w^j}(u) du,$$

$$\mathcal{A}^{\mathbb{C}\text{ov}_2}_{w^j,w^k,T,t',T'} \equiv \frac{1}{\sqrt{TT'}} \sum_{i=1}^{2} \int_{t'-T}^{t'+T'} F(u) \mathcal{D}^{\mathbb{C}\text{ov}_2,i}_{w^j}(u) du,$$

for $j = k$ with $\left\{\left\{\mathcal{D}^{\mathbb{C}\text{ov}_1,i}_{w^j}(u)\right\}_{i=1,..,6}, \left\{\mathcal{D}^{\mathbb{C}\text{ov}_2,i}_{w^j}(u)\right\}_{i=1,2}\right\}_{j=V,M}$ derived in Propositions E.2 and E.3 and $F(u) \equiv \min\{T, t'+T'-u\} - \max\{0, t'-u\}$, and as

$$\mathcal{A}^{\mathbb{C}\text{ov}_1}_{w^j,w^k,T,t',T'} \equiv \frac{1}{\sqrt{TT'}} \sum_{i=1}^{6} \int_{t'-T}^{t'+T'} F(u) \mathcal{G}^{\mathbb{C}\text{ov}_1,i}_{w^j,w^k}(u) du,$$

$$\mathcal{A}^{\mathbb{C}\text{ov}_2}_{w^j,w^k,T,t',T'} \equiv \frac{1}{\sqrt{TT'}} \sum_{i=1}^{2} \int_{t'-T}^{t'+T'} F(u) \mathcal{G}^{\mathbb{C}\text{ov}_2,i}_{w^j,w^k}(u) du,$$

for $j \neq k$ with $\left\{\left\{\mathcal{G}^{\mathbb{C}\text{ov}_1,i}_{w^j,w^k}(u)\right\}_{i=1,..,6}, \left\{\mathcal{G}^{\mathbb{C}\text{ov}_2,i}_{w^j,w^k}(u)\right\}_{i=1,2}\right\}_{j,k\in\{V,M\}}$ derived in Proposition E.7.

Proof: To show the proposition, we need to show

$$\mathbb{C}\text{ov}\left(\int_t^{t+T}\hat{w}_u^j dR_u, \int_{t'}^{t'+T'}\hat{w}_u^k dR_u\right) = \sqrt{TT'}\left(\mathcal{A}^{\mathbb{C}\text{ov}_1}_{w^j,w^k,T,t',T'} + \mathcal{A}^{\mathbb{C}\text{ov}_2}_{w^j,w^k,T,t'T'}\right). \tag{E.41}$$

Proceeding as in the proof of Lemma E.1 and using $t' \geq t+T$, we find

$$\begin{aligned}
&\mathbb{C}\text{ov}\left(\int_t^{t+T}\hat{w}_u^j dR_u, \int_{t'}^{t'+T'}\hat{w}_u^k dR_u\right)\\
&= \int_{u=t}^{t+T}\int_{u'=t'}^{t'+T'} \mathbb{C}\text{ov}\left(\hat{w}_u^j dR_u, \hat{w}_{u'}^k dR_{u'}\right)\\
&= \int_{u=t}^{t+T}\int_{u'=t'}^{t'+T} \left\{\mathbb{C}\text{ov}\left[\hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u'}^k \mathbb{E}_{u'}(dR_{u'})\right] + \mathbb{E}\left[\hat{w}_u^j \mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u'}^k \mathbb{E}_{u'}(dR_{u'}))\right]\right\}
\end{aligned} \tag{E.42}$$

To compute (E.42), we proceed as in the proof of Lemma E.2. Equation (E.42) becomes

$$\begin{aligned}
&\int_{u=t}^{t+T}\int_{s=t'-u}^{t'+T'-u} \left\{\mathbb{C}\text{ov}\left[\hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s})\right] + \mathbb{E}\left[\hat{w}_u^j \mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s}))\right]\right\}\\
&= \int_{s=t'-(t+T)}^{t'+T'-t}\int_{u=\max\{t,t'-s\}}^{\min\{t+T,t'+T'-s\}} \left\{\mathbb{C}\text{ov}\left[\hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s})\right]\right.\\
&\quad \left.+\mathbb{E}\left[\hat{w}_u^j \mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s}))\right]\right\}\\
&= \int_{s=t'-(t+T)}^{t'+T'-t}\int_{u=\max\{t,t'-s\}}^{\min\{t+T,t'+T'-s\}} \frac{1}{du}\left\{\mathbb{C}\text{ov}\left[\hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s})\right]\right.\\
&\quad \left.+\mathbb{E}\left[\hat{w}_u^j \mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s}))\right]\right\} du\\
&= \int_{s=t'-(t+T)}^{t'+T'-t}\int_{u=\max\{t,t'-s\}}^{\min\{t+T,t'+T'-s\}} \frac{1}{dt}\left\{\mathbb{C}\text{ov}\left[\hat{w}_t^j \mathbb{E}_t(dR_t), \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s})\right]\right.\\
&\quad \left.+\mathbb{E}\left[\hat{w}_t^j \mathbb{C}\text{ov}_t(dR_t, \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s}))\right]\right\} du\\
&= \int_{s=t'-(t+T)}^{t'+T'-t} \left(\min\{t+T, t'+T'-s\} - \max\{t, t'-s\}\right)\frac{1}{dt}\left\{\mathbb{C}\text{ov}\left[\hat{w}_t^j \mathbb{E}_u(dR_t), \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s})\right]\right.\\
&\quad \left.+\mathbb{E}\left[\hat{w}_t^j \mathbb{C}\text{ov}_t(dR_t, \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s}))\right]\right\}\\
&= \int_{u=t'-T}^{t'+T'} \left(\min\{t+T, t'+T'+t-u\} - \max\{t, t'+t-u\}\right)\frac{1}{dt}\left\{\mathbb{C}\text{ov}\left[\hat{w}_t^j \mathbb{E}_u(dR_t), \hat{w}_u^k \mathbb{E}_u(dR_u)\right]\right.
\end{aligned}$$

$$+\mathbb{E}\left[\hat{w}_t^j \mathbb{C}\text{ov}_t(dR_t, \hat{w}_u^k \mathbb{E}_u(dR_u))\right]\Big\}$$

$$= \int_{u=t'-T}^{t'+T'} \left(\min\{T, t'+T'-u\} - \max\{0, t'-u\}\right) \frac{1}{dt} \Big\{\mathbb{C}\text{ov}\left[\hat{w}_t^j \mathbb{E}_u(dR_t), \hat{w}_u^k \mathbb{E}_u(dR_u)\right]$$

$$+\mathbb{E}\left[\hat{w}_t^j \mathbb{C}\text{ov}_t(dR_t, \hat{w}_u^k \mathbb{E}_u(dR_u))\right]\Big\}$$

$$= \int_{u=t'-T}^{t'+T'} F(u) \frac{1}{dt} \Big\{\mathbb{C}\text{ov}\left[\hat{w}_t^j \mathbb{E}_u(dR_t), \hat{w}_u^k \mathbb{E}_u(dR_u)\right] + \mathbb{E}\left[\hat{w}_t^j \mathbb{C}\text{ov}_t(dR_t, \hat{w}_u^k \mathbb{E}_u(dR_u))\right]\Big\}, \quad \text{(E.43)}$$

where the first and sixth steps follow from the change of variable $s = u' - u$, the second step follows by changing the order of the integrals, the third step follows because the covariance in the first term and the expectation in the second term are unconditional and depend only on s, and the last step follows from the definition of $F(u)$. Combining (E.43) with Lemmas E.2, E.5 and E.6, we find (E.41). ∎

Proposition E.9 computes the weights of value and momentum in their unconditionally optimal (mean-variance maximizing) combination over investment horizon T. The proposition assumes symmetric assets.

Proposition E.9. *Suppose $\eta = \boldsymbol{1}'$ and $\Sigma = \hat{\sigma}^2(I + \omega \boldsymbol{1}\boldsymbol{1}')$. The weights of value and momentum in their combination that maximizes an unconditional mean-variance objective over investment horizon T are*

$$\hat{y}^V = \frac{1}{a} \frac{SR_{w^V,T} - SR_{w^M,T}\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)}{1 - \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)^2} \sqrt{\frac{T}{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right)}}, \quad \text{(E.44)}$$

$$\hat{y}^M = \frac{1}{a} \frac{SR_{w^M,T} - SR_{w^V,T}\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)}{1 - \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)^2} \sqrt{\frac{T}{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)}}. \quad \text{(E.45)}$$

Proof: Consider an investor at time t with horizon T, who can invest in the riskless asset, the index η and the strategies (w_t^V, w_t^M). The investor's optimization problem is as in Lemma C.2, except that the budget constraint (C.10) is replaced by

$$\Delta W_{t+T} = \hat{\hat{x}}_t \int_t^{t+T} \eta dR_u + \hat{y}_t^V \int_t^{t+T} \hat{w}_u^V dR_u + \hat{y}_t^M \int_t^{t+T} \hat{w}_u^M dR_u. \quad \text{(E.46)}$$

Substituting ΔW_{t+T} from (E.46) and setting $\mathcal{I}_t = \emptyset$, we can write the investor's objective (C.7) as

$$\begin{aligned}
&\hat{\hat{x}}\mathbb{E}\left(\int_t^{t+T} \eta dR_u\right) + \hat{y}^V\mathbb{E}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right) + \hat{y}^M\mathbb{E}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)\\
&-\frac{a}{2}\left[\hat{\hat{x}}^2\mathbb{V}\text{ar}\left(\int_t^{t+T} \eta dR_u\right) + (\hat{y}^V)^2\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right) + (\hat{y}^M)^2\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)\right.\\
&+2\hat{\hat{x}}\hat{y}^V\mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^V dR_u\right) + 2\hat{\hat{x}}\hat{y}^M\mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)\\
&\left.+2\hat{y}^V\hat{y}^M\mathbb{C}\text{ov}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)\right]. \qquad \text{(E.47)}
\end{aligned}$$

The proof of Lemma C.2 implies that the first and second covariances in (E.47) are zero if $\mathbb{C}\text{ov}(\eta dR_u, \hat{w}_{u'}^j dR_{u'}) = 0$ for $u < u'$ and $j = V, M$. The proof of Lemma E.1 implies

$$\begin{aligned}
\mathbb{C}\text{ov}(\eta dR_u, \hat{w}_{u'}^j dR_{u'}) &= \mathbb{C}\text{ov}\left[\eta\mathbb{E}_u(dR_u), \hat{w}_{u'}^j\mathbb{E}_{u'}(dR_{u'})\right] + \mathbb{E}\left[\eta\mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u'}^j\mathbb{E}_{u'}(dR_{u'}))\right]\\
&= \mathbb{E}\left[\eta\mathbb{C}\text{ov}_u(dR_u, \hat{w}_{u'}^j\mathbb{E}_{u'}(dR_{u'}))\right]\\
&= \frac{1}{du}\left(f + \frac{k\Delta}{\eta\Sigma\eta'}\right)\left\{\mathbb{E}\left[\eta\Lambda_{u'}\mathbb{C}\text{ov}_u(dR_u, w_{u'}^j\Sigma p_f')\right] + \mathbb{E}\left[\eta w_{u'}^j\Sigma p_f'\mathbb{C}\text{ov}_u(dR_u, \Lambda_{u'})\right]\right\},
\end{aligned} \qquad \text{(E.48)}$$

where the second step follows because (C.6) implies $\eta\mathbb{E}_u(dR_u) = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\eta\Sigma\theta' dt$, which is constant over time, and the third step follows from the proof of Lemma E.4. Since (3.6) and (B.10) imply that $\mathbb{C}\text{ov}_u(dR_u, \Lambda_{u'})$ is collinear to $\Sigma p_f'$, the second term in (E.48) is zero because $\eta\Sigma p_f' = 0$. Since (B.10), (B.11) and (D.19) imply that $\mathbb{C}\text{ov}_u(dR_u, w_{u'}^V\Sigma p_f')$ is a linear combination of $\Sigma p_f'$ and $\Sigma^2 p_f'$, the first term in (E.48) is zero for $j = V$ because $\eta\Sigma p_f' = 0$ and because for symmetric assets, Lemma C.5 implies $\eta\Sigma^2 p_f' = 0$. Since (A.2), (B.10) and (D.48) imply that $\mathbb{C}\text{ov}_u(dR_u, w_{u'}^M\Sigma p_f')$ is a linear combination of $\Sigma p_f'$ and $\Sigma^2 p_f'$, the first term in (E.48) is zero for $j = M$ because $\eta\Sigma p_f' = \eta\Sigma^2 p_f' = 0$.

Setting the first and second covariances in (E.47) to zero, we can simplify (E.47) to

$$\begin{aligned}
&\hat{\hat{x}}\mathbb{E}\left(\int_t^{t+T} \eta dR_u\right) + \hat{y}^V\mathbb{E}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right) + \hat{y}^M\mathbb{E}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)\\
&-\frac{a}{2}\left[\hat{\hat{x}}^2\mathbb{V}\text{ar}\left(\int_t^{t+T} \eta dR_u\right) + (\hat{y}^V)^2\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right) + (\hat{y}^M)^2\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)\right.
\end{aligned}$$

$$+2\hat{y}^V\hat{y}^M\mathbb{C}\text{ov}\left(\int_t^{t+T}\hat{w}_u^V dR_u, \int_t^{t+T}\hat{w}_u^M dR_u\right)\Bigg].$$

The first-order conditions over $\hat{y}_t^V$ and $\hat{y}_t^M$ are

$$\mathbb{E}\left(\int_t^{t+T}\hat{w}_u^V dR_u\right) = a\left[\hat{y}^V\mathbb{V}\text{ar}\left(\int_t^{t+T}\hat{w}_u^V dR_u\right) + \hat{y}^M\mathbb{C}\text{ov}\left(\int_t^{t+T}\hat{w}_u^V dR_u, \int_t^{t+T}\hat{w}_u^M dR_u\right)\right], \tag{E.49}$$

$$\mathbb{E}\left(\int_t^{t+T}\hat{w}_u^M dR_u\right) = a\left[\hat{y}^V\mathbb{C}\text{ov}\left(\int_t^{t+T}\hat{w}_u^M dR_u, \int_t^{t+T}\hat{w}_u^M dR_u\right) + \hat{y}^M\mathbb{V}\text{ar}\left(\int_t^{t+T}\hat{w}_u^M dR_u\right)\right], \tag{E.50}$$

respectively. Solving the linear system of (E.49) and (E.50), we find (E.44) and (E.45). We normalize $\hat{y}^V$ and $\hat{y}^M$ by setting $a\sqrt{\frac{\mathbb{V}\text{ar}\left(\int_t^{t+T}\hat{w}_u^V dR_u\right)}{T}} = a\sqrt{\frac{\mathbb{V}\text{ar}\left(\int_t^{t+T}\hat{w}_u^M dR_u\right)}{T}} = 1.$ ∎

Proposition E.10 computes the weights of value and momentum in the combination that best approximates the strategy that is optimal over investment horizon T. We construct the approximating combination by minimizing the unconditional variance of the difference in returns over horizon T between that combination and the optimal strategy. The proposition assumes symmetric assets.

Proposition E.10. *Suppose $\eta = \boldsymbol{1}'$ and $\Sigma = \hat{\sigma}^2(I + \omega\boldsymbol{11}')$. The weights (λ^V, λ^M) of value and momentum in the combination that minimizes*

$$\mathbb{V}\text{ar}\left[\int_t^{t+T} w_u dR_u - \left(\lambda^\eta\int_t^{t+T}\eta dR_u + \lambda^V\int_t^{t+T}\hat{w}_u^V dR_u + \lambda^M\int_t^{t+T}\hat{w}_u^M dR_u\right)\right], \tag{E.51}$$

where w_t is the optimal strategy over investment horizon T derived in Section 6.1, are

$$\lambda^V = \left[\mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u\right) - \mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u\right)\right.$$
$$\left.\times\mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^V dR_u, \int_t^{t+T}\hat{w}_u^M dR_u\right)\right]\frac{1}{1-\mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^V dR_u, \int_t^{t+T}\hat{w}_u^M dR_u\right)^2}\sqrt{\frac{\mathbb{V}\text{ar}\left(\int_t^{t+T} w_u dR_u\right)}{\mathbb{V}\text{ar}\left(\int_t^{t+T}\hat{w}_u^V dR_u\right)}}, \tag{E.52}$$

$$\lambda^M = \left[\mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u\right) - \mathbb{C}\text{orr}\left(\int_t^{t+T}\hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u\right)\right.$$

$$\times \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)\Bigg] \frac{1}{1-\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)^2} \sqrt{\frac{\mathbb{V}\text{ar}\left(\int_t^{t+T} w_u dR_u\right)}{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)}}. \tag{E.53}$$

Proof: The first-order conditions from minimizing (E.51) over λ^V and λ^M are

$$\mathbb{C}\text{ov}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u\right) = \lambda^\eta \mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^V dR_u\right)$$

$$+ \lambda^V \mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right) + \lambda^M \mathbb{C}\text{ov}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right), \tag{E.54}$$

$$\mathbb{C}\text{ov}\left(\int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u\right) = \lambda^\eta \mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)$$

$$+ \lambda^V \mathbb{C}\text{ov}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right) + \lambda^M \mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right), \tag{E.55}$$

respectively. Since with symmetric assets Proposition E.9 implies

$$\mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^V dR_u\right) = \mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right) = 0,$$

(E.54) and (E.55) imply (E.52) and (E.53).

To translate the weights λ^V and λ^M to weights $\hat{y}^V$ and $\hat{y}^M$ as in Proposition E.9, we multiply them by the weight $\hat{y}$ given to the optimal strategy w_t. Proceeding as in Proposition E.9, we find

$$\mathbb{C}\text{ov}\left(\int_t^{t+T} \eta dR_u, \int_t^{t+T} w_u dR_u\right) = 0$$

because $\mathbb{C}\text{ov}_u(dR_u, w_{u'}\Sigma p_f')$ is collinear to $\Sigma p_f'$. Therefore, maximization of (C.7) yields

$$\hat{y} = \frac{1}{a} \frac{SR_{w,T}}{\sqrt{\mathbb{V}\text{ar}\left(\int_t^{t+T} w_u dR_u\right)}}.$$

The resulting weights $\hat{y}^V$ and $\hat{y}^M$ are

$$\hat{y}^V = \hat{y}\lambda^V = \frac{1}{a}\Bigg[\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u\right) - \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u\right)$$

$$\times \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)\Bigg] \frac{SR_{w,T}}{1-\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)^2} \sqrt{\frac{T}{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right)}}, \tag{E.56}$$

$$\hat{y}^M = \hat{y}\lambda^M = \frac{1}{a}\Bigg[\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u\right) - \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u\right)$$

$$\times \mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)\Bigg] \frac{SR_{w,T}}{1-\mathbb{C}\text{orr}\left(\int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u\right)^2} \sqrt{\frac{T}{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)}}. \tag{E.57}$$

Equations (E.56) and (E.57) are analogous to (E.44) and (E.45) in Proposition E.9. We normalize $\hat{y}^V$ and $\hat{y}^M$ by setting $a\sqrt{\frac{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^V dR_u\right)}{T}} = a\sqrt{\frac{\mathbb{V}\text{ar}\left(\int_t^{t+T} \hat{w}_u^M dR_u\right)}{T}} = 1$. ∎

F Sensitivity Analysis

Table F.I reports moments derived in Sections 5 and 6 in the following cases: baseline, where parameter values are as in Table I; lookback window for momentum equal to one year instead of seven months; fraction of asset return variance generated by fund flows equal to 10% instead of 15%; fund flows as fraction of fund holdings smaller by 50% than in the baseline (spread in quarterly FIT between top and bottom stock deciles sorted based on FIT equal to 22.27% × 0.5); and active share of residual supply portfolio equal to 20% instead of 10%. When deviating from the baseline to meet a calibration target, we choose parameter values to meet all remaining targets in Table I.

Table F.I indicates that many of the patterns shown in Sections 5 and 6 are robust across cases. In particular: (i) the Sharpe ratio of value, which is stable across cases when horizon is short (infinitesimal), drops somewhat when horizon increases (to five years), and rises significantly when horizon increases further (to twenty years); (ii) the Sharpe ratio of momentum, which is less stable than value's across cases when horizon is short, drops significantly when horizon increases, and becomes essentially flat when horizon increases further; (iii) the Sharpe ratio of value is more volatile than that of momentum, especially for long horizons; (iv) value and momentum are modestly negatively correlated for short horizons and modestly positively for long horizons; (v) the value spread is positively correlated with value's Sharpe ratio for short horizons and strongly so for long horizons; (vi) the value spread is slightly negatively correlated with momentum's Sharpe ratio for short horizons but strongly positively correlated for long horizons; (vii) the value-momentum correlation is strongly positively correlated with value's Sharpe ratio for short horizons but slightly

Table F.I: Sensitivity analysis.

Moment (%)	**Base**	**1-yr Mom**	**10% Flows**	**Vol $\times$ 0.5**	**20% AS**
SR_{w^V}	27.05	27.05	26.17	26.76	27.77
$SR_{w^V,5}$	23.00	23.00	25.16	23.26	27.25
$SR_{w^V,20}$	38.43	38.43	39.11	38.77	45.87
$\mathbb{V}\text{ar}\left(SR_{w^V,t}\right)$	63.25	63.25	39.18	61.50	56.49
$\mathbb{V}\text{ar}\left(SR_{w^V,t,5}\right)$	40.88	40.88	33.74	40.43	39.42
SR_{w^M}	53.66	51.28	27.57	51.75	45.24
$SR_{w^M,5}$	34.28	29.86	21.64	33.54	31.45
$SR_{w^M,20}$	33.04	28.37	21.08	32.38	30.57
$\mathbb{V}\text{ar}\left(SR_{w^M,t}\right)$	46.58	45.72	30.58	45.49	43.19
$\mathbb{V}\text{ar}\left(SR_{w^M,t,5}\right)$	6.52	7.54	6.55	6.48	6.28
$\mathbb{C}\text{orr}\left(dR_t^V, dR_t^M\right)$	-12.20	-16.81	-13.01	-12.51	-15.68
$\mathbb{C}\text{orr}\left(R_{t,t+5}^V, R_{t,t+5}^M\right)$	11.68	12.43	8.61	11.15	8.85
$\mathbb{C}\text{orr}\left(VS_t, SR_{w^V,t}\right)$	26.00	26.00	41.25	26.46	29.94
$\mathbb{C}\text{orr}\left(VS_t, SR_{w^V,t,5}\right)$	97.81	97.81	98.68	97.66	97.24
$\mathbb{C}\text{orr}\left(VS_t, SR_{w^M,t}\right)$	-8.13	-15.16	-11.77	-8.49	-11.79
$\mathbb{C}\text{orr}\left(VS_t, SR_{w^M,t,5}\right)$	87.99	82.77	90.01	87.81	88.08
$\mathbb{C}\text{orr}\left(\mathbb{C}\text{orr}_t(dR_t^V, dR_t^M), SR_{w^V,t}\right)$	86.02	85.32	82.63	85.71	83.56
$\mathbb{C}\text{orr}\left(\mathbb{C}\text{orr}_t(dR_t^V, dR_t^M), SR_{w^V,t,5}\right)$	-3.56	-7.37	-0.77	-3.90	-7.52
$\mathbb{C}\text{orr}\left(\mathbb{C}\text{orr}_t(dR_t^V, dR_t^M), SR_{w^M,t}\right)$	41.34	39.99	52.75	41.77	45.71
$\mathbb{C}\text{orr}\left(\mathbb{C}\text{orr}_t(dR_t^V, dR_t^M), SR_{w^M,t,5}\right)$	18.47	25.50	20.08	18.09	15.94

negatively correlated for long horizons; and (viii) the value-momentum correlation is positively correlated with momentum's Sharpe ratio, especially for short horizons.

When the lookback window of momentum increases to one year, momentum's short-horizon correlations with value and the value spread become more negative. This is because momentum with a long lookback window becomes more similar to the opposite of a value strategy: it buys assets with a long history of good performance, which trade on average at a high price relative to fundamental value.

When flows account for a smaller fraction of asset return variance, momentum's Sharpe ratio decreases significantly. The intuition goes back to the momentum-generating mechanism in the model. Long-horizon investors buy assets with poor recent and expected future performance because they do not want to run the risk that by waiting and buying later the assets cease to be underpriced. Since mispricing is caused by flows, it becomes less volatile when flows generate smaller price variation. Therefore, long-horizon investors bear less risk by waiting, causing prices of assets with poor recent performance to drop fast rather than more gradually, and momentum to become less profitable. With momentum becoming less profitable at the beginning of the flow cycle, value becomes less unprofitable at that stage of the cycle. Therefore, its Sharpe ratio becomes less volatile and more correlated with the value spread over short horizons. Changes in other parameters have weaker effects on the moments in Table F.I.

G VAR Calculations

We compute Sharpe ratios and correlations for a general VAR in which the logarithmic returns R_{Vt} of HML and R_{Mt} of UMD evolve jointly with N predictor variables $(Y_{1t}, .., Y_{Nt})$ according to

$$X_{t+1} = A + BX_t + \epsilon_{t+1}, \tag{G.1}$$

where $X_t \equiv (R_{Vt}, R_{Mt}, Y_{1t}, .., Y_{Nt})'$, A is a $(N+2) \times 1$ constant vector, B is a $(N+2) \times (N+2)$ constant matrix, and ϵ_{t+1} is a $(N+2) \times 1$ random vector with covariance matrix Σ.

We first compute the expectation $\mathbb{E}_X$ of X_t. Taking expectations of both sides of (G.1), we find

$$\mathbb{E}_X = A + B\mathbb{E}_X \Rightarrow \mathbb{E}_X = (I - B)^{-1}A. \tag{G.2}$$

We next compute the covariance matrix $\Sigma_X \equiv \mathbb{C}\text{ov}\left(X_t, X_t'\right)$ of X_t. Iterating (G.1) from minus infinity to t, we find

$$X_t = (I + B + B^2 + ..)A + \epsilon_t + B\epsilon_{t-1} + B^2\epsilon_{t-2} + .. \tag{G.3}$$

Taking covariances of both sides in (G.3), we find

$$\Sigma_X = \Sigma + B\Sigma B' + B^2\Sigma\left(B^2\right)' + ..$$

$$\Rightarrow \Sigma_X = \Sigma + B\Sigma_X B'. \tag{G.4}$$

Equation (G.4) yields a linear system of scalar equations in the elements of Σ_X.

Consider next the sum $X_{t+1}+..+X_{t+k}$, whose first two elements are the cumulative logarithmic returns of HML and UMD. The expectation of $X_{t+1}+..+X_{t+k}$ is $k\mathbb{E}_X$. We next compute the covariance matrix $\Sigma_{Xk} \equiv \mathbb{C}\text{ov}\left(X_{t+1}+..+X_{t+k},(X_{t+1}+..+X_{t+k})'\right)$ of $X_{t+1}+..+X_{t+k}$. Iterating (G.1) from $t+1$ to $t+k$ for $k \geq 1$, we find

$$X_{t+k} = \left(I+B+..+B^{k-2}\right)A + B^{k-1}X_{t+1} + B^{k-2}\epsilon_{t+2} + B^{k-3}\epsilon_{t+3} + .. + \epsilon_{t+k}. \tag{G.5}$$

Summing (G.5) from $t+1$ to $t+k$, we find

$$\begin{aligned} X_{t+1}+..+X_{t+k} = &\left[I+(I+B)+..+\left(I+B+..+B^{k-2}\right)\right]A + \left(I+B+..+B^{k-1}\right)X_{t+1} \\ &+ \left(I+B+..+B^{k-2}\right)\epsilon_{t+2} + \left(I+B+..+B^{k-3}\right)\epsilon_{t+3} + .. + \epsilon_{t+k}. \end{aligned} \tag{G.6}$$

Taking covariances of both sides in (G.6) and noting

$$I+B+..+B^m = (I-B)^{-1}\left(I-B^{m+1}\right),$$

we find

$$\begin{aligned} \Sigma_{Xk} = &(I-B)^{-1}\left(I-B^k\right)\Sigma_X\left[(I-B)^{-1}\left(I-B^k\right)\right]' \\ &+ (I-B)^{-1}\left(I-B^{k-1}\right)\Sigma\left[(I-B)^{-1}\left(I-B^{k-1}\right)\right]' \\ &+ (I-B)^{-1}\left(I-B^{k-2}\right)\Sigma\left[(I-B)^{-1}\left(I-B^{k-2}\right)\right]' + .. + \Sigma. \end{aligned} \tag{G.7}$$

We finally compute the covariance matrix $\Sigma_{Xk\ell} \equiv \mathbb{C}\text{ov}\left(X_{t-(\ell-1)}+..+X_t,(X_{t+1}+..+X_{t+k})'\right)$. Since (G.5) implies

$$\mathbb{C}\text{ov}\left(X_{t+1},X'_{t+k}\right) = \Sigma_X\left(B^{k-1}\right)',$$

the covariance matrix $\Sigma_{Xk\ell}$ is

$$\Sigma_{Xk\ell} = \Sigma_X\left[\left(B+B^2+..+B^k\right) + \left(B^2+B^3+..+B^{k+1}\right) + .. + \left(B^\ell+B^{\ell+1}+..+B^{\ell+k-1}\right)\right]'$$

$$= \Sigma_X \left[B(I-B)^{-2} \left(I - B^{\ell} \right) \left(I - B^{k} \right) \right]'. \tag{G.8}$$

The annualized Sharpe ratio of value over horizon k is

$$\frac{k(\mathbb{E}_X)_1}{\sqrt{(\Sigma_{Xk})_{11}\, k}}. \tag{G.9}$$

The annualized Sharpe ratio of momentum over horizon k is

$$\frac{k(\mathbb{E}_X)_2}{\sqrt{(\Sigma_{Xk})_{22}\, k}}. \tag{G.10}$$

If periods in the VAR are months, then (G.9) and (G.10) must be multiplied by $\sqrt{12}$. The correlation between value and momentum returns over horizon k is

$$\frac{(\Sigma_{Xk})_{12}}{\sqrt{(\Sigma_{Xk})_{11}\,(\Sigma_{Xk})_{22}}}. \tag{G.11}$$

The correlation between the return of value over lookback window ℓ and over horizon k is

$$\frac{(\Sigma_{Xk\ell})_{11}}{\sqrt{(\Sigma_{Xk})_{11}\,(\Sigma_{X\ell})_{11}}}. \tag{G.12}$$

The correlation between the return of momentum over lookback window ℓ and over horizon k is

$$\frac{(\Sigma_{Xk\ell})_{22}}{\sqrt{(\Sigma_{Xk})_{22}\,(\Sigma_{X\ell})_{22}}}. \tag{G.13}$$

The correlation between the return of value over lookback window ℓ and the return of momentum over horizon k is

$$\frac{(\Sigma_{Xk\ell})_{12}}{\sqrt{(\Sigma_{Xk})_{22}\,(\Sigma_{X\ell})_{11}}}. \tag{G.14}$$

The correlation between the return of momentum over lookback window ℓ and the return of value over horizon k is

$$\frac{(\Sigma_{Xk\ell})_{21}}{\sqrt{(\Sigma_{Xk})_{11}\,(\Sigma_{X\ell})_{22}}}. \tag{G.15}$$