Corner Polyhedron and Intersection Cuts

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Abstract

Four decades ago, Gomory introduced the corner polyhedron as a relaxation of a mixed integer set in tableau form and Balas introduced intersection cuts for the corner polyhedron. A recent paper of Andersen, Louveaux, Weismantel and Wolsey has generated a renewed interest in the corner polyhedron and intersection cuts. We survey these two approaches and the recent developments in multi-row cuts. We stress the importance of maximal lattice-free convex sets and of the so-called infinite relaxation.

1 Introduction

In a recent paper [6], Andersen, Louveaux, Weismantel and Wolsey study a mixed integer linear programming (MILP) model in tableau form, where the basic variables are free integer variables and the nonbasic variables are continuous and nonnegative. This model is important because it arises as a relaxation of any MILP, and can be used to generate cut. It preserves some of the complexity of general MILPs but it is sufficiently simplified that one can prove interesting results about it. In particular, Andersen, Louveaux, Weismantel and Wolsey investigate the case of two rows (and two integer variables). They study the model from a geometric point of view and show that, besides nonnegativity constraints, the facet defining inequalities can be derived from splits, triangles and quadrilaterals.

This elegant result has sparked a renewed interest in the work of Gomory [60] and Gomory and Johnson [62] on the corner polyhedron, and of Balas on intersection cuts generated from convex sets [10], dating back to the early 1970s.

Split cuts are a classical example of intersection cuts. They are equivalent [77] to Gomory's mixed integer (GMI) cuts [58], which are generated from a single equation. Most cutting planes currently implemented in software are split cuts, such as GMI cuts from tableau rows, mixed integer rounding inequalities [73] and lift-and-project cuts [12]. A flurry of current research investigates intersection cuts derived from multiple rows of the tableau.

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This survey covers both classical and recent results. It starts by introducing the work of Gomory [60] and Gomory and Johnson [62] on corner polyhedra (Section 2), and the work of Balas on intersection cuts (Section 3). Proofs are given when they can provide insight. For example, we present the proof of the equivalence between intersection cuts and valid inequalities for the corner polyhedron in the special case of pure integer programs. We then show that intersection cuts have a nice description in the language of convex analysis, using the notion of gauge function. The survey stresses the connection with maximal lattice-free convex sets (Section 4). Intersection cuts generated from such sets give rise to minimal valid inequalities, and therefore are particularly important. Lovász [72] showed that maximal lattice-free convex sets are polyhedra. This implies that the corresponding intersection cuts have a very simple formula. These formulas are best studied in the context of the so-called infinite relaxations. In the pure integer case (Section 5), we are back to the model of Gomory and Johnson [62]. Arguably, one of the deepest results for this model is the Gomory-Johnson 2-slope theorem [64]. We give a complete proof of this result. The connection between minimal valid inequalities and maximal lattice-free convex sets is particularly elegant in the context of the continuous infinite relaxation (Section 6). The mixed case is considered in Section 7. We present a geometric perspective on integer lifting, when starting from minimal valid inequalities of the continuous infinite relaxation. Section 8 contains recent results on rank and closures, and Section 9 discusses very briefly the recent computational experience with multi-row cuts.

2 Corner polyhedron

We consider a mixed integer linear set

$$\begin{aligned} &Ax = b \\ &x_j \in \mathbb{Z} \quad \text{for } j = 1, \dots, p \\ &x_j \geq 0 \quad \text{for } j = 1, \dots, n \end{aligned} \tag{1}$$

where $p \leq n$, the matrix $A \in \mathbb{Q}^{m \times n}$ and the column vector $b \in \mathbb{Q}^m$. We assume that A has full row rank m. Given a feasible basis B, let $N = \{1, \dots, n\} \setminus B$ index the nonbasic variables. We rewrite the system $Ax = b$ as

$$x_i \quad = \quad \bar{b}_i - \textstyle\sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for } i \in B \tag{2}$$

where $\bar{b}_i \geq 0,\ i \in B$. The corresponding basic solution is $\bar{x}_i = \bar{b}_i, i \in B,\ \bar{x}_j = 0, j \in N$. If $\bar{b}_i \in \mathbb{Z}$ for all $i \in B \cap \{1, \dots, p\}$, then $\bar{x}$ is a feasible solution to (1).

If this is not the case, we address the problem of finding valid inequalities for the set (1) that are violated by the point $\bar{x}$. Typically, $\bar{x}$ is an optimal solution of the linear programming (LP) relaxation of an MILP having (1) as feasible set.

The key idea is to work with the corner polyhedron introduced by Gomory [59, 60], which is obtained from (1) by dropping the nonnegativity restriction on all the basic variables x_i, $i \in B$. Note that in this relaxation we can drop the constraints $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for all $i \in B \cap \{p+1, \dots, n\}$ since these variables x_i are continuous and only appear in one equation and no other constraint. Therefore from now on we assume that all basic variables in (2) are integer variables, i.e. $B \subseteq \{1, \dots, p\}$.

Therefore the relaxation of (1) introduced by Gomory is

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1, \ldots, p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array} \tag{3}$$

The convex hull of the feasible solutions to (3) is called the *corner polyhedron* relative to the basis B and it is denoted by corner(B). Any valid inequality for the corner polyhedron is valid for the set (1).

Let $P(B)$ be the linear relaxation of (3). $P(B)$ is a polyhedron whose vertices and extreme rays are simple to describe, a property that will be useful in generating valid inequalities for corner(B). Indeed, $x_i = \bar{b}_i$, for $i \in B$, $x_j = 0$, for $j \in N$ is the unique vertex of $P(B)$. The recession cone of $P(B)$ is defined by the following system.

$$\begin{array}{rcll} x_i & = & - \sum_{j\in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Since the projection of this cone onto $\mathbb{R}^N$ is defined by the inequalities $x_j \geq 0$, $j \in N$ and variables x_i, $i \in B$ are defined by the above equations, its extreme rays are the vectors satisfying at equality all but one nonnegativity constraints. Thus there are $|N|$ extreme rays, $\bar{r}^j$ for $j \in N$, defined by

$$\bar{r}^j_h = \begin{cases} -\bar{a}_{hj} & \text{if } h \in B, \\ 1 & \text{if } j = h, \\ 0 & \text{if } h \in N \setminus \{j\}. \end{cases} \tag{4}$$

Remark 2.1. *The vectors $\bar{r}^j$, $j \in N$ are linearly independent. Hence $P(B)$ is an $|N|$-dimensional polyhedron whose affine hull is defined by the equations $x_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x_j$ for $i \in B$.*

The rationality assumption of the matrix A will be used in the proof of the next lemma.

Lemma 2.2. *If the affine hull of $P(B)$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then* corner$(B)$ *is an $|N|$-dimensional polyhedron. Otherwise* corner(B) *is empty.*

Proof. Since corner(B) is contained in the affine hull of $P(B)$, corner(B) is empty when the affine hull of $P(B)$ contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

Next we assume that the affine hull of $P(B)$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and we show that corner(B) is an $|N|$-dimensional polyhedron. We first show that corner(B) is nonempty.

Let $x' \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ belong to the affine hull of $P(B)$. Then $x'_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x'_j$ for $i \in B$.

Let N^- be the subset of indices in $j \in N$ such that $x'_j < 0$. If N^- is empty, x' belongs to corner(B). Let $D \in \mathbb{Z}^+$ be such that $D\bar{a}_{ij} \in \mathbb{Z}$ for all $i \in B$ and $j \in N^-$. Define the point x'' as follows

$$x''_j = x'_j,\ j \in N \setminus N^-; \quad x''_j = x'_j - D\lfloor \frac{x'_j}{D} \rfloor,\ j \in N^-; \quad x''_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x''_j,\ i \in B.$$

By construction, $x''_j \geq 0$ for all $j \in N$ and x''_i is integer for every $i = 1, \ldots, p$. Since x'' satisfies $x''_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x''_j$, x'' belongs to corner(B). This shows that corner(B) is nonempty.

Since $P(B)$ is a rational polyhedron, the recession cones of $P(B)$ and corner(B) coincide by Meyer's theorem [74]. Since the dimension of both $P(B)$ and its recession cone is $|N|$ and corner$(B) \subseteq P(B)$, the dimension of corner(B) is $|N|$. $\square$

Example 2.3. *Consider the pure integer program*

$$
\begin{array}{rcl}
\max \frac{1}{2}x_2 + x_3 & & \\
x_1 + x_2 + x_3 & \leq & 2 \\
x_1 - \frac{1}{2}x_3 & \geq & 0 \\
x_2 - \frac{1}{2}x_3 & \geq & 0 \\
x_1 + \frac{1}{2}x_3 & \leq & 1 \\
-x_1 + x_2 + x_3 & \leq & 1 \\
x_1, x_2, x_3 & \in & \mathbb{Z} \\
x_1, x_2, x_3 & \geq & 0.
\end{array} \tag{5}
$$

This problem has 4 feasible solutions $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ *and* $(1,1,0)$, *all satisfying* $x_3 = 0$. *These four points are shown in the* (x_1, x_2)*-space in Figure 1.*

We first write the problem in standard form (1) by introducing continuous slack or surplus variables $x_4, \ldots, x_8$. *Solving the LP relaxation, we obtain*

$$
\begin{array}{rcllll}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_4 & = & 0 & -\frac{1}{2}x_6 & +\frac{3}{2}x_7 & +\frac{1}{2}x_8 \\
x_5 & = & 0 & +\frac{1}{2}x_6 & -\frac{1}{2}x_7 & +\frac{1}{2}x_8.
\end{array}
$$

The optimal basic solution is $x_1 = x_2 = \frac{1}{2}$, $x_3 = 1$, $x_4 = \ldots = x_8 = 0$.

Relaxing the nonnegativity of the basic variables and dropping the two constraints relative to the continuous basic variables x_4 *and* x_5, *we obtain the formulation (3) for this example:*

$$
\begin{array}{rcllll}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_1, x_2, x_3 & \in & \mathbb{Z} & & & \\
x_6, x_7, x_8 & \geq & 0. & & &
\end{array} \tag{6}
$$

Let $P(B)$ *be the linear relaxation of* (6). *The projection of* $P(B)$ *in the space of original variables* x_1, x_2, x_3 *is a polyhedron with unique vertex* $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. *The extreme rays of its recession cone are* $v^1 = (\frac{1}{2}, \frac{3}{2}, -1)$, $v^2 = (-\frac{3}{2}, -\frac{1}{2}, -1)$ *and* $v^3 = (\frac{1}{2}, -\frac{1}{2}, -1)$.

In Figure 1, the shaded region (both light and dark) is the intersection of $P(B)$ *with the plane* $x_3 = 0$. *Let* P *be the polyhedron defined by the inequalities of* (5) *that are satisfied at equality by the point* $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. *The intersection of* P *with the plane* $x_3 = 0$ *is the dark shaded region. Thus* P *is strictly contained in* $P(B)$. *This is usually the case when the basis is degenerate, which is the case here, and which is a frequent occurrence in integer programming.*

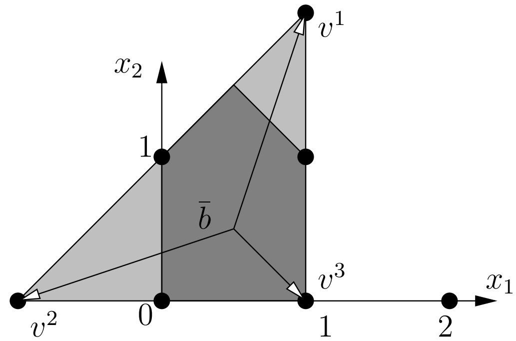

Figure 1: Intersection of the corner polyhedron with the plane $x_3 = 0$

Gomory [59] gave conditions that guarantee that optimizing over the corner polyhedron produces an optimal solution of the underlying MILP. This is known as the Asymptotic Theorem.

We say that a valid inequality $\sum_{j\in N} \gamma_j x_j \geq \delta$ for corner(B) is *trivial* if it is implied by the nonnegativity constraints $x_j \geq 0$, $j \in N$, that is, if $\gamma_j \geq 0$ for all $j \in N$ and $\delta \leq 0$. The inequality is said *nontrivial* otherwise

Lemma 2.4. *Assume* corner(B) *is nonempty. Every nontrivial valid inequality for* corner(B) *can be written in the form* $\sum_{j\in N} \gamma_j x_j \geq 1$ *where* $\gamma_j \geq 0$, $j \in N$.

Proof. Since every basic variable is a linear combination of nonbasic ones, every valid inequality for corner(B) can be written as $\sum_{j\in N} \gamma_j x_j \geq \delta$ in terms of the nonbasic variables x_j for $j \in N$ only. We argue next that $\gamma_j \geq 0$ for all $j \in N$. Indeed, if $\gamma_{j^*} < 0$ for some $j^* \in N$, then consider $\bar{r}^{j^*}$ defined in (4). We have $\sum_{j\in N} \gamma_j \bar{r}^{j^*}_j = \gamma_{j^*} < 0$, hence $\min\{\sum_{j\in N} \gamma_j x_j \ : \ x \in \text{corner}(B)\}$ is unbounded, because $\bar{r}^{j^*}$ is in the recession cone of corner(B).

If $\delta \leq 0$, the inequality $\sum_{j\in N} \gamma_j x_j \geq \delta$ is trivial, hence $\delta > 0$ and we may assume without loss of generality that $\delta = 1$. Thus every nontrivial valid inequality for corner(B) can be written in the form $\sum_{j\in N} \gamma_j x_j \geq 1$ where $\gamma_j \geq 0$, $j \in N$. ☐

Since variables x_i, $\in B$ are free integer variables, (3) can be reformulated as follows

$$\begin{array}{rcll} \sum_{j\in N} \bar{a}_{ij} x_j & \equiv & \bar{b}_i \mod 1 & \text{for } i \in B \\ x_j & \in & \mathbb{Z} & \text{for } j \in \{1,\ldots,p\} \cap N \\ x_j & \geq & 0 & \text{for } j \in N. \end{array} \tag{7}$$

This point of view was extensively studied by Gomory and Johnson [60, 61, 62, 63, 64, 65, 66]. We will come back to it in Section 5.

3 Intersection cuts

We describe a paradigm introduced by Balas [10] for constructing inequalities that are valid for the corner polyhedron and that cut off the basic solution $\bar{x}$.

Consider a closed convex set $C \subseteq \mathbb{R}^n$ such that the interior of C contains the point $\bar{x}$. (Recall that $\bar{x}$ belongs to the interior of C if C contains an n-dimensional ball centered at $\bar{x}$. This implies that C is full-dimensional). Assume that the interior of C contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. In particular C does not contain any feasible point of (3) in its interior. For each of the $|N|$ extreme rays of corner(B), define

$$\alpha_j = \max\{\alpha \geq 0 : \ \bar{x} + \alpha \bar{r}^j \in C\}. \tag{8}$$

Since $\bar{x}$ is in the interior of C, $\alpha_j > 0$. When the half-line $\{\bar{x} + \alpha \bar{r}^j : \alpha \geq 0\}$ intersects the boundary of C, then α_j is finite, the point $\bar{x} + \alpha_j \bar{r}^j$ belongs to the boundary of C and the semi-open segment $\{\bar{x} + \alpha \bar{r}^j, \ 0 \leq \alpha < \alpha_j\}$ is contained in the interior of C. When $\bar{r}_j$ belongs the recession cone of C, we have $\alpha_j = +\infty$. Define $\frac{1}{+\infty} = 0$. The inequality

$$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1 \tag{9}$$

is the *intersection cut* of corner(B) defined by C.

Theorem 3.1. (Balas [10]) *Let $C \subset \mathbb{R}^n$ be a closed convex set whose interior contains the point $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. The intersection cut* (9) *defined by C is a valid inequality for* corner(B).

Proof. The set of points of the linear relaxation $P(B)$ of corner(B) that are cut off by (9) is $S := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i = 1, \ldots, q$, $x_j \geq 0$, $j \in N$, $\sum_{j \in N} \frac{x_j}{\alpha_j} < 1\}$. We will show that S is contained in the interior of C. Since the interior of C does not contain a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, the result will follow.

Consider polyhedron $\bar{S} := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i = 1, \ldots, q$, $x_j \geq 0$, $j \in N$, $\sum_{j \in N} \frac{x_j}{\alpha_j} \leq 1\}$. By Remark 2.1, $\bar{S}$ is a $|N|$-dimensional polyhedron with vertices $\bar{x}$ and $\bar{x} + \alpha_j \bar{r}^j$ for α_j finite and extreme rays $\bar{r}^j$ for $\alpha_j = +\infty$. Since the vertices of $\bar{S}$ that lie on the hyperplane $\{x \in \mathbb{R}^n : \sum_{j \in N} \frac{x_j}{\alpha_j} = 1\}$ are the points $\bar{x} + \alpha_j \bar{r}^j$ for α_j finite, every point in S can be expressed as a convex combination of points in the segments $\{\bar{x} + \alpha \bar{r}^j, \ 0 \leq \alpha < \alpha_j\}$ for α_j finite, plus a conic combination of extreme rays $\bar{r}^j$, for $\alpha_j = +\infty$. Since, by definition of α_j, the interior of C contains the segments $\{\bar{x} + \alpha \bar{r}^j, \ 0 \leq \alpha < \alpha_j\}$ for α_j finite and the rays $\bar{r}^j$ for $\alpha_j = +\infty$ belong to the recession cone of C, the set S is contained in the interior of C. □

We say that a valid inequality $\sum_{j \in N} \gamma_j x_j \geq 1$ for corner(B) *dominates* a valid inequality $\sum_{j \in N} \gamma'_j x_j \geq 1$ for corner(B) if every point $x \in \mathbb{R}^n$ such that $x_j \geq 0$, $j \in N$, that satisfies the second also satisfies the first. Note that $\sum_{j \in N} \gamma_j x_j \geq 1$ dominates $\sum_{j \in N} \gamma'_j x_j \geq 1$ if and only if $\gamma_j \leq \gamma'_j$ for all $j \in N$.

Remark 3.2. *Let C_1, C_2 be two closed convex sets whose interiors contain $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. If C_1 is contained in C_2, then the inequality* (9) *relative to C_2 dominates the inequality* (9) *relative to C_1.*

A closed convex set C whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is *maximal* if C is not strictly contained in a closed convex set with the same properties. Any closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is contained in a maximal such set [20]. This property and Remark 3.2 imply that it is enough to consider intersection cuts defined by maximal closed convex sets whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

A set $K \subset \mathbb{R}^p$ that contains no point of $\mathbb{Z}^p$ in its interior is called $\mathbb{Z}^p$*-free*.

Remark 3.3. *One way of constructing a closed convex set C whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is the following. In the space $\mathbb{R}^p$, construct a $\mathbb{Z}^p$-free closed convex set K whose interior contains the orthogonal projection of $\bar{x}$ onto $\mathbb{R}^p$. The cylinder $C = K \times \mathbb{R}^{n-p}$ is a closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.*

Example 3.4. *Consider the following 4-variable mixed-integer set*

$$\begin{aligned} x_1 &= b_1 + a_{11}y_1 + a_{12}y_2 \\ x_2 &= b_2 + a_{21}y_1 + a_{22}y_2 \\ x &\in \mathbb{Z}^2 \\ y &\geq 0 \end{aligned} \tag{10}$$

where the rays $r^1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, r^2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \in \mathbb{R}^2$ are not colinear and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \notin \mathbb{Z}^2$.

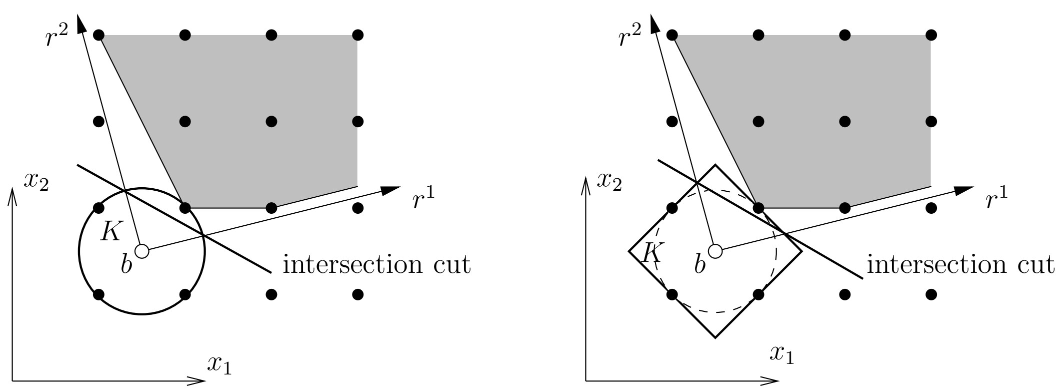

Figure 2: Intersection cuts

Figure 2 represents problem (10) in the space of the variables x_1, x_2. The set of feasible points $x \in \mathbb{R}^2$ for the linear relaxation of (10) is the cone with apex b and extreme rays r^1, r^2. The feasible points $x \in \mathbb{Z}^2$ for (10) are represented by the black dots in this cone. The shaded region represents the corner polyhedron in the (x_1, x_2)-space. The figure depicts two examples of lattice-free convex sets $K \subset \mathbb{R}^2$ containing b in their interior.

Because there are two nonbasic variables in this example, the intersection cut can be represented by a line in the space of the basic variables, namely the line passing through the intersection points p^1, p^2 of the boundary of K with the half lines $\{b + \alpha r^1 : \alpha \geq 0\}$, $\{b + \alpha r^2 : \alpha \geq 0\}$.

The coefficients α_1, α_2 defining the intersection cut $\frac{y_1}{\alpha_1} + \frac{y_2}{\alpha_2} \geq 1$ are obtained by $\alpha_j = \frac{\|p^j - b\|}{\|r^j\|}$, $j = 1, 2$. Note that the intersection cut on the right dominates the one on the left (as stated in Remark 3.2), since the lattice-free set on the right contains the one on the left, and thus the coefficients $\frac{1}{\alpha_j}$ are smaller.

Example 3.5. *(Intersection cut defined by a split)*

Given $\pi \in \mathbb{Z}^p$ and $\pi_0 \in \mathbb{Z}$, let $K := \{x \in \mathbb{R}^p : \pi_0 \leq \pi x \leq \pi_0 + 1\}$. Since for any $\bar{x} \in \mathbb{Z}^p$ either $\pi\bar{x} \leq \pi_0$ or $\pi\bar{x} \geq \pi_0 + 1$, K is a $\mathbb{Z}^p$-free convex set. Furthermore it is easy to verify that if the entries of π are relatively prime, both hyperplanes $\{x \in \mathbb{R}^p : \pi x = \pi_0\}$ and $\{x \in \mathbb{R}^p : \pi x = \pi_0 + 1\}$ contain points in $\mathbb{Z}^p$. Therefore K is a maximal $\mathbb{Z}^p$-free convex set in this case. Consider the cylinder $C := K \times \mathbb{R}^{n-p} = \{x \in \mathbb{R}^n : \pi_0 \leq \sum_{j=1}^p \pi_j x_j \leq \pi_0 + 1\}$. By Remark 3.3, C is a $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set. Such a set C is called a split.

Given a corner polyhedron corner(B)*, let $\bar{x}$ be the unique vertex of its linear relaxation $P(B)$. If $\bar{x}_j \notin \mathbb{Z}$ for some $j = 1, \ldots, p$, there exist π, π_0 such that $\pi_0 < \sum_{j=1}^p \pi_j \bar{x}_j < \pi_0 + 1$. Then the split C contains $\bar{x}$ in its interior. We apply formula (8) to C. Define $\epsilon := \pi\bar{x} - \pi_0$. Since $\pi_0 < \pi\bar{x} < \pi_0 + 1$, we have $0 < \epsilon < 1$. Also, for $j \in N$, define scalars:*

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi\bar{r}^j} & \text{if } \pi\bar{r}^j < 0, \\ \frac{1-\epsilon}{\pi\bar{r}^j} & \text{if } \pi\bar{r}^j > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{11}$$

where $\bar{r}^j$ is defined in (4).

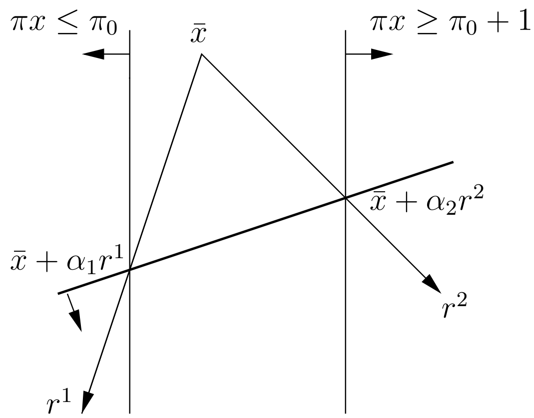

Figure 3: Intersection cut defined by a split

As indicated earlier, the interpretation of α_j is the following. Consider the half-line $\bar{x} + \alpha\bar{r}^j$, where $\alpha \geq 0$, starting from $\bar{x}$ in the direction $\bar{r}^j$. The value α_j is the largest $\alpha \geq 0$ such that $\bar{x} + \alpha\bar{r}^j$ belongs to C. In other words, when the above half-line intersects one of the hyperplanes $\pi x = \pi_0$ or $\pi x = \pi_0 + 1$, this intersection point $\bar{x} + \alpha_j\bar{r}^j$ defines α_j (see Figure 3) and when the direction $\bar{r}^j$ is parallel to the hyperplane $\pi x = \pi_0$, $\alpha_j = +\infty$. The intersection cut defined by the split C is given by:

$$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1. \tag{12}$$

Intersection cuts defined by splits seem to play a particularly important role when it comes to describing corner(B). As an example, for the edge relaxation of the stable set problem, Campelo and Cornuéjols [29] showed that every nontrivial facet defining inequality for corner(B) is an intersection cut defined by a split. Andersen and Weismantel [8] showed that intersection cuts defined by splits are the most desirable when it comes to minimizing the number of nonzero coefficients in the cut.

Example 3.6. *(Gomory Mixed Integer cuts from the tableau [58])*

Balas [10] showed that the GMI cuts derived from rows of the simplex tableau (2) *are intersection cuts defined by splits.*

Consider a corner polyhedron corner(B) *described by the system* (3). *Let* $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ *be an equation where* $\bar{b}_i$ *is fractional. Let* $f_0 = \bar{b}_i - \lfloor \bar{b}_i \rfloor$ *and* $f_j = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$. *Define* $\pi_0 := \lfloor \bar{b}_i \rfloor$, *and for* $j = 1, \ldots, p$,

$$\pi_j := \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in N \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in N \text{ and } f_j > f_0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

For $j = p+1, \ldots, n$, *we define* $\pi_j := 0$. *Note that* $\pi_0 < \pi \bar{x} < \pi_0 + 1$.

Next we derive the intersection cut defined by the split $C := \{x \in \mathbb{R}^n : \pi_0 \leq \pi x \leq \pi_0 + 1\}$ *following Example 3.5. We compute* α_j *using formula* (11), *where* $j \in N$. *We have*

$$\epsilon = \pi \bar{x} - \pi_0 = \sum_{i \in B} \pi_i \bar{x}_i - \pi_0 = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f_0.$$

Let $j \in N$. *Using* (4) *and* (13), *we obtain* $\pi \bar{r}^j = \pi_j \bar{r}^j_j + \pi_i \bar{r}^j_i$ *since* $\bar{r}^j_h = 0$ *for all* $h \in N \setminus \{j\}$ *and* $\pi_h = 0$ *for all* $h \in B \setminus \{i\}$. *Therefore*

$$\pi \bar{r}^j = \begin{cases} \lfloor \bar{a}_{ij} \rfloor - \bar{a}_{ij} & = -f_j & \text{if } 1 \leq j \leq p \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil - \bar{a}_{ij} & = 1 - f_j & \text{if } 1 \leq j \leq p \text{ and } f_j > f_0, \\ -\bar{a}_{ij} & & \text{if } j \geq p+1. \end{cases} \tag{14}$$

Now α_j *follows from formula* (11). *Therefore the intersection cut* (12) *defined by the split* C *is*

$$\sum_{\substack{j \in N,\, j \leq p \\ f_j \leq f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \in N,\, j \leq p \\ f_j > f_0}} \frac{1 - f_j}{1 - f_0} x_j + \sum_{\substack{p+1 \leq j \leq n \\ \bar{a}_{ij} > 0}} \frac{\bar{a}_{ij}}{f_0} x_j - \sum_{\substack{p+1 \leq j \leq n \\ \bar{a}_{ij} < 0}} \frac{\bar{a}_{ij}}{1 - f_0} x_j \geq 1. \tag{15}$$

This is the GMI cut.

The Gomory formula looks complicated, and it may help to think of it as an inequality of the form

$$\sum_{j=1}^{p} \pi(\bar{a}_{ij}) x_j + \sum_{j=p+1}^{n} \psi(\bar{a}_{ij}) x_j \geq 1$$

where the functions π *and* ψ, *associated with the integer and continuous variables respectively, are*

$$\begin{aligned} \pi(a) :=& \quad \min\{\tfrac{f}{f_0}, \tfrac{1-f}{1-f_0}\} \quad \text{where } f = a - \lfloor a \rfloor \\ \psi(a) :=& \quad \max\{\tfrac{a}{f_0}, \tfrac{-a}{1-f_0}\}. \end{aligned} \tag{16}$$

Properties of functions π *and* ψ, *that yield valid inequalities for* corner(B) *are described in Section 7.*

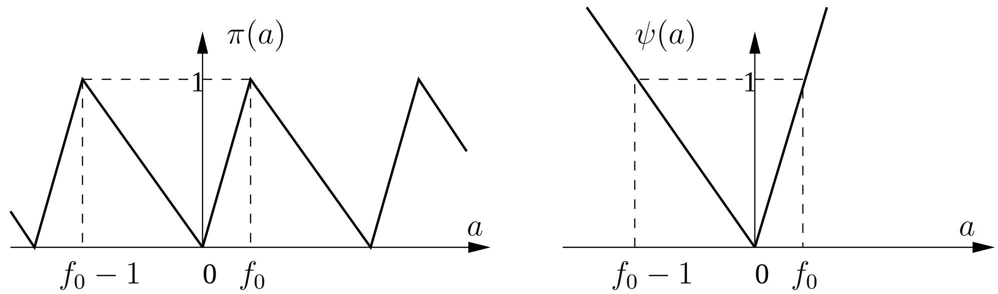

Figure 4: Gomory functions π and ψ

3.1 Equivalence between intersection cuts and valid inequalities for the corner polyhedron

Theorem 3.1 shows that intersection cuts are valid for corner(B). The following theorem provides a converse statement, namely that corner(B) is defined by the intersection cuts. We assume here that corner(B) is nonempty.

Theorem 3.7. *Every nontrivial facet defining inequality for* corner(B) *is an intersection cut.*

Proof. We prove the theorem in the pure integer case (see [30] for the general case). Consider a nontrivial valid inequality for corner(B). By Lemma 2.4 it is of the form $\sum_{j \in N} \gamma_j x_j \geq 1$ with $\gamma_j \geq 0$, $j \in N$. We show that it is an intersection cut.

Consider the polyhedron $S = P(B) \cap \{x \in \mathbb{R}^n : \sum_{j \in N} \gamma_j x_j \leq 1\}$. Since $\sum_{j \in N} \gamma_j x_j \geq 1$ is a valid inequality for corner(B), all points of $\mathbb{Z}^p \cap S$ satisfy $\sum_{j \in N} \gamma_j x_j = 1$.

Since $P(B)$ is a rational polyhedron, $P(B) = \{x \in \mathbb{R}^n : Cx \leq d\}$ for some integral matrix C and vector d. Let

$$T = \{x \in \mathbb{R}^n : Cx \leq d + 1, \sum_{j \in N} \gamma_j x_j \leq 1\}.$$

We first show that T is a $\mathbb{Z}^p$-free convex set. Assume that the interior of T contains an integral point $\tilde{x}$. That is, $\tilde{x}$ satisfies all inequalities defining T strictly. Since $Cx \leq d+1$ is an integral system, then $C\tilde{x} \leq d$ and $\sum_{j \in N} \gamma_j \tilde{x}_j < 1$. This contradicts the fact that all points of $\mathbb{Z}^p \cap S$ satisfy $\sum_{j \in N} \gamma_j x_j = 1$.

Since $\bar{x}$ belongs to S and $\sum_{j \in N} \gamma_j \bar{x}_j = 0$, T is a $\mathbb{Z}^p$-free convex set containing $\bar{x}$ in its interior. Note that the intersection cut defined by T is $\sum_{j \in N} \gamma_j x_j \geq 1$. □

3.2 The gauge function

Intersection cuts have a nice description in the language of convex analysis. Let $K \subseteq \mathbb{R}^n$ be a closed, convex set with the origin in its interior. A standard concept in convex analysis [69, 79] is that of *gauge* (sometimes called Minkowski function), which is the function γ_K defined by

$$\gamma_K(r) = \inf\{t > 0 : \frac{r}{t} \in K\}, \quad \text{for all } r \in \mathbb{R}^n.$$

It is the smallest scalar $t > 0$ such that $\frac{r}{t}$ belongs to K. Since the origin is in the interior of K, $\gamma_K(r) < +\infty$ for all $r \in \mathbb{R}^n$.

The coefficients α_j of the intersection cut defined in (8) can be expressed in terms of the gauge of $K := C - \bar{x}$, namely $\frac{1}{\alpha_j} = \gamma_K(\bar{r}^j)$.

Remark 3.8. *The intersection cut defined by the $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set C is precisely $\sum_{j \in N} \gamma_K(\bar{r}^j) x_j \geq 1$, where $K := C - \bar{x}$.*

A function $g : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $g(r^1) + g(r^2) \geq g(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$. The function g is *positively homogeneous* if $g(\lambda r) = \lambda g(r)$ for every $r \in \mathbb{R}^n$ and every $\lambda > 0$. A function $g : \mathbb{R}^n \to \mathbb{R}$ is *sublinear* if g is subadditive and positively homogeneous.

Note that if $g : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous, then $g(0) = 0$. Indeed, for any $t > 0$, we have that $g(0) = g(t0) = tg(0)$, which implies that $g(0) = 0$.

Lemma 3.9. *Given a closed convex set K with the origin in its interior, the gauge γ of K is a nonnegative sublinear function.*

Proof. It follows from the definition of gauge that γ is positively homogeneous and nonnegative.

Since K is a closed convex set, γ is a convex function. We now show that γ is subadditive. We have that $\gamma(r^1) + \gamma(r^2) = 2(\gamma(\frac{r^1}{2}) + \gamma(\frac{r^2}{2})) \geq 2\gamma(\frac{r^1+r^2}{2}) = \gamma(r^1 + r^2)$, where the equalities follow by positive homogeneity and the inequality follows by convexity. □

Remark 3.10. *A sublinear function $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and convex.*

Proof. Let g be a sublinear function. The convexity of g follows from $\frac{1}{2}(g(r^1) + g(r^2)) = g(\frac{r^1}{2}) + g(\frac{r^2}{2}) \geq g(\frac{r^1+r^2}{2})$ for every $r^1, r^2 \in \mathbb{R}^n$, where the equality follows by positive homogeneity and the inequality by subadditivity. Every convex function is continuous, see e.g. Rockafellar [79]. □

Lemma 3.11. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative sublinear function and let $K = \{x \in \mathbb{R}^n : g(x) \leq 1\}$. Then K is a closed convex set with the origin in its interior and g is the gauge of K.*

Proof. By Remark 3.10, g is convex. Therefore K is a closed convex set. Since the interior of K is $\{x \in \mathbb{R}^n : g(x) < 1\}$ and $g(0) = 0$, the origin is in the interior of K.

Let $x \in \mathbb{R}^n$. If the ray $\{tx : t \geq 0\}$ intersects the boundary of K, let $t^* > 0$ be such that $g(t^* x) = 1$. Since g is positively homogeneous, $g(x) = \frac{1}{t^*} = \inf\{t > 0 : \frac{x}{t} \in K\}$. If the ray $\{tx : t \geq 0\}$ does not intersect the boundary of K, since g is nonnegative and positively homogeneous, then $g(tx) = 0$ for all $t > 0$. Hence $g(x) = 0 = \inf\{t > 0 : \frac{x}{t} \in K\}$. □

4 Maximal lattice-free convex sets

For a good reference on lattices and convexity, we recommend Barvinok [15]. Here we will only work with the integer lattice $\mathbb{Z}^p$. By Remark 3.2, the best possible intersection cuts are the ones defined by full-dimensional *maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex sets* in $\mathbb{R}^n$, that is, full-dimensional subsets of $\mathbb{R}^n$ that are convex, their interior contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and are maximal with respect to inclusion with the above two properties.

Lemma 4.1. *Let C be a maximal $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$-free convex set and let K be its projection onto $\mathbb{R}^p$. Then K is a maximal $\mathbb{Z}^p$-free convex set and $C = K \times \mathbb{R}^{n-p}$.*

Proof. Let K' be the projection of C onto $\mathbb{R}^p$. Then C is contained in the set $K' + (\{0\}^p \times \mathbb{R}^{n-p})$. Since C is a $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set, K' is a $\mathbb{Z}^p$-free convex set. Let K be a maximal $\mathbb{Z}^p$-free convex set containing K'. Then the set $K \times \mathbb{R}^{n-p}$ is a $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$-free convex set and $C \subseteq K' \times \mathbb{R}^{n-p} \subseteq K \times \mathbb{R}^{n-p}$. Since C is maximal, these three sets coincide and the result follows. □

The above lemma shows that it suffices to study maximal $\mathbb{Z}^p$-free convex sets. Lovász [72] shows that these sets are polyhedra with a lattice point in the relative interior of each of their facets.

Theorem 4.2. (Lovász [72]) *A set $K \subset \mathbb{R}^p$ is a full-dimensional maximal lattice-free convex set if and only K is a polyhedron of the form $K = P + L$ where P is a polytope, L is a rational linear space,* $\dim(K) = \dim(P) + \dim(L) = p$, *$K$ does not contain any lattice point in its interior and there is a lattice point in the relative interior of each facet of K.*

We prove the theorem under the assumption that K is a bounded set. The complete proof of the above theorem appears in [20].

Proof of Theorem 4.2 in the bounded case. Assume K is bounded. Then there exist vectors l, u in $\mathbb{Z}^p$ such that K is contained in the box $B = \{x \in \mathbb{R}^p : l_i \leq x_i \leq u_i, \ i = 1 \dots p\}$. For each $y \in B \cap \mathbb{Z}^p$, since K is a lattice-free convex set, there exists an half-space $\{x \in \mathbb{R}^p : \ a_y x \leq b_y\}$ containing K such that $a_y y = b_y$ (separation theorem for convex sets [15]). Since B is a bounded set, $B \cap \mathbb{Z}^p$ is a finite set. Therefore

$$P = \{x \in \mathbb{R}^p : \ l_i \leq x_i \leq u_i, \ i = 1 \dots p, \ a_y x \leq b_y, \ y \in B \cap \mathbb{Z}^p\}$$

is a polytope. By construction P is lattice-free and $K \subseteq P$. Therefore $K = P$ by maximality of K.

We now show that each facet of K contains a lattice point in its relative interior. Assume by contradiction that facet F_t of K does not contain a point of $\mathbb{Z}^p$ in its relative interior. Let $a_t x \leq b_t$ be the inequality defining F_t. Given $\varepsilon > 0$, let K' be the polyhedron defined by the same inequalities that define K except the inequality $\alpha_t x \leq \beta_t$ that has been substituted with the inequality, $\alpha_t x \leq \beta_t + \varepsilon$. Since the recession cones of K and K' coincide, K' is a polytope. Since K is a maximal lattice-free convex set and $K \subset K'$, K' contains points in $\mathbb{Z}^p$ in its interior. Since K' is a polytope, the number of points in $K' \cap \mathbb{Z}^p$ is finite. Hence there exists one such point minimizing $\alpha_t x$, say z. Let K'' be the polyhedron defined by the same inequalities that define K except the inequality $\alpha_t x \leq \beta_t$ that has been substituted with the inequality $\alpha_t x \leq \alpha_t z$. By construction, K'' does not contain any point of $\mathbb{Z}^p$ in its interior and properly contains K, contradicting the maximality of K. □

Doignon [54], Bell [26] and Scarf [80] show the following.

Theorem 4.3. *Any full-dimensional maximal lattice-free convex set $K \subseteq \mathbb{R}^p$ has at most 2^p facets.*

Proof. By Theorem 4.2, each facet F contains an integral point x^F in its relative interior. If there are more than 2^p facets, then there exist two distinct facets F, F' such that x^F and $x^{F'}$ are congruent modulo 2. Now their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral and it is in the interior of K, contradicting the fact that K is lattice-free. $\square$

In $\mathbb{R}^2$, Theorem 4.3 implies that full-dimensional maximal lattice-free convex sets have at most 4 facets. Using Theorem 4.2, one can show that they are either:

1. Splits: namely sets of the form $\{x \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$, where $\pi_0, \pi_1, \pi_2 \in \mathbb{Z}$ and π_1, π_2 are coprime;
2. Triangles with an integral point in the relative interior of each side and no integral point in the interior of the triangle;
3. Quadrilaterals with an integral point in the relative interior of each side and no integral point in the interior of the quadrilateral.

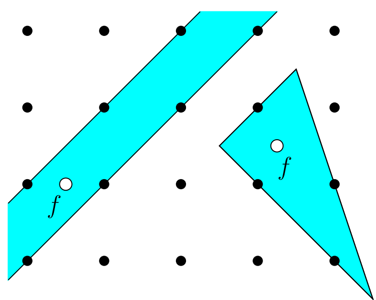

Figure 5: Maximal lattice-free convex sets with nonempty interior in $\mathbb{R}^2$

The recent renewed interest in intersection cuts and the corner polyhedron was sparked by Andersen, Louveaux, Weismantel and Wolsey [6] who proved that, when $|B| = p = 2$, the intersection cuts defined by splits, triangles and quadrilaterals describe corner(B) completely.

Cornuéjols and Margot [34] characterize exactly which splits, triangles and quadrilaterals produce intersection cuts that are facets of corner(B), again in the case when $|B| = p = 2$.

Andersen, Louveaux and Weismantel [4] generalize the 2-row model to include upper bounds on the nonbasic variables and show that new intersection cuts are needed, such as intersection cuts defined by pentagons.

Recall that a polyhedron is integral if all its minimal faces contain integral vectors. Del Pia and Weismantel [43] show that the convex hull of a mixed integer set can be obtained with inequalities derived from integral lattice-free polyhedra. Averkov, Wagner and Weismantel [9] show that that in fixed dimension, up to unimodular transformations, there exist a finite number of maximal polyhedra (with respect to inclusion), among the integral lattice-free polyhedra.

Properties of maximal lattice-free convex sets in dimension $p \geq 3$ were studied by Scarf [81] and Andersen, Wagner and Weismantel [7]. In particular, Scarf shows that, in $\mathbb{R}^3$, maximal lattice-free convex sets with exactly one integral point in the relative interior of each facet have the property that these integral points all lie on two consecutive lattice hyperplanes.

By Remark 3.2, undominated intersection cuts are defined by maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex sets containing $\bar{x}$ in their interior. By Lemma 4.1 and Theorem 4.2, these sets are polyhedra of the type $K \times \mathbb{R}^{n-p}$, where K is a lattice-free polyhedron in $\mathbb{R}^p$. The next theorem shows how to compute the coefficients of the intersection cut from the facet description of K.

Lemma 4.4. *Let $K = \{x \in \mathbb{R}^p : \sum_{h=1}^p d_h^i (x_h - \bar{x}_h) \leq 1,\ i = 1, \ldots, t\}$ be a $\mathbb{Z}^p$-free polyhedron. The coefficients α_j, $j \in N$, defining the intersection cut* (9) *defined by $K \times \mathbb{R}^{n-p}$, are given by*

$$\frac{1}{\alpha_j} = \max_{i=1,\ldots,t} \sum_{h=1}^p d_h^i \bar{r}_h^j.$$

Proof. Since $\alpha_j = \max\{\alpha \geq 0 : \ \bar{x} + \alpha \bar{r}^j \in K \times \mathbb{R}^{n-p}\}$ and $K \times \mathbb{R}^{n-p} = \{x \in \mathbb{R}^n : \sum_{h=1}^p d_h^i (x_h - \bar{x}_h) \leq 1,\ i = 1, \ldots, t\}$, then $\frac{1}{\alpha_j} = \max\{0,\ \sum_{h=1}^p d_h^i \bar{r}_h^j,\ i = 1, \ldots, t\}$.

Since $K \times \mathbb{R}^{n-p}$ is contained in a maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set, by Theorem 4.2, the recession cone of K has dimension smaller than p, hence it has empty interior. Therefore the system of strict inequalities $\sum_{h=1}^p d_h^i r_h < 0$, $i = 1, \ldots, t$ admits no solution. This shows that $\max_{i=1,\ldots,t} \sum_{h=1}^p d_h^i \bar{r}_h^j \geq 0$. □

Let $r^j \in \mathbb{R}^p$ denote the restriction of $\bar{r}^j \in \mathbb{R}^n$ to the first p components. Lemma 4.4 states that intersection cuts are of the form $\sum_{j \in N} \psi(r^j) x_j \geq 1$, where $\psi : \mathbb{R}^p \to \mathbb{R}_+$ is defined by

$$\psi(r) := \max_{i=1,\ldots,t} \sum_{h=1}^p d_h^i r_h. \tag{17}$$

Given a fixed positive integer p and $\bar{b}_i$, $i \in B$ in (3), define a *valid function* to be any function $\psi : \mathbb{R}^p \to \mathbb{R}_+$ such that $\sum_{j \in N} \psi(r^j) x_j \geq 1$ is valid for corner(B) for any number of continuous variables and any choice of $\bar{a}_{ij}$, $i \in B, j \in N$, where r^j is the restriction of the vector $\bar{r}^j$ defined in (4) to $i = 1, \ldots, p$. A valid function ψ is *minimal* if there exists no ψ' distinct from ψ such that $\psi' \leq \psi$.

Since undominated intersection cuts are defined by maximal $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex sets containing $\bar{x}$ in their interior, minimal valid functions are of the form (17).

A function $\psi : \mathbb{R}^p \to \mathbb{R}$ is *piecewise linear* if $\mathbb{R}^p$ can be partitioned into a finite number of polyhedral regions such that the restriction of ψ to the interior of each of these regions is an affine function.

Corollary 4.5. *Every minimal valid function is sublinear and piecewise linear.*

In Section 5 we consider a model with an infinite number of integer variables. We will see that minimal valid functions can be more complicated for such a model.

5 Infinite relaxation

Lemma 4.4 gives a formula for computing the coefficients of an intersection cut, namely $\frac{1}{\alpha_j} = \psi(r^j)$, where ψ is the function defined in (17). Note that the function ψ does not depend on the number of nonbasic variables and on the vectors r^js. Any function with such properties can therefore be used as a "black box" to generate cuts from the tableau of any integer program. The next three sections are devoted to gaining a better understanding of such functions. Gomory and Johnson [62] introduced a convenient setting for the study of these functions, which we introduce next.

Consider problem (3) when all variables x_j are integer for $j \in N$. This problem can be stated as

$$\begin{array}{rcll} f_i + \sum_{j\in N} r_i^j x_j & \in & \mathbb{Z} & \text{for } i = 1, \dots, q \\ x_j & \in & \mathbb{Z}_+ & \text{for } j \in N. \end{array} \tag{18}$$

Gomory and Johnson [62] suggested relaxing the space of variables x_j, $j \in N$, to an infinite-dimensional space, where the variables x_r are defined for any $r \in \mathbb{R}^q$. We obtain the *infinite relaxation*

$$\begin{array}{rcll} f + \sum_{r\in\mathbb{R}^q} r x_r & \in & \mathbb{Z}^q & \\ x_r & \in & \mathbb{Z}_+ & \text{for all } r \in \mathbb{R}^q \\ x & & & \text{has a finite support.} \end{array} \tag{19}$$

By x *has finite support*, we mean $x_r > 0$ for a finite number of $r \in \mathbb{R}^q$. Every problem of the type (18) can be obtained from (19) by setting to 0 all but a finite number of variables. This is why x is restricted to have finite support in the above model. Furthermore the study of model (19) yields information on (18) that are independent on the data in (18).

Denote by G_f the set of feasible solutions to (19). Note that $G_f \neq \emptyset$ since $x_r = 1$ for $r = -f$ and $x_r = 0$ otherwise, is a feasible solution to (19). A function $\pi : \mathbb{R}^q \to \mathbb{R}$ is *valid* if $\pi \geq 0$ and the linear inequality

$$\textstyle\sum_{r\in\mathbb{R}^q} \pi(r) x_r \geq 1 \tag{20}$$

is satisfied by all feasible solutions of (19).

The relevance of the above definition rests on the fact that any valid function π yields a valid inequality for the original integer program (18) by restricting the inequality (20) to the space r^j , $j \in N$.

The nonnegativity assumption in the definition of valid function might, however, seem artificial. If we removed such assumption, then there could be valid functions taking negative values. However, any valid function should be nonnegative over rational vectors. Indeed, let π be a function such that (20) holds for every $x \in G_f$, and suppose $\pi(\tilde{r}) < 0$ for some $\tilde{r} \in \mathbb{Q}^q$. Let $D \in \mathbb{Z}^+$ such that $D\tilde{r}$ is an integral vector, and let $\bar{x}$ be a feasible solution of G_f (for example $\bar{x}_r = 1$ for $r = -f$, $\bar{x}_r = 0$ otherwise). Let $\tilde{x}$ be defined by $\tilde{x}_{\tilde{r}} := \bar{x}_{\tilde{r}} + MD$ where M is a positive integer, and $\tilde{x}_r := \bar{x}_r$ for $r \neq \tilde{r}$. It follows that $\tilde{x}$ is a feasible solution of G_f. We have $\sum \pi(r)\tilde{x}_r = \sum \pi(r)\bar{x}_r + \pi(\tilde{r})MD$. Choose the integer M large enough, namely $M > \frac{\sum \pi(r)\bar{x}_r - 1}{D|\pi(\tilde{r})|}$. Then $\sum \pi(r)\tilde{x}_r < 1$, contradicting the fact that $\tilde{x}$ is feasible.

Thus, since data in mixed integer programming problem are rational and valid functions should be nonnegative over rational vectors, one is only interested in nonnegative valid function.

A valid function $\pi : \mathbb{R}^q \to \mathbb{R}$ is *minimal* if there is no valid function $\pi' \neq \pi$ such that $\pi'(r) \leq \pi(r)$ for all $r \in \mathbb{R}^q$. It can be shown that, indeed, for every valid function π, there exists a minimal valid function π' such that $\pi' \leq \pi$. We remark that minimal valid function only take values between 0 and 1.

A function $\pi : \mathbb{R}^q \to \mathbb{R}$ is *periodic* if $\pi(r) = \pi(r + w)$, for every $w \in \mathbb{Z}^q$. Therefore a periodic function is entirely defined by its values in $[0, 1[^q$.

Lemma 5.1. *If π is a minimal valid function, then π is periodic and $\pi(0) = 0$.*

Proof. Suppose π is not periodic. Then $\pi(\tilde{r}) > \pi(\tilde{r} + w)$ for some $\tilde{r} \in \mathbb{R}^q$ and $w \in \mathbb{Z}^q$. Define the function π' as follows:

$$\pi'(r) := \begin{cases} \pi(\tilde{r} + w) & \text{if } r = \tilde{r} \\ \pi(r) & \text{if } r \neq \tilde{r}. \end{cases}$$

We show that π' is valid. Consider any $\bar{x} \in G_f$. Let $\tilde{x}$ be defined as follows:

$$\tilde{x}_r := \begin{cases} \bar{x}_r & \text{if } r \neq \tilde{r},\ \tilde{r} + w \\ 0 & \text{if } r = \tilde{r} \\ \bar{x}_{\tilde{r}} + \bar{x}_{\tilde{r}+w} & \text{if } r = \tilde{r} + w. \end{cases}$$

Since $\bar{x} \in G_f$ and $w\bar{x}_{\tilde{r}+w} \in \mathbb{Z}^q$, we have that $\tilde{x} \in G_f$. Furthermore $\sum \pi'(r)\bar{x}_r = \sum \pi(r)\tilde{x}_r \geq 1$. This proves that the function π is periodic.

If $\bar{x}$ is a feasible solution of G_f, then so is $\tilde{x}$ defined by $\tilde{x}_r := \bar{x}_r$ for $r \neq 0$, and $\tilde{x}_0 = 0$. Therefore, if π is valid, then π' defined by $\pi'(r) = \pi(r)$ for $r \neq 0$ and $\pi'(0) = 0$ is also valid. Since π is minimal and nonnegative, it follows that $\pi(0) = 0$. □

Lemma 5.2. *If π is a minimal valid function, then π is subadditive.*

Proof. Let $r^1, r^2 \in \mathbb{R}^q$. We need to show $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$. Define the function π' as follows.

$$\pi'(r) := \begin{cases} \pi(r^1) + \pi(r^2) & \text{if } r = r^1 + r^2 \\ \pi(r) & \text{if } r \neq r^1 + r^2. \end{cases}$$

We show that π' is valid. Consider any $\bar{x} \in G_f$. Define $\tilde{x}$ as follows

$$\tilde{x}_r := \begin{cases} \bar{x}_{r^1} + \bar{x}_{r^1+r^2} & \text{if } r = r^1 \\ \bar{x}_{r^2} + \bar{x}_{r^1+r^2} & \text{if } r = r^2 \\ 0 & \text{if } r = r^1 + r^2 \\ \bar{x}_r & \text{otherwise.} \end{cases}$$

Using the definitions of π' and $\tilde{x}$, it is easy to verify that

$$\sum_r \pi'(r)\bar{x}_r = \sum_r \pi(r)\tilde{x}_r. \tag{21}$$

Furthermore we have $f + \sum r\bar{x}_r = f + \sum r\tilde{x}_r \in \mathbb{Z}^q$. Since $\tilde{x} \geq 0$, this implies that $\tilde{x} \in G_f$.

Since π is valid, this implies $\sum_r \pi(r)\tilde{x}_r \geq 1$. Therefore, by (21), $\sum \pi'(r)\bar{x}_r \geq 1$. Thus π' is valid. Since π is minimal, we get $\pi(r^1 + r^2) \leq \pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$. □

Note that any minimal valid function π must satisfy $\pi(r) \leq 1$ for all $r \in \mathbb{R}^q$. Furthermore, it must satisfy $\pi(-f) = 1$. It follows from subadditivity that $\pi(r) + \pi(-f-r) \geq \pi(-f) = 1$. A function $\pi : \mathbb{R}^q \to \mathbb{R}$ is said to satisfy the *symmetry condition* if $\pi(r) + \pi(-f-r) = 1$ for all $r \in \mathbb{R}^q$.

Lemma 5.3. *If π is a minimal valid function, then π satisfies the symmetry condition.*

Proof. Suppose there exists $\tilde{r} \in \mathbb{R}^q$ such that $\pi(\tilde{r}) + \pi(-f-\tilde{r}) \neq 1$. Since π is valid, $\pi(\tilde{r}) + \pi(-f-\tilde{r}) = 1+\delta$ where $\delta > 0$. Note that, since $\pi(r) \leq 1$ for all $r \in \mathbb{R}^q$, it follows that $\pi(\tilde{r}) > 0$.

Define the function π' as follows:

$$\pi'(r) := \begin{cases} \frac{1}{1+\delta}\pi(\tilde{r}) & \text{if } r = \tilde{r}, \\ \pi(r) & \text{if } r \neq \tilde{r}, \end{cases} \qquad r \in \mathbb{R}^q.$$

We show that π' is valid. Consider any $\bar{x} \in G_f$. Note that

$$\sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r = \sum_{\substack{r \in \mathbb{R}^q \\ r \neq \tilde{r}}} \pi(r)\bar{x}_r + \frac{1}{1+\delta}\pi(\tilde{r})\bar{x}_{\tilde{r}}$$

If $\bar{x}_{\tilde{r}} = 0$ then $\sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r = \sum_{r \in \mathbb{R}^q} \pi(r)\bar{x}_r \geq 1$ because π is valid. If $\bar{x}_{\tilde{r}} \geq (1+\delta)/\pi(\tilde{r})$ then $\sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r \geq 1$. Thus we can assume that $1 \leq \bar{x}_{\tilde{r}} < (1+\delta)/\pi(\tilde{r})$. It follows

$$\begin{aligned} \sum_{r \in \mathbb{R}^q} \pi'(r)\bar{x}_r &= \sum_{\substack{r \in \mathbb{R}^q \\ r \neq \tilde{r}}} \pi(r)\bar{x}_r + \pi(\tilde{r})(\bar{x}_{\tilde{r}} - 1) + \pi(\tilde{r}) - \frac{\delta}{1+\delta}\pi(\tilde{r})\bar{x}_{\tilde{r}} \\ &\geq \pi(-f-\tilde{r}) + \pi(\tilde{r}) - \delta \\ &\geq 1 + \delta - \delta = 1, \end{aligned}$$

where the inequality $\sum_{r \in \mathbb{R}^q r \neq \tilde{r}} \pi(r)\bar{x}_r + \pi(\tilde{r})(\bar{x}_{\tilde{r}} - 1) \geq \pi(-f-\tilde{r})$ follows by the subadditivity of π (Lemma 5.2). Therefore π' is valid, contradicting the minimality of π. $\square$

Theorem 5.4. (Gomory and Johnson [62]) *Let $\pi : \mathbb{R}^q \to \mathbb{R}$ be a nonnegative function. Then π is a minimal valid function if and only if $\pi(0) = 0$, π is periodic, subadditive and satisfies the symmetry condition.*

Proof. Lemmas 5.1, 5.2 and 5.3 prove the necessity.

Assume now that $\pi(0) = 0$, π is periodic, subadditive and satisfies the symmetry condition.

We first show that π is valid. The symmetry condition implies $\pi(0) + \pi(-f) = 1$. Since $\pi(0) = 0$, we have $\pi(-f) = 1$. Any $\bar{x} \in G_f$ satisfies $\sum r\bar{x}_r = -f + w$ for some $w \in \mathbb{Z}^q$. We have that $\sum \pi(r)\bar{x}_r \geq \pi(\sum r\bar{x}_r) = \pi(-f+w) = \pi(-f) = 1$, where the inequality comes from subadditivity and the second to last equality comes from periodicity. Thus π is valid.

If π is not minimal, there exists a valid function $\pi' \leq \pi$ such that $\pi'(\tilde{r}) < \pi(\tilde{r})$ for some $\tilde{r} \in \mathbb{R}^q$. Then $\pi(\tilde{r}) + \pi(-f-\tilde{r}) = 1$ implies $\pi'(\tilde{r}) + \pi'(-f-\tilde{r}) < 1$, contradicting the validity of π'. $\square$

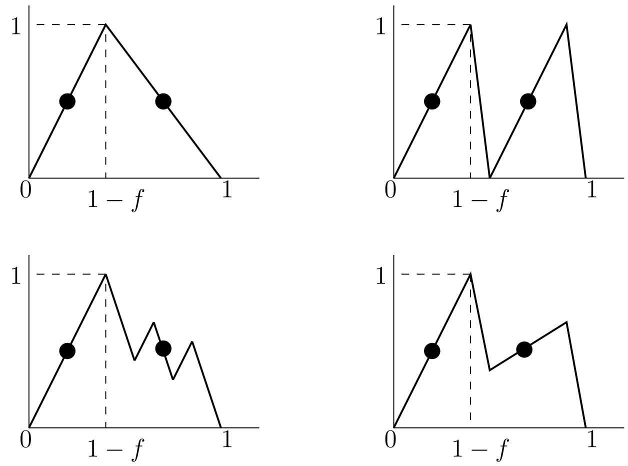

Figure 6: Examples of minimal valid functions ($q = 1$)

Example 5.5. *Figure 6 gives examples of minimal valid functions for model* (19) *with* $q = 1$. *The functions are represented in the interval* $[0, 1]$ *and are defined elsewhere by periodicity. Note the symmetry relative to the points* $(\frac{1-f}{2}, \frac{1}{2})$ *and* $(1 - \frac{f}{2}, \frac{1}{2})$. *Checking subadditivity is a nontrivial task. Gomory, Johnson and Evans [65] showed that, for a continuous nonnegative piecewise linear function, it is enough to check that* $\pi(a)+\pi(b) \geq \pi(a+b)$ *and* $\pi(a)+\pi(b-a) \geq \pi(b)$ *at all the breakpoints* a, b *of the function. More examples of minimal valid functions for* (19) *with* $q = 1$ *are given by Miller, Li and Richard [75]. Their examples are also continuous nonnegative piecewise linear functions. The situation is more complicated when* $q = 2$ *in model* (19). *Dey and Richard [47] initiated such a study.*

Let π be a minimal valid function. Thus π is subadditive by Lemma 5.2. Denote by $E(\pi)$ the set of all possible inequalities $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ that are satisfied as an equality.

Lemma 5.6. *Let* π *be a minimal valid function. Assume* $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, *where* π_1 *and* π_2 *are valid functions. Then* π_1 *and* π_2 *are minimal functions and* $E(\pi) \subseteq E(\pi_1) \cap E(\pi_2)$.

Proof. Suppose π_1 is not minimal. Let $\pi_1' \neq \pi$ be a valid function, such that $\pi_1' \leq \pi_1$. Then $\pi' = \frac{1}{2}\pi_1' + \frac{1}{2}\pi_2$ is a valid function, distinct from π, and $\pi' \leq \pi$. This contradicts the minimality of π.

Suppose $E(\pi) \not\subseteq E(\pi_1) \cap E(\pi_2)$. We may assume $E(\pi) \not\subseteq E(\pi_1)$. That is, there exist r_1, r_2 such that $\pi(r_1)+\pi(r_2) = \pi(r_1+r_2)$ and $\pi_1(r_1)+\pi_1(r_2) > \pi_1(r_1+r_2)$. Since π_2 is minimal, it is subadditive and therefore $\pi_2(r_1)+\pi_2(r_2) \geq \pi_2(r_1+r_2)$. This contradicts the assumption that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. □

A valid function π is *extreme* if it cannot be expressed as a convex combination of two distinct valid functions. That is, $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ implies $\pi = \pi_1 = \pi_2$.

Example 5.7. *The four functions of Figure 6 are extreme. For the first three, this will follow from the two-slope theorem (see next section). The proof of extremality for the last function is more complicated (see [64]). We remark that extreme functions are not always continuous.*

Indeed, Dey, Richard, Li and Miller [49] show that the for $0 < 1 - f < .5$*, the following discontinuous function of Figure 7 is extreme.*

$$\pi(r) := \begin{cases} \frac{r}{1-f} & \text{for } 0 \le r \le 1-f \\ \frac{r}{2-f} & \text{for } 1-f < r < 1 \end{cases}$$

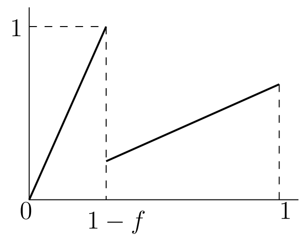

Figure 7: A discontinuous extreme valid function

Gomory and Johnson [64] conjectured that extreme valid functions are always piecewise linear. Basu, Conforti, Cornuéjols and Zambelli [19] disprove this conjecture. However Corollary 4.5 shows that the Gomory-Johnson conjecture is "almost" true, and that pathologies only arise when we consider an infinite number of integer variables.

5.1 The two-slope theorem

We now examine extreme functions $\pi : \mathbb{R} \to \mathbb{R}$ for the single row problem ($q = 1$ in model (19)). Our goal in this section is to prove the Gomory-Johnson two-slope theorem [64]. A useful tool for showing that a valid function is extreme is the Interval Lemma. The version stated here was proven in [19], and it is a variant of the Interval Lemma stated in Gomory and Johnson [64]. They prove the lemma under the assumption that the function in the statement is continuous, whereas we only require the function to be bounded one every interval. Other variants of the Interval Lemma that do not require the function to be continuous have been given by Dey et al. [49]. The proof we give is in the same spirit of the solution of Cauchy's Equation (see for example Chapter 2 of Aczél [1]).

Lemma 5.8 (Interval lemma)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function bounded on every bounded interval. Let* $a_1 < a_2$ *and* $b_1 < b_2$ *be four rational numbers. Consider the sets* $A := [a_1, a_2]$*,* $B := [b_1, b_2]$ *and* $A + B := [a_1 + b_1, a_2 + b_2]$*. If* $f(a) + f(b) = f(a + b)$ *for all* $a \in A$ *and* $b \in B$*, then* f *is an affine function in each of the set* A*,* B *and* $A + B$*, and it has the same slope in each of these sets.*

Proof. We first show the following.

Claim 1. Let $a \in A$*, and let* $\varepsilon > 0$ *such that* $b_1 + \varepsilon \in B$*. For every nonnegative integer* p *such that* $a + p\varepsilon \in A$*, we have* $f(a + p\varepsilon) - f(a) = p(f(b_1 + \varepsilon) - f(b_1))$*.*

For $h = 1, \ldots, p$, by hypothesis $f(a+h\varepsilon)+f(b_1) = f(a+h\varepsilon+b_1) = f(a+(h-1)\varepsilon)+f(b_1+\varepsilon)$. Thus $f(a+h\varepsilon) - f(a+(h-1)\varepsilon) = f(b_1+\varepsilon) - f(b_1)$, for $h = 1, \ldots, p$. By summing the above p equations, we obtain $f(a + p\varepsilon) - f(a) = p(f(b_1 + \varepsilon) - f(b_1))$. This concludes the proof of Claim 1.

Let $\bar{a}, \bar{a}' \in A$ such that $\bar{a} - \bar{a}' \in \mathbb{Q}$ and $\bar{a} > \bar{a}'$. Define $c := \frac{f(\bar{a}) - f(\bar{a}')}{\bar{a} - \bar{a}'}$.

Claim 2. For every $a, a' \in A$ such that $a - a' \in \mathbb{Q}$, we have $f(a) - f(a') = c(a - a')$.

We may assume $a > a'$. Choose a positive rational ε such that $\bar{a} - \bar{a}' = \bar{p}\varepsilon$ for some integer $\bar{p}$, $a - a' = p\varepsilon$ for some integer p, and $b_1 + \varepsilon \in B$. By Claim 1,

$$f(\bar{a}) - f(\bar{a}') = \bar{p}(f(b_1 + \varepsilon) - f(b_1)) \quad \text{and} \quad f(a) - f(a') = p(f(b_1 + \varepsilon) - f(b_1)).$$

Dividing the last equality by $a - a' = p\varepsilon$ and the second to last by $\bar{a} - \bar{a}' = \bar{p}\varepsilon$, we obtain

$$\frac{f(b_1 + \varepsilon) - f(b_1)}{\varepsilon} = \frac{f(\bar{a}) - f(\bar{a}')}{\bar{a} - \bar{a}'} = \frac{f(a) - f(a')}{a - a'} = c.$$

Thus $f(a) - f(a') = c(a - a')$. This concludes the proof of Claim 2.

Claim 3. For every $a \in A$, $f(a) = f(a_1) + c(a - a_1)$.

Let $\delta(x) = f(x) - cx$. We show that $\delta(a) = \delta(a_1)$ for all $a \in A$ and this proves the claim. Since f is bounded on every bounded interval, δ is bounded over A, B and $A + B$. Let M be a number such that $|\delta(x)| \leq M$ for all $x \in A \cup B \cup (A + B)$.

Suppose by contradiction that, for some $a^* \in A$, $\delta(a^*) \neq \delta(a_1)$. Let N be a positive integer such that $|N(\delta(a^*) - \delta(a_1))| > 2M$.
By Claim 2, $\delta(a^*) = \delta(a)$ for every $a \in A$ such that $a^* - a$ is rational. Thus there exists $\bar{a}$ such that $\delta(\bar{a}) = \delta(a^*)$, $a_1 + N(\bar{a} - a_1) \in A$ and $b_1 + \bar{a} - a_1 \in B$. Let $\bar{a} - a_1 = \varepsilon$. By Claim 1,

$$\delta(a_1 + N\varepsilon) - \delta(a_1) = N(\delta(b_1 + \varepsilon) - \delta(b_1)) = N(\delta(a_1 + \varepsilon) - \delta(a_1)) = N(\delta(\bar{a}) - \delta(a_1))$$

Thus $|\delta(a_1 + N\varepsilon) - \delta(a_1)| = |N(\delta(\bar{a}) - \delta(a_1))| = |N(\delta(a^*) - \delta(a_1))| > 2M$, which implies $|\delta(a_1 + N\varepsilon)| + |\delta(a_1)| > 2M$, a contradiction. This concludes the proof of Claim 3.

By symmetry between A and B, Claim 3 implies that there exists some constant c' such that, for every $b \in B$, $f(b) = f(b_1) + c'(b - b_1)$. We show $c' = c$. Indeed, given $\varepsilon > 0$ such that $a_1 + \varepsilon \in A$ and $b_1 + \varepsilon \in B$, $c\varepsilon = f(a_1 + \varepsilon) - f(a_1) = f(b_1 + \varepsilon) - f(b_1) = c'\varepsilon$, where the second equality follows from Claim 1.
Therefore, for every $b \in B$, $f(b) = f(b_1) + cf(b - b_1)$. Finally, since $f(a) + f(b) = f(a + b)$ for every $a \in A$ and $b \in B$, it follows that for every $w \in A + B$, $f(w) = f(a_1 + b_1) + c(w - a_1 - b_1)$. □

A function $\pi : [0, 1] \to \mathbb{R}$ is *piecewise linear* if there are finitely many values $0 = r_0 < r_1 < \ldots < r_k = 1$ such that the function is of the form $\pi(r) = a_j r + b_j$ in interval $]r_j, r_{j+1}[$, for $j = 0, \ldots, k - 1$. The *slopes* of a piecewise linear function are the different values of a_j for $j = 1, \ldots, k$. Note that a piecewise linear function $\pi : [0, 1] \to \mathbb{R}$ is continuous if and only if, for $j = 1, \ldots, k - 1$, $a_j r_j + b_j = a_{j+1} r_j + b_{j+1}$. Alternatively, π is continuous if and only if $\pi(r) = a_j r + b_j$ in the closed interval $[r_j, r_{j+1}]$, for $j = 0, \ldots, k - 1$.

Theorem 5.9. *(Gomory-Johnson two-slope theorem) Let $\pi : \mathbb{R} \to \mathbb{R}$ be a minimal valid function. If the restriction of π to the interval $[0, 1]$ is a continuous piecewise linear function with only two slopes, then π is extreme.*

Proof. Consider valid functions π_1, π_2 such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. By Lemma 5.6, π_1 and π_2 are minimal valid functions. Since π, π_1, π_2 are minimal, by Theorem 5.4 they are nonnegative and $\pi(0) = \pi_1(0) = \pi_2(0) = 0$, $\pi(1) = \pi_1(1) = \pi_2(1) = 0$, $\pi(1-f) = \pi_1(1-f) = \pi_2(1-f) = 1$. We will prove $\pi = \pi_1 = \pi_2$. We recall that minimal valid functions can only take values between 0 and 1, thus π, π_1, π_2 are bounded everywhere.

Assume w.l.o.g. that the slopes of π are distinct in consecutive intervals and let s^+ and s^- be the positive and negative slopes of π. Therefore $\pi(r) = s^+ r$ for $0 \leq r \leq r_1$ and $\pi(r) = \pi(r_{k-1}) + s^-(r - r_{k-1})$ for $r_{k-1} \leq r \leq r_k = 1$. Therefore π has slope s^+ in interval $[r_i, r_{i+1}]$ if i is even and slope s^- if i is odd.

We next show the following. π_1, π_2 *are continuous piecewise linear functions with two slopes. In intervals* $[r_i, r_{i+1}]$*, i even,* π_1, π_2 *have positive slopes* s_1^+, s_2^+*. In intervals* $[r_i, r_{i+1}]$*, i odd,* π_1, π_2 *have negative slopes* s_1^-, s_2^-*.*

Let $i \in \{0, \ldots, k\}$. Assume first i even. Let ϵ be a sufficiently small rational number and define $A = [0, \epsilon]$, $B = [r_i, r_{i+1} - \epsilon]$. Then $A + B = [r_i, r_{i+1}]$ and π has slope s^+ in all three intervals. Since $\pi(0) = 0$, then $\pi(a) + \pi(b) = \pi(a + b)$ for every $a \in A$ and $b \in B$. By Lemma 5.6, $\pi_1(a) + \pi_1(b) = \pi_1(a + b)$ and $\pi_2(a) + \pi_2(b) = \pi_2(a + b)$ for every $a \in A$ and $b \in B$. Thus, by the Interval lemma (Lemma 5.8), π_1 and π_2 are affine functions in each of the closed intervals A, B and $A + B$, where π_1 has positive slope s_1^+ and π_2 has positive slope s_2^+ in each of these sets. The proof for the case i odd is identical, only one needs to choose intervals $A = [r_i + \epsilon, r_{i+1}]$, $B = [1 - \epsilon, 1]$ and use the fact that $\pi(1) = 0$. This shows that, for i even, $\pi_1(r) = \pi_1(r_j) + s_1^+(r - r_j)$ and $\pi_2(r) = \pi_2(r_j) + s_2^+(r - r_j)$, while, for i odd, $\pi_1(r) = \pi_1(r_j) + s_1^-(r - r_j)$ and $\pi_2(r) = \pi_2(r_j) + s_2^-(r - r_j)$. In particular π_1 and π_2 are continuous piecewise linear functions.

Define L_ℓ^+ and L_r^+ as the sum of the lengths of the intervals of positive slope included in $[0, 1 - f]$ and $[1 - f, 1]$, respectively. Define L_ℓ^- and L_r^- as the sum of the lengths of the intervals of negative slope included in $[0, 1 - f]$ and $[1 - f, 1]$, respectively. Note that $L_\ell^+ > 0$ and $L_r^- > 0$.

By the above claim, since $\pi(0) = \pi_1(0) = \pi_2(0) = 0$, $\pi(1) = \pi_1(1) = \pi_2(1) = 0$ and $\pi(1 - f) = \pi_1(1 - f) = \pi_2(1 - f) = 1$, it follows that the vectors (s^+, s^-), (s_1^+, s_1^-), (s_2^+, s_2^-) all satisfy the above system

$$
\begin{aligned}
L_\ell^+ \sigma^+ + L_\ell^- \sigma^- &= 1 \\
L_r^+ \sigma^+ + L_r^- \sigma^- &= -1.
\end{aligned}
$$

Suppose the constraint matrix of the above system is singular. Then the vector (L_r^+, L_r^-) is a multiple of (L_ℓ^+, L_ℓ^-), so it must be a nonnegative multiple, but this is impossible since the right-hand-side of the two equations are one positive and one negative. Thus the constraint matrix is nonsingular, so the system has a unique solution. This implies that $\sigma^+ = s^+ = s_1^+ = s_2^+$ and $\sigma^- = s^- = s_1^- = s_2^-$, and therefore $\pi = \pi_1 = \pi_2$. □

The two-slope theorem can be used to show that many families of valid inequalities are extreme, such as GMI cuts, the 2-step MIR inequality of Dash and Günlük [39] and, more generally, the n-step MIR inequalities of Kianfar and Fathi [70].

The Gomory-Johnson two-slope theorem applies to the single row problem ($q = 1$ in model (19)). Cornuéjols and Molinaro [35] proved a three-slope theorem for the two-row problem ($q = 2$ in model (19)).

6 Continuous Infinite Relaxation

We consider the following model, where all nonbasic variables $y_j,\ j \in N$ are continuous.

$$\begin{array}{rcll} f_i + \sum_{j\in N} r_i^j y_j & \in & \mathbb{Z} & \text{for } i = 1,\dots,q \\ y_j \geq 0 & & & \text{for } j \in N. \end{array} \tag{22}$$

Borozan and Cornuéjols [28] and Basu, Conforti, Cornuéjols and Zambelli [20] studied the *continuous infinite relaxation*, obtained from (22) by augmenting the space of variables $y_j,\ j \in N$, to an infinite-dimensional space $\{y_r,\ r \in \mathbb{R}^q\}$.

$$\begin{array}{rcll} f + \sum_{r\in\mathbb{R}^q} r y_r & \in & \mathbb{Z}^q & \\ y_r \geq 0 & & & \text{for all } r \in \mathbb{R}^q \\ y & & & \text{has a finite support.} \end{array} \tag{23}$$

Denote by R_f the set of feasible solutions to (23). A function $\psi : \mathbb{R}^q \to \mathbb{R}$ is *valid* for R_f if the linear inequality

$$\sum_{r\in\mathbb{R}^q} \psi(r) y_r \geq 1 \tag{24}$$

is satisfied by all vectors in R_f. Any valid function ψ yields a valid inequality for the mixed-integer set (22) by restricting the inequality (24) to the variables $y_{r^j},\ j \in N$.

A valid function $\psi :\ \mathbb{R}^q \to \mathbb{R}$ for R_f is *minimal* if there is no valid function $\psi' \neq \psi$ such that $\psi'(r) \leq \psi(r)$ for all $r \in \mathbb{R}^q$. It can be shown that, indeed, for every valid function ψ for R_f, there exists a minimal valid function ψ' such that $\psi' \leq \psi$.

Lemma 6.1. *If $\psi :\ \mathbb{R}^q \to \mathbb{R}$ is a minimal valid function for R_f then the following hold*

i) ψ is sublinear;

ii) ψ is nonnegative.

Proof. To prove $i)$, we need to show that ψ is subadditive and positively homogeneous. The proof that ψ is subadditive is identical to the proof of Lemma 5.2. We now show that ψ is positively homogeneous.

Suppose there exists $\tilde{r} \in \mathbb{R}^q$ and $\lambda > 0$ such that $\psi(\lambda\tilde{r}) \neq \lambda\psi(\tilde{r})$. Without loss of generality we may assume that $\psi(\lambda\tilde{r}) < \lambda\psi(\tilde{r})$, else we can consider $\lambda\tilde{r}$ instead of $\tilde{r}$ and λ^{-1} instead of λ. Define a function ψ' as follows.

$$\psi'(r) := \begin{cases} \lambda^{-1}\psi(\lambda\tilde{r}) & \text{if } r = \tilde{r} \\ \psi(r) & \text{if } r \neq \tilde{r}. \end{cases}$$

We will show that ψ' is valid. Consider any $\bar{y} \in R_f$. Define $\tilde{y}$ as follows

$$\tilde{y}_r := \begin{cases} 0 & \text{if } r = \tilde{r} \\ \bar{y}_{\lambda\tilde{r}} + \lambda^{-1}\bar{y}_{\tilde{r}} & \text{if } r = \lambda\tilde{r} \\ \bar{y}_r & \text{otherwise.} \end{cases}$$

Using the definitions of ψ' and $\tilde{y}$, it is easy to verify that

$$\sum_r \psi'(r)\bar{y}_r = \sum_r \psi(r)\tilde{y}_r.$$

Furthermore we have $f + \sum r\bar{y}_r = f + \sum r\tilde{y}_r \in \mathbb{Z}^q$. Since $\tilde{y} \geq 0$, this implies that $\tilde{y} \in R_f$.

Since ψ is valid, we have that $\sum_r \psi(r)\tilde{y}_r \geq 1$. Therefore $\sum \psi'(r)\bar{y}_r \geq 1$. This shows that ψ' is valid, contradicting the fact that ψ is minimal. Therefore ψ is positively homogeneous.

ii) We first prove that $\psi(r) \geq 0$ for every $r \in \mathbb{Q}^q$. Suppose $\psi(\tilde{r}) < 0$ for some $\tilde{r} \in \mathbb{Q}^q$. Let $D \in \mathbb{Z}^+$ such that $D\tilde{r}$ is an integral vector, and let $\bar{y}$ be a feasible solution of R_f (for example $\bar{y}_r = 1$ for $r = -f$, $\bar{y}_r = 0$ otherwise). Let $\tilde{y}$ be defined by $\tilde{y}_{\tilde{r}} := \bar{y}_{\tilde{r}} + MD$ where M is a positive integer, and $\tilde{y}_r := \bar{y}_r$ for $r \neq \tilde{r}$. It follows that $\tilde{y}$ is a feasible solution of R_f. We have $\sum \psi(r)\tilde{y}_r = \sum \psi(r)\bar{y}_r + \psi(\tilde{r})MD$. Choose the integer M large enough, namely $M > \frac{\sum \psi(r)\bar{y}_r - 1}{D|\psi(\tilde{r})|}$. Then $\sum \psi(r)\tilde{y}_r < 1$, contradicting the fact that $\tilde{y}$ is feasible.

Since ψ is a continuous function that is nonnegative over $\mathbb{Q}^q$, and $\mathbb{Q}^q$ is dense in $\mathbb{R}^q$, then ψ is nonnegative over $\mathbb{R}^q$. $\square$

Lemma 6.2. *Let B be a $\mathbb{Z}^q$-free closed convex set with f in its interior. Let ψ be the gauge of $B - f$. Then ψ is a valid function.*

Proof. By Lemma 3.9, ψ is sublinear. Consider $y \in R_f$. Then $\sum ry_r = \bar{x} - f$, for some $\bar{x} \in \mathbb{Z}^q$.

$$\sum \psi(r)y_r = \sum \psi(ry_r) \geq \psi\left(\sum ry_r\right) = \psi(\bar{x} - f) \geq 1$$

where the first equality follows by positive homogeneity, the first inequality by subadditivity and the last from the fact that B is a $\mathbb{Z}^q$-free convex set. $\square$

Given a nonnegative sublinear function ψ, let

$$B_\psi = \{x \in \mathbb{R}^q \,:\, \psi(x - f) \leq 1\}.$$

Theorem 6.3. *If ψ is a minimal valid function, then B_ψ is a maximal $\mathbb{Z}^q$-free convex set containing f in its interior, and ψ is the gauge of $B_\psi - f$.*

Proof. Since ψ is a minimal valid function, by Lemma 6.1, ψ is a nonnegative sublinear function. By Lemma 3.11, B_ψ is a closed convex set with f in its interior and ψ is the gauge of $B_\psi - f$.

Since ψ valid, $\psi(\bar{x} - f) \geq 1$ for every $\bar{x} \in \mathbb{Z}^q$, thus B_ψ is a $\mathbb{Z}^q$-free convex set.

We only need to prove that B_ψ is a maximal $\mathbb{Z}^q$-free convex set. Suppose not, and let B' be a Z^q-free convex set properly containing B_ψ. Let ψ' be the gauge of $B' - f$. Then by definition of gauge $\psi' \leq \psi$, and $\psi' \neq \psi$ since $B' \neq B_\psi$. By Lemma 6.2 ψ' is a valid function, a contradiction to the minimality of ψ. $\square$

Theorem 6.4. *Let B be a maximal $\mathbb{Z}^q$-free convex set containing f in its interior. Let $\psi : \mathbb{R}^q \to \mathbb{R}$ be the gauge of $B - f$. Then ψ is a minimal valid function for R_f.*

Proof. By Lemma 6.2 ψ is valid. Suppose there exists a minimal valid function ψ' such that $\psi' \leq \psi$ and $\psi' \neq \psi$. Then $B_{\psi'}$ is a $\mathbb{Z}^q$-free convex set and $B_{\psi'} \supset B_\psi$. Since $B = B_\psi$, this contradicts the maximality of B. $\square$

As earlier, we define a valid function ψ to be extreme if it cannot be written as the convex combination of two distinct valid functions. The next theorem exhibits a correspondence between extreme inequalities for the infinite model (23) and extreme inequalities for the finite problem (22). This theorem appears in [20] and is very similar to a result of Dey and Wolsey [52].

Theorem 6.5. *Let B be a maximal lattice-free convex set in $\mathbb{R}^q$ with f in its interior. Let $L = \text{lin}(B)$ and let $P = B \cap (f + L^\perp)$. Then $B = P + L$, L is a rational space, and P is a polytope. Let $v^1, \dots, v^k$ be the vertices of P, and $r^{k+1}, \dots, r^{k+h}$ be a rational basis of L. Define $r^j = v^j - f$ for $j = 1, \dots, k$. Let $R_f(r^1, \dots, r^{k+h})$ denote the set of solutions to (22) where $N = \{1, \dots, k+h\}$.*

Then the inequality $\sum_{r\in\mathbb{R}^q} \psi_B(r)s_r \geq 1$ is extreme for R_f if and only if the inequality $\sum_{j=1}^k s_j \geq 1$ is extreme for $\text{conv}(R_f(r^1, \dots, r^{k+h}))$.

It could be argued that, for an infinite model such as (23), one should consider valid inequalities (24) where ψ takes values in $\mathbb{R}\cup\{+\infty\}$ instead of just $\mathbb{R}$. Indeed, valid inequalities of this type exist for (23). For this reason, Borozan and Cornuéjols [28] consider valid functions ψ that take values in $\mathbb{R}\cup\{+\infty\}$. However Zambelli [82] showed that the extension to $\mathbb{R}\cup\{+\infty\}$ is never needed for the finite model (22), in the sense that the coefficients of every valid inequality for (22) are always defined by some finite valid function. This result together with Theorem 6.5 justifies the choice that we made in this section to define ψ with its values in $\mathbb{R}$.

Model (23) was extended to the case where $f + \sum_{r\in\mathbb{R}^q} ry_r \in \mathbb{Z}^q \cap P$ for some rational polyhedron [11, 21, 25, 52, 57, 67]. In this model, minimal valid inequalities are still of the form (24) but now they may have negative coefficients $\psi(r)$. Many of the key results still hold in this more general model. In particular it is proven in [21] that the maximal S-free convex sets are polyhedra, and that there is a one-to-one correspondence between maximal S-free convex sets and minimal inequalities. These results have been further extended by Moran and Dey [76] to the case where P is any convex set.

7 The mixed integer infinite relaxation

We consider here the following infinite mixed integer set:

$$\begin{array}{lll} f + \sum_{r\in\mathbb{R}^q} rx_r + \sum_{r\in\mathbb{R}^q} ry_r & \in & \mathbb{Z}^q \\ x_r \in \mathbb{Z}_+ & & \text{for all } r \in \mathbb{R}^q \\ y_r \geq 0 & & \text{for all } r \in \mathbb{R}^q \\ x, y & & \text{have a finite support.} \end{array} \tag{25}$$

We denote with M_f the set of feasible solutions to (25). Note that the infinite relaxation G_f is the set $\{x : (x, 0) \in M_f\}$ and the continuous infinite relaxation R_f is the set $\{y : (0, y) \in M_f\}$.

A function (π, ψ) where $\pi : \mathbb{R}^q \to \mathbb{R}$ and $\psi : \mathbb{R}^q \to \mathbb{R}$ is *valid* for M_f if $\pi \geq 0$ and the linear inequality

$$\sum_{r\in\mathbb{R}^q} \pi(r)x_r + \sum_{r\in\mathbb{R}^q} \psi(r)y_r \geq 1 \tag{26}$$

is satisfied by all vectors in M_f. If (π, ψ) is a valid function, then π is a valid function for G_f and ψ is a valid function for R_f.

A valid function (π, ψ) for M_f is *minimal* if there is no valid function (π', ψ'), distinct from (π, ψ), where $\pi' \leq \pi$ and $\psi' \leq \psi$. It can be shown that, indeed, for every valid function (π, ψ), there exists a minimal valid function (π', ψ') such that $\pi' \leq \pi$ and $\psi' \leq \psi$.

Lemma 7.1. *Let (π, ψ) be a minimal valid function for M_f. Then:*

i) $\pi(r) \leq \psi(r)$ *for every* $r \in \mathbb{R}^q$

ii) ψ *is a nonnegative sublinear function.*

Proof. Let (π, ψ) be a minimal valid function for M_f. Assume $\pi(r^*) > \psi(r^*)$ for some $r \in \mathbb{R}^q$. Let (π', ψ) be the function defined as

$$\pi'(r) := \begin{cases} \psi(r) & \text{for } r = r^* \\ \pi(r) & \text{for } r \in \mathbb{R}^q \setminus \{r^*\} \end{cases}$$

Given $(x, y) \in M_f$, define

$$x'_r := \begin{cases} 0 & \text{for } r = r^* \\ x_r & \text{for } r \in \mathbb{R}^q \setminus \{r^*\} \end{cases} \qquad y'_r := \begin{cases} x_r + y_r & \text{for } r = r^* \\ y_r & \text{for } r \in \mathbb{R}^q \setminus \{r^*\} \end{cases}$$

It is immediate to check that $(x', y') \in M_f$ and that $\sum_{r \in \mathbb{R}^q} \pi'(r) x_r + \sum_{r \in \mathbb{R}^q} \psi(r) y_r = \sum_{r \in \mathbb{R}^q} \pi(r) x'_r + \sum_{r \in \mathbb{R}^q} \psi(r) y'_r \geq 1$. This shows that (π', ψ) is a valid function. This contradicts the minimality of (π, ψ) and $i)$ is proven.

The same proof as the one in Lemma 6.1 shows that ψ is a nonnegative sublinear function $\mathbb{R}^q \to \mathbb{R}$ and any such function satisfies $\psi(0) = 0$. □

The next theorem, due to Johnson [66], shows that in a minimal valid function (π, ψ) for M_f, the function ψ is uniquely determined by π.

Theorem 7.2. *Let (π, ψ) be a valid function for M_f. The function (π, ψ) is minimal for M_f if and only if π is a minimal valid function for G_f and ψ is defined by*

$$\psi(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \quad \textit{for every } r \in \mathbb{R}^q. \tag{27}$$

Proof. Using the same arguments as in the proofs of Lemmas 5.1, 5.2 and 5.3 it can be shown that, if (π, ψ) is minimal, then the function $\pi : \mathbb{R}^q \to \mathbb{R}$ is periodic, subadditive, and satisfies the symmetry condition and $\pi(0) = 0$. By Theorem 5.4, π is a minimal valid function for G_f.

Therefore, we only need to show that, given a valid function (π, ψ) for M_f such that π is minimal for G_f, (π, ψ) is a minimal valid function for M_f if and only if ψ is defined by (27).

Let us define the function ψ' by

$$\psi'(r) = \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \quad \text{for every } r \in \mathbb{R}^q.$$

We will show that ψ' is well defined, (π, ψ') is valid for M_f, and that $\psi' \leq \psi$. This will imply that (π, ψ) is minimal if and only if $\psi = \psi'$, and the statement will follow.

We now show that ψ' is well defined. This amounts to showing that the lim sup in (27) is always finite. We recall that

$$\limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon} := \lim_{\alpha\to 0^+} \sup\{\frac{\pi(\epsilon r)}{\epsilon} \,:\, 0<\epsilon\leq\alpha\} = \inf_{\alpha>0} \sup\{\frac{\pi(\epsilon r)}{\epsilon} \,:\, 0<\epsilon\leq\alpha\}.$$

Let ψ'' be a function such that $\psi''\leq\psi$ and (π,ψ'') is a minimal valid function for M_f (as mentioned earlier, such a function exists).

By Lemma 7.1, $\pi\leq\psi''$ and ψ'' is a sublinear function. Thus, for every $\epsilon>0$ and every $r\in\mathbb{R}^q$, it follows that

$$\frac{\pi(\epsilon r)}{\epsilon} \leq \frac{\psi''(\epsilon r)}{\epsilon} = \psi''(r)$$

thus

$$\limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \leq \psi''(r).$$

This shows that ψ' is well defined and $\psi'\leq\psi''\leq\psi$.

It follows easily from the definition of ψ' and the definition of lim sup that ψ' is sublinear.

We conclude the proof by showing that (π,ψ') is valid for M_f. Let $(\bar{x},\bar{y})\in M_f$.

Suppose by contradiction that

$$\sum_{r\in\mathbb{R}^q}\pi(r)\bar{x}_r + \sum_{r\in\mathbb{R}^q}\psi'(r)\bar{y}_r = 1-\delta$$

where $\delta>0$.

Let $\bar{r}=\sum_{r\in\mathbb{R}^q} r\bar{y}_r$. By definition of ψ', it follows that, for some $\bar{\alpha}>0$ sufficiently small,

$$\frac{\pi(\epsilon\bar{r})}{\epsilon} < \psi'(\bar{r})+\delta \quad \text{for all } 0<\epsilon\leq\bar{\alpha}. \tag{28}$$

Choose $D\in\mathbb{Z}$ such that $D\geq\bar{\alpha}$, and define, for all $r\in\mathbb{R}^q$,

$$\tilde{x}_r = \begin{cases} \bar{x}_r & r\neq\frac{\bar{r}}{D} \\ \bar{x}_r + D & r=\frac{\bar{r}}{D} \end{cases}$$

Note that all entries of $\tilde{x}$ are nonnegative integers and that $\sum_{r\in\mathbb{R}^q} r\tilde{x}_r = \sum_{r\in\mathbb{R}^q} r\bar{x}_r + \sum_{r\in\mathbb{R}^q} r\bar{y}_r$, thus $\tilde{x}$ is in G_f. Now

$$\begin{array}{rcll} \sum_{r\in\mathbb{R}^q}\pi(r)\tilde{x}_r & = & \sum_{r\in\mathbb{R}^q}\pi(r)\bar{x}_r + \frac{\pi(D^{-1}\bar{r})}{D^{-1}} & \\ & < & \sum_{r\in\mathbb{R}^q}\pi(r)\bar{x}_r + \psi'(\bar{r})+\delta & \text{(by (28) because } D^{-1}\leq\bar{\alpha}) \\ & \leq & \sum_{r\in\mathbb{R}^q}\pi(r)\bar{x}_r + \sum_{r\in\mathbb{R}^q}\psi'(r)\bar{y}_r+\delta = 1, & \text{(by sublinearity of } \psi') \end{array}$$

contradicting the fact that π is valid for G_f. □

Lemma 7.3. *Let (π,ψ) be a minimal valid function for M_f. Let $B_\psi=\{x\in\mathbb{R}^q \,:\, \psi(x-f)\leq 1\}$. Then ψ is the gauge of the lattice-free convex set $B_\psi - f$.*

Proof. By Lemma 7.1 ψ is a nonnegative sublinear function and $\psi(0)=0$ and by Lemma 3.11. ψ is the gauge of B_ψ. Since ψ is a valid function for R_f, we have that $\psi(\bar{x}-f)\geq 1$ for every $\bar{x}\in\mathbb{Z}^q$. Therefore B_ψ is a lattice-free convex set. □

If (π, ψ) is a minimal valid function, by Theorem 7.2, π is a minimal function for G_f. However ψ is not in general a minimal valid function for R_f. Indeed, consider the four functions $\pi_1, \dots, \pi_4$ of Figure 6. These functions are extreme for G_f. Let s_i^+ be the positive slope of π_i at 0 and s_i^- be the negative slope at 1 (or at 0, since the function is periodic). By Theorem 7.2, for each of these functions, the function ψ_i for which (π_i, ψ_i) is minimal for M_f is the function $\psi_i(r) := \{s_i^+ r \text{ for } r \geq 0,\ \ s_i^- r \text{ for } r \leq 0\}$.

The positive slopes are identical, while the largest negative slope is s_1^-, thus ψ_1 is pointwise smaller than the other functions. Indeed, ψ_1 is the only minimal function for R_f.

Lemma 7.4. *Let (π, ψ) be a minimal valid function for M_f. Given $r^* \in \mathbb{R}^q$, if*

$$\psi(r^*) + \psi(\bar{z} - f - r^*) = \psi(\bar{z} - f) = 1 \textit{ for some } \bar{z} \in \mathbb{Z}^q,$$

then $\pi(r^*) = \psi(r^*)$.

Proof. Given $\bar{z} \in \mathbb{Z}^q$, define

$$x_r := \begin{cases} 1 & \text{for } r = r^* \\ 0 & \text{for } r \in \mathbb{R}^q \setminus \{r^*\} \end{cases} \qquad y_r := \begin{cases} 1 & \text{for } r = \bar{z} - f - r^* \\ 0 & \text{for } r \in \mathbb{R}^q \setminus \{\bar{z} - f - r^*\} \end{cases}$$

It is straightforward to check that $(x, y) \in M_f$. Therefore we have

$$1 \leq \pi(r^*) + \psi(\bar{z} - f - r^*) \leq \psi(r^*) + \psi(\bar{z} - f - r^*) = \psi(\bar{z} - f) = 1 \tag{29}$$

where the first inequality follows from the fact that $(x, y) \in M_f$ and that (π, ψ) is a valid function for M_f, the second inequality follows because, by Lemma 7.1 $\pi(r^*) \leq \psi(r^*)$. Now (29) implies $\pi(r^*) = \psi(r^*)$. □

In [18] it is proven that if r^* does not satisfy Property (29), then $\pi'(r^*) < \psi(r^*)$ for some minimal valid function (π', ψ).

Lemma 7.5. *Let (π, ψ) be a valid function for M_f and let $\pi'(r) = \inf_{w \in \mathbb{Z}^q} \pi(r + w)$. Then (π', ψ) is a valid function for M_f.*

Proof. The fact that (π', ψ) is a valid function for M_f can be shown by the same argument of the proof of Lemma 5.1. □

Corollary 7.6. *Let ψ be a valid function for R_f and let $\pi'(r) = \inf_{w \in \mathbb{Z}^q} \psi(r + w)$. Then (π', ψ) is a valid function for M_f.*

Proof. Given a minimal valid function ψ for R_f, (ψ, ψ) is a valid function for M_f. Apply Lemma 7.5 to (ψ, ψ). □

The function π' defined in Corollary 7.6 is a *trivial lifting* of ψ [13, 62].

Let (π, ψ) be a minimal valid function for M_f where in addition ψ is a minimal function for R_f. We exhibit an example where the minimality of ψ completely determines the function π.

Example 7.7. *(Dey and Wolsey [50, 51]) Let $q = 2$. Consider the maximal lattice-free triangle $K = \text{conv}(\binom{0}{0}, \binom{2}{0}, \binom{0}{2})$, assume that f is a point in the interior of K (see Figure 8) and let ψ_K be the gauge of $K - f$. By Theorem 6.4, ψ_K is a minimal valid function for R_f. For each of the three points $z_1 = \binom{1}{0}$, $z_2 = \binom{0}{1}$, $z_3 = \binom{1}{1}$ on the boundary of K, we have that $\psi_K(z_i - f) = 1$. For $i = 1, 2, 3$, define $R(z_i) = \{r \in \mathbb{R}^2 : \psi_K(r) + \psi_K(z_i - f - r) = \psi_K(z_i - f) = 1\}$. Since K is a maximal lattice-free convex set, the function ψ_K is given by (17). Therefore regions $R(z_1)$, $R(z_2)$, $R(z_3)$ are the three grey quadrilaterals.*

Let (π, ψ_K) be a minimal function valid for M_f and let $R = R(Z_1) \cup R(Z_2) \cup R(Z_3)$. Lemma 7.4 shows that $\pi(r) = \psi_K(r)$ for every $r \in R$. By Lemma 7.3 π is a minimal valid function for G_f and by Lemma 5.1 π is periodic.

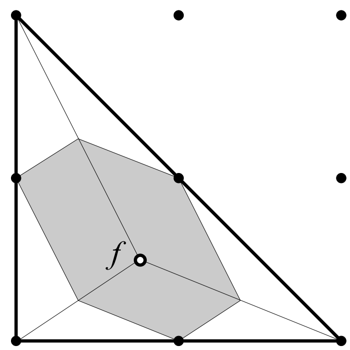

Figure 8: Lattice free triangle giving an inequality with a unique minimal lifting. The shaded region depicts $f + R$.

For $r \in \mathbb{R}^2$, $r - \lfloor r \rfloor$ is the unit box $[0,1] \times [0,1]$. We show that every point in $[0,1] \times [0,1]$ can be translated by an integral vector into $f + R$. Note that $[0,1] \times [0,1] \setminus R$ is the union of the two triangles $\text{conv}(\binom{0}{0}, \binom{1}{0}, \frac{f}{2})$ and $\text{conv}(\binom{0}{0}, \binom{0}{1}, \frac{f}{2})$. The first one can be translated into R by adding the vector $\binom{0}{1}$ and the second can be translated into R by adding the vector $\binom{1}{0}$. The above argument shows that integral translations of R cover $\mathbb{R}^2$. Since the area of R is equal to 1, integral translations of R actually define a tiling of $\mathbb{R}^2$. This discussion implies that the trivial lifting can be computed efficiently, since, for any $d \in \mathbb{R}^2$, it gives a construction for an integral vector w^d such that $d + w^d \in R_K$.

This geometric perspective on lifting was extended to general q by Conforti, Cornuéjols and Zambelli [31], Basu, Campelo, Conforti, Cornuéjols and Zambelli [18], and Basu, Cornuéjols and Köppe [22]. Dey and Wolsey [53] combine the trivial lifting approach described above with traditional sequential lifting. Dey and Richard [48] give facet defining inequalities for the infinite relaxation. Richard and Dey [78] give a comprehensive survey on the group theoretic approach.

8 Closures

Let $Q = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ be a rational polyhedron and let $S = Q \cap \{x \in \mathbb{R}^n : x_j \in \mathbb{Z},\ j = 1, \dots p\}$ be a mixed integer set. (The corner polyhedron is a mixed integer set of this type). Given a family $\mathcal{F}$ of inequalities that are valid for S, the closure of Q (with respect to $\mathcal{F}$) is the convex set defined by the system of inequalities associated with Q plus all the inequalities in $\mathcal{F}$. The main questions that are investigated in this section are the polyhedrality of the closure of Q, the comparative strength of various closures, and whether $\text{conv}(S)$ can be obtained with a finite number of applications of the closure operation.

8.1 Split closure and split rank

Given $\pi \in \mathbb{Z}^p$ and $\pi_0 \in \mathbb{Z}$, consider the split $C := \{x \in \mathbb{R}^n : \pi_0 \leq \sum_{j=1}^p \pi_j x_j \leq \pi_0 + 1\}$. Define Q_{π,π_0} to be the convex hull of $Q \setminus \text{int}(C)$, where $\text{int}(C)$ denotes the interior of C. Equivalently, $Q_{\pi,\pi_0} = \text{conv}(Q^{\leq} \cup Q^{\geq})$ where $Q^{\leq} = Q \cap \{x \in \mathbb{R}^n : \sum_{j=1}^p \pi_j x_j \leq \pi_0\}$ and $Q^{\geq} = Q \cap \{x \in \mathbb{R}^n : \sum_{j=1}^p \pi_j x_j \geq \pi_0 + 1\}$. We have $S \subseteq Q_{\pi,\pi_0} \subseteq Q$.

Define the *split closure of* Q to be

$$\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{p+1}} Q_{\pi,\pi_0}.$$

Although it follows from Balas' work on disjunctive programming that Q_{π,π_0} is a polyhedron, this does not imply that the split closure of Q is a polyhedron. A fundamental result of Cook, Kannan and Schrijver [32] is the following.

Theorem 8.1. *Let $Q = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ be a rational polyhedron and let $S = Q \cap \{x \in \mathbb{R}^n : x_j \in \mathbb{Z}^p,\ j = 1, \dots p\}$ be a mixed integer set. Then the split closure of Q is a rational polyhedron.*

Andersen, Louveaux and Weismantel [5] define the facet-width of a polyhedron and show that the closure obtained by adding cuts derived from lattice-free polyhedra with bounded facet-width is polyhedral.

Andersen, Cornuéjols and Li [2] show that the split closure of Q is identical to the intersection over all bases of all intersection cuts defined by splits.

Theorem 8.2. *Let $Q = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ be a rational polyhedron and let $S = Q \cap \{x \in \mathbb{R}^n : x_j \in \mathbb{Z}^p,\ j = 1, \dots p\}$ be a mixed integer set. Let $\alpha x \leq \beta$ be a facet defining inequality for the split closure of Q. Then there exists a (possibly infeasible) basis B of $Ax = b$ such that $\alpha x \leq \beta$ is an intersection cut of* $\text{corner}(B)$ *defined by a split.*

Dash, Günluk and Raack [41] give a short proof of the above result.

8.2 Triangle and quadrilateral closures

Basu, Bonami, Cornuéjols and Margot [16] consider model (3) when $|B| = p = 2$. As before, $P(B)$ denotes the linear relaxation of (3).

Let $S \subset \mathbb{R}^n$ denote the *split closure* of $P(B)$. Define the *triangle closure* T of $P(B)$ to be the subset of $\mathbb{R}^n$ satisfying all intersection cuts defined by maximal $\mathbb{Z}^2$-free triangles, and the *quadrilateral closure* Q to be the set satisfying all intersection cuts defined by maximal $\mathbb{Z}^2$-free quadrilaterals.

Lovász' theorem and Theorem 3.7 imply that

$$\text{corner}(B) = S \cap T \cap Q.$$

One can show that $T \subseteq S$ and $Q \subseteq S$. This may seem counter-intuitive because some split inequalities are facets of corner(B). However, any split can be obtained as the limit of a sequence of $\mathbb{Z}^2$-free triangles. Consequently, the set obtained by intersecting *all* intersection cuts defined by triangles is contained in the split closure. The same observation holds for quadrilaterals.

Both the triangle closure and the quadrilateral closure are good approximations of corner(B) in the following sense.

Theorem 8.3.
corner$(B) \subseteq T \subseteq 2\,$corner$(B)$ *and*
corner$(B) \subseteq Q \subseteq 2\,$corner$(B)$.

On the other hand, the split closure is not always a good approximation of corner(B).

Theorem 8.4. *For any $\alpha > 1$, there is a choice of data in* (3) *such that* $S \not\subseteq \alpha\,$corner(B).

The above theorem is a worst-case result. A probabilistic analysis of split inequalities was performed by He, Ahmed and Nemhauser [68]. They show that split inequalities are better on average (in a precisely defined sense) than intersection cuts defined by triangles. Del Pia, Wagner, Weismantel [42] analyze the benefit of adding a non-split inequality on top of the split closure. Using a different probabilistic approach, Basu, Cornuéjols and Molinaro [24] show that the split closure is a good approximation of corner(B) on average.

Dash, Dey and Günlük, [38] proved an intriguing result for model (3) when $|B| = p = 2$. They showed that corner(B) is defined entirely by disjunctive cuts from *crooked cross disjunctions* of the form $\{x \in \mathbb{R}^2 : \ \pi^1 x \le \pi^1_0,\ (\pi^2 - \pi^1)x \le \pi^2_0 - \pi^1_0\} \vee \{x \in \mathbb{R}^2 : \ \pi^1 x \le \pi^1_0,\ (\pi^2 - \pi^1)x \ge \pi^2_0 - \pi^1_0 + 1\} \vee \{x \in \mathbb{R}^2 : \ \pi^1 x \ge \pi^1_0 + 1,\ \pi^2 x \le \pi^2_0\} \vee \{x \in \mathbb{R}^2 : \ \pi^1 x \ge \pi^1_0 + 1,\ \pi^2 x \ge \pi^2_0 + 1\}$ where $\pi^1, \pi^2 \in \mathbb{Z}^2$ and $\pi^1_0, \pi^2_0 \in \mathbb{Z}$. In other words, intersection cuts defined by splits, triangles and quadrilaterals are all implied by this family of disjunctive cuts. This work was inspired by Balas and Qualizza who observed that disjunctive cuts from simple cross disjunctions already imply intersection cuts defined by quadrilaterals and by several types of triangles.

8.3 Intersection cuts with infinite split rank

Let the 0-*split closure* of Q be Q itself. For $t = 1, 2, 3, \ldots$, the t-*split closure* of Q is obtained by taking the split closure of the $(t-1)$-split closure of Q. It is known that if S is a pure integer set (i.e. $p = n$), then conv(S) coincides with the t-split closure, for some finite t. However, if S is a mixed-integer set, it may happen that conv(S) is strictly contained in the t-split closure, for any finite t [32].

Let $ax \ge a_0$ be a valid inequality for S and let t be the smallest nonnegative integer such that the inequality is valid for the t-split closure of Q. The value t is the *split rank* of the inequality with respect to Q.

Intersection cuts can have arbitrarily large split rank. To illustrate this, consider the following example introduced by Cook, Kannan and Schrijver [32].

Consider the polytope $P := \{(x_1, x_2, y) \in \mathbb{R}^3_+ : \ x_1 \geq y, \ x_2 \geq y, \ x_1 + x_2 + 2y \leq 2\}$, and the mixed integer linear set $S := \{(x_1, x_2, y) \in P \, : \, x_1, x_2 \in \mathbb{Z}\}$.

Cook, Kannan and Schrijver showed that the split rank of the inequality $y \leq 0$ is not finite. Yet it is an intersection cut. Indeed, by adding slack or surplus variables, the system defining P is equivalent to

$$\begin{array}{rcl} -x_1 + y + s_1 & = & 0 \\ -x_2 + y + s_2 & = & 0 \\ x_1 + x_2 + 2y + s_3 & = & 2 \\ x_1, x_2, y, s_1, s_2, s_3 & \geq & 0. \end{array}$$

The tableau relative to the basis B defining the vertex $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $y = \frac{1}{2}$ $s_1 = s_2 = s_3 = 0$ is

$$\begin{array}{rcl} x_1 & = & \frac{1}{2} + \frac{3}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ x_2 & = & \frac{1}{2} - \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{4}s_3 \\ y & = & \frac{1}{2} - \frac{1}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ & & x_1, x_2, y, s_1, s_2, s_3 \geq 0. \end{array}$$

Since y is a continuous basic variable, we drop the corresponding tableau row. The corner polyhedron corner(B) is the convex hull of the points satisfying

$$\begin{array}{rcl} x_1 & = & \frac{1}{2} + \frac{3}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ x_2 & = & \frac{1}{2} - \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{4}s_3 \\ & & s_1, s_2, s_3 \geq 0 \\ & & x_1, x_2 \in \mathbb{Z} \end{array}$$

The extreme rays of corner(B) are the vectors $(\frac{3}{4}, -\frac{1}{4}, 1, 0, 0)$, $(-\frac{1}{4}, \frac{3}{4}, 0, 1, 0)$ and $(-\frac{1}{4}, -\frac{1}{4}, 0, 0, 1)$. Let K be the triangle conv$\{(0,0), (2,0), (0,2)\}$, and $C = K \times \mathbb{R}^3$. Since K is $\mathbb{Z}^2$-free, C defines an intersection cut. The largest α such that $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0) + \alpha(\frac{3}{4}, -\frac{1}{4}, 1, 0, 0)$ belongs to C is $\alpha_1 = 2$, the largest α such that $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0) + \alpha(-\frac{1}{4}, \frac{3}{4}, 0, 1, 0)$ belongs to C is $\alpha_2 = 2$ and the largest α such that $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0) + \alpha(-\frac{1}{4}, -\frac{1}{4}, 0, 0, 1)$ belongs to C is $\alpha_3 = 2$. The intersection cut defined by C is therefore $\frac{1}{2}s_1 + \frac{1}{2}s_2 + \frac{1}{2}s_3 \geq 1$. Since $y = \frac{1}{2} - \frac{1}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3$, the intersection cut is equivalent to $y \leq 0$. Adding this single inequality to the formulation of P, we obtain conv(S). Yet $y \leq 0$ does not have finite split rank [32]. This example has been generalized by Li and Richard [71].

Dey and Louveaux [46] study the split rank of intersection cuts for problems with two integer variables (model (3) where $|B| = p = 2$). Surprisingly, they show that all intersection cuts have finite split rank except for the ones defined by lattice-free triangles with integral vertices and an integral point in the middle of each side. The triangle K defined above is of this type.

Basu, Cornuéjols and Margot [23] extend this result to more than two integer variables. To state their theorem, we first need to define the 2-hyperplane property.

A set S of points in $\mathbb{R}^p$ is *2-partitionable* if either $|S| \leq 1$ or there exists a partition of S into *nonempty* sets S_1 and S_2 and a split such that the points in S_1 are on one of its boundary hyperplanes and the points in S_2 are on the other. We say that a polytope is *2-partitionable* if its integer points are 2-partitionable.

Let P be a rational $\mathbb{Z}^p$-free polytope in $\mathbb{R}^p$ and let P_I be the convex hull of the integer points in P. We say that P has the *2-hyperplane property* if every face of P_I that is not contained in a facet of P is 2-partitionable. Note that one of the faces of P_I is P_I itself, thus, if P has the 2-hyperplane property, P_I must be 2-partitionable.

Consider model (3) where $|B| = p$. Let $r^j \in \mathbb{R}^p$ denote the restriction of $\bar{r}^j \in \mathbb{R}^n$ to $\{1, \ldots, p\}$. For $j \in N$, let L_j be the half-line $L_j := \{\bar{b} + \lambda r^j : \lambda \geq 0\}$.

Theorem 8.5. *Consider model* (3) *where* $|B| = p$. *Let* P *be a rational* $\mathbb{Z}^p$*-free polytope in* $\mathbb{R}^p$ *containing* $\bar{b}$ *in its interior. Assume that each vertex of* P *belongs to at least one of the half-lines* L_j *for* $j \in N$. *The intersection cut defined by* P *has finite split rank if and only if* P *has the 2-hyperplane property.*

Dey [44] gives more results on lower bounds of the split rank of intersection cuts.

A different perspective on closures is proposed by Andersen, Louveaux and Weismantel [5], who also give a certificate for infeasibility of mixed integer linear sets in the spirit of Farkas' lemma [3].

9 Computations

Recent computational experiments [14, 17, 40, 45, 55] test the effectiveness of intersection cuts in practice, particularly those defined by splits but also those derived from multiple rows.

Fischetti and Monaci [56] consider the mixed integer problem obtained from a MILP by removing all the constraints that are not binding at the optimal vertex of the LP relaxation. (This is a stronger relaxation than the corner polyhedron). Using classical instances from the MIPLIB library, they show that the optimal value of this relaxation provides a very good bound of the original MILP value, although solving this relaxation using standard branch-and-cut codes is difficult.

Balas and Saxena [14] and Dash, Günlük and Lodi [40] compare the gap closed when optimizing over the split closure $z_S - z_{LP}$ to the total integrality gap $z_{IP} - z_{LP}$. A striking outcome of these experiments is that the split closure produces a good approximation of the integer hull in practice, closing 70 to 80% of the gap $z_{IP} - z_{LP}$ on average, on MIPLIB instances. It should be noted that variance is high, the gap closed ranging from 0 in some cases and 100% in others. However, optimizing over the split closure is extremely time consuming.

In practice, one would like to generate a good set of split cuts quickly. A very effective approach is to use Gomory's mixed integer cut formula (15) applied to the rows of the optimal LP tableau. Over the set of MIPLIB instances, these cuts close already 24% of the gap $z_{IP} - z_{LP}$ on average, again with a high variance. Other split cuts, such as MIR, lift-and-project, reduce-and-split, typically improve the gap closed to 40% or more. We just mention a few recent studies along these lines here [27], [36], [37].

Some initial results have been obtained on intersection cuts from multiple rows (Basu, Bonami, Cornuéjols and Margot [17], Dey, Lodi, Tramontani and Wolsey [45], Espinoza [55]).

The work of Dey, Lodi, Tramontani and Wolsey [45] indicates that, on 2-row and 5-row multidimensional knapsack problems, intersection cuts defined by triangles close significantly more gap that GMI cuts from the tableau. Unfortunately, the number of triangle cuts generated is orders of magnitude larger. This raises the issue of cut generation. How does one select a few deep multi-row cuts? Espinoza [55] generated intersection cuts defined by cross-polytopes (also called octahedra) with some success, and Basu, Bonami, Cornuéjols and Margot [17] generated cuts defined by triangles obtained from a degenerate basis with disappointing results. Overall, the jury is still out on the practical usefulness of multi-row cuts.

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