Minimally Infeasible Set Partitioning Problems with Balanced Constraints

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Abstract

We study properties of systems of linear constraints that are minimally infeasible with respect to some subset S of constraints (i.e. systems that are infeasible, but become feasible upon removal of any constraint in S). We then apply these results and a theorem of Conforti, Cornuéjols, Kapoor, and Vušković, to a class of $0, 1$ matrices, for which the linear relaxation of the set partitioning polytope $LSP(A) = \{x \mid Ax = \mathbf{1}, x \geq 0\}$ is integral. In this way we obtain combinatorial properties of those matrices in the class that are minimal (w.r.t. taking row submatrices) with the property that the set partitioning polytope associated with them is infeasible.

1 Introduction

Determining if a system $Ax = \mathbf{1}$ has a $0, 1$ solution, where A is a $0, 1$ matrix (i.e. finding a feasible solution for the *set partitioning problem*) is NP-complete in general. When the matrix A is *balanced*, however, the problem can be formulated as a linear program [2], and is therefore polynomial. Furthermore, under the assumption that A is balanced, if the set partitioning problem is infeasible, this fact can be shown by a simple combinatorial certificate [6], which is an extension of Hall's condition for the existence of a perfect matching in a bipartite graph. It is therefore natural to look for a combinatorial algorithm that either finds a solution to a given set partitioning problem with balanced constraints, or the certificate of infeasibility for such problem. Finding such an algorithm, however, seems to be difficult.

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In order to achieve a better understanding of those balanced matrices for which the set partitioning problem is infeasible, we characterize the matrices which are minimal with such property, meaning that they do not admit a solution to the set partitioning problem, while all proper row-submatrices do (we call such matrices *minimally non partitionable*).

Since, in the balanced case, finding a solution to the set partitioning problem is equivalent to finding a basic solution to a linear program, we start by looking at general systems of linear constraints which are infeasible. If a system of linear constraints has no feasible solution, obviously there exists a subset of constraints that is still infeasible, and is minimal with such property. More generally, in Section 2 we study systems of linear constraints which are not feasible, but that admit a solution whenever we remove a constraint from a specified subset S of the rows (we call such systems *minimally infeasible with respect to* S).

In Section 3, we apply results from the previous section and a theorem of Conforti, Cornuéjols, Kapoor, and Vušković [6] to obtain combinatorial properties of minimally non partitionable balanced matrices. We also show that these matrices essentially characterize all systems of constraints of the form $Ax \sim \mathbf{1}$, $x \geq 0$ (where $Ax \sim \mathbf{1}$ denotes a system of equations and inequalities with constraint matrix A and right hand side $\mathbf{1}$) that are minimally infeasible with respect to the rows of A, when A is balanced.

2 Infeasible systems of linear inequalities

We study linear systems of equations and inequalities that are infeasible.

Given an integer n, we denote by $[n]$ the set $\{1, \ldots, n\}$. Given an $m \times n$ matrix A with entries a_{ij}, $i \in [m]$, $j \in [n]$, we will denote by a^i the ith row of A, and by a_j the jth column of A. Also, for any $i \in [m]$ we will denote with A^i the $(m-1) \times n$ submatrix of A obtained by removing the ith row. Given a subset S of $[m]$, we will denote $\bar{S} = [m] \setminus S$, and by A_S the row-submatrix of A induced by the rows in S.

Given a matrix A and a vector b we denote by $Ax \sim b$ a system

$$\begin{cases} a^i x \leq b_i & i \in S_1 \\ a^i x \geq b_i & i \in S_2 \\ a^i x = b_i & i \in S_3 \end{cases}$$

for some partition S_1, S_2, S_3 of the rows of A.

Given a system $Ax \sim b$, if $\bar{A}$ is a submatrix of A and $\bar{b}$ is the restriction of b to the rows of $\bar{A}$, we denote by $\bar{A}x \sim \bar{b}$ a system where the direction of the inequalities is consistent with the direction in $Ax \sim b$.

Given a system $Ax \sim b$ and a subset S of the rows of A, we say that $Ax \sim b$ is *minimally infeasible with respect to* S if $Ax \sim b$ has no solution, but $A^i x \sim b^i$ is feasible for every row index $i \in S$; a solution of such system is a *mate* of row a^i. When $S = [m]$ we say that the system is *minimally infeasible*.

An *orientation of* $Ax \sim b$ *with respect to* S is obtained from $Ax \sim b$ by substituting every equation $a^i x = b_i$, $i \in S$, with either $a^i x \leq b_i$ or $a^i x \geq b_i$.

A *restriction with respect to* S *of* $Ax \sim b$ is a system obtained from $Ax \sim b$ by substituting some of the inequalities $a^i x \leq b_i$ or $a^i x \geq b_i$, $i \in S$, with $a^i x = b_i$.

Theorem 2.1 *Let $Ax \sim b$ be a minimally infeasible system with respect to S.*

(i) Every restriction w.r.t. S of $Ax \sim b$ is also minimally infeasible w.r.t. S.

(ii) $Ax \sim b$ admits a unique orientation w.r.t. S, say $Ax \overset{*}{\sim} b$, that is infeasible. Furthermore, $Ax \overset{*}{\sim} b$ is minimally infeasible w.r.t. S.

(iii) $\mathrm{rk}(A_S) \geq |S| - 1$. *In particular* $\mathrm{rk}(A_S) = |S| - 1$ *if and only if $A_S x \sim b_S$ is minimally infeasible, while* $\mathrm{rk}(A_S) = |S|$ *if and only if $A_S x \sim b_S$ is feasible.*

Proof: (i) Let $R \subseteq S$ be the set of constraints that are set to equality. The proof is by induction on $|R|$, the case $|R| = 0$ being trivial. We only need to show that, for any $h \in R$, the system $A^h x \sim b^h$, $a^h x = b_h$, denoted by $Ax \sim' b$, is minimally infeasible w.r.t. S, since we can then apply induction to $Ax \sim' b$ and $R \setminus \{h\}$. Consider the case where the hth constraint in $Ax \sim b$ is of the form $a_h x \leq b_h$ (the case $a_h x = b_h$ is trivial, while the case $a_h x \geq b_h$ is analogous).

Since $Ax \sim b$ is infeasible, also $Ax \sim' b$ is infeasible. Thus we only need to show that $A^i x \sim' b^i$ has a solution for every $i \in S$. Let x^i and x^h be solutions to $A^i x \sim b^i$ and $A^h x \sim b^h$, respectively. Clearly, x^h is a solution for $A^h x \sim' b^h$ and $a^h x^h > b_h$, and if $a^h x^i = b_h$, then x^i is a solution for $A^i x \sim' b^i$. Thus we may assume $i \neq h$ and $a^h x^i - b^h < 0$. Given $\alpha = a^h x^h - b_h > 0$ and $\beta = b_h - a^h x^i > 0$, the vector $y = \frac{\alpha}{\alpha+\beta} x^i + \frac{\beta}{\alpha+\beta} x^h$, is a solution to $A^i x \sim' b^i$.

(ii) Let $I \subseteq S$ be the set of constraints of $Ax \sim b$ with index in S that are of the form $a^i x = b_i$, $i \in I$. For every $i \in I$, let x^i be a mate of a^i. Clearly, for every $i \in I$, $a^i x^i \neq b_i$, else x^i would be a solution for $Ax \sim b$. Denote by $Ax \overset{*}{\sim} b$ the orientation of $Ax \sim b$ obtained by substituting, for every $i \in I$, the equation $a^i x = b_i$ with the inequality $a^i x \leq b_i$ if $a^i x^i > b_i$, and with the inequality $a^i x \geq b_i$ if $a^i x^i < b_i$. We show that $Ax \overset{*}{\sim} b$ is infeasible.

Suppose not and let $\bar{x}$ be a solution. Let J be the set containing all $i \in I$ such that $\lambda_i = a^i \bar{x} - b_i \neq 0$. For every $i \in J$, let $\mu_i = |a^i x^i - b_i|$. Thus $a^i x^i = b_i - \mu_i \frac{\lambda_i}{|\lambda_i|}$ for every $i \in J$, by construction of $Ax \overset{*}{\sim} b$ and since $\bar{x}$ is a solution to such system. One may readily verify that

$$y = \frac{\bar{x} + \sum_{i \in J} \frac{|\lambda_i|}{\mu_i} x^i}{1 + \sum_{i \in J} \frac{|\lambda_i|}{\mu_i}}$$

satisfies $Ay \sim b$, a contradiction. Since, for every $i \in S$, x^i satisfies $A^i x \overset{*}{\sim} b^i$, $Ax \overset{*}{\sim} b$ is minimally infeasible w.r.t. S.

For the uniqueness, suppose there exist two distinct orientations $Ax \sim' b$ and $Ax \sim'' b$ of $Ax \sim b$ that are infeasible. W.l.o.g., there exists $j \in S$ such that the jth constraint of $Ax \sim' b$ is $a^j x \leq b^j$, while the jth constraint of $Ax \sim'' b$ is $a^j x \geq b^j$. Thus both systems

$a^j x \le b^j$, $A^j x = b^j$ and $a^j x \ge b^j$, $A^j x = b^j$ are infeasible, which is a contradiction, since x^j must satisfy one of them.

(iii) Let T be a set of constraints such that $A_T x \sim b_T$ is minimally infeasible. Thus $S \subseteq T$. Also, by (i), $A_T x = b_T$ is minimally infeasible. For every $i \in T$, $A_T^i x = b_T^i$ has a solution, thus $\mathrm{rk}(A_T^i | b_T^i) = \mathrm{rk}(A_T^i)$. Therefore, for every $i \in T$, we have

$$\mathrm{rk}(A_T | b_T) \le \mathrm{rk}(A_T^i | b_T^i) + 1 = \mathrm{rk}(A_T^i) + 1 \le \mathrm{rk}(A_T) + 1. \tag{1}$$

Since $A_T x = b_T$ has no solution, $\mathrm{rk}(A_T | b_T) = \mathrm{rk}(A_T) + 1$, and equality holds throughout in (1). In particular, the rows of (A_T, b_T) are linearly independent, thus $\mathrm{rk}(A_T) = |T| - 1$ and $\mathrm{rk}(A_S) \ge |S| - 1$. To conclude, if $A_S x \sim b_S$ is minimally infeasible, then $S = T$ and $\mathrm{rk}(A_S) = |S| - 1$, while if $A_S x \sim b_S$ is feasible, then $\mathrm{rk}(A_S) = \mathrm{rk}(A_S | b_S) = |S|$. □

As an aside, Theorem 2.1 (i) yields the following elementary proof of Farkas Lemma.

Lemma 2.2 (Farkas Lemma) *The system $Ax \le b$ is infeasible if and only if the system $uA = 0$, $ub < 0$, $u \ge 0$ is feasible.*

Proof: For the necessity, assume $uA = 0$, $ub < 0$, $u \ge 0$ is feasible: then $0 = uAx \le ub < 0$ for every x such that $Ax \le b$, a contradiction.

For the sufficiency, let $Ax \le b$ be an infeasible system. We assume that $Ax \le b$ is minimally infeasible. (Our assumption is justified since we may consider a minimally infeasible subsystem of $Ax \le b$, and set to 0 all the u_i's corresponding to the other inequalities.)

Since $Ax \le b$ is minimally infeasible, then by Theorem 2.1(i) $Ax = b$ is minimally infeasible so, by elementary linear algebra, $uA = 0$, $ub < 0$ is feasible. Let u be such a vector. It suffices to show $u \ge 0$. Suppose $I = \{i : u_i < 0\}$ is nonempty. This shows that the system $A_I x \ge b_I$, $A_{\bar{I}} x \le b_{\bar{I}}$ is infeasible, since the vector u' defined by $u'_i = |u_i|$, $i \in [n]$, satisfies $u' \binom{-A_I}{A_{\bar{I}}} = 0$, $u' \binom{-b_I}{b_{\bar{I}}} < 0$, $u' \ge 0$. But by Theorem 2.1 (ii) $Ax \le b$ is the unique orientation of $Ax = b$ that is infeasible, a contradiction. □

Lemma 2.3 *Let $Ax \sim b$ be a minimally infeasible system w.r.t. S. For every $i \in S$, let x^i be a solution to $A^i x \sim b^i$ satisfying $A_S^i x = b_S^i$. The vectors x^i, $i \in S$ are affinely independent.*

Proof: By Theorem 2.1 (i), the system $A_S x = b_S$, $A_{\bar{S}} x \sim b_{\bar{S}}$ is minimally infeasible w.r.t. S, thus vectors x^i, $i \in S$ as in the statement exist. W.l.o.g., assume $S = \{1, \ldots, s\}$. Let $\lambda_1, \ldots, \lambda_s$ be multipliers, not all zeroes, such that $\sum_{i=1}^s \lambda_i = 0$. Let $y = \sum_{i=1}^s \lambda_i x^i$. It suffices to show $y \neq 0$. Clearly

$$A_S y = \sum_{i=1}^{s} \lambda_i A_S x^i = \begin{bmatrix} \lambda_1(a^1 x^1 - b_1) + b_1(\sum_{i=1}^s \lambda_i) \\ \vdots \\ \lambda_s(a^s x^s - b_s) + b_s(\sum_{i=1}^s \lambda_i) \end{bmatrix} = \begin{bmatrix} \lambda_1(a^1 x^1 - b_1) \\ \vdots \\ \lambda_s(a^s x^s - b_s) \end{bmatrix}.$$

Since $a^i x^i - b_i \neq 0$ for every $i \in S$, and $\lambda_j \neq 0$ for some $j \in S$, then $Ay \neq 0$, therefore $y \neq 0$. Thus $x^1, \ldots, x^s$ are affinely independent. □

A system $Ax \sim b$ is *irreducible* if it is minimally infeasible and for every proper column submatrix A' of A the system $A'x \sim b$ is not minimally infeasible. This means that there exists a constraint that can be removed from $A'x \sim b$ and the system thus obtained is still infeasible.

Theorem 2.4 *If $Ax \sim b$ is minimally infeasible, then every $m \times (m-1)$ submatrix of A with full column rank is irreducible.*

Thus, if $Bx \sim b$ is irreducible, then B is an $m \times (m-1)$ matrix. Furthermore, every submatrix B^i, $i \in [m]$, is nonsingular.

Proof: From the proof of Theorem 2.1(iii), $\mathrm{rk}(A^i) = m - 1$ for every $i \in [m]$. From standard linear algebra, if $\bar{A}$ is a column submatrix of A formed by $m-1$ linearly independent columns, then $\bar{A}^i$ is square and nonsingular for every $i \in [m]$, therefore $\bar{A}^i x = b^i$ has a (unique) solution. Since $Ax \sim b$ is infeasible, $\bar{A}x \sim b$ is infeasible, thus $\bar{A}x \sim b$ is minimally infeasible. Since $\mathrm{rk}(\bar{A}) = m-1$, then by Theorem 2.1(iii) $\bar{A}$ is irreducible. $\square$

The *reverse system* $Ax \overset{r}{\sim} b$ of $Ax \sim b$ is obtained by substituting, for every $i \in S$, each inequality $a^i x \leq b_i$ with $a^i x \geq b_i$, and each inequality $a^i x \geq b_i$ with $a^i x \leq b_i$

Corollary 2.5 *If the system $Ax \sim b$ is irreducible then the reverse system of its unique minimally infeasible orientation defines a full dimensional simplex whose vertices are the unique mates of $Ax = b$. Conversely, if $Ax \leq b$ is a system with no redundant constraints that defines a full dimensional simplex, then $Ax \geq b$ is an irreducible system.*

Proof: Let $Ax \overset{*}{\sim} b$ be the unique orientation of $Ax \sim b$ that is minimally infeasible and let $Ax \overset{r}{\sim} b$ be the reverse system of $Ax \overset{*}{\sim} b$. Let $x^1, \ldots, x^m$ be vectors satisfying $A^i x^i = b^i$, $i \in [m]$. By the construction of $Ax \overset{*}{\sim} b$ in the proof of Theorem 2.1 (ii), $x^1, \ldots, x^m$ satisfy $Ax \overset{r}{\sim} b$. By Theorem 2.4, $x^1, \ldots, x^m$ are the unique vertices of the polytope defined by $Ax \overset{r}{\sim} b$. By Lemma 2.3, $x^1, \ldots, x^m$ are affinely independent, thus $Ax \overset{r}{\sim} b$ is a full-dimensional simplex.
The converse of the statement is obvious. $\square$

3 Some infeasible set partitioning systems

Given a $0,1$ matrix A, the *set partitioning polytope* is

$$SP(A) = \mathrm{conv}\{x \mid Ax = \mathbf{1},\ x \geq 0,\ x \text{ integral}\}.$$

Its linear relaxation is the polytope

$$LSP(A) = \{x \mid Ax = \mathbf{1},\ x \geq 0\}$$

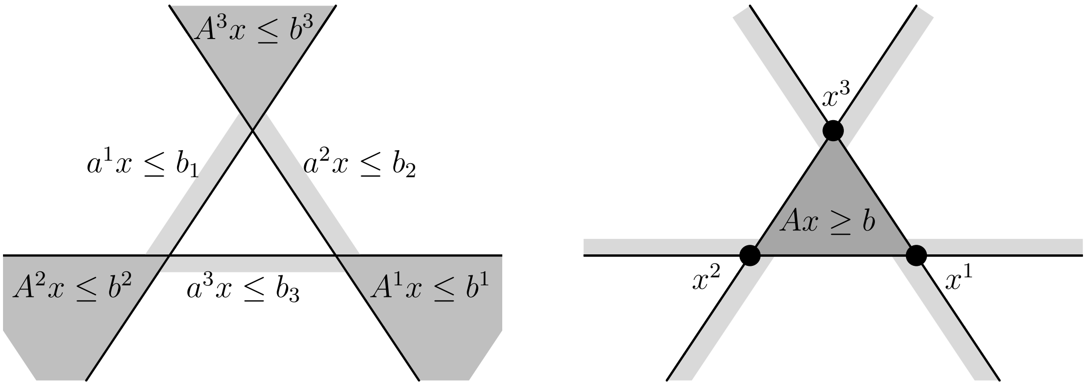

Figure 1: An irreducible system and the associated simplex

An $m \times n$ $0,1$ matrix A is *partitionable* if $SP(A)$ is nonempty, *non-partitionable* otherwise. A is *minimally non-partitionable* (MNP) if A is non-partitionable, but A^i is partitionable for every $i \in [m]$. Clearly, a matrix is non-partitionable if and only if it contains a row submatrix that is MNP.

Since determining if A is partitionable is an NP-complete problem, it appears to be hard to give a short certificate for non-partitionability. We restrict ourselves to a class of matrices for which $SP(A)$ is an integer polytope, i.e. $SP(A) = LSP(A)$, and admits a combinatorial certificate for non-partitionability.

3.1 Balanced matrices

We say that a $0,1$ matrix A is *balanced* if no square submatrix of A of odd order has precisely two ones in each row and in each column. The following theorem is due to Berge [2].

Theorem 3.1 *Let A be an $m \times n$ balanced matrix, and let (S_1, S_2, S_3) be a partition of $[m]$. Then*

$$P(A) = \{x \in \mathbb{R}^n \ : \ \begin{array}{l} a^i x \le 1 \text{ for } i \in S_1 \\ a^i x \ge 1 \text{ for } i \in S_2 \\ a^i x = 1 \text{ for } i \in S_3 \\ x \ge \mathbf{0}\} \end{array}$$

is an integral polytope.

It follows from Theorem 3.1 that $SP(A) = LSP(A)$ for every balanced matrix A. Therefore a balanced matrix A is minimally non-partitionable if and only if $LSP(A)$ is empty but $LSP(A^i)$ is nonempty for every $i \in [m]$, that is, if $Ax = \mathbf{1}$, $x \ge 0$ is minimally infeasible w.r.t. the rows of A.

Conforti, Cornuéjols, Kapoor and Vušković [6] showed the following certificate that characterizes non-partitionable balanced matrices and extends Hall's condition for the existence of a perfect matching in a bipartite graph.

Theorem 3.2 *Let A be a balanced matrix. $Ax = \mathbf{1}$, $x \geq 0$ is infeasible if and only if the rows of A contain two disjoint subsets R and B such that $|R| < |B|$ and every column of A contains at least as many 1s in the rows of R as in the rows of B.*

In order to give the reader some intuition on the result, we show the "easy direction" of the statement, namely that if the rows of A admit sets R and B with the above properties, then $Ax = \mathbf{1}$, $x \geq 0$ is infeasible. Let u be the vector satisfying $u_i = 1$ if $i \in R$, $u_i = -1$ if $i \in B$, $u_i = 0$ otherwise If x satisfies $Ax = \mathbf{1}$, $x \geq 0$, then $0 \leq uAx = u\mathbf{1} < 0$, a contradiction.

Note that if A is a $0,1$ matrix that contains disjoint subsets satisfying the condition of Theorem 3.2, then $LSP(A) = SP(A) = \emptyset$.

Using Theorems 3.1 and 3.2, we give properties of MNP balanced matrices.

Corollary 3.3 *Let A be an MNP balanced matrix. Then the rows of A can be partitioned into sets R and B such that $|B| = |R| + 1$ and every column of A contains at least as many 1s in the rows of R as in the rows of B. In particular, A has an odd number of rows.*

Proof: Let m be the number of rows of A. Since A is MNP, $[m]$ contains disjoints subsets R and B as in the statement of Theorem 3.2. Suppose $[m] \setminus (R \cup B) \neq \emptyset$ or $|B| \geq |R| + 2$. In the former case, choose $i \in [m] \setminus (R \cup B)$, else choose $i \in B$. Let $B' = B \setminus i$. Clearly, $|B'| > |R|$ and every column of A^i contains at least as many ones in the rows of R as in the rows of B', thus, by Theorem 3.2, $A^i x = \mathbf{1}$, $x \geq 0$ is infeasible, a contradiction. $\square$

Throughout the rest of the paper, whenever A is an $m \times n$ balanced MNP matrix, we always denote by B and R the two subsets of the rows satisfying the properties stated in Corollary 3.3 (we will in fact show that R and B are unique). We call the rows in B the *blue rows* of A, and the rows in R the *red rows* of A. Given any entry a_{ij} of A, we say that a_{ij} is *blue* (resp. *red*), if a^i is blue (resp. red).

3.2 Mates of MNP balanced matrices

Given an MNP matrix A and a row a^i of A, a $0,1$ vector $\bar{x}$ is a *mate* of a^i if $\bar{x}$ satisfies $A^i \bar{x} = \mathbf{1}$. If $\bar{x}$ is a mate of some row of A, we say that $\bar{x}$ is a mate for A. By definition, each row of A has at least one mate. If A is balanced, we say that $\bar{x}$ is a *blue mate* (resp. *red mate*) if a^i is a blue row (resp. red row). (It should be noted that this definition of mate is more restrictive than the definition we gave in Section 2 for general systems, since here $\bar{x}$ is required to be integral.)

We will identify the mate $\bar{x}$ with the subset of columns whose characteristic vector is $\bar{x}$. Thus we will view mates indifferently as vectors or sets, and when we say that a column is contained in the mate $\bar{x}$, we mean that $\bar{x}$ has a nonzero entry in that column.

Throughout the rest of the paper, we assume that the matrix A has no columns with all zeroes.

Lemma 3.4 *Let A be a balanced MNP matrix, and $\bar{x}$ be a mate of row a^i. The following hold:*

(i) If a^i is blue, then $a^i\bar{x} = 0$ and every column in $\bar{x}$ contains as many red 1s as blue 1s.

(ii) If a^i is red, then $a^i\bar{x} \geq 2$. Furthermore, equality holds if and only if every column in $\bar{x}$ contains as many red 1s as blue 1s.

Proof: (i) The following chain of inequalities holds

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j \leq \sum_{j=1}^{n}(\sum_{h\in R} a_{hj} - \sum_{h\in B\setminus i} a_{hj})\bar{x}_j = \sum_{h\in R} a^h\bar{x} - \sum_{h\in B\setminus i} a^h\bar{x} = |R| - |B| + 1 = 0$$

where the first inequality holds since each column of A has at least as many red ones as blue ones. Thus equality holds throughout and the columns in $\bar{x}$ contain as many red 1s as blue 1s.
(ii) Similarly to (i), we have

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j \geq \sum_{j=1}^{n}(\sum_{h\in B} a_{hj} - \sum_{h\in R\setminus i} a_{hj})\bar{x}_j = \sum_{h\in B} a^h\bar{x} - \sum_{h\in R\setminus i} a^h\bar{x} = |B| - |R| + 1 = 2$$

where equality holds throughout if and only if the columns in $\bar{x}$ contain as many red 1s as blue 1s. □

Proposition 3.5 *Let A be an MNP balanced matrix. The sets R and B are unique with the properties in the statement of Corollary 3.3.*

Proof: By Lemma 3.4, B must be the set of rows that are orthogonal to their mates. □

We say that a mate $\bar{x}$ of a red row a^i is *good* if $a^i\bar{x} = 2$.

Lemma 3.6 *Let A be an MNP balanced matrix, S be a subset of R, and y be a nonnegative integral vector such that*

$$\begin{aligned} a^i y = {} & |S| && \textit{for } i \in [m] \setminus S \\ a^i y = {} & |S| + 1 && \textit{for } i \in S. \end{aligned} \qquad (2)$$

Then there exist $|S|$ good mates of distinct rows of S, $x^1, \ldots, x^{|S|}$, such that $y = \sum_{i=1}^{|S|} x^i$.

Proof: The proof is by induction on $s = |S|$. If $s = 0$, then $y = 0$, since $Ay = 0$ and each column of A has at least a nonzero entry. Assume $s \geq 1$. Let

$$P(S) = \left\{ x \in \mathbb{R}^n : \begin{array}{rcl} a^i x & = & 1 \text{ for } i \in [m] \setminus S \\ a^i x & \geq & 1 \text{ for } i \in S \\ x & \geq & 0 \end{array} \right\} \tag{3}$$

By definition, $y/s \in P(S)$, hence y/s is the convex combination of vertices of $P(S)$; that is, there exist vertices $y^1, \ldots, y^t$ of $P(S)$ and positive coefficients $\lambda_1, \ldots, \lambda_t$, such that $y/s = \sum_{i=1}^t \lambda_i y^i$ and $\sum_{i=1}^t \lambda_i = 1$. We will prove that the $y^1, \ldots, y^t$ are good mates of rows of S. By Theorem 3.1, $P(S)$ is an integral polyhedron, hence $y^1, \ldots, y^t$ are integral vectors.

For any $x \in P(S)$, define the *excess* of x, $\varepsilon(x)$, as

$$\varepsilon(x) = \mathbf{1}^T A_S x - s.$$

Clearly, $\varepsilon(y/s) = 1$. Also, since A is non-partitionable, $\varepsilon(y^i) \geq 1$ for $i = 1, \ldots, t$. Thus

$$1 = \varepsilon(y/s) = \sum_{i=1}^t \lambda_i \varepsilon(y^i) \geq \sum_{i=1}^t \lambda_i = 1,$$

hence equality holds throughout, and $\varepsilon(y^i) = 1$ for every $i = 1, \ldots, t$. This means that $y^1, \ldots, y^t$ are good mates of rows in S.

Let $z = y - y^1$ and a^h be the row of which y^1 is mate. Since y is integral and nonnegative, and y^1 is a $0,1$ vector, $y^1 \leq y$, thus $z \geq 0$. Also, $a^i z = s - 1$ for any $i \in [m] \setminus (S \setminus \{h\})$, and $a^i z = s$ for any $i \in S \setminus \{h\}$. By applying the inductive hypothesis to $S \setminus \{h\}$ and z, there exist $|S| - 1$ good mates of distinct rows of $S \setminus \{h\}$, $x^2, \ldots, x^s$ such that $z = x^2 + \ldots + x^s$. Therefore, given $x^1 = y^1$, $x^1, \ldots, x^s$ are good mates of pairwise distinct rows of S, and $y = \sum_{i=1}^s x^i$. □

The following is an analogous of Lemma 3.6

Lemma 3.7 *Let A be an MNP balanced matrix, S be a subset of B, and y be a nonnegative integral vector such that*

$$\begin{array}{rcl} a^i y = & |S| & \textit{for } i \in [m] \setminus S \\ a^i y = & |S| - 1 & \textit{for } i \in S. \end{array}$$

Then there exist $|S|$ mates of distinct rows of S, $x^1, \ldots, x^{|S|}$, such that $y = \sum_{i=1}^{|S|} x^i$.

Proof: The proof is by induction on $s = |S|$. If $s = 0$, then $y = 0$, since $Ay = 0$ and each column of A has at least a nonzero entry. Assume $s \geq 1$. Let

$$P(S) = \left\{ x \in \mathbb{R}^n : \begin{array}{rcl} a^i x & = & 1 \text{ for } i \in [m] \setminus S \\ a^i x & \leq & 1 \text{ for } i \in S \\ x & \geq & 0 \end{array} \right\}$$

By definition, $y/s \in P(S)$, hence y/s is the convex combination of vertices of $P(S)$; that is, there exist vertices $y^1, \ldots, y^t$ of $P(S)$ and positive coefficients $\lambda_1, \ldots, \lambda_t$, such that $y/s = \sum_{i=1}^t \lambda_i y^i$ and $\sum_{i=1}^t \lambda_i = 1$. We will prove that the $y^1, \ldots, y^t$ are mates of rows of S. By Theorem 3.1, $P(S)$ is an integral polyhedron, hence $y^1, \ldots, y^t$ are integral vectors.

For any $x \in P(S)$, define the *deficiency* of x, $\delta(x)$, as

$$\delta(x) = s - \mathbf{1}^T A_S x.$$

Clearly, $\delta(y/s) = 1$. Also, since A non-partitionable, $\delta(y^i) \geq 1$ for $i = 1, \ldots, t$. Thus

$$1 = \delta(y/s) = \sum_{i=1}^{t} \lambda_i \delta(y^i) \geq \sum_{i=1}^{t} \lambda_i = 1,$$

hence equality holds throughout, and $\delta(y^i) = 1$ for every $i = 1, \ldots, t$. This means that $y^1, \ldots, y^t$ are mates of rows in S. Let $z = y - y^1$ and a^h be the row of which y^1 is mate. Since y is integral and nonnegative and y^1 is a $0, 1$ vector, $y^1 \leq y$, thus $z \geq 0$. Also, $a^i z = s - 1$ for any $i \in [m] \setminus (S \setminus \{h\})$, and $a^i z = s - 2$ for any $i \in S \setminus \{h\}$. By applying the inductive hypothesis to $S \setminus \{h\}$ and z, there exist $|S| - 1$ good mates of distinct rows of $S \setminus \{h\}$, $x^2, \ldots, x^s$ such that $z = x^2 + \ldots + x^s$. Therefore, given $x^1 = y^1$, $x^1, \ldots, x^s$ are mates of pairwise distinct rows of S, and $y = \sum_{i=1}^s x^i$. □

From now on, whenever A is a balanced MNP matrix, we assume $B = \{b_1, \ldots, b_{k+1}\}$, and $R = \{r_1, \ldots, r_k\}$. Also, we denote by n the number of columns.

Theorem 3.8 *Let A be an MNP balanced matrix.*

Let $x^{b_1}, \ldots, x^{b_{k+1}}$ be arbitrarily chosen mates of $a^{b_1}, \ldots, a^{b_{k+1}}$. Then there exist good mates $x^{r_1}, \ldots, x^{r_k}$ of $a^{r_1}, \ldots, a^{r_k}$, respectively, such that $\sum_{i=1}^{k+1} x^{b_i} = \sum_{i=1}^{k} x^{r_i}$. In particular, every red row of A has a good mate contained in $x^{b_1} \cup \ldots \cup x^{b_{k+1}}$.

Let $x^{r_1}, \ldots, x^{r_k}$ be arbitrarily chosen good mates of $a^{r_1}, \ldots, a^{r_k}$. Then there exist mates $x^{b_1}, \ldots, x^{b_{k+1}}$ of $a^{b_1}, \ldots, a^{b_{k+1}}$, respectively, such that $\sum_{i=1}^{k} x^{r_i} = \sum_{i=1}^{k+1} x^{b_i}$.

Proof: Let $\beta = \sum_{i=1}^{k+1} x^{b_i}$. By Lemma 3.4(i),

$$\begin{aligned} a^i\beta = &\quad k \quad &\text{for } i \in B \\ a^i\beta = &\quad k+1 \quad &\text{for } i \in R \end{aligned} \tag{4}$$

thus, by Lemma 3.6, there exist good mates $x^{r_1}, \ldots, x^{r_k}$ of $a^{r_1}, \ldots, a^{r_k}$, respectively, such that $\beta = \sum_{i=1}^{k} x^{r_i}$.

A similar argument, using Lemma 3.7, shows the second part of the statement. □

Corollary 3.9 *Let A be an MNP balanced matrix and A' be the column submatrix of A induced by the columns of A with as many red* 1*s as blue* 1*s. Then A' is MNP,* $\mathrm{rk}(A') = 2k$*, while* $\mathrm{rk}(A) = 2k + 1$ *if and only if A contains a column with more red* 1*s than blue* 1*s. In particular, $Ax = \mathbf{1}$ has a solution if and only if $rk(A) = 2k + 1$.*

Proof: Since A is MNP, A' is non-partitionable. For every $b_i \in B$, there is a mate x^{b_i} of a^{b_i}. Lemma 3.4(i) implies that x^{b_i} does not contain any column with more red than blue nonzero entries, hence $\cup_{i=1}^{k+1} x^{b_i}$ is included in the columns of A'. By Theorem 3.8, every red row of A has a good mate contained in $\cup_{i=1}^{k+1} x^{b_i} \subseteq A'$. Thus the restrictions of $x^{b_1}, \ldots, x^{b_{k+1}}, x^{r_1}, \ldots, x^{r_k}$ to the columns of A' are mates of A', hence A' is MNP.

By Theorem 2.1 (iii), $\mathrm{rk}(A') \geq 2k$. Since the sum of the blue rows of A' minus the sum of the red rows of A' is zero, then $\mathrm{rk}(A') = 2k$. If a column of A has more red 1s than blue 1s, then the sum of the red 1s minus the sum of the blue 1s of such column is nonzero, therefore $\mathrm{rk}(A) > \mathrm{rk}(A') = 2k$, thus $\mathrm{rk}(A) = 2k+1$. Finally, by Theorem 2.1 (iii), since $Ax = \mathbf{1}, x \geq 0$ is minimally infeasible w.r.t. the rows of A, $Ax = \mathbf{1}$ has a solution if and only if $\mathrm{rk}(A) = 2k+1$. $\square$

Theorem 3.10 *Let A be an MNP balanced matrix such that the sum of the red rows equals the sum of the blue rows. Then*

$$P = \left\{ x \, : \begin{array}{rcl} A_B x & \leq & \mathbf{1} \\ A_R x & \geq & \mathbf{1} \\ x & \geq & 0 \end{array} \right\} = \mathrm{conv}\{m \, : \, m \text{ is a mate of } A\}$$

Proof: Let $\bar{x}$ be a vertex of P. Since every column of A has as many red 1s as blue 1s,

$$k \leq \sum_{i \in R} a^i \bar{x} = \sum_{i \in B} a^i \bar{x} \leq k+1, \tag{5}$$

where the first inequality follows from $A_R x \geq \mathbf{1}$, while the last follows from $A_B x \leq \mathbf{1}$.

Since A is balanced, $\bar{x}$ is a $0,1$ vector, so one of the two inequalities in (5) is satisfied at equality, so $A^i \bar{x} = \mathbf{1}$ for some row i. $\square$

Notice, however, that Theorem 3.10 does not hold in general, when A contains a column with strictly more red 1s than blue 1s. For example, the matrix

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} B \\ R \\ B \\ R \\ B \end{array}$$

is balanced and MNP, but $(0,0,0,0,1)$ is a vertex of the polytope P, as defined in the statement, that is not a mate of A.

3.3 Critical MNP balanced matrices

We say that a $0,1$ matrix A is *critical* if A is MNP, but any column submatrix of A is not MNP.

Theorem 3.11 *Let A be a critical balanced matrix. The following hold:*

(*i*) *The sum of the blue rows of A equals the sum of the red rows of A.*

(*ii*) *All red mates for A are good.*

(*iii*) *Each row of A has a unique mate.*

(*iv*) *Each column of A is contained in as many red mates as blue mates.*

Proof: (i) Follows immediately from Corollary 3.9.
(ii) Follows immediately from (i) and from Lemma 3.4.
(iii) Choose an integral nonnegative vector $\beta = (\beta_1, \ldots, \beta_n)$ such that

$$\begin{aligned} a^i\beta = &\quad k \qquad \text{for } i \in B \\ a^i\beta = &\quad k+1 \qquad \text{for } i \in R, \end{aligned}$$

so that $\beta_1, \ldots, \beta_n$ is highest possible in the lexicographical order. (Notice that such a choice of β exists, since the vector $\sum_{i=1}^{k+1} x^{b_i}$, where $x^{b_1}, \ldots, x^{b_{k+1}}$ are mates of $a^{b_1}, \ldots, a^{b_{k+1}}$, satisfies the above system.)

By Lemma 3.7, there exist mates of $a^{b_1}, \ldots, a^{b_{k+1}}$, say $x^{b_1}, \ldots, x^{b_{k+1}}$, respectively, such that $\beta = \sum_{i=1}^{k+1} x^{b_i}$. Also, by Theorem 3.8, there exist good mates $x^{r_1}, \ldots, x^{r_k}$ of $a^{r_1}, \ldots, a^{r_k}$, respectively, such that $\beta = \sum_{i=1}^{k} x^{r_i}$. Observe that all components of β are strictly positive, else the submatrix of A induced by the columns in which β is positive would be MNP, contradicting the fact that A is critical.

Suppose that there exists a row of A, say a^h, that has a mate $\bar{x} \neq x^h$. Suppose h is chosen so that the index j such that $\bar{x}_j \neq x^h_j$ is smallest possible.

If $\bar{x}_j = 1$ and $x^h_j = 0$, let $\beta' = \beta - x^h + \bar{x}$, else let $\beta' = \beta - \bar{x} + x^h$. Since β is strictly positive, $\beta' \geq 0$, furthermore $\beta_i = \beta'_i$ for every $i < j$ (since j is smallest possible), and $\beta_j < \beta'_j$. Therefore β' contradicts the maximality assumption on β.
(iv) By Theorem 3.8 and part (iii), the sum of the red mates equals the sum of the blue mates. □

If A is a balanced critical matrix, by Theorem 3.11 (iii) we may univocally define the vectors $m^1, \ldots, m^{2k+1}$ to be the unique mates of rows $a^1, \ldots, a^{2k+1}$. Let $M(A) = (m_{ij})$ be the $(2k+1) \times n$ matrix where $m_{ij} = m^i_j$. We call $M(A)$ the *mate matrix* of A. Also, we say that m^i is a blue (resp. red) row of $M(A)$ if $i \in B$ (resp. $i \in R$).

One might wonder whether it is true that, provided that A is balanced, $M(A)$ is balanced as well. However, this is false, as shown by the following example.

$$A = \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right] B \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] R \end{array} \quad ; \quad M(A) = \left[\begin{array}{cccccccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \qquad (6)$$

In the example, A is a 9×8 matrix, $|R| = 4$, and $|B| = 5$. The sum of the red rows equals the sum of the blue rows, thus $Ax = \mathbf{1}$ is infeasible. Also, the rows of $M(A)$ are mates of the rows of A, and A is full column rank, thus A is critical. It is easy to check that A is balanced, whereas $M(A)$ is not balanced, since it contains a 3×3 unbalanced submatrix.

Theorem 3.12 *Let A be a critical balanced matrix.*

(i) The sum of the blue rows of $M(A)$ equals the sum of the red rows of $M(A)$.

(ii) For every $j \in [n]$, $a_j \cdot m_j = 1$. For every $j \neq h \in [n]$, $\sum_{i \in B} a_{ij} m_{ih} = \sum_{i \in R} a_{ij} m_{ih}$.

(iii) A consists of $2k$ linearly independent columns.

(iv) $M(A)$ is critical and the rows of A are its unique mates.

Proof: (i) Follows directly from Theorem 3.11(iv).
(ii) Let $B_j = \{i \in B \,|\, a_{ij} = 1\}$ and $R_j = \{i \in R \,|\, a_{ij} = 1\}$. By Theorem 3.11(i), $|R_j| = |B_j|$. Clearly, $m_{ij} = 0$ for every $i \in B_j$, since $a^i \cdot m^i = 0$.
Let $\gamma = \sum_{i \in B_j} m^i + u^j$, where u^j is the vector with 1 in the jth entry, and zero elsewhere. Thus $\gamma_j = 1$ and

$$\begin{array}{rll} a^i \gamma = & |R_j| & \text{for } i \notin R_j \\ a^i \gamma = & |R_j| + 1 & \text{for } i \in R_j. \end{array}$$

By Lemma 3.6 and Theorem 3.11(iii), $\gamma = \sum_{i \in R_j} m^i$.

In particular, since $\gamma_j = 1$, there exists exactly one $s \in R_j$ such that $m_{sj} = 1$, while $m_{ij} = 0$ for every $i \in R_j \setminus \{s\}$. Thus $a_j \cdot m_j = 1$. Finally, for every $h \neq j$ in $[n]$, $\sum_{i \in B} a_{ij} m_{ih} = \sum_{i \in B_j} m_{ih} = \gamma_h = \sum_{i \in R_j} m_{ih} = \sum_{i \in R} a_{ij} m_{ih}$.
(iii) The following chain of equalities holds:

$$n = \sum_{j=1}^{n} \sum_{i=1}^{2k+1} a_{ij} m_{ij} = \sum_{i=1}^{2k+1} a^i \cdot m^i = \sum_{i \in R} 2 = 2k$$

where the first equality follows from (ii) and the third equality follows from the fact that $a^i \cdot m^i = 0$ for $i \in B$, and $a^i \cdot m^i = 2$ for $i \in R$. Since, by Corollary 3.9, $\text{rk}(A) = 2k$, A has full column rank.

(iv) By (i), $M(A)x = \mathbf{1}$ is infeasible, so it is minimally infeasible, thus, by Theorem 2.1 (iii), $\text{rk}(M(A)) \geq 2k$. Since $M(A)$ has $2k$ columns, $M(A)$ must be critical. $\square$

Theorem 3.13 *Let A be a critical balanced matrix. Then both systems $A_B x \geq \mathbf{1}$, $A_R x \leq \mathbf{1}$ and $M(A)_B x \geq \mathbf{1}$, $M(A)_R x \leq \mathbf{1}$ are irreducible.*

Furthermore, $A_B x \leq \mathbf{1}$, $A_R x \geq \mathbf{1}$ defines a simplex whose vertices are $m^1, \dots, m^{2k+1}$, and $M(A)_B x \leq \mathbf{1}$, $M(A)_R x \geq \mathbf{1}$ defines a simplex whose vertices are $a^1, \dots, a^{2k+1}$.

Proof: By Farkas Lemma, $A_B x \geq \mathbf{1}$, $A_R x \leq \mathbf{1}$ (resp. $M(A)_B x \geq \mathbf{1}$, $M(A)_R x \leq \mathbf{1}$) is infeasible, since $\mathbf{1}^T\binom{-A_B}{A_R} = 0$ (resp. $\mathbf{1}^T\binom{-M(A)_B}{M(A)_R} = 0$) and $\mathbf{1}^T\binom{-\mathbf{1}_B}{\mathbf{1}_R} = |R| - |B| < 0$. Since A and $M(A)$ are critical, $A_B x \geq \mathbf{1}$, $A_R x \leq \mathbf{1}$ and $M(A)_B x \geq \mathbf{1}$, $M(A)_R x \leq \mathbf{1}$ are minimally infeasible. Since A and $M(A)$ are $(2k+1) \times 2k$ matrices, by Theorem 2.1(iii) $A_B x \geq \mathbf{1}$, $A_R x \leq \mathbf{1}$ and $M(A)_B x \geq \mathbf{1}$, $M(A)_R x \leq \mathbf{1}$ are irreducible.
The second part of the statement follows immediately from Corollary 2.5. $\square$

Recall from Theorem 2.4 that if $Ax \sim b$ is a minimally infeasible system, where A is an $m \times n$ matrix, then $\text{rk}(A) = m - 1$ and, for every $m \times (m-1)$ submatrix A' of A with full column rank, $A'x \sim b$ is an irreducible system.

If A is a $(2k+1) \times n$ MNP balanced matrix, Theorem 3.12 shows that we can choose a $(2k+1) \times 2k$ submatrix A' of A, with full column rank, such that A' is critical. However, not every $(2k+1) \times 2k$ submatrix of A with full column rank is critical, even if its columns have as many blue 1s as red 1s. For example, consider the following MNP (but not critical) balanced matrix.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} B \\ R \\ B \\ R \\ B \end{array}$$

By Corollary 3.9, the system $Ax = \mathbf{1}$ is infeasible, therefore it is minimally infeasible. The submatrices obtained by removing columns 2 or 5 are critical matrices, while if we remove column 3 we obtain the submatrix $\bar{A}$ with full column rank which is not critical but for which the system $\bar{A}x = \mathbf{1}$ is irreducible. Indeed, if $\bar{A}$ was critical, then the system $\bar{A}^i x = \mathbf{1}$ would have a nonnegative solution for every row i, however the only solution of $\bar{A}^5 x = \mathbf{1}$ is $(1, 1, 1, -1)$. The same happens if we remove column 4.

Proposition 3.14 *Let A be a critical balanced matrix, and $j \in [2k]$.*

If $S = \{i \in [2k+1] \mid m_{ij} = 0\}$, then the matrix $\bar{A}$ obtained from A_S by deleting column j is MNP.

If $T = \{i \in [2k+1] \mid a_{ij} = 0\}$, then the matrix $\bar{M}$ obtained from $M(A)_T$ by deleting column j is MNP.

Proof: By Theorem 3.11(i) and Theorem 3.12(ii), the sum of the blue rows of $\bar{A}$ equals the sum of the red rows of $\bar{A}$, thus $\bar{A}x = \mathbf{1}$ is infeasible. On the other hand, if we denote by $\bar{m}^i$ the vector obtained from m^i by removing the jth component, then, for every $i \in S$, $\bar{A}^i \bar{m}^i = A^i_S m^i = \mathbf{1}$.

The argument for the second part of the statement is analogous. □

3.4 Minimal infeasibility of more general systems

In this section, we consider general systems of the form $Ax \sim \mathbf{1}$, $x \geq 0$, where A is a balanced matrix. We show that systems of this form that are minimally infeasible w.r.t. the rows of A are essentially characterized in terms of MNP matrices.

Theorem 3.15 *Let A be an $m \times n$ balanced matrix and (S_1, S_2, S_3) be a partition of $[m]$. The system*

$$\begin{array}{rcll} a^i x & \leq & 1 & i \in S_1 \\ a^i x & \geq & 1 & i \in S_2 \\ a^i x & = & 1 & i \in S_3 \\ x & \geq & 0 & \end{array}$$

denoted by $Ax \sim \mathbf{1}$, $x \geq 0$, is minimally infeasible w.r.t. the rows of A if and only if A is MNP, $S_1 \subseteq R$, and $S_2 \subseteq B$.

Proof: By Theorem 2.1 (i), $Ax = \mathbf{1}$, $x \geq 0$ is minimally infeasible w.r.t. the rows of A, therefore A is MNP. By Theorem 2.1 (ii), there exists a unique orientation $Ax \overset{*}{\sim} \mathbf{1}$ of $Ax \sim \mathbf{1}$ such that $Ax \overset{*}{\sim} \mathbf{1}$, $x \geq 0$ is minimally infeasible w.r.t. the rows of A. By the construction of such orientation in the proof of Theorem 2.1 (ii), $Ax \overset{*}{\sim} \mathbf{1}$ is the system $A_B x \geq \mathbf{1}$, $A_R x \leq \mathbf{1}$, thus $S_1 \subseteq R$, and $S_2 \subseteq B$. □

4 Questions, examples, and counterexamples

Question 4.1 *Give a combinatorial algorithm that, given a $0,1$ matrix A, determines, in polynomial time, one of the following outcomes:*

1. *A is not balanced.*

2. *Two subsets R and B satisfying the conditions of Theorem 3.2.*

3. *A $0,1$ vector x such that $Ax = \mathbf{1}$.*

Note that outcomes 2 and 3 are mutually exclusive, while Theorems 3.1 and 3.2 ensure that, if A is balanced, one of 2 or 3 must occur (however, such an algorithm may produce outcomes 2 or 3 even if the matrix is not balanced).

The following argument yields a polynomial time algorithm that correctly produces one of the above outcomes.

Given a $0,1$ *matrix* A*, find a vertex* v *of* $Ax = \mathbf{1}$, $x \geq 0$. *If* v *exists and is fractional,* A *is not balanced, otherwise, if* v *exists and is integral, outcome 3 holds. If* v *does not exist, repeat iteratively the above procedure to matrices obtained from* A *by removing one row at the time, until eventually we obtain a row submatrix* $\bar{A}$ *of* A *such that* $\bar{A}x = \mathbf{1}$, $x \geq 0$ *is minimally infeasible w.r.t. the rows of* $\bar{A}$.

For every row i *of* $\bar{A}$*, let* v^i *be a vertex of* $\bar{A}^i x = \mathbf{1}$, $x \geq 0$. *If* v^i *is fractional for some* i*,* A *is not balanced. Otherwise, let* B *be the set of rows* i *such that* $a^i v^i = 0$*, and* R *be the set of rows* i *such that* $a^i v^i \geq 2$. *Check if* R *and* B *satisfy the conditions of Theorem 3.2. If so, outcome 2 holds, otherwise, by Lemma 3.4,* A *is not balanced.*

By "combinatorial algorithm" we mean an algorithm that uses only addition, subtraction and comparison, so in particular we do not allow general purpose algorithms to solve systems of linear inequalities.

It is observed in [7] that, using an algorithm of Cameron and Edmonds [3], one can construct an easy, polynomial time, combinatorial algorithm that takes as input a $0,1$ matrix A and a positive integer k, and outputs one of the following:

1. A square submatrix of A of odd order with two ones per row and per column (hence a certificate that A is not balanced).

2. A partition of the columns of A into k sets, such that, for each row i, if row i has less than k 1s, then row i has at most a 1 in each of the k sets, otherwise row i has at least a 1 in each of the k sets.

Note that, if A is a balanced matrix with exactly k ones in each row, such an algorithm partitions the columns of A into k sets whose incidence vectors are solutions of $Ax = \mathbf{1}$.

It would be interesting to have an algorithm of the same type to solve Question 4.1.

Question 4.2 *Let* A *be a MNP balanced matrix. Is it true that the support of every good mate of* A *is contained in the column set of some critical column submatrix of* A*?*

The following are all the critical balanced matrices with at most 5 rows, up to permuting rows and columns.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Question 4.3 *Provide a construction that generates all critical balanced matrices.*

Question 4.4 *Let A be a critical balanced matrix. Is it true that A has a column with exactly two ones?*

In general, it is not true that, if A is balanced and critical, $M(A)$ has a column with exactly two 1s, as shown in the following example. Notice that A is balanced, but $M(A)$ is not, since the submatrix of $M(A)$ indexed by rows 1, 6, 8 and columns 2, 5, 6 is a 3×3 matrix with two 1s per row and column.

$$A = \begin{array}{cc} \begin{bmatrix} 1&0&0&0&0&0&1&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&1&0&0&0\\ 0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0\\ 0&0&0&1&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&1&1\\ 0&0&0&1&0&0&0&0&0&1 \end{bmatrix} & \begin{matrix} B\\ R\\ B\\ R\\ B\\ R\\ B\\ R\\ B\\ R\\ B \end{matrix} \end{array} \quad M(A) = \begin{bmatrix} 0&1&0&1&0&1&0&0&1&0\\ 1&1&0&1&0&1&0&0&1&0\\ 1&0&0&1&0&1&0&0&1&0\\ 1&0&1&1&0&1&0&0&1&0\\ 1&0&1&0&0&1&0&1&0&1\\ 1&0&1&0&1&1&0&1&0&1\\ 1&0&1&0&1&0&0&1&0&1\\ 0&1&0&0&1&0&1&1&0&1\\ 0&1&0&0&1&0&1&0&0&1\\ 0&1&0&0&1&0&1&0&1&1\\ 0&1&0&0&1&0&1&0&1&0 \end{bmatrix} \tag{7}$$

Question 4.5 *Is it true that every $(2k+1) \times 2k$ critical balanced matrix contains a $(2k-1) \times (2k-2)$ submatrix which is critical?*

Let A be the matrix in (7). The only critical 9×8 critical submatrix of A is the following submatrix $\bar{A}$, obtained removing rows 4 and 8 (which are both red), and columns 3 and 8. Notice that the bicoloring of $\bar{A}$ is not the restriction of the bicoloring of A

$$\bar{A} = \begin{array}{cc} \begin{bmatrix} 1&0&0&0&0&1&0&0\\ 1&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&1\\ 0&0&1&0&0&0&0&1 \end{bmatrix} & \begin{matrix} B\\ R\\ B\\ R\\ B\\ R\\ B\\ R\\ B \end{matrix} \end{array}$$

This disproves the conjecture that the rows and columns of every critical $(2k+1) \times 2k$ balanced matrix can be ordered so that, for every $1 \leq h \leq k$, the submatrix A_h induced by the first $2h+1$ rows and the first $2h$ columns is critical and the bicoloring of A_h is the restriction of the bicoloring of A. Notice that this conjecture, if true, would have provided a strengthening of the statement in Question 4.5.

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