Disjoint Paths in Arborescences

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Abstract

An arborescence in a digraph is a tree directed away from its root. A classical theorem of Edmonds characterizes which digraphs have λ arc-disjoint arborescences rooted at r. A similar theorem of Menger guarantees λ strongly arc disjoint rv-paths for every vertex v, where "strongly" means no two paths contain a pair of symmetric arcs.

We prove that if a directed graph D contains two arc-disjoint spanning arborescences rooted at r, then D contains two such arborences with the property that for every node v the paths from r to v in the two arborences satisfy Menger's theorem.

1 Introduction

Given a digraph $D = (V, A)$ and a subset S of V, define $\Delta^-_D(S)$ to be the subset of A with the head in S and the tail in $V \setminus S$ and $\delta^-_D(S) = |\Delta^-_D(S)|$. Let $\Delta^+_D(S) = \Delta^-_D(V \setminus S)$, $\delta^+_D(S) = |\Delta^+_D(S)|$.

Let r be a node of D. An *arborescence rooted at* r is a subgraph $F = (V(F), E(F))$ of D which contains r, is connected and $\delta^-_F(r) = 0$, while $\delta^-_F(v) = 1$ for every other node of $V(F)$. The arborescence F is *spanning* if $V(F) = V$.

The following are two basic results on graph connectivity:

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Theorem 1 (Edmonds [1]) *A digraph $D = (V, A)$ with a specified node r contains λ pairwise arc-disjoint spanning arborescences rooted at r if and only if $\delta^-_D(S) \geq \lambda$ for every $\emptyset \neq S \subseteq V \setminus r$.*

Two arcs are *symmetric* if they have the same endnodes but have opposite orientations. In a digraph two paths are *strongly arc-disjoint* if they are arc-disjoint and they do not contain a pair of symmetric arcs.

Theorem 2 (Menger [7]) *A digraph $D = (V, A)$ with two specified nodes r and v contains λ pairwise strongly arc-disjoint paths from r to v if and only if $\delta^-_D(S) \geq \lambda$ over all $S \subseteq V \setminus r$ with $v \in S$.*

The following conjecture, if true, provides a strengthening of both Theorems 1 and 2:

Conjecture 1 *A digraph $D = (V, A)$ with a specified node r contains λ pairwise arc-disjoint spanning arborescences rooted at r such that, for every $v \in V \setminus r$, the λ paths from r to v in each of these arborescences are strongly arc-disjoint if and only if $\delta^-_D(S) \geq \lambda$ for every $\emptyset \neq S \subseteq V \setminus r$.*

Note that Conjecture 1 does not require the λ arborescences to be strongly arc-disjoint.

Conjecture 1 obviously implies Theorem 1. That it implies Theorem 2 can be seen as follows: Let $D' = (V, A')$ be obtained from D by adding λ arcs from v to each node $x \in V \setminus \{r, v\}$. Then $\delta^-_D(S) \geq \lambda$ over all $S \subseteq V \setminus r$ with $v \in S$ if and only if $\delta^-_{D'}(S) \geq \lambda$ over all $S \subseteq V \setminus r$ and D contains λ pairwise strongly arc-disjoint paths from r to v if and only if D' contains λ pairwise arc-disjoint spanning arborescences rooted at r such that, for every $v \in V \setminus r$, the λ paths from r to v in each of these arborescences are strongly arc-disjoint.

Although we cannot settle Conjecture 1 in the general case, we give below a proof when $\lambda = 2$.

There is a known conjecture (see [2], [6]) that is an undirected counterpart of Conjecture 1. Given an undirected graph $G = (V, E)$ and a subset $S \neq \emptyset$ of V, let $\Delta_G(S)$ be the set of edges of E with one endnode in S and the other in $V \setminus S$ and $\delta_G(S) = |\Delta_G(S)|$.

Conjecture 2 *An undirected graph $G = (V, E)$ with a specified node r contains λ spanning trees such that, for every $v \in V \setminus r$, the λ paths from r to v in each of these trees are pairwise edge-disjoint if and only if $\delta_G(S) \geq \lambda$ for every $\emptyset \neq S \subsetneq V$.*

Indeed, Conjecture 2 is a special case of Conjecture 1. To see this, given a graph $G = (V, E)$ construct a digraph $D = (V, A)$ on the same node set by introducing a pair of symmetric arcs $(u, v), (v, u)$ for every edge uv of G. Given λ spanning arborescences in D satisfying Conjecture 1, the corresponding λ spanning trees in G satisfy Conjecture 2. So Conjecture 1 implies Conjecture 2. In fact, the two conjectures are equivalent if all arcs in D come in symmetric pairs. Again, Conjecture 2 has been proved only for $\lambda = 2$ using depth first search [6].

Similar results are known for the case where "strongly arc-disjoint paths" is replaced by "internally disjoint paths" in Conjecture 1 (where two paths are *internally disjoint* if they have no node in common, except possibly the ends). Whitty [8] proved the internally-disjoint version of the Conjecture for $\lambda = 2$. A simpler proof is due to Huck [4]. Recently Huck [5] found a counterexample to the internally-disjoint version of the Conjecture when $\lambda > 2$.

2 Proof of Conjecture 1 for $\lambda = 2$

If G contains two arc-disjoint spanning arborescences F_1, F_2 rooted at r, then, for all $S \subseteq V \setminus r$ and $i = 1, 2$, $|\Delta^-_D(S) \cap A(F_i)| \geq 1$, thus $\delta^-_D(S) \geq 2$.

For the converse, from Theorem 1 we may assume w.l.o.g. that the digraph $D = (V, A)$ is the union of two arc-disjoint spanning arborescences rooted at r, that is $\delta^-_D(r) = 0$, $\delta^-_D(v) = 2$ for every $v \in V \setminus r$, and $\delta^-_D(S) \geq 2$ for every $S \subseteq V \setminus r$. So the arcs of D are partitioned in pairs having the same head. Arcs in the same pair are *mates*. We may also assume w.l.o.g. that $\Delta^+_D(r)$ consists of two parallel arcs, say a and a' with r' as head. If not, we may add a new node $\bar{r}$ and two parallel arcs from $\bar{r}$ to r; one can easily verify that the case $\lambda = 2$ of Conjecture 1 holds for the new digraph D' with specified node $\bar{r}$ if and only if it holds for D with specified node r.

Given an arborescence $F = (V(F), A(F))$ of D, let $D \setminus F = (V, A \setminus A(F))$. Assume now that F satisfies the following

Property 1 $\delta^-_{D\setminus F}(S) \geq 1$ *for every* $S \subseteq V \setminus r$.

(That is, $D \setminus F$ contains a spanning arborescence.)

A subset of $V \setminus r$ is *critical* if it satisfies Property 1 with equality; the unique arc of $D \setminus F$ entering a critical set is said *special*. Since $\delta^-_D(v) = 2$, every node v in $V(F) \setminus r$ belongs to a critical set.

By submodularity of function $\delta^-(\cdot)$, if S and S' are critical sets and $S \cap S' \neq \emptyset$, then $S \cap S'$ and $S \cup S'$ are also critical. So if e is a special arc, there is a unique maximal critical set $S_e(F)$ entered by e.

Claim 1 *Let $e = (u, v)$ and $e' = (u', v')$ be two special arcs. If $u' \in S_e(F)$ then $S_{e'}(F) \subsetneq S_e(F)$.*

Proof of Claim 1. If $u' \in S_e(F)$ then $S_e(F) \cup S_{e'}(F)$ is critical and is entered by e. Since $S_e(F)$ is maximal, then $S_e(F) = S_e(F) \cup S_{e'}(F)$. Since $u' \notin S_{e'}(F)$, then $S_{e'}(F) \subsetneq S_e(F)$. $\diamond$

A *boundary node* is a node $v \in V(F)$ connected by an arc (v, w) to a node $w \notin V(F)$.

Let $|V| = n$ and let $F_1, \ldots, F_{n-1}$ be arborescences rooted at r constructed as follows:

> *Let F_1 be the arborescence with $V(F_1) = \{r, r'\}$, $A(F_1) = a$ and $i = 1$.*
>
> *While $i < n - 1$, among all sets $S_e(F_i)$ that contain a boundary node $v \in S_e(F_i)$, pick one which is inclusionwise minimal and let (v, w) be an arc such that $w \notin V(F_i)$. Let F_{i+1} be obtained from F_i by adding node w and arc (v, w), set $i = i + 1$.*

We prove that F_{n-1} can indeed be constructed by the above rule and that $F = F_{n-1}$ and $F' = D \setminus F$ satisfy Conjecture 1. Note that by construction, F_1 satisfies Property 1 and r is not a boundary node.

Assume F_i, $i < n - 1$ satisfies Property 1. So F_i contains at least one boundary node. Since every node in $V(F_i) \setminus r$ belongs to a critical set, the above procedure can be carried out to construct F_{i+1}.

We now show that if F_i satisfies Property 1, then F_{i+1} satisfies Property 1. This is equivalent to showing that the arc (v, w) added to F_i by the above procedure is not special.

Let $S_e(F_i)$ be the minimal critical set containing v. Assume (v, w) is special. Then by Claim 1, $S_{(v,w)}(F_i) \subsetneq S_e(F_i)$. Let $S_N = S_{(v,w)}(F_i) \setminus V(F_i)$ and $S_F = S_{(v,w)}(F_i) \cap V(F_i)$. Both S_N and S_F are nonempty since $w \notin V(F_i)$ and $S_{(v,w)}(F_i)$ is critical. Furthermore S_N is not a critical set, for it does not contain any node in $V(F_i)$. So there exists one arc (y, z), where $y \in S_F$ and

$z \in S_N$. Thus y is a boundary node in $S_{(v,w)}(F_i)$ and $S_{(v,w)}(F_i) \subsetneq S_e(F_i)$, contradicting the minimality of $S_e(F_i)$.

This shows that F and F' are arc-disjoint spanning arborescences of D.

We finally show that for every node z the two rz-paths in F and F' can not contain a pair of symmetric arcs.

Assume there exists a node z such that the rz-paths P_z^F and $P_z^{F'}$ in F and F' contain one of the arcs (u, v) and (v, u) respectively. Let (u', v) be the mate of (u, v) (obviously $(u', v) \in P_z^{F'}$), let (v, w) be the arc in P_z^F with v as tail (possibly $u' = r$ or $w = z$) and assume $(v, w) \in A(F_{i+1}) \setminus A(F_i)$.

Let $u' = z_0, v = z_1, u = z_2, \ldots, z_{m-1}, z_m = z$ the $u'z$-subpath of $P_z^{F'}$. Since $w \notin V(F_i)$ both arcs entering z are in $D \setminus F_i$ and $z \notin V(F_i)$. Since $u \in V(F_i)$ there exist two nodes z_k, z_{k+1} of lowest index such that z_k is in $V(F_i)$ and z_{k+1} is not (clearly, $k \geq 2$). Then z_k is a boundary node for F_i.

Since, for $1 \leq j \leq k$, all sets $\{z_j\}$ are critical, then all arcs (z_{j-1}, z_j) are special, and each set $S_{(z_{j-1},z_j)}(F_i)$ contains the head z_j of the next arc. By Claim 1, for $2 \leq j \leq k$, $S_{(z_{j-1},z_j)}(F_i) \subsetneq S_{(z_{j-2},z_{j-1})}(F_i)$. So $S_{(z_{k-1},z_k)}(F_i) \subsetneq S_e(F_i)$ and contains the bundary node z_k, contradicting the minimality of $S_e(F_i)$.

□

The construction in the proof can be implemented in polynomial time. Gabow [3] gave a $O(\lambda^2 n^2)$ algorithm to find λ arc-disjoint arborescences in a digraph D, thus we may find two arc-disjoint spanning arborescences of D in time $O(n^2)$, and assume D is just the union of such arborescences. We claim that our construction can be implemented, on such D, in time $O(n^2)$ as well.

Notice that, at the ith iteration, if $e = (u, v)$ is a special arc such that v is the unique boundary node in $S_e(F_i)$, then $S_e(F_i)$ is inclusionwise minimal with such property; in fact, if for some special arc e', $S_{e'}(F_i) \subseteq S_e(F_i)$ contains a boundary node, then $v \in S_{e'}(F_i)$ and $u \notin S_{e'}(F_i)$, so $e' = e$.

Also, for any special arc e, if we denote by $R_i(e)$ the set of nodes reachable from r in $D \setminus (A(F_i) \cup \{e\})$, $S_e(F_i) = V \setminus R_i(e)$.

In order to implement the construction in the proof, we need to show how to compute, at the ith iteration, a minimal $S_e(F_i)$ containing a boundary node.

Start from any boundary node v_0, let (u_0, v_0) be the special arc entering v_0, compute $R_i(u_0, v_0)$. Suppose we have computed $R_i(u_j, v_j)$, where v_j is a boundary node and (u_j, v_j) is a critical arc, $0 \leq j \leq |V(F_i)|$.

If $S_{(u_j,v_j)}(F_i) = V \setminus R_i(u_j, v_j)$ does not contain any boundary node except v_j, then $S_{(u_j,v_j)}(F_i)$ is minimal containing a boundary node.

Otherwise, choose a boundary node $v_{j+1} \neq v_j$ in $V \setminus R_i(u_j, v_j)$, and let (u_{j+1}, v_{j+1}) be the unique special arc entering v_{j+1}. Compute the set R' of nodes reachable from $R_i(u_j, v_j)$ in $D \setminus (A(F_i) \cup \{(u_{j+1}, v_{j+1})\})$, and let $R_i(u_{j+1}, v_{j+1}) := R_i(u_j, v_j) \cup R'$.

Clearly, this procedure takes linear time at each iteration, and there are $n - 1$ iterations, so the total running time is $O(n^2)$.

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