London Taught Course Centre

2018 examination

Graph Theory

Instructions to candidates

This open-book exam has 2 questions. Some parts are harder than others, and you should not expect to be able to answer all questions completely. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed. You are also allowed to use any theorems you find, provided they are properly referenced.

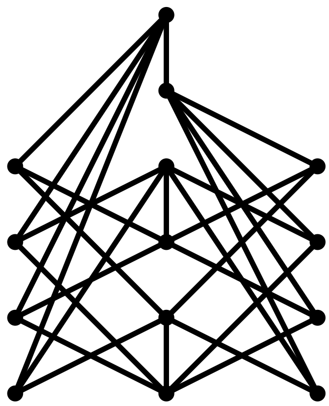
1 The complexity class EXP consists of all languages $\mathcal{L}$ for which there exists a Turing Machine deciding $\mathcal{L}$ in time 2^{n^k}, for some $k = k(\mathcal{L})$. The complexity class NEXP is the class of all languages $\mathcal{L}$ for which there exists a nondeterministic Turing Machine accepting $\mathcal{L}$ in time 2^{n^k}, for some $k = k(\mathcal{L})$.

(a) Explain why NP $\subseteq$ EXP.

(b) Can you modify the proof of Savitch's Theorem to show that EXP $=$ NEXP?

2 (a) Show that for each $\alpha > 0$ there is a constant $C(\alpha)$ such that the following holds. If G is an n-vertex graph with minimum degree at least αn, then either G contains C_7 as a (not necessarily induced) subgraph or it is $C(\alpha)$-colourable. *Hint: first show that any graph which does not contain a four-vertex path is* 3*-colourable.*

(b) Now prove the same statement with the following graph F replacing C_7.



To better understand F, observe that it consists of a matching of size three (the middle six vertices) and eight more vertices on the sides, each of which is adjacent to one end of each matching edge; and the eight vertices have each of the eight possible adjacencies under that condition.

(c) Show that for any C and any $f(n)$ such that $f(n)/n \to 0$ as $n \to \infty$, for all sufficiently large n there exists an n-vertex graph G which does not contain F, whose minimum degree is at least $f(n)$, and which is not C-colourable.