LTCC Course: Graph Theory 2024/25
Solutions to Exercises 2

Exercise 1.
We claim that, for a graph G with c components, v vertices and e edges, any embedding of G in the plane has f faces, where $v - e + f = 1 + c$.

We proceed by induction on c, noting first that the result is true for the graph with $c = 0$ (since then also $v = 0$, $e = 0$, but $f = 1$).

Given a graph G with at least one component H, consider any embedding of G in the plane. Then the whole of H lies in one face F of the embedding of $G - H$. The embedding of $G - H$ has $1 + c(G-H) - v(G-H) + e(G-H)$ faces, by the induction hypothesis. The embedding of H inside the face F has (not counting the exterior face) $1 - v(H) + e(H)$ faces. The total number of faces of the embedding is thus $2 + c(G-H) - v(G-H) - v(H) + e(G-H) + e(H) = 1 + c(G) - v(G) + e(G)$, as claimed.

The result now follows by induction.

Exercise 2.
The key thing here is to see that the identification of boundary segments of the $4k$-gon identifies all the corners as a single vertex. (You should check that, within each section of four segments, all five corners are identified.) So the number of vertices is 1, as is the number of faces. The number of edges is $2k$, as the boundary segments are identified in pairs. Thus $v - e + f = 2 - 2k$, in line with the Euler-Poincaré formula.

Exercise 3.
(a) We claim that G has K_3 as a minor if and only if it contains a cycle.

If G does contain a cycle, then we can show that K_3 is a minor of G by removing all edges and vertices not on the cycle, and then (if necessary) repeatedly contracting edges to reduce the length of the cycle until we are left with. $K_3 = C_3$.

If G has a K_3 minor, then there are three disjoint connected sets of vertices V_1, V_2, V_3 with an edge e_{ij} between each pair (V_i, V_j). Now we can find (possibly trivial) paths inside each V_i connecting the endpoints of the two edges e_{ij} in that set. Joining these paths with the e_{ij} gives a cycle.

So the graphs with no K_3 minor are exactly the forests.

(b) Similarly to part (a), a graph with no C_k minor, where C_k is cycle on k vertices, is exactly one with no cycle of length k or greater.

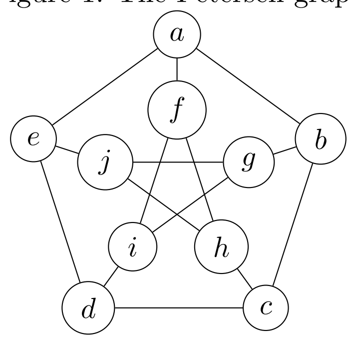
In particular, if G has no C_4 minor, then all cycles of G are triangles. Another way to say this is that every *block* (maximal 2-connected subgraph) of G is either an edge or a triangle.

Exercise 4.

(a) – The Petersen graph P has 10 vertices, and its shortest cycle is of length 5. (Convince yourself about this.) Using this last fact, we see that, in a putative embedding in the plane with f_i i-sided faces, we have $2e = \sum_i if_i \geq 5f$. Thus Euler's formula gives $10 = 5v - 5e + 5f \leq 5v - 3e$, so $e \leq \frac{1}{3}(5v - 10) = 40/3$. Therefore $e \leq 13$. But the Petersen graph has 15 edges.

– The Petersen graph is often drawn like this:

Figure 1: The Petersen graph



It's fairly easy to find $K_{3,3}$ as a topological minor. We can take c, e, i to be the vertices of one partition class, and a, d, j to be the vertices of the other, for example. Most of the paths we use are the obvious single-edge paths; the non-obvious ones are jhc and jgi, and abc and afi.

– Again from the above drawing, it's obvious that contracting af, bg, ch, di and ej gives a K_5-minor.

(b) Our proof in the first part of (a) above shows that we have to delete at least 2 edges to make P planar. It's not obvious from the above drawing that we can manage with as few as two edges removed. But in fact we can: removing fi and ch works, for example. To see this, observe that once ch is removed, we can move the vertex h up and left till its remaining edges no longer create crossings. The only crossing left is the one between fi and gj, so removing fi next indeed gives a planar graph.

Exercise 5.

Let H be C_3, the cycle on 3 vertices, and let G be C_4, the cycle on 4 vertices. Then by suppressing one vertex of degree two in G we get H, so $H \leq_T G$. But G is bipartite, whereas H is clearly not bipartite.

Exercise 6.

Let $\mathcal{G}$ be the class of all graphs for which the number of edges in every connected component is at most the number of vertices. Note that for a connected component $C = (V(C), E(C))$ of a graph G we must have $|E(C)| \geq |V(C)| - 1$. Moreover, for a connected component C we have $|E(C)| = |V(C)| - 1$ if and only if C is a tree. From this it is easy to see that $|E(C)| \leq |V(C)|$ if and only if C is a tree (and then $|E(C)| = |V(C)| - 1$) or C has exactly one cycle (and then $|E(C)| = |V(C)|$).

So for every graph in $\mathcal{G}$ we have that every component is a tree or a connected graph with exactly one cycle. And if we remove a vertex, or remove or contract an edge, then we only need to consider what happens with the component containing that vertex or edge.

If C is a component that is a tree, then removing a vertex of degree one leaves a smaller tree, while removing a vertex of degree more than one gives a number of smaller trees. Similarly, if C is a component with one cycle, then removing a vertex not on the cycle leaves one component with one cycle, and possibly some smaller trees. And removing a vertex on the cycle leaves one or more parts that are all trees.

If C is a component that is a tree, then removing an edge gives two smaller trees. Similarly, if C is a component with one cycle, then removing an edge from the cycle will transform C to a tree; while removing any other edge will leave one part with a cycle and one part that is a smaller tree.

If C is a component that is a tree, then contracting an edge gives a smaller tree. Similarly, if C is a component with one cycle, then contracting an edge from the cycle will leave a smaller component with one shorter cycle (if the original cycle had length at least 4) or transforms C to a tree (if the original cycle had length 3); while contracting any other edge will leave one component with the same unique cycle.

So we'll see that for every graph in $\mathcal{G}$ we have that removing a vertex or an edge, or contracting an edge, will result in a graph that is still in $\mathcal{G}$. This shows that $\mathcal{G}$ is closed under taking minors.

Any forbidden minor for $\mathcal{G}$ must be a graph that is not in $\mathcal{G}$, hence it must have components with at least two cycles. Moreover, a minimal minor has just one component. A bit of trial and error seems to indicate that there are two different smallest graph with two cycles:

– G_1 with 4 vertices and 5 edges (there is only one such graph);
– G_2 with 5 vertices and 6 edges, formed by joining two triangles.

I'll leave it to you to prove that these are in fact exactly the minimal forbidden minors of $\mathcal{G}$. For this you must prove that every graph *not* in $\mathcal{G}$ has at least on of G_1 or G_2 as a minor.

Exercise 7.
For a natural number $k \geq 3$, define the graph H_k as follows: Start with the cycle C_k on k vertices, say with vertices $x_1, \ldots, x_k$ and edges $x_i x_{i+1}$ (where we set $k+1=1$). Now add new vertices $y_1, \ldots, y_k$ and add the edges $x_i y_i$ and $y_i x_{i+1}$, for $i = 1, \ldots, k$ (again, setting $k+1=1$).

We will show that the sequence $H_3, H_4, \ldots$ has the property that $H_k \not\leq_T H_\ell$ for all $k < \ell$. To do this, observe that for any two vertices x_i, x_j in some H_k we have that there are four edge disjoint paths between x_i and x_j in H_k. But if we remove one vertex or one edge from H_k, then this property is no longer satisfied. And once this property is no longer satisfied, then no vertex removal, edge removal, or suppression of a vertex of degree two can bring it back.

So the only operation that we can use in order to transform H_ℓ to H_k for some $\ell > k$ is suppressing a vertex of degree two. But also doing this as the first operation will destroy one of the edge disjoint paths between x_i and x_j.

So no topological minor of H_ℓ can be equal to H_k for $k < \ell$, completing the proof.