Exercises and Solutions (Lecture 4, LTCC Course: Graph Theory)

1. *(a) Prove Lemma 5. Hint: first evaluate $\mathbb{E}e^{tX}$ for $t > 0$ a fixed constant, apply Markov's inequality, and optimise t. Start with the case of independent identical Bernoulli variables (i.e. $p_1 = p_2 = \dots$. For the general case, you might want to look up 'Jensen's inequality'.*
(b) Try to generalise this to prove the martingale concentration inequality (1) after Lemma 9. The difficult part is figuring out how to evaluate $\mathbb{E}e^{tX}$.

Answer.
(a) Since $X = Y_1 + \cdots + Y_n$ is a sum of independent Bernoulli random variables, we have $\mathbb{E}e^{tX} = \prod_{i=1}^n \mathbb{E}e^{tY_i} = \prod_{i=1}^n \big(1 - p_i + p_i e^t\big)$, where p_i is the probability that $Y_i = 1$. If all the p_i are identical and equal to p, this is equal to $\big(1 - p + pe^t\big)^n$. If they are not all identical, but their mean is p, we still have $\mathbb{E}e^{tX} \le \big(1-p+pe^t\big)^n$. Probably the way to see this is to take the logarithm (which turns the product into a sum) and observe that for x, $t > 0$, the function $\log\big(1 - x + xe^t\big)$ is a concave function of x, hence by Jensen's inequality the sum is maximised when all terms are equal.

Now since e^{tx} is a monotone increasing function of x when $t > 0$, we have

$$\mathbb{P}\big(X \ge (1+\delta)\mathbb{E}X\big) = \mathbb{P}\big(e^{tX} \ge e^{t(1+\delta)\mathbb{E}X}\big) \le \frac{\mathbb{E}e^{tX}}{e^{t(1+\delta)\mathbb{E}X}}\,,$$

where the inequality is Markov's inequality. We just need to find a value of t which makes this suitably small (which should be positive since we assume $t > 0$ a couple of times). It turns out that choosing t such that

$$e^t = \frac{(1+\delta)(n - \mathbb{E}X)}{n - (1+\delta)\mathbb{E}X}$$

works; the calculation is not too hard, and works since for $0 < \delta < 3/2$ we have $(1+\delta)\ln(1+\delta) - \delta \ge \frac{1}{3}\delta^2$ (In fact, it's true for a slightly larger range).

For the lower bound, we use the same method to calculate $\mathbb{E}e^{t(n-X)}$ and the same application of Markov's inequality to show that $n - X$ is not likely to be larger than $n - (1-\delta)\mathbb{E}X$. This time, we choose t such that

$$e^t = \frac{n - \mathbb{E}X + \delta\mathbb{E}X}{(n - \mathbb{E}X)(1-\delta)}\,.$$

(b) The idea is to prove by induction on n that $\mathbb{E}e^{tX} \le \big(1 - \frac{s}{n} + \frac{s}{n}e^t\big)^n$. Given this, the result follows by the identical use of Markov's inequality and (the same) optimisation of t. The $n = 1$ base of this induction is trivial. For the induction step, let X_0' be the random variable $X - Y_1$ induced on the probability space given by $Y_1 = 0$ with the conditional measure, and X_1' the same random variable on the (disjoint) space with $Y_1 = 1$ and the conditional measure. Observe that

$$\mathbb{E}e^{tX} = \mathbb{P}(Y_1 = 0)\mathbb{E}e^{tX_0'} + e^t\mathbb{P}(Y_1 = 1)\mathbb{E}e^{tX_1'}$$

and the two random variables X_0' and X_1' are sums of $n-1$ sequentially dependent random variables whose observed expectations are both almost surely at most $s - \mathbb{P}(Y_1 = 1)$. By induction we obtain

$$\mathbb{E}e^{tX} \le \big(1 - \mathbb{P}(Y_1 = 1) + e^t\mathbb{P}(Y_1 = 1)\big)\big(1 - \tfrac{s-\mathbb{P}(Y_1=1)}{n-1} + \tfrac{s-\mathbb{P}(Y_1=1)}{n-1}e^t\big)^{n-1}\,,$$

and again using Jensen's inequality (exactly as before) we obtain the desired upper bound.

2. *For $k \in \mathbb{N}$, a graph $G = (V, E)$ has* Property S_k *if, for every pair (A, B) of disjoint k-element subsets of V, there is a vertex x of the graph that is adjacent to every vertex of A and no vertex of B.*

(a) Find a graph with property S_1.
(b) Show that, for each $k \in \mathbb{N}$, there is a graph with property S_k.

Answer.
(a) The smallest graph with this property is the 5-cycle. However, as we're about to prove, most graphs work!
(b) Take a random graph $G(n, 1/2)$. For specific sets A and B of size k, the probability that there is no vertex (outside A and B) which is adjacent to all of A and none of B is $\left(1 - 2^{-2k}\right)^{n-2k}$. This is because the event that any individual vertex x does the job has probability 2^{-2k}, and these $n - 2k$ events are independent.

Now, call a pair (A, B) *bad* if there is no $x \in V(G) \setminus (A \cup B)$ adjacent to all of A and none of B. The expected number of bad pairs is

$$\binom{n}{k}\binom{n-k}{k}\left(1 - 2^{-2k}\right)^{n-2k} \leq n^{2k}\left(1 - 2^{-2k}\right)^{n-2k}.$$

For each fixed k, this expression tends to 0 as $n \to \infty$. Choose n_0 to make the expected number of bad pairs less than 1. Then there must be a graph on n_0 vertices with no bad pair, and this graph has property S_k.

3. *A k-uniform* hypergraph *is a pair $H = (V, E)$, where V is a set of* vertices*, and E is a family of k-element subsets of V. (So a 2-uniform hypergraph is just a graph.) A hypergraph $H = (V, E)$ has* Property B *if V can be partitioned into two subsets V_1 and V_2 in such a way that no edge is entirely contained within one of the two sets.*
(a) Show that, if $H = (V, E)$ is a k-uniform hypergraph with $|E| < 2^{k-1}$, then H has property B.
(b) Show that, if $H = (V, E)$ is a k-uniform hypergraph such that each edge in E intersects at most d others, and $e(d + 1) \leq 2^{k-1}$, then H has property B.

Answer.
(a) Take each vertex of V and put it into V_1 or V_2, each with probability $1/2$. The probability that an edge, with k vertices, lies entirely within one of the two sets is 2^{1-k}. The expected number of edges lying entirely within one of the two sets is $|E|2^{1-k} < 1$. Therefore there is some partition in which there is no edge lying entirely within one of the two sets, which is what we wanted.
(b) I reckon "apply the Local Lemma" is sufficient!

4. (a) *Let $p = n^{-t}$, for $0 < t < 1$, and let k be a fixed natural number. Write down an expression for the expected number of k-cliques in $G(n, p)$. Hence show that, if $t > 2/(k - 1)$, the probability that $G(n, p)$ contains a k-clique tends to zero as $n \to \infty$.*

It is also true that, if $t < 2/(k - 1)$, then the probability that $G(n, p)$ contains a k-clique tends to one as $n \to \infty$: to prove this, one needs to work with the variance of the number of k-cliques.
(b) *Let H denote the graph on five vertices a, b, c, d, e with seven edges: a, b, c, d form a clique, and de is also an edge. For $p = n^{-7/10}$, find the expected number of copies of H in $G(n, p)$. What is*

$$\lim_{n \to \infty} \mathbb{P}(G(n, p) \textit{ contains a copy of } H)?$$

(c) *There is a parameter $b(H)$ of graphs such that, if $t > b(H)$, then the probability that $G(n, p)$ contains a copy of H as a subgraph tends to zero, while if $t < b(H)$ then this probability tends to 1. Based on the calculations in this question, what do you think this parameter $b(H)$ might be?*

Answer.
(a) The expected number of k-cliques is

$$\binom{n}{k} p^{\binom{k}{2}} \leq n^k n^{-tk(k-1)/2} = n^{k(1-t(k-1)/2)}.$$

So the expected number of k-cliques tends to zero if $t > 2/(k-1)$, and as usual this implies that the probability of existence of a k-clique tends to zero.
(b) When counting the number of copies of H in a graph, if we have one copy, we don't count permuting the labels a, b, c as giving us a separate copy. However, we do count a 5-clique as giving us 20 different copies. Adopting other conventions will only affect the constant factors, and won't obscure the main point.

The expected number of copies of H is

$$n(n-1)\binom{n-2}{3} p^7 = (1+o(1))\frac{1}{6} n^5 p^7 = (1+o(1))\frac{1}{6} n^{5-49/10} = (1+o(1))\frac{1}{6} n^{1/10}.$$

Of course, this is very large. However, the expected number of 4-cliques in $G(n,p)$ is at most $n^4 p^6 = n^{-1/5}$. This means that the probability that $G(n,p)$ contains a 4-clique tends to zero as $n \to \infty$. If G contains no 4-clique, then it certainly contains no copy of H.

This is no paradox. The probability that $G(n,p)$ contains a 4-clique is indeed of order $n^{-1/5}$. However, if $G(n,p)$ does contain a 4-clique, then each of its vertices d will be adjacent to about $np = n^{3/10}$ other vertices that can play the role of e, and so $G(n,p)$ will contain on the order of $n^{3/10}$ copies of H. So the expected number of copies of H is seen again to be at least $n^{-1/5} n^{3/10} = n^{1/10}$.
(c) To get the expected number of copies of H to be greater than 1, we need $n^{|V(H)|} p^{|E(H)|} \gg 1$, which means $|V(H)| - t|E(H)| > 0$, if $p = n^{-t}$. This suggests that we should be interested in $b'(H) = |V(H)|/|E(H)|$. It is certainly true that, if $t > b'(H)$, then there are unlikely to be any copies of H, whereas if $t < b'(H)$ then the expected number of copies of H is large.

However, (b) should warn us that it's not *that* simple. A better proposal is

$$b(H) = \min_{H' \subseteq H} |V(H')|/|E(H')|,$$

where the minimum is over all subgraphs H' of H. This proposal turns out to be right: see Bollobás, *Random Graphs*.

5. *Set* $p = n^{-2/5}$*, and consider a random graph* $G = G(n,p)$*.*
(a) Show that the degree of any fixed vertex v *has a Binomial distribution, and find an upper bound on the probability that this degree is greater than or equal to* $n^{2/3}$*. [You may need to look up some estimates on the tails of the distribution of a Binomial random variable.]*
(b) Show that the probability that the maximum degree of G *is at most* $n^{2/3}$ *is at least* $2/3$*.*
(c) Show that, with probability at least $2/3$*, for every pair* (U, V) *of subsets of* $V(G)$*, with* $|U|, |V| \geq n^{1/2}$*, there is an edge from* U *to* V*.*
(d) What can you deduce from (b) and (c)?

Answer.
(a) The degree of a vertex v is the sum, over all other vertices u, of the indicator function of the event that uv is an edge. This means that the degree of v is a sum of Bernoulli (0-1 valued) random variable with probability p of being 1, so the degree of v is a Binomial random variable X with parameters $(n-1, p)$. So its mean is $p(n-1) \leq n^{3/5}$. The variance is also about $n^{3/5}$, so deviations of greater than about $n^{3/10}$ from its mean are unlikely.

To get a more precise answer, let us use a version of the *Chernoff bound* which is different from the one seen in the lecture.

Theorem 1. *Let X be a binomial random variable with mean μ. Then, for all $t \geq 0$:*
(a) $\Pr(X \geq \mu + t) \leq \exp(-t^2/2(\mu + t/3))$*;*
(b) $\Pr(X \leq \mu - t) \leq \exp(-t^2/2\mu)$*.*

Here, for instance, we can take $t = \frac{1}{2}n^{2/3}$: the probability that $d(v)$ is greater than $(n-1)p+t$ is at most $\exp(-t^2/2(\mu+t/3)) \leq \exp(-t) = \exp(-n^{2/3}/2)$, for n large enough. Hence the probability that $d(v)$ is as large as $n^{2/3}$ is at most this large.

Cruder estimates can still give bounds that are perfectly good enough for this purpose.

(b) The probability that *there is* a vertex with degree at least $n^{2/3}$ is at most n times the probability that one particular vertex has this large a degree, which is therefore at most $ne^{-n^{2/3}/2}$, which is certainly at most $1/3$ (for n large enough).

(c) The probability that some particular pair (U, V) of disjoint sets of size $\lceil n^{1/2} \rceil$ is "bad" (spans no edge) is at most $(1-p)^n$, since there are at least n pairs of potential edges, and this is therefore an upper bound on the probability that none of them are in the graph.

Now, the expected number of bad pairs is at most

$$\binom{n}{n^{1/2}}^2 (1 - n^{-2/5})^n \leq \left(en/n^{1/2}\right)^{2n^{1/2}} e^{-n^{-2/5}n} \leq \exp\left(2n^{1/2}\log n - n^{3/5}\right) < 1/3,$$

again at least provided n is large enough. Note that we were crude where we could be, but we didn't compromise on the key part of the count, which is the power of n in the exponent.

(d) Of course, this means that there is at least one graph of maximum degree at most $n^{2/3}$ such that there is an edge between every pair (U, V) of sets of at least $n^{1/2}$ vertices.

You should see that we could make this argument a whole lot tighter, and get a stronger result. Let me emphasise again that, if you want to construct a sequence of n-vertex graphs with the properties above, you'll have a tough time.

6. (a) *Try to prove Lemma 9. You should find that the only difficult part is to prove that most edges are in the 'right' number of triangles. You will not be able to prove that T_{uv} behaves nicely for every edge $uv \in G$: you will need to assume both that uv happened to lie in about the 'right' number of triangles in G, and that 'most' of those triangles share edges with about the 'right' number of triangles in G. If you make this assumption, you should be able to modify the argument given to show that T_{uv} is likely to be about the 'right' size. Then you will need to show that there cannot be too many edges of G which don't satisfy the assumption.*

(b) *Try to prove the special case of Theorem 8 from Lemma 9. The difficulty here is to find out how to set constants in order to make the argument work.*

(c) *Try to prove Theorem 8—or try to understand the argument given in e.g. Alon and Spencer!*

Answer.
For a solution please consult the section on the "Rödl nibble" in Alon and Spencer.