Exercises and Solutions (Lecture 5, LTCC Course: Graph Theory)

1. *Fill in the (geometric) details in the proof of Theorem 2.4.*

Answer.
First, what does it mean to say that a set of four points in general position do not form a convex quadrilateral? Consider the convex hull of the set of four points: this has at least three extreme points. If there are four extreme points, then we do get a convex quadrilateral: if there are only three, then the fourth point is inside the convex hull of the others.

The other assertion that needs to be verified is that, if we place 5 points in the plane in general position, then some 4 of them form a convex 4-gon. Suppose not, and consider the convex hull of the set of 5 points: this must be a triangle. Now consider the line through the two points in the interior of the triangle. Two corners of the triangle lie on the same side of this line. Now these two points, together with the two interior points, form a convex 4-gon.

2. *Let S be an infinite set of points in the plane. Show that there is an infinite subset A of S such that either no three points of A are on a line, or all points of A are on a line.*

Answer.
Colour the triples of points from S *red* if the three points lie on a line, and *blue* if they do not. Now apply Ramsey's Theorem. Either: (a) there is an infinite subset A such that all triples are red, or (b) there is an infinite subset A such that all triples are blue. In case (b), no three points of A lie on a line. In case (a), consider the line L through any pair x, y of points from A: as each triple $\{x, y, z\}$ is red, all other points z also lie on L.

3. *The Ramsey number $R_k(3)$ is the minimum number n of vertices such that, if the edges of K_n are coloured with k colours, there is always a monochromatic triangle. Show that $R_k(3) \leq k(R_{k-1}(3) - 1) + 2$. [Hint: if you don't know the classic proof that $R_2(3) \leq 6$, find and read that first.]*

Deduce that $R_k(3) \leq \lfloor ek! \rfloor + 1$.

Answer.
Set $n = k(R_{k-1}(3) - 1) + 2$. We need to prove that, however we k-colour the edges of K_n, we get a monochromatic triangle.

Take any vertex x of K_n. As there are $n-1 > k(R_{k-1}(3)-1)$ edges of K_n incident with x, there is some colour (say red) such that there are more than $R_{k-1}(3) - 1$ red edges – in other words, at least $R_{k-1}(3)$ red edges – incident with x. Let A be the set of vertices joined to x by a red edge. If there is some red edge yz between two vertices of A, then xyz forms a red triangle. If not, then the complete graph with vertex set A has its edges coloured with the remaining $k - 1$ colours: the definition of $R_{k-1}(3)$ now ensures that there is a monochromatic triangle inside A.

For the second part, we first note that the result is true for $k = 1$. For an inductive proof, we wish to argue that

$$R_k(3) \leq k(R_{k-1}(3) - 1) + 2 \leq k(\lfloor e(k-1)! \rfloor + 1 - 1) + 2 \leq \lfloor ek! \rfloor + 1.$$

So we need to check that $k\lfloor e(k-1)! \rfloor + 1 \leq \lfloor ek! \rfloor$.

The trick (surely something you might think of trying), is to write out

$$k!e = k! \sum_{i=0}^{\infty} \frac{1}{i!} = \sum_{i=0}^{k} \frac{k!}{i!} + \sum_{i=k+1}^{\infty} \frac{1}{(k+1)(k+2)\cdots i}.$$

The first sum is an integer, and the second is less than $\sum_{j=1}^{\infty} \frac{1}{(k+1)^j} = 1/k \leq 1$. So $\lfloor ek! \rfloor = \sum_{i=0}^{k} \frac{k!}{i!}$, for $k \geq 1$.

Now we're done, as, for $k \geq 2$,

$$k\lfloor e(k-1)! \rfloor + 1 = k \sum_{i=0}^{k-1} \frac{(k-1)!}{i!} + 1 = \sum_{i=0}^{k-1} \frac{k!}{i!} + \frac{k!}{k!} = \lfloor ek! \rfloor.$$

4. *Let $B_{n,p}$ be a random bipartite graph, with two vertex classes V_1 and V_2 each of size n. So each pair of vertices in different classes is joined by an edge with probability p.*
(a) Show that, for all $\varepsilon > 0$, $p > 0$,

$$\mathbb{P}((V_1, V_2) \text{ is an } \varepsilon\text{-regular pair in } B_{n,p}) \to 1 \text{ as } n \to \infty.$$

(b) Show that, for any bipartite graph H,

$$\mathbb{P}(B_{n,p} \text{ contains a copy of } H \text{ as a subgraph}) \to 1 \text{ as } n \to \infty.$$

[Hint: take many disjoint subsets of the vertex set, each of size $|V(H)|$.]

Answer.
(a) Call a pair (A, B), with $A \subseteq V_1$, $B \subseteq V_2$, each of size at least εn, "bad" if the number of edges of the random bipartite graph between A and B is outside the range

$$(p - \varepsilon/2)|A|\,|B|, (p + \varepsilon/2)|A|\,|B|.$$

We would like to show that the expected number of bad pairs tends to zero.

If we can do this, then we'll be done: in any graph with no bad pairs,

$$\left| \frac{e(A,B)}{|A|\,|B|} - \frac{e(V_1,V_2)}{|V_1|\,|V_2|} \right| \leq \left| \frac{e(A,B)}{|A|\,|B|} - p \right| + \left| \frac{e(V_1,V_2)}{|V_1|\,|V_2|} - p \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for every A and B, so (V_1, V_2) is an ε-regular pair.

The number of pairs (A, B) with $A \subseteq V_1$, $B \subseteq V_2$, each of size at least εn, is certainly at most 2^{2n}. So it's enough to prove that, for any fixed pair (A, B), the probability that it is bad is $o(4^{-n})$.

The number of edges between A and B is a binomial random variable with parameters $(|A|\,|B|, p)$. We're interested in knowing the probability that it deviates from its expectation $|A|\,|B|p$ by as much as $\varepsilon|A|\,|B|/2$. We also know that $|A|\,|B| \geq \varepsilon^2 n^2$.

This is actually true with quite a bit to spare. You can do this via direct estimates of binomial coefficients, but let's use the Chernoff bounds, as in the previous set of answers.

Theorem 0.1. *Let X be a binomial random variable with mean μ. Then, for all $t \geq 0$:*
(a) $\Pr(X \geq \mu + t) \leq \exp(-t^2/2(\mu + t/3))$;
(b) $\Pr(X \leq \mu - t) \leq \exp(-t^2/2\mu)$.

Suppose that $\varepsilon < p < 1 - \varepsilon$. Then we apply the Chernoff bounds with $\mu = |A|\,|B|p$ and $t = \varepsilon|A|\,|B|/2$, so $\mu + t/3 \leq 3\mu/2$, and obtain

$$\Pr(|X - \mu| \geq t) \leq 2\exp(-t^2/3\mu) = 2\exp(-\varepsilon^2|A|\,|B|/12p) \leq 2\exp(\varepsilon^4 n^2/12),$$

which is indeed $o(4^n)$.

(b) This is easier. Given H, take a bipartition of H, with t vertices in one class and $s \leq t$ in the other. Now take $r = \lfloor n/t \rfloor$ disjoint sets $A_1, \dots, A_r$ of vertices of size t in V_1, and r disjoint sets $B_1, \dots, B_r$ of vertices of size s in V_2. The probability that the graph on each vertex set $A_i \cup B_i$ contains a copy of H is at least $p^{st} > 0$. The probability that none of the r sets $A_i \cup B_i$ contains a copy of H is at most $(1 - p^{st})^r$, which tends to zero as $n \to \infty$. Notice that p, s, t are all fixed, while $r \simeq n/t$ tends to infinity with n.

5. *Suppose G is a bipartite graph, with vertex classes V_1 and V_2, each of size n. Suppose also that the maximum degree of G is at most $\varepsilon^2 n$. Show that the pair (V_1, V_2) is ε-regular.*

Answer.
Let $A \subseteq V_1$ and $B \subseteq V_2$ be sets of vertices of size at least εn. The number $e(A, B)$ of edges between A and B is at most $|A|\varepsilon^2 n$, so the density $e(A, B)/|A|\,|B|$ is at most $\varepsilon^2 n/|B| \leq \varepsilon$. So the two numbers $e(A, B)/|A|\,|B|$ and $e(V_1, V_2)/n^2$ both lie between 0 and ε, and thus their difference is at most ε.

Thus the pair is ε-regular.

6. *Let G_n be the following bipartite graph. The vertex set of G_n is $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$. The edges are given by $x_i y_j \in E(G_n)$ if $i < j$.*

Fix $\varepsilon > 0$. For each value of n, find an explicit ε-regular partition of G_n into at least three and at most (say) $10/\varepsilon$ parts.

Answer.
If $3 \leq n \leq 10/\varepsilon$, the partition into singletons suffices, so we may assume that $n > 10\varepsilon$.

The idea here is that each vertex class V_i should consist of vertices with "similar" degrees. Let's suppose our classes are to have size t, and we are to have k of them. A natural plan is to form classes $X_\ell = \{x_j : (\ell - 1)t < j \leq \ell t\}$ and $Y_\ell = \{y_j : (\ell - 1)t < j \leq \ell t\}$, so long as $\ell t \leq n$. So we'll have $k = 2\lfloor n/t \rfloor$. Any vertices not in any of these classes will go into the class V_0: we'll need to ensure this is small. This is achieved whenever $t \leq \varepsilon n$, so that $|V_0| < 2\varepsilon n = \varepsilon |V(G_n)|$.

Pairs of the form (X_ℓ, X_m) and (Y_ℓ, Y_m) form empty graphs, which are certainly ε-regular for any $\varepsilon > 0$. Also, pairs (X_ℓ, Y_m) with $\ell > m$ are empty, while pairs (X_ℓ, Y_m) with $\ell < m$ are complete, so all of these pairs are ε-regular.

Pairs of the form (X_ℓ, Y_ℓ) are not ε-regular, and there's nothing that can be done about this. But this is acceptable, as long as the number $k/2$ of such pairs is at most $\varepsilon\binom{k}{2}$; so we need $k \geq 1/\varepsilon + 1$. We will have this provided $n/t \geq 1/2\varepsilon + 3/2$: we can certainly assume $\varepsilon \leq 1$, so it's enough to ensure that $n/t \geq 2/\varepsilon$, which will also satisfy $t \leq \varepsilon n$, as we needed to get V_0 small.

Thus we may set $t = \lfloor \varepsilon n/2 \rfloor > \varepsilon n/3$, and hence $k \leq 6\varepsilon$, which is fine.

(The fiddling around at the end is only necessary in uninteresting cases where n is fairly small, or ε is not very small. Generally, we can see that we just need the number of parts to be between about $1/\varepsilon$ and $10/\varepsilon$, and if we set $t \simeq \varepsilon n/2$ then we'll get k to be about $4/\varepsilon$, which is fine.)