

# Bundling, Rationing, and Dispersion Strategies in Private and Common Value Auctions\*

Ulf Axelson

SIFR and SSE

E-mail: [ulf.axelson@sifr.org](mailto:ulf.axelson@sifr.org)

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## **Abstract**

I study optimal selling strategies by a multi-product seller who is confined to using a standard auction format but has leeway in bundling products and choosing aggregate and individual quantities offered to buyers. I show how decisions of bundling, rationing, and dispersing the allocation depend on whether the product is of private or common value, how high demand is relative to supply, and what auction mechanism the seller uses.

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\*Preliminary and incomplete.

# 1. Introduction

This paper studies packaging and rationing choices faced by a multi-product, multi-unit seller when buyers have private information about tastes. In this environment, a profit maximizing seller has an incentive to design the selling mechanism in such a way as to extract as much surplus as possible from buyers. There are (at least) three instruments at the sellers disposal to accomplish this:

- *Bundling*: A multi-product seller has a choice between selling each product separately or tying the purchase of one product to the purchase of another. Examples of bundling in product markets abound - a notorious example in later years being the strategy of Microsoft to bundle software with their sale of operating systems. Another example, studied by Stigler [18], is the practice of movie theaters to bundle ticket sales for consecutive shows (“block booking”). In financial markets, the practice of pooling assets and selling a single security with claims to all of them is commonplace for loans and stocks.
- *Rationing*: The seller has two ways of restricting the aggregate quantity sold. One way is to withhold some of the quantity from the market. Another way is to set a reserve price below which no purchase can be made. In the monopoly pricing literature, this is the classical way for a monopolist to increase profits at the expense of efficiency.
- *Dispersion*: On top of deciding whether differentiated products should be bundled or not, the seller also has to decide what quantity of each product a bundle should contain. Should products be sold unit by unit, or in larger packages? I call this a *dispersion* choice because within an auction context it involves deciding whether to spread out the quantity sold over a larger set of winning bidders or whether to keep the allocation concentrated. In consumer markets, the choice of package size is an important marketing variable. In financial markets, the practice of dispersion is widespread in, for example, the allocation of initial public offerings, where bidders often get rewarded less than their demanded quantity even though they bid a higher price than the market clearing price.

I argue that regardless of the bargaining or market power of the seller, he will typically have a lot of leeway in using these three instruments (except possibly for setting a reserve price). Of course, if the seller has complete latitude over the choice of selling mechanism, these instruments will all be embedded in a larger mechanism design choice which also

involves constructing the optimal auction format. I argue, however, that it is often reasonable to assume that the seller may be constrained to using a standard auction when selling his products. There are several reasons for this. First, the format may be predetermined by the market and is not easily adaptable to every single seller. Second, designing a complicated novel mechanism can introduce a lot of confusion on the part of buyers. Third, the seller may not be completely sure how to design the mechanism - it may depend in an intricate way on assumptions about the distribution of buyers and may not be robust to small changes in the market environment. For all of these reasons, I pay special attention to three standard auction formats: Discriminatory price auctions, in which bidders submit sealed bids and winning bidders pay their bids, uniform price auction, in which winning bidders all pay the bid of the pivotal bidder, and ascending price auctions, in which bidders drop out as the price rises until demand equals supply. I also discuss mechanisms that are more attractive to the seller (when they exist), and study how bundling, rationing, and dispersion strategies vary with the auction format.

A further goal of the paper is to characterize how these strategies vary with the product characteristics. In particular, I distinguish between private value products such as consumer goods where the valuation of a buyer is not affected by the valuation of others, and common value products such as financial assets where everyone has the same intrinsic value of consumption but have differing information about the true value of the asset.

The set-up of the paper is as follows: A single risk-neutral seller has  $I$  independent products to sell and a fixed quantity  $\bar{Q}$  of each. There are  $N$  risk-neutral bidders, each of whom gets a signal about each product. The valuation contains a private and a common value component, and I assume that valuations across products are independent (thus, there are no complementarities between products). I also assume that buyers have a constant marginal value of consumption up to a quantity threshold normalized at one, after which they have zero marginal value of consumption.

The results of the paper can be summarized as follows. First, I study the independent private value case. With independent private values all the standard auction formats are revenue equivalent, and are often optimal mechanisms. I first show that the seller wants to concentrate the allocation as much as possible when values are private; that is, zero dispersion is optimal. Thus, winning bidders get allocated a full unit. This is natural, since the highest bidders are the ones with the highest marginal valuation of the products.

I also reaffirm the standard monopoly pricing result that some rationing is optimal. The seller always wants to set a non-zero reserve price to increase overall payments, even at the

expense of not always selling the full quantity. I show that setting an optimal reserve price is a better instrument for rationing than withholding quantity from the market ex ante. Indeed, once the optimal reserve price is set, the seller does not withhold any quantity as long as bids are above the reserve price. On the other hand, if the seller cannot set a reserve price, withholding some quantity ex ante is typically optimal.

I go on to show that the choice of whether to bundle products or sell them separately depends critically on the ratio of supply to “effective” demand, where effective demand is the expected number of bidders with a positive valuation. When the supply is small relative to demand, separate sales are typically optimal, while bundling is typically optimal when the supply is large relative to demand (cf Palfrey [16] who developed similar results for the case of one unit). The intuition is that when the supply is small relative to demand, the pivotal price setting bidder will likely be closer to the top of the distribution, while when supply is large he will likely be closer to the bottom. Therefore, since bundling will tend to push valuations towards the center of the distribution, it is beneficial in the second case but not in the first.

I also show how efficiency relates to the optimal choices of the seller. The choice of optimal concentration is efficient, while rationing and bundling introduces inefficiencies. A reserve price excludes some bidders with positive valuations, and bundling may cause a bidder with a high valuation for one product to be excluded if he has a sufficiently low valuation for the other product. Interestingly, bundling tends to reduce the first inefficiency by lowering the optimal reserve price, and so there is no obvious efficiency ranking between bundled and separate sales. I show that when the number of products is sufficiently large and the supply is sufficiently large, bundled sales may be both optimal and efficient relative to separate sales. In fact, in the limit as the number of products grows large and the supply is as big as the effective demand, maximal bundling will both extract the total consumer surplus and be socially efficient. On the other hand, if supply is sufficiently small, separate sales are optimal and efficient. In the middle region, there are typically situations where the optimal bundling strategy is inefficient.

I next study correlated private values. Most of the results for independent private values goes through in this case. Maximal concentration is still optimal, as is setting a non-zero reserve price. However, the reserve price is now a less efficient instrument for rationing, and I show that it can be beneficial to restrict quantity ex ante even when the optimal reserve price is set. The intuition for this is that with correlated values, bidders do not expect to face the same distribution of competitors independent of their own signal. Thus, ideally a reserve

price should not be fixed, but should change with the conditional distribution of bidders - when all values are relatively low, the reserve should be lower than otherwise. Decreasing the aggregate quantity supplied approximates this type of conditional rationing. By reducing the supply, the pivotal bidder's valuation is increased, but only relative to the conditional distribution of valuations.

The bundling decision will again depend in the same way on supply relative to demand, but will now be affected by which auction format is being used. With correlated private values, revenue equivalence still holds between uniform and ascending price auctions, but as was established in Milgrom and Weber [14], discriminatory price auctions perform worse. Since bundling tends to decrease valuation differences between bidders, it also decreases the revenue differences between auction formats. Therefore, bundling will typically be a substitute for choosing a better auction format, and I show that if the seller is forced to use discriminatory price auctions, he will be more tempted to bundle. On the other extreme, if the seller could utilize the optimal mechanism design, bundling would never occur. With correlated values the seller can typically extract all surplus in a separate sale if he uses the optimal mechanism, as has been shown by Myerson [15], Cremer and McLean [4], and McAfee McMillan and Reny [10]. Thus, there is no room for either bundling, rationing, or dispersion in this case (indeed, any of those strategies would reduce revenues).

For common value products, much of the analysis above changes dramatically. With common values, all the standard auction formats can be revenue ranked, with ascending price auctions being (weakly) superior to uniform price auction which in turn are (weakly) superior to discriminatory price auctions. Selling strategies will differ between formats, especially between the discriminatory price auction and the other two formats. I first show that maximal concentration is typically optimal in discriminatory price auctions, but may or may not be in uniform and ascending price auctions. In the private value case, concentration was optimal because higher bidders had higher marginal values. With common values, a higher bidder has a higher *signal*, but everyone has the same valuation function ex post (when all information is revealed). It is therefore not too surprising that dispersion can be optimal. In a uniform price auction, the price is set by the pivotal bidder, who takes into account the information that he is pivotal when formulating his bid. When the allocation is concentrated, the pivotal bidder tends to have a higher signal than most of his competitors, which increases the winner's curse and hence decreases his bid. By dispersing the concentration, the pivotal bidder will tend to have a lower signal, but the winner's curse will also diminish. Which of these effects is stronger is not clear, and I show circumstances in which either can dominate.

In the discriminatory price auction, however, bidders use strategic bid-shading, which reduces revenues to the seller. This strategic behavior is exacerbated when the allocation is dispersed, as bidders do not have to worry as much about not winning. This explains why maximal concentration is typically optimal in discriminatory price auctions.

Rationing strategies also differ in common value auctions. I show that zero rationing (setting a zero reserve price and offering the whole quantity) is often optimal in uniform and ascending price auction, especially as the auction grows large. As the auction grows large (in that the number of bidders and the quantity both grow large), the underpricing in uniform and ascending price auction goes to zero. There is therefore no need for restricting the quantity sold. On the other hand, in discriminatory price auctions the underpricing does *not* go to zero as the auction grows large because of the strategic bid-shading discussed above. Therefore, a non-zero reserve price and restricted quantity ex ante is typically optimal.

The choice of bundling decision will also be determined by different driving forces than in private value auctions. I first show that, regardless of the size of supply relative to demand, bundling is optimal in all auction formats if the seller has enough products to bundle. The intuition is simple: Bidders' information about the average quality in a large bundle tends to converge to the mean, so that there is no private information about the mean in the limit. Therefore, the underpricing per unit goes to zero as the number of products goes to infinity. On the other hand, I show that if we think of holding the number of products fixed, and let the auction grow large, separate sales become optimal in ascending and uniform price auctions under some distributional assumptions. As the auction grows large, the underpricing goes to zero regardless of the bundling strategy, but the convergence rate is faster for separate sales. This is because bidders tend to have less complex information about an individual asset, and the informational distance between the pivotal losing bidder and the winning bidders therefore tends to disappear quicker. For discriminatory price auctions, a similar result holds if the allocation is very concentrated, because only then does the underpricing go to zero as the auction grows large. When the supply is bigger, bundling is typically more attractive in discriminatory price auctions even as the auction grows large.

The last point again illustrates the fact that bundling tends to be a substitute for an inferior auction mechanism. However, I show that this is not a general theorem. In fact, there are situations when bundling is optimal in ascending price auctions when separate sales are optimal in uniform price auctions, even though the ascending price auction is superior.

Finally, I study the situation when products have both private and common value components. Using a valuation function that is additively separable in the private and common

value components, I show that each component can be analyzed separately as in the discussion above, and that the optimal choice of strategy reduces to choosing the strategy that maximizes the sum of the two components. The weight on each part will depend on the auction format being used and how large the auction is. In particular, since the underpricing per unit sold goes to zero on the common value part as the auction grows large if the ascending or the uniform price auction is used, the bundling and dispersion decisions will be completely determined by considering the private value component. On the other hand, rationing will decrease the revenue raised on the common value part even in large auctions, and so there will be less of a tendency for rationing than if the products were pure private value. Since the discriminatory price auction does not feature zero underpricing on the common value part unless the allocation is extremely concentrated, common value considerations will be more important in discriminatory price auctions.

### **Literature Review (\*Incomplete\*)**

Many aspects of this problem have already been studied. It should be stated up front that especially the results for independent private values have close parallels in existing literature. A stylized categorization of the results can be done as follows:

*Private values, single unit, single product:* Myerson [15] studied the mechanism design problem showing that first price, second price, and english auctions are all revenue equivalent and optimal, under some conditions. For affiliated private values, Myerson [15] and later Cremer and McLean [4] also showed that all surplus could in fact be extracted.

*Private values, multiple unit, single product:* This problem has been studied by Harris and Raviv [6] and Maskin and Riley [9]. For unit demand, revenue equivalence still holds, and ascending, uniform, and discriminatory price auctions are still optimal (with the correct reserve price). With downward sloping demand curves, these auctions are not optimal. Rather, a non-linear posted pricing schedule is optimal. Even in the competitive limit, the monopolist will produce too little. Also, Maskin and Riley show that for the consumer with top tastes, marginal utility is set equal to marginal cost, and for all other types, marginal utility is higher than marginal cost. In the regular case, if the hazard rate is nonincreasing, then marginal utility is decreasing in type, showing that relatively lower types consume too little - there is concentration to the top. That is, the allocation is concentrated relative to first best. Note that this is not necessarily the case, however it is the case for the highest type and for the parametrization we consider in this paper. If types are independent but utilities depend on all other types, then the same condition hold for a non-decreasing hazard rate. Jackson and Kremer [7] also provide an example of revenue-increasing bundling in

private value auctions when marginal utilities are decreasing.

*Private values, single unit, multiple products:* Palfrey [16] shows, for the case of exogenously given auctions and independent products, that bundling is optimal with two bidders, but not with many bidders. Chakraborty [3] sharpens some of these results to give exact conditions for when there is a critical number of bidders  $N$  above which separate sales are optimal and below which pooling is optimal. However, Armstrong shows that this is no longer true when the optimal mechanism is used. With the optimal mechanism, which allows for endogenous bundling as a function of bids, allowing some probability of bundling is optimal even for two bidders. In this paper, I show that Armstrong’s result is specific to his binary set-up, and that bundling is often optimal with two bidders and one good even when the optimal reserve price is set.

## 2. Model Set-Up

A single risk-neutral seller has  $I$  differentiated products to sell. Each product  $i$  is of quality  $Z_i$ , where  $Z_i$  is a random variable. I assume that qualities are distributed identically and independently according to the distribution function  $G(z)$ . The seller’s supply of each product is  $\bar{Q}$ . There are  $N$  risk neutral bidders. Each bidder  $n$  privately observes a signal  $X_{i,n}$  about each product  $i$ . The value  $V$  that a bidder attaches to consuming a bundle containing  $q_1, \dots, q_I$  units of the  $I$  products is given by

$$V(q_1, \dots, q_I, X_{1,n}, \dots, X_{I,n}, Z_1, \dots, Z_I) = \sum_{i=1}^I \min(q_i, 1) (tX_{i,n} + (1-t)Z_i) \quad (2.1)$$

Thus, marginal utility is constant up to a quantity normalized to one, after which it is zero.  $X_{i,n}$  is the *private value* component and  $Z_i$  the *common value* component of the valuation. The parameter  $t \in [0, 1]$  measures the relative importance of these components, with  $t = 1$  corresponding to the pure private value case and  $t = 0$  corresponding to the pure common value case. I assume that products are neither substitutes nor complements, so that the utility of consuming product  $i$  is independent of the consumption of product  $j$ .

Signals are independent across products, and I assume that conditional on a quality realization  $Z_i = z$ , signals about product  $i$  are distributed identically and independently across bidders according to the distribution function  $F(x|z)$  with density  $f(x|z)$  with support in  $[\underline{x}, \bar{x}]$ . Sellers do not know the realization of any random variables, but bidders can make an inference about  $Z_i$  from observing the signal  $X_{i,n}$ . To be precise, I assume that



the monotone likelihood ratio property holds:

$$\text{MLRP } \frac{f(x|z)}{f(x'|z)} \geq \frac{f(x|z')}{f(x'|z')} \text{ if } x > x', z > z'$$

The MLRP assumption captures the notion that a higher signal is correlated with a higher value, and implies together with the valuation function in 2.1 that values are *affiliated* (see Milgrom and Weber [14]).

I assume that the seller is faced with an exogenously specified auction format when selling his products, but that he has leeway in choosing the quantity he puts on the market and the way products are bundled, both in terms of number of distinct products and in terms of the quantity of each product sold to winning bidders. More specifically, the seller has the following choice variables:

-A partition  $\Pi = \{\Pi_1, \dots, \Pi_J\}$  of the  $I$  assets, such that  $J$  separate auctions are held, one for each bundle of products. I call this a *bundling* strategy.

-A choice of aggregate quantity  $Q_j \leq \bar{Q}$  for each bundle to put on the market and a choice of reserve price  $r_j$  below which no bundle is sold. I call this a *rationing* strategy.

-A choice of individual quantity  $q_j$  allocated to each winning bidder, or, equivalently, a choice of  $K_j = \frac{Q_j}{q_j} \in \{1, \dots, N\}$  of how many bidders are allocated a bundle, subject to the lowest of them paying at least  $r_j$ . I call this a *dispersion* strategy.

Thus, the auction for bundle  $\Pi_j$  will be a multi-unit auction of  $K_j$  identical items, each valued by bidder  $n$  at

$$V_j \left( q_j, \{X_{i,n}\}_{i \in \Pi_j}, \{Z_{i,n}\}_{i \in \Pi_j} \right) = \min(q_j, 1) \left( t \sum_{i \in \Pi_j} X_{i,n} + (1-t) \sum_{i \in \Pi_j} Z_{i,n} \right)$$

I study the following standard auction formats (which, as is discussed in more detail below, are in increasing order of attractiveness to the seller):

1. *Discriminatory price auction (DP)*: Each bidder submits a sealed bid  $b_n$  and bundles are awarded to the  $K$  highest bidders subject to  $b_n \geq r$ . Winning bidders pay their bid.
2. *Uniform price auction (UP)*: Each bidder submits a sealed bid  $b_n$  and bundles are awarded to the  $K$  highest bidders subject to  $b_n \geq r$ . Winning bidders pay the bid of the  $K + 1$ :st highest bidder or  $r$ , whichever is larger.
3. *Ascending price auction (AP)*: The price starts at  $r$  and is increased continuously throughout the auction, and bidders decide at each moment whether to stay in the

auction or drop out (irreversibly). Once all but  $K$  bidders have dropped out, the auction stops and the remaining bidders are rewarded a bundle at the stopping price.

Given an auction format  $A \in \{DP, UP, AP\}$ , denote the revenues in auction  $j$  by  $\pi_{jA}$ . The seller's maximization problem is then given as follows:

$$\max_{\Pi, \{Q_j \leq \bar{Q}, r_j, K_j\}_{j=1}^J} \sum_{j=1}^J E(\pi_{jA} | K_j, Q_j, r_j) \quad (2.2)$$

I will also discuss how the results for the standard auction formats above differ from the solutions where the seller has more power to choose an optimal mechanism. In particular, one can think of two increasingly powerful levels of mechanism design accessible to the seller:

4. *Optimal mechanism given a bundling strategy:* Once the bundling strategy is picked, revenue maximizing auctions are run under the restriction that a winner must be awarded the same quantity of each product in bundle  $\Pi_j$ .
5. *Unrestricted optimal mechanism.*

It is worth pointing out several restrictions imposed on the seller in Problem 2.2 relative to mechanisms 4 and 5. Relative to mechanism 4, the standard auction formats impose a restriction that all winning bidders are awarded the same quantities. This turns out not to be an important restriction in most circumstances (see below), mainly due to the step-like nature of the valuation function. More importantly, there is a severe restriction in how payments and allocations are allowed to depend on the bids of competing bidders. This is unimportant in the case of independent private values, but as has been shown by Myerson [15], Cremer and McLean [4] and McAfee, McMillan and Reny [10] it is typically possible to extract all bidder surplus in an optimal mechanism when bidders have correlated valuations. In that case, allocations will be efficient if separate auctions for each product is run and strategic components such as bundling, rationing, and dispersion have no bite.

Furthermore, relative to mechanism 5 the four other mechanisms have a restriction that either the allocation of product  $i$  to bidder  $n$  is independent of  $X_{i'n}$  (if products  $i$  and  $i'$  are in different bundles), or as dependent on  $X_{i'n}$  as  $X_{in}$  (if they are in the same bundle). This is not necessarily a feature of a revenue-maximizing mechanism. In fact, McAfee, McMillan, and Whinston show in a monopoly pricing set-up that it is typically optimal to have the allocation of  $i$  depend positively on  $X_{i'n}$  but less strongly so than on  $X_{in}$ , even in the case of independent product values and independent private values. The solution to 2.2 will,

however, be qualitatively similar to the solution to the less restricted problem, and will often come quite close to achieving the same level of revenues for the seller.

## 2.1. Equilibria of the auctions

To characterize equilibria for the three standard auction formats in the auction of bundle  $\Pi_j$ , denote by  $X_n$  and  $Z$  the sum of signals and qualities for the bundle, and by  $V(q, X_n, Z)$  the valuation of the bundle:

$$\begin{aligned} X_n &\equiv \sum_{i \in \Pi_j} X_{i,n} \\ Z &\equiv \sum_{i \in \Pi_j} Z_i \\ V(q, X_n, Z) &\equiv \min(q, 1) (tX_n + (1-t)Z) \end{aligned}$$

Equilibria for the three auction formats can then be found without any further restrictions on the signal- and asset structure for the case of private values using the one-dimensional aggregate signals  $X_n$ . In the case of common values ( $t < 1$ ), however, equilibria can be quite difficult to characterize, and a symmetric equilibrium may even fail to exist (see Jackson ). The difficulty stems from the multi-dimensionality of the signals: It is in general not possible to construct a sufficient statistic for the signals  $\{X_{i,n}\}_{i \in \Pi_j}$  that also satisfies MLRP with respect to  $Z$ . To circumvent these problems, I will frequently use the following distributional assumption on signals and qualities when  $t < 1$ :

$$\mathbf{D1} \quad Z_i \in \{0, 1\}, X_{i,n} \in \{0, 1\}, P(X_{i,n} = 1 | Z_i = 1) = p > P(X_{i,n} = 1 | Z_i = 0) = q$$

Many of the results in the paper hold without invoking this assumption.

To allow for mixed strategies when signals are discretely distributed, denote by  $s_n \in [0, 1]$  a random draw from a uniform distribution for each bidder that is independent of all other variables. Also, denote by  $Y_1 \geq Y_2 \geq \dots \geq Y_N$  the  $N$  order statistics of the sample of signals  $\{X_i\}_{i=1}^N$ , and by  $\tilde{Y}_1 \geq \tilde{Y}_2 \geq \dots \geq \tilde{Y}_N$  the order statistics of the ‘‘augmented’’ signals  $\{\{X_i, s_i\}\}_{i=1}^N$  ordered lexicographically so that  $\{X_i, s_i\} > \{X_j, s_j\}$  if  $X_i > X_j$  or if  $X_i = X_j$  and  $s_i > s_j$ . Similarly, define by  $Y_K^{-n}$  and  $\tilde{Y}_K^{-n}$  the  $K$  :th order statistic of the same samples excluding  $\{X_n\}$  and  $\{X_n, s_n\}$  respectively. The goal is now to find a bid-function  $b_A(x, s)$  for auction format  $A$  that describes bidding by an agent with signals  $X_n = x$  and  $s_n = s$  and constitutes a symmetric Bayesian Nash equilibrium. This can be done by a relatively

straight-forward extension of the analysis in Milgrom and Weber [14] to the multi-unit case allowing for discretely distributed signals. Define a function  $v(x, y, s)$  by

$$v(x, y, s) = E \left( V(1, X_n, Z) \mid X_n = x, \tilde{Y}_K^{-n} = \{y, s\} \right)$$

Then the equilibrium bid functions are given in the following proposition.

**Proposition 1.** *Suppose  $t = 1$  or  $\Pi_j$  is a singleton or assumption D1 holds, and the auction has parameters  $\{Q, r, K\}$ . In all auction formats,  $b_A(x, s) < r$  for  $\{x, s\} < \{x^*, s^*\}$  where the screening level  $\{x^*, s^*\}$  is defined as*

$$\{x^*, s^*\} = \inf \left\{ \{x, s\} \mid E \left( V(q, X_n, Z) \mid X_n = x, \tilde{Y}_K^{-n} < \{x, s\} \right) \geq r \right\}$$

The equilibrium bid function for the discriminatory price auction for  $\{x, s\} \geq \{x^*, s^*\}$  is given by

$$b_{DP}(x, s) = rL(x^*, s^* \mid x, s) + \min(q, 1) \int_{\{x^*, s^*\}}^{\{x, s\}} v(\alpha, \alpha, t) dL(\alpha, t \mid x, s)$$

where

$$L(\alpha, t \mid x, s) = \exp \left( - \int_{\{\alpha, t\}}^{\{x, s\}} \frac{f_{\tilde{Y}_K^{-n}}(\{y, u\} \mid X_n = y)}{F_{\tilde{Y}_K^{-n}}(\{y, u\} \mid X_n = y)} d\{y, u\} \right)$$

The equilibrium bid function for the uniform price auction for  $\{x, s\} \geq \{x^*, s^*\}$  is given by

$$b_{UP}(x, s) = \min(q, 1) * v(x, x, s)$$

The equilibrium bid function for the ascending price auction is defined recursively as follows for  $\{x, s\} \geq \{x^*, s^*\}$ : Suppose  $M < N - K$  bidders do not participate at  $r$ . Define  $b^0(x, s \mid M)$  as

$$b^0(x, s \mid M) = E \left( V(q, X_n, Z) \mid X_n = x, \tilde{Y}_K^{-n}, \dots, \tilde{Y}_{N-M-1}^{-n} = \{x, s\}, \tilde{Y}_{N-M}^{-n} < \{x^*, s^*\} \right)$$

If the price reaches  $b^0(x, s \mid M)$  without anyone else dropping out, drop out. If someone drops out at  $p_1 < b_{AP}^0(x, s \mid M)$ , define  $b^1(x, s \mid M, p_1)$  as

$$\begin{aligned} b^1(x, s \mid M, p_1) &= E \left( V(q, X_n, Z) \mid X_n = x, \tilde{Y}_K^{-n}, \dots, \tilde{Y}_{N-M-2}^{-n} = \{x, s\}, \right. \\ &\quad \left. b^0(\tilde{Y}_{N-1-M}^{-n}) = p_1, \tilde{Y}_{N-M}^{-n} < \{x^*, s^*\} \right) \end{aligned}$$

If the price reaches  $b^1(x, s | M, p_1)$  without anyone else dropping out, drop out. Continue similarly until the price reaches

$$b^k(x, s | M, p_1, \dots, p_k) = E \left( V(q, X_n, Z) \mid X_n = x, \tilde{Y}_K^{-n}, \dots, \tilde{Y}_{N-k-M-1}^{-n} = \{x, s\}, \right. \\ \left. b^{k-1}(\tilde{Y}_{N-k-M}^{-n}) = p_k, \dots, b^0(\tilde{Y}_{N-1-M}^{-n}) = p_1 \right)$$

where  $k$  more bidders have dropped out at prices  $p_k \geq p_{k-1} \geq \dots \geq p_1$  and drop out when the price reaches  $b^k(x, s | M, p_1, \dots, p_k)$  if noone else has dropped out.

Proof. In Appendix.

Note that because the value function is additively separable in the private and common value component, so is the bid function for all auction formats. Therefore, we can think of the auction as two separate auction, one with a pure private value component and one with a pure common value component, as long as we make sure that the screening level introduced by the reserve price is the same as in the composite auction. This is the approach taken in sections 3, 4, and 5 below. Section 6 then revisits the mixed case.

### 3. Independent Private Values

Suppose  $t = 1$  and values are independently distributed. Proposition 1 shows that bids in the uniform and ascending price auction are identical on the private value component, but differ on the common values component, while the discriminatory price auction differs on both. Thus, uniform and ascending price auctions raise exactly the same revenues when values are private. In fact, as has been established for similar cases elsewhere, with independent private values the discriminatory price auction raises the same expected revenues as well.

**Lemma 1.** *Given  $\{\Pi_j, \{Q, r\}, q\}$ , all three auction formats are revenue equivalent.*

Proof. Omitted. Maskin and Riley [9] have proved revenue equivalence for the continuous case. It is relatively straight-forward to extend the proof to the discrete case.<sup>1</sup> ■

It follows from Lemma 1 that the choice of bundling, rationing, and dispersion strategies are not dependent on the particular auction format. I start by analyzing the dispersion strategy and show that maximal concentration is always optimal in standard private value auctions.

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<sup>1</sup>However, it should be noted that revenue equivalence only holds in the discrete case if bids over the whole interval  $[\underline{X}, \overline{X}]$  are allowed. When values are discrete, it is typically optimal for the seller not to allow bids over certain intervals (see Maskin and Riley () and Armstrong ()).

**Proposition 2.**  $q^* = 1$  for all auction formats.

Proof. Clearly, we can rule out  $q^* > 1$ , since we can always reduce  $Q$  instead of allocating extra quantity to bidders with zero marginal valuation. Given revenue equivalence, we can confine study to the uniform price auction. From Proposition 1, the price per unit sold will be  $\max(Y_{K+1}, r)$  and the quantity sold will be  $q \sum_{i=1}^K 1_{Y_i \geq r} = Q \frac{\sum_{i=1}^K 1_{Y_i \geq r}}{K}$ . Suppose  $q < 1$ , and suppose you increase quantity to  $q' = \frac{Q}{K-1}$ . Then, the price per unit sold will be  $\max(Y_K, r) \geq \max(Y_{K+1}, r)$  and the quantity sold will be  $Q \frac{\sum_{i=1}^{K-1} 1_{Y_{i,N} \geq r}}{K-1} \geq Q \frac{\sum_{i=1}^K 1_{Y_{i,N} \geq r}}{K}$ . Hence, expected revenue is increased. ■

Following Myerson [15], define the “virtual utility”  $J(x)$  as

$$J(x) \equiv x - \frac{1 - F(x)}{f(x)}$$

A regularity assumption that I will sometimes invoke is that  $J(x)$  is increasing:

**Regularity**  $F(x)$  is regular if the virtual utility  $J(x)$  is increasing in  $x$ .

This is a standard assumption in the mechanism design literature (see Myerson [15] and Maskin and Riley [9]).

Define the set  $J^0$  as follows:

$$J^0 = \begin{cases} J^{-1}(0) & J^{-1}(0) \neq \emptyset \\ \{0\} & J^{-1}(0) = \emptyset \end{cases}$$

That is,  $J^0$  contains the solutions to the equation  $J^{-1}(0) = 0$  or  $\{0\}$  if no such solution exists. The following proposition derives the optimal reserve price  $r^*$  and shows that when there is a chance of bidders’ valuation being below the valuation of the seller, a reserve price strictly higher than the valuation of the seller is optimal.

**Proposition 3.**  $r^* \in J^0$ . If  $\underline{x} \leq 0$ , there exists  $\varepsilon > 0$  such that  $r^* > \varepsilon$  for all  $Q, N$ .

Proof. Following Maskin and Riley, one can express the expected revenues (with  $q = 1$ ) as

$$E(\pi) = E \left( \sum_{n=1}^Q 1_{Y_n \geq r} J(Y_n) \right) \quad (3.1)$$

Clearly, if  $J(x)$  is positive for all  $x$ , setting  $r$  to zero is optimal. If  $J(x)$  is negative somewhere,  $r$  should be set at a point where  $J$  crosses 0 from below. Such a point exists, since  $J(\bar{x}) = \bar{x} > 0$ . This proves the first part of the proposition. The second part follows from the

fact that  $J(x)$  is negative at  $x \leq 0$ , so that the smallest solution to  $J(x) = 0$  is strictly positive. ■

The following proposition shows that under very mild assumptions, it is always optimal to sell the full quantity  $\bar{Q}$  if the seller can set the optimal reserve price. Thus, rationing through the reserve price is always more efficient than rationing through the quantity put on the market. However, if the seller does not have enough power to credibly set a reserve price, decreasing the aggregate quantity can serve as a substitute.

**Proposition 4.** *If  $J(x) = 0$  has at most one solution,  $Q^* = \bar{Q}$  if the optimal reserve price is set. If the seller is forced to set  $r = 0$ , there exists situations such that  $Q^* < \bar{Q}$  even when  $\underline{x} = 0$  and  $\bar{Q} < N$ .*

*Proof.* If  $J(x) = 0$  has at most one solution, it is positive above  $r^*$ . Hence, Expression 3.1 is strictly increasing in  $Q$ . Now suppose the seller is forced to set  $r = 0$ . Then, if  $\frac{\bar{Q}}{N} \rightarrow 1$  as  $N \rightarrow \infty$ , Expression 3.1 will be decreasing in  $Q$  at  $\bar{Q}$  as  $N \rightarrow \infty$ , since  $J(0) < 0$ . ■

I end the analysis of dispersion and rationing strategies by relating the standard auction formats to the optimal mechanism given a bundling decision (format 4 above). As has been established in Maskin and Riley [9], regularity actually implies that the standard auction formats are optimal under the optimal rationing and dispersion strategies.

**Proposition 5.** *(Maskin and Riley) If  $F(x)$  is regular, all auction formats with  $q^* = 1, r^* = J^0, Q^* = \bar{Q}$  are optimal mechanisms.*

For non-regular distributions, Maskin and Riley show that the standard auction formats perform worse than an ascending price auction with appropriately chosen discrete jumps in the price increases, in which bidders who drop out just before a jump point get to share any allocation that is left after distributing to remaining bidders. This will in effect create a type of dispersion, since  $0 < q < 1$  for a range of bidder valuations. This should be contrasted with the result in 2, which states that for the standard auction formats, dispersion is never valuable.

I now turn to analyzing the bundling choice  $\Pi$ . In light of Proposition 2, we can take  $q$  given as one. The following two propositions show that the bundling decision will depend on the ratio of demand to supply. The smaller the supply  $Q$  is relative to the number of bidders, the more beneficial separate sales will be, and vice versa.

**Proposition 6.** *Suppose the auction is regular and that  $\underline{x} \geq 0$ . Then, there exists a  $\beta < 1$  such that for all  $K, N$  there exists  $I$  such that if  $\frac{K}{N} \geq \beta$ , bundling is optimal.*

Proof. I start by establishing the following Lemma:

**Lemma 2.** *Suppose the auction is regular. Then, given  $K, N$  expected revenue per unit  $\pi(K, N)$  is bounded by*

$$E(Y_{K+1}) \leq \pi(K, N) \leq \begin{cases} \frac{1-F(r)}{K/N} r & \frac{K}{N} \geq 1 - F(r) \\ (1-F)^{-1}(K/N) & \frac{K}{N} < 1 - F(r) \end{cases}$$

Proof.  $E(Y_{K+1})$  is the expected profit in a uniform price auction without reserve, and hence serves as a lower bound for expected revenues. Now compare  $\pi(aK, aN)$  to  $\pi(K, N)$  for  $a$  some integer larger than one. Then,  $\pi(aK, aN) \geq \pi(K, N)$ , since running  $a$  auctions identical to the  $K, N$  auction yields  $\frac{a\pi(K, N)}{a} = \pi(K, N)$  in expected revenues per unit. As  $a$  goes to infinity, revenue in the optimal auction is given as in the right side of the Lemma.

■

It is now enough to show that  $\pi(K, N)$  is below the mean for  $\frac{K}{N} \geq \beta$  for some  $\beta$ . Suppose  $\beta = 1$ . Then,

$$\begin{aligned} \pi(K, N) &\leq (1 - F(r)) r \\ &= \int_r^{\bar{x}} r f(x) dx \\ &< \int_{\underline{x}}^{\bar{x}} x f(x) dx \end{aligned}$$

Thus, there exists a  $\beta < 1$  such that the statement holds. ■

**Proposition 7.** *Suppose  $Q$  is given. Then there exists an  $M$  such that for all  $I, \underline{x}, N \geq M$ , separate sales dominate bundling.*

Proof. <<Extending Palfrey>>

Remark: These Propositions extend results in Palfrey [16] to the case of multiple units and optimal reserve prices. He show that pooling is always optimal with two bidders when there is no reserve. This result is modified when the optimal reserve price is taken into account. Armstrong [1], for example, has shown that separate sales are always optimal for a binary distribution when  $K = 1$ , even when  $N = 2$ . However, even with an optimal reserve price, it is often the case that bundling dominates separate sales when  $N = 2$ . For example, for one can sharpen the for symmetric unimodal distributions, .



One can sharpen the propositions by making stronger distributional assumptions. For example, if  $f(x)$  is symmetric and unimodal and  $\underline{x} \geq 0$ , one can show that bundling dominates separate sales if and only if  $\frac{K+.5}{N} > .5$  for any  $I$ .

The intuition behind propositions 6 and 7 is relatively straightforward. When the supply is large relative to demand, the pivotal price-setting bidder tends to have a relatively low valuation. Therefore, bundling independent products helps by pulling all valuations closer to the mean. The opposite is true when the supply is small relative to demand, since the pivotal bidder then tends to have an above average valuation.

The following proposition builds on the same intuition by showing that the lower the support of the valuation function, the more likely it is that separate sales are optimal when the optimal reserve price is set. If the optimal reserve price is close to the top of the support, the effective supply relative to demand is low regardless of the actual supply.

**Proposition 8.** *Suppose  $N, Q$  are given, and the support of  $f(x)$  is  $[\underline{x}, \underline{x} + b]$  with  $f(x) > \varepsilon > 0$  for all  $x \in [\underline{x}, \underline{x} + b]$ . Then there exists an  $a > -b$  such that separate sales are optimal for all  $I$  and all  $\underline{x}$  less than  $a$ .*

Proof. Straight-forward. ■

In Palfrey's analysis, separate sales are always more efficient than bundling, even though the seller only prefers separate sales for a large enough number of bidders. This depends crucially on the fact that there is no reserve price. When there is no reserve price (and no other type of rationing or dispersion), separate sales are efficient as the products are always allocated to the highest value users. However, when the optimal reserve price is set, there is an additional inefficiency associated with separate sales in that the screening level is typically higher than for bundled sales. This inefficiency can be so high that bundling is more efficient, especially when the number of distinct products in the bundle is large. The following proposition shows that when the supply is large relative to demand and the number of distinct products is large enough, bundling is typically both optimal and efficient relative to separate sales. The opposite holds when supply is small relative to demand. For intermediate cases, it can well be that the optimal bundling strategy is inefficient relative to some other available bundling strategy.

**Proposition 9.** *Suppose  $\underline{x} = 0$  and the optimal reserve is set. Then there exists  $\underline{\beta}, \bar{\beta}, \underline{I}$  satisfying  $0 < \underline{\beta} < \bar{\beta} < 1$  such that bundling is optimal and creates a larger consumer surplus than separate sales if  $\frac{K}{N} > \bar{\beta}$  and  $I > \underline{I}$ , and separate sales are optimal and create a larger consumer surplus than bundling if  $\frac{K}{N} < \underline{\beta}$ . In general, there are situations when*

*bundling is optimal but inefficient relative to separate sales, and separate sales are optimal but inefficient relative to bundling.*

#### 4. Affiliated Private Values

With affiliated private values, revenue equivalence still holds between uniform and ascending price auctions, as can be seen by inspecting the bid functions in Proposition 1. However, the discriminatory price auction typically performs worse than the uniform and ascending price auctions, as is established in the following proposition.

**Lemma 3.** *The ascending and uniform price auctions are revenue equivalent given a fixed reserve price  $r$ , and are superior to the first price auction.*

Proof. Bids in the uniform price auction are equal to the valuation, while in the ascending price auction it is optimal to stay in the auction up until your valuation. With a continuous bid support, this will lead to equal revenues. That the discriminatory price auction is worse is shown in Milgrom and Weber for the unit supply case, but can easily be extended to the multi-unit supply case. ■

Just as in the case for independent private values, maximal concentration is optimal for all standard auction formats.

**Proposition 10.**  *$q^* = 1$  is optimal in all auction formats.*

Proof. Same as before for ascending and uniform price auctions. <<To be finished>> .

■

Also, just like in the independent case, there will always be some rationing through a reserve price if the lower end of the support is zero. However, a fixed reserve price is no longer a powerful enough rationing instrument to assure  $Q = \bar{Q}$ . The intuition for this is that with affiliated values, bidders do not expect to face the same distribution of competitors independent of their own signal. Thus, ideally a reserve price should not be fixed, but should change with the conditional distribution of bidders - when all values are relatively low, the reserve should be lower than otherwise. Decreasing the aggregate quantity supplied approximates this type of conditional rationing. By reducing the supply, the pivotal bidder's valuation is increased, but only relative to the conditional distribution of valuations.

**Proposition 11.** *If  $\underline{x} \leq 0$ , there exists  $\varepsilon > 0$  such that  $r^* > \varepsilon$  for all  $Q, N$ . There exists situations in which  $Q < \bar{Q}$  even when the optimal reserve price is set.*

Equivalents of Propositions 6, 7, 8, and 9 can be established relatively easily for the affiliated case. A major difference is that the bundling decision will be different for different auction formats. In particular, we can show that bundling will often be optimal in discriminatory price auctions when it is not in uniform or ascending price auctions.

**Proposition 12.** *As  $I$  goes to infinity, bundling is always optimal in discriminatory price auctions if it is optimal in uniform and ascending price auctions.*

**Proof.** In the pooled auction, revenues per variety converge to the average valuation in the different auction formats. There is a cut off level  $N^*$  above which separate sales are better for uniform price auctions. Since for separate sales, discriminatory price auctions generate strictly lower revenues, pooling should be at least as valuable for discriminatory price auctions. ■

It is also worth discussing the relation between the standard auction formats and more powerful mechanisms as in 4 and 5 above. In fact, as was shown in Myerson [15] and Cremer and McLean [4], it is possible to extract all bidder surplus with the optimal mechanism. Thus, rationing and dispersion is inconsequential. Separate sales together with one of these mechanisms is always optimal. Thus, there is a negative correlation between the amount of bundling and the attractiveness of the mechanism to the seller. Proposition 12 can be strengthened to state that if maximal bundling is optimal in a certain auction format when  $I$  is big enough, then it is also optimal in all auction formats that are revenue inferior.

## 5. Pure Common Values

With pure common values, ascending price auctions are revenue superior to uniform price auctions, and both are revenue superior to discriminatory price auctions.

**Lemma 4.** *With pure common values, the auction formats are ranked as follows:  $E(\pi_{AP}) \geq E(\pi_{UP}) \geq E(\pi_{DP})$*

**Proof.** Extension of Milgrom and Weber. ■

A special case of this revenue ranking is when the auction grows large in the sense that  $\bar{Q}, N \rightarrow \infty$ . Then, it can be shown that the underpricing per product disappears in uniform and ascending price auctions, but not in discriminatory price auctions unless the allocation is highly concentrated. (Compare Theorem 5 of Jackson and Kremer [7] for a similar result.)

**Lemma 5.** *As  $K, N \rightarrow \infty$ ,  $\frac{K}{N} \rightarrow \beta < 1$ , underpricing per unit goes to zero in uniform and ascending price auctions. In discriminatory price auctions, the underpricing per unit goes to zero if and only if the probability of bidders with the highest signal getting an allocation is less than one. Otherwise, the underpricing per unit is strictly bounded from zero and is increasing in  $\beta$ .*

**Proof.** (Sketch) Suppose the equilibrium is strictly increasing. Suppose  $\frac{K}{N} \rightarrow \beta$  with  $1 - F(H - 1 | 0) > \beta$ . Then, there will always be some bidders with the highest signal that do not get any allocation, regardless of the realization of  $z$ . These bidders will get exactly zero revenue. But from indifference over the mixing region, this implies that they get zero revenue whatever they bid. The aggregate profits must be zero as well.

Suppose  $1 - F(H - 1 | 0) < \beta$  but  $1 - F(H - 1 | 1) > \beta$ , so that it is possible that all bidders with the highest signal get an allocation, but it only happens when the asset is worthless. Then, there is a positive measure of bidders with the highest signal who expect to get zero surplus; hence, all of them expect to get zero surplus.

Now suppose that all bidders with the highest signal always get the asset; and that some fraction of bidders who do not get the highest signal also always get the asset. There is a price  $k$  above which you can be assured of getting the asset. It has to be the case that

$$k < E(z|x)$$

where  $x < H - 1$ . In the case of two signals, therefore,  $k < E(z|x = 0)$ . This implies that there will be underpricing.

In the case of more signals, proceed as follows: No clientele can get a negative expected gain. Set the expected gain to the bidders with the second highest signal to zero. Suppose they always get the asset; then, they should pay  $E(z|x)$ . But then, the highest bidders will gain, because they can pay the same.

With continuous signal, we would get the result that for any allocation which makes bidders below the upper bound get the asset with positive probability, then underpricing per asset will not go to zero. It remains to show what happens when all high signal bidders do not get an allocation at the highest  $z$ , but do at some  $z > 0$ . ■

**Corollary 1.** *Suppose  $N \rightarrow \infty$ ,  $\frac{Q}{N} \rightarrow \beta < 1$ . Then,  $q^* \rightarrow 1$  in discriminatory price auctions.*

As the corollary suggests, maximal concentration is typically optimal in discriminatory price auctions.

**Conjecture 1.**  $q^* = 1$  in discriminatory price auctions

However, for uniform and ascending price auctions, dispersion is often optimal. The following proposition delineates the optimal dispersion strategies as the auction grows large.

**Proposition 13.** *There exists situations in which  $q^* < 1$  for uniform and ascending price auctions. Suppose signals  $X$  and quality  $Z$  are discretely distributed, and that  $\beta < P(X = \bar{x} | Z = \underline{z})$ . Then, if  $F(x|z) - F(x|z') > \varepsilon > 0 \forall x, z, z \neq z'$ , there exists  $a > 0$  such that for all  $P(X = \bar{x} | Z = \underline{z}) \leq a$ , we have  $q^* \rightarrow C < 1$  in uniform price auctions as  $N \rightarrow \infty$ ,  $\frac{Q}{N} \rightarrow \beta$ .*

Proof. In Appendix.

**Proposition 14.** *Suppose signals  $X$  are discretely distributed and quality  $Z$  is continuously distributed, and that  $\beta < P(X = \bar{x} | Z = \underline{z})$ . Then,  $q^* \rightarrow 1$  in uniform price auctions as  $N \rightarrow \infty$ ,  $\frac{Q}{N} \rightarrow \beta$ .*

Proof. In Appendix.

I now study the rationing strategy  $\{Q, r\}$ .

**Proposition 15.** *With uniform and ascending price auctions,  $r \rightarrow 0$  and  $Q \rightarrow \bar{Q}$  as  $N \rightarrow \infty$ ,  $\frac{\bar{Q}}{N} \rightarrow \beta < 1$ . In discriminatory price auctions, there exists situations where  $r > 0$  and  $Q < \bar{Q}$  even as  $N \rightarrow \infty$ .*

Proof. Since underpricing per unit goes to zero as  $N \rightarrow \infty$ , it is strictly better to sell more (as long as the value is above zero), so  $Q \rightarrow \bar{Q}$ ,  $r \rightarrow 0$ . <<To be finished>> ■

I now turn to the bundling decision.

I first show the following: In the pure common values case, and for a finite fixed number of bidders and units of each asset, bundling will always be optimal if the seller has access to enough different products. This result holds under very general specifications of the underlying signal distribution and the correlation structure across products.

**Proposition 16.** *Holding  $N$  and  $K$  fixed, bundling is optimal in all auction formats as  $I$  goes to infinity.*

**Proof.** The proposition is almost trivial once we note that revenue is homogeneous of degree 1 in  $Z$ . From homogeneity, we can then look at an auction of  $w_I \equiv \frac{\sum_{i=1}^I z_i}{I}$  to get the expected revenue per asset. Since  $w_I \rightarrow_{a.s.} c$ , the auction is virtually identical to an auction under certainty, where the only possible symmetric equilibrium is for everyone to bid  $c$  in all auction

forms. Formally, the proposition follows immediately from the equilibrium bidding functions in Proposition 16 and from the fact that  $E(w_I | x, y_1 = x) \rightarrow_{a.s.} c$  for a set  $x$  of measure one. However, not surprisingly, the proposition holds under much more general conditions than the ones necessary for deriving the equilibrium bidding functions. In particular, we do not need any of the distributional assumptions on values and signals that are necessary for aggregation as long as a symmetric equilibrium exists. To illustrate this, we prove the proposition for second price auctions under no other assumption than that of existence of a symmetric equilibrium  $b(x)$ . It is enough to show that there exists an  $\underline{I}$  such that for all  $I > \underline{I}$ ,  $b(x)$  is above  $c - \varepsilon$  with probability  $1 - \delta$  for arbitrary  $\varepsilon, \delta > 0$ . Suppose this was not true so that for each  $\underline{I}$ , there exists an  $I > \underline{I}$  such that  $b(x) < c - \varepsilon$  for a set  $x$  of measure at least  $\delta^2$ . Then, any bidder can increase his expected profits by bidding  $\widehat{b}(x) = c - \varepsilon$  for all  $x$  such that  $b(x) < c - \varepsilon$ . By doing this, he now wins the auction on a set of positive measure where he didn't win before, and he wins at a price lower than or equal to  $c - \varepsilon$ . His conditional expectation of the value of the security on this set goes to  $c$  as  $I$  goes to infinity, except for a set of measure 0. Since losses are bounded above by  $c - \varepsilon$  on this set of measure 0, this change in strategy gives strictly higher expected profits. Proofs for uniform and ascending price auctions are almost identical. ■

The intuition for this result is that as the bundle grows large, bidders' information about the average quality in the bundle tend to converge, so that they cannot utilize their informational rents. More surprising, the following proposition shows that if we let the degree of competition grow while holding the number of products fixed, separate sales are optimal *even if supply is large relative to demand*.

**Proposition 17.** *Suppose assumption D1 holds. Then, holding  $I$  fixed, separate sales become optimal in uniform and ascending price auctions as  $N \rightarrow \infty$ ,  $K/N \rightarrow \beta$  for  $\beta < 1$ .*

**Proof.** In Appendix.

The result in Proposition 17 is exclusive to uniform and ascending price auctions. The following Proposition shows that if the bundle can be made large enough, bundling is optimal in discriminatory auctions as long as the supply is not picked up exclusively by bidders with the highest signal.

**Proposition 18.** *There exists  $\underline{I}$  such that for all  $N, K$  such that  $\frac{K}{N} > P(X = \bar{x})$ , bundling is optimal in the discriminatory price auction for all  $I \geq \underline{I}$ .*

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<sup>2</sup>The measure of the set is given by  $\sum_{x: b(x) < c - \varepsilon} \sum_{\mathbf{z}} P(\mathbf{x} | \mathbf{z}) P(\mathbf{z})$ .

**Proof.** Since the underpricing does not go to zero with  $N$  for discriminatory price auctions, the existence of an  $\underline{I}$  independent of  $N$  can be established in the same way as in the proof of Proposition 16. ■

Propositions 16, 17 and 18 should be contrasted with the private value case in Propositions 6 and 7. For common values, bundling can be optimal regardless of the ratio of supply to demand, as can separate sales. In the case of uniform and ascending price auction, it is the degree of diversification relative to how large the auction is that decides whether bundling or separate sales are optimal. In the discriminatory price auction, it is the degree of diversification relative to the degree of competition that is the deciding factor.

**Proposition 19.** *Suppose assumption D1 holds and  $\bar{Q} = 1$ . Then, the underpricing for a bundle with  $I$  products goes to zero with  $N (1 - pq^{I-1})^N$  for uniform and discriminatory price auctions, and at least as fast for ascending price auctions. Separate sales become optimal in all auction formats as  $N \rightarrow \infty$ , and the rate of convergence of all auction formats is the same.*

The Propositions above establish that bundling is typically more powerful in auction formats that are less attractive to the seller, as the discriminatory versus uniform, or an auction without reserve relative to one with the optimal reserve, or a standard auction format relative to the optimal mechanism. However, as the following proposition shows, it is not a general rule that bundling is used more in less attractive auction formats.

**Proposition 20.** *Under assumption D1, if bundling is optimal for uniform price auctions, it is also optimal for ascending price auctions. There exists situations in which bundling is optimal in ascending price auctions where it is not in uniform price auctions.*

**Proof.** For separate sales, revenues are the same for uniform and ascending price auctions. For bundled sales, revenues in ascending price auctions are strictly higher. Therefore, at a point of indifference between bundling and not in uniform price auction, bundling must be strictly better for ascending price auctions. ■

## 6. Mixed Common and Private Values

Putting the results from sections 4 and 5 together, we see that for discriminatory price auctions, almost all results go in the same direction as the results for affiliated private values. However, there may be an increased tendency to bundle if there are enough products. For

the other auction formats, that converge on the underpricing, the impact of the common value component will be largest for small  $N$ . For large  $N$ , the common value component does not get affected much by the choice of strategy relative to the private value component. Thus, any small private value component will dictate the choice of strategies. In particular, there will be no dispersion, and the bundling decision will be based on how high demand is relative to supply. However, the rationing strategy may get affected - since the rationing strategy will in fact affect the revenues raised on the common value part.

## **7. Conclusion**

<<To be written.>>



# APPENDICES

## Proof of Proposition 1

First I establish that under assumption D1, there exists a sufficient statistic for the multidimensional signal that satisfies MLRP:

**Lemma 6.** *Suppose assumption D1 holds. Then,  $X_n$  is a sufficient statistic for  $\{X_{i,n}\}_{i \in \Pi_j}$  and satisfies MLRP with respect  $Z$ .*

Proof. Denote the sum of signals by  $X \equiv \sum_{i \in J} X_i$  and the sum of values by  $Z \equiv \sum_{i \in J} Z_i$ .

It is easy to show that  $X$  is a sufficient statistic for  $Z$  using the formal definition of a sufficient statistic<sup>3</sup>, but it is also obvious from the symmetry of signals and assets: To predict how many of the assets take on value one (that is, to predict  $Z$ ) it doesn't matter for which assets you got a high signal, only how many high signals you got (that is,  $X$ ). It remains to show that  $X$  satisfies MLRP with respect to  $Z$ , so that signals can be ranked unambiguously from high to low according to the value of  $X$  when bundle  $\Pi_j$  is auctioned.

We show this by induction. First, we show that under assumption A1, MLRP holds for 1 asset. This follows from  $p > q$ :

$$\begin{aligned} \frac{P(x = 1 | z = 1)}{P(x = 0 | z = 1)} &= \frac{p}{1 - p} \\ &> \frac{q}{1 - q} \\ &= \frac{P(x = 1 | z = 0)}{P(x = 0 | z = 0)} \end{aligned}$$

Next, we show that if MLRP holds for  $K - 1$  assets, then it holds for  $K$  assets. Denote by  $P(x | z \text{ from } K)$  the probability of observing a sum of signals  $x$  about  $K$  assets given that the sum of  $K$  assets is equal to  $z$ . It is enough to show that, given our induction hypothesis,

$$\frac{P(x | z \text{ from } K)}{P(x | z + 1 \text{ from } K)} > \frac{P(x + 1 | z \text{ from } K)}{P(x + 1 | z + 1 \text{ from } K)} \quad \forall x, z \leq K - 1 \quad (.1)$$

These local conditions imply the global MLRP condition. Note that we can write, for

---

<sup>3</sup>See, for example, DeGroot [?].

$1 \leq z \leq K - 1$  and  $x \leq K - 1$  :<sup>4</sup>

$$P(x|z \text{ from } K) = \frac{K}{K-x} \left( (1-p) \frac{z}{K} P(x|z-1 \text{ from } K-1) + (1-q) \frac{(K-z)}{K} P(x|z \text{ from } K-1) \right)$$

The left hand side of .1 then becomes

$$\frac{P(x|z \text{ from } K)}{P(x|z+1 \text{ from } K)} = \frac{(1-p) z \frac{P(x|z-1 \text{ from } K-1)}{P(x|z \text{ from } K-1)} + (1-q)(K-z)}{(1-p)(z+1) + (1-q)(K-z-1) \frac{P(x|z+1 \text{ from } K-1)}{P(x|z \text{ from } K-1)}}$$

and, for  $x \leq K - 2$ , the right hand side of .1 becomes

$$\frac{P(x+1|z \text{ from } K)}{P(x+1|z+1 \text{ from } K)} = \frac{(1-p) z \frac{P(x+1|z-1 \text{ from } K-1)}{P(x+1|z \text{ from } K-1)} + (1-q)(K-z)}{(1-p)(z+1) + (1-q)(K-z) \frac{P(x+1|z+1 \text{ from } K-1)}{P(x+1|z \text{ from } K-1)}}$$

From our induction hypothesis,  $\frac{P(x+1|z-1 \text{ from } K-1)}{P(x+1|z \text{ from } K-1)} < \frac{P(x|z-1 \text{ from } K-1)}{P(x|z \text{ from } K-1)}$  and  $\frac{P(x+1|z+1 \text{ from } K-1)}{P(x+1|z \text{ from } K-1)} > \frac{P(x|z+1 \text{ from } K-1)}{P(x|z \text{ from } K-1)}$ . This proves  $\frac{P(x|z \text{ from } K)}{P(x|z+1 \text{ from } K)} > \frac{P(x+1|z \text{ from } K)}{P(x+1|z+1 \text{ from } K)}$ , as long as  $1 \leq z \leq K - 1$  and  $x \leq K - 2$ . We also need to show that .1 holds for  $z = 0$  and  $x = K - 1$ . For  $z = 0$  we have:

$$\begin{aligned} P(x|z=1 \text{ from } K) &= \binom{K-1}{x-1} q^{x-1} (1-q)^{K-x} p + \binom{K-1}{x} q^x (1-q)^{K-1-x} (1-p) \\ P(x|z=0 \text{ from } K) &= \binom{K}{x} q^x (1-q)^{K-x} \end{aligned}$$

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<sup>4</sup>This is derived as follows:

$$P(x|z \text{ from } K) = \frac{P(x \cap x_K = 0 | z \text{ from } K)}{P(x_K = 0 | x)}$$

Here,  $P(x_K = 0 | x)$  is the probability of observing a zero signal on the  $K$ :th asset conditional on observing  $x$  ones on all  $K$  assets, and is given by

$$P(x_K = 0 | x) = \frac{K-x}{K}$$

$P(x \cap x_K = 0 | z \text{ from } K)$  is the probability of observing  $x$  ones on  $K$  assets and zero on the  $K$ :th asset given that  $z$  assets are one. It is calculated as:

$$\begin{aligned} P(x \cap x_K = 0 | z \text{ from } K) &= P(z_K = 1 | z \text{ from } K) P(x_K = 0 | z_K = 1) P(x|z-1 \text{ from } K-1) \\ &\quad + P(z_K = 0 | z \text{ from } K) P(x_K = 0 | z_K = 0) P(x|z \text{ from } K-1) \\ &= (1-p) z P(x|z-1 \text{ from } K-1) + (1-q)(K-z) P(x|z \text{ from } K-1) \end{aligned}$$

so that

$$\frac{P(x|z=1 \text{ from } K)}{P(x|z=0 \text{ from } K)} = \frac{x}{K} \frac{p}{q} + \frac{K-x}{K} \frac{1-p}{1-q}$$

This is increasing in  $x$ , since  $\frac{1-p}{1-q} > \frac{p}{q}$ . It is similarly straightforward to show that .1 holds for  $x = K - 1$ . ■

The rest of the Proposition is proved by straight-forward extension of the results in Milgrom and Weber [14].//To be completed//. ■

**Proof of Propositions 13 and 14:**

**Lemma 7.** *The rate of convergence is increasing in  $\frac{\gamma(F(x^{**}))}{\gamma(1-\beta)}$*

Proof: We have that

$$\begin{aligned} f_{Y_{K+1}}(x|z) &= \frac{N!}{K!(N-K-1)!} F(x|z)^{N-K-1} (1-F(x|z))^K f(x|z) \\ &= \frac{N!}{K!(N-K)!} F(x|z)^{N-K} (1-F(x|z))^K \frac{f(x|z)}{F(x|z)} (N-K) \\ &= \sqrt{N} \frac{N!}{K!(N-K)!} \gamma(1-\beta)^N \left( \frac{\gamma(F(x|z))}{\gamma(1-\beta)} \right)^N \frac{f(x|z)}{F(x|z)} \sqrt{N} (1-\beta) \end{aligned}$$

Note that

$$\lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \sqrt{N} \frac{N!}{K!(N-K)!} \gamma(1-\beta)^N = C > 0$$

for some constant  $C$ . This follows from the fact that for a binomial distribution  $X \sim \text{Bin}[N, \beta]$ , we have that

$$\theta \equiv \sqrt{N} \left( \frac{X}{N} - \beta \right) \rightarrow N(0, \beta(1-\beta))$$

so that, for some  $\varepsilon(N) > 0$  such that  $\lim_{N \rightarrow \infty} \varepsilon(N) = 0$  and  $\lim_{N \rightarrow \infty} \sqrt{N} \varepsilon(N) = \infty$ , we have that

$$\frac{P\left(-\frac{\varepsilon(N)}{2} \leq \theta \leq \frac{\varepsilon(N)}{2}\right)}{\varepsilon(N)} \rightarrow \frac{1}{\sqrt{2\pi\beta(1-\beta)}}$$

(the normal distribution evaluated at 0). But we can also write this as

$$\begin{aligned} \frac{P\left(-\frac{\varepsilon(N)}{2} \leq \theta \leq \frac{\varepsilon(N)}{2}\right)}{\varepsilon(N)} &= \frac{\sum_{X=-\sqrt{N}\frac{\varepsilon(N)}{2}+K}^{-\sqrt{N}\frac{\varepsilon(N)}{2}+K} P(X)}{\varepsilon(N)} \\ &\approx \sqrt{N} P(K) \end{aligned}$$

which gives the result. Hence,  $f_{Y_{K+1}}(x|z)$  goes to zero at a rate proportional to

$$\left( \frac{\gamma(F(x|z))}{\gamma(1-\beta)} \right)^N \frac{f(x|z)}{F(x|z)} \sqrt{N} (1-\beta)$$

Since  $f(x|z) > 0$  and for  $\beta < 1$ , this is dominated by the behavior of  $\left( \frac{\gamma(F(x|z))}{\gamma(1-\beta)} \right)^N$ . ■

**Lemma 8.** *The convergence rate is in the interval  $\left[ \max \left[ \frac{\gamma(F(x^{**}|0))}{\gamma(1-\beta)}, \frac{\gamma(F(x^{**}|I))}{\gamma(1-\beta)} \right], \max_z \left[ \frac{\gamma(F(x^{**}|z))}{\gamma(1-\beta)} \right] \right]$*

**Lemma 9.** *If  $P(x > H - 1)$  is small enough, some rationing is optimal.*

Proof. For  $\beta \rightarrow 0$ , we have convergence proportional to  $F(H - 1|I)$ . This follows since

$$\frac{F(H - 1|I)^{1-\beta} (1 - F(H - 1|I))^\beta}{(1 - \beta)^{1-\beta} \beta^\beta} \rightarrow F(H - 1|I)$$

as  $\beta \rightarrow 0$ . For  $P(x > H - 1)$  small enough, this is close to one. For  $\beta$  lower, it is enough to show that  $F(x|z) - F(x|z - 1) > \varepsilon$  for all  $z$ . If this is so, we must have

$$\frac{\gamma(F(x^*|z))}{\gamma(1-\beta)} < 1 \quad \forall z$$

so that the convergence rate is strictly faster. This is so since  $F(x^*|z)$  will be strictly smaller than  $1 - \beta$ . But for  $z$  discrete, this will be the case. ■

**Lemma 10.** *If  $P(x > H - 1) > 0$ , and  $z$ 's are on a fine enough grid so that  $F(x|z) - F(x|z - 1) < \varepsilon$  for all  $z, x$ , then rationing is not optimal.*

Proof: If  $F(x|z) - F(x|z - 1)$  are sufficiently close together,  $\frac{\gamma(F(x^*|z))}{\gamma(1-\beta)}$  is arbitrarily close to one, and will be larger than  $F(H - 1|1)$ . Hence, rationing is not optimal. ■

**Proof of Proposition :**

We begin by establishing a couple of lemmas that will be useful in proving the proposition.

**Lemma 11.**

$$\lim_{N, K \rightarrow \infty, \frac{K}{N} \rightarrow \beta} f_{Y_{K+1}}(x|z) = 0 \quad \forall x \text{ such that } F(x|z) \neq (1 - \beta)$$

**Lemma 12.**

$$\lim_{N, K \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \frac{f_{Y_{K+1}}(x|z)}{f_{Y_{K+1}}(y|u)} = 0 \quad \text{if } \gamma(F(x|z)) < \gamma(F(y|u))$$

where  $\gamma(F)$  is defined as

$$\gamma(x) \equiv F^{1-\beta} (1-F)^\beta$$

Proof: We can write  $f_{Y_{K+1}}(x|z)$  as

$$\begin{aligned} f_{Y_{K+1}}(x|z) &= \frac{N!}{K!(N-K-1)!} F(x|z)^{N-K-1} (1-F(x|z))^K f(x|z) \\ &= \left( F(x|z)^{1-\beta} (1-F(x|z))^\beta \right)^N \frac{N!}{K!(N-K-1)!} F(x|z) f(x|z) \\ &= \gamma(F(x|z))^N \frac{N!}{K!(N-K-1)!} F(x|z) f(x|z) \end{aligned}$$

Therefore, we have that

$$\lim_{N, K \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \frac{f_{Y_{K+1}}(x|z)}{f_{Y_{K+1}}(y|u)} = \left( \frac{\gamma(F(x|z))}{\gamma(F(y|u))} \right)^N \frac{F(x|z) f(x|z)}{F(y|u) f(y|u)}$$

If  $\gamma(F(x|z)) < \gamma(F(y|u))$ , this goes to zero with  $N$ , which proves Lemma 12.

Lemma 11 is then proved as follows. Since  $\gamma(F)$  has a unique maximum at  $F = 1 - \beta$ , it follows that  $\frac{f_{Y_{K+1}}(x|z)}{f_{Y_{K+1}}(y|z)}$  goes to infinity for  $x$  such that  $F(x|z) = 1 - \beta$  and  $y \neq x$ . Now suppose, contrary to the claim in the lemma, that

$$\lim_{N, K \rightarrow \infty, \frac{K}{N} \rightarrow \beta} f_{Y_{K+1}}(y|z) = \varepsilon \quad (.2)$$

for some  $\varepsilon > 0$  and for  $y$  such that  $F(y|z) \neq 1 - \beta$ . Also, suppose without loss of generality that  $y < x$ . Then, since  $\gamma(F)$  is concave, it follows that

$$\gamma(F(y|z)) < \gamma\left(F\left(\frac{y+x}{2}|z\right)\right) < \gamma(F(x|z))$$

But this implies that for all  $y' \in [\frac{y+x}{2}, x]$ ,  $f_{Y_{K+1}}(y'|z)$  goes to infinity from Lemma 12 and expression .2. This in turn implies that

$$\int_{y'=0}^H f_{Y_{K+1}}(y'|z) dy' \rightarrow \infty > 1$$

which is a contradiction. Hence,  $f_{Y_{K+1}}(y|z)$  must go to zero. ■

**Lemma 13.** *The hazard rate  $\frac{1-F(x|z)}{f(x|z)}$  is strictly increasing in  $z$  for  $x < H - 1$  and is independent of  $z$  for  $x \geq H - 1$ .*

Proof. We have that

$$\frac{1 - F(x|z)}{f(x|z)} = \int_x^H \frac{f(y|z)}{f(x|z)} dy$$

Since MLRP assumes that  $\frac{f(y|z)}{f(x|z)}$  is strictly increasing in  $z$  for  $y \geq H - 1$  and  $x < H - 1$ , and is constant in  $z$  for  $x \geq H - 1$ , the result follows. ■

To prove proposition ?, divide the bundle  $Z_I$  into two monotonically increasing claims  $a_1(Z_I) = \min(Z_I, I - 1)$  and  $a_2(Z_I) = \max(0, Z_I - (I - 1))$ . Then, from Lemma ?, we have that the expected underpricing  $U(Z_I)$  satisfies

$$\begin{aligned} U(Z_I) &= U(a_1(Z_I) + a_2(Z_I)) \\ &= U(a_1(Z_I)) + U(a_2(Z_I)) \end{aligned}$$

with

$$U(a_i(Z_I)) \geq 0 \quad i = 1, 2$$

Note that  $Z_1 = a_2(Z_1)$ . To prove the proposition, it is sufficient to show that  $U(Z_1)$  goes to zero with  $N$  faster than  $U(a_2(Z_I))$  for  $I > 1$ .

The underpricing  $U(a_2(Z_I))$  in uniform price auctions is given by

$$\begin{aligned} U(a_2(Z_I)) &= E(a_2(Z_I)) - E(\pi(a_2(Z_I))) \\ &= \int_0^H (E(a_2(Z_I) | Y_{K+1} = x) - E(a_2(Z_I) | Y_{K+1} = x, Y_K = x)) f_{Y_{K+1}}(x) dx \\ &= \int_0^{H-1} (P(I | Y_{K+1} = x) - P(I | Y_{K+1} = x, Y_K = x)) f_{Y_{K+1}}(x) dx \\ &= P(I) \int_0^{H-1} f_{Y_{K+1}}(x | I) A(x) dx \end{aligned} \tag{.3}$$

where  $A(x)$  is defined as

$$A(x) \equiv 1 - \frac{P(I | Y_{K+1} = x, Y_K = x)}{P(I | Y_{K+1} = x)}$$

and the third equality above follows from the fact that for  $x$  in the highest equivalence interval  $x \in [H - 1, H)$ ,  $P(I | Y_{K+1} = x) = P(I | Y_{K+1} = x, Y_K = x)$  since the event  $[Y_{K+1} = x]$  is

informationally equivalent to  $[Y_{K+1} = Y_K = \dots = Y_1 = x]$  on the highest equivalence interval.

The following Lemma delineates some properties of the integrand  $f_{Y_{K+1}}(x|I)A(x)$  in expression .3 that are used below:

**Lemma 14.** For  $x \in [0, H - 1)$ ,  $f_{Y_{K+1}}(x|I)A(x)$  is strictly positive and goes to zero with  $\min(f_{Y_{K+1}}(x|I - 1), f_{Y_{K+1}}(x|I))$  :

$$\lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \frac{f_{Y_{K+1}}(x|I)A(x)}{\min(f_{Y_{K+1}}(x|I - 1), f_{Y_{K+1}}(x|I))} = C > 0$$

for some bounded constant  $C$ .

Proof: First rewrite  $A(x)$  as

$$\begin{aligned} 1 - \frac{P(I|Y_{K+1} = x, Y_K = x)}{P(I|Y_{K+1} = x)} &= 1 - \frac{\frac{f_{Y_{K+1}, Y_K}(x, x|I)P(I)}{f_{Y_{K+1}, Y_K}(x, x)}}{\frac{f_{Y_{K+1}}(x|I)P(I)}{f_{Y_{K+1}}(x)}} \\ &= 1 - \frac{\frac{\frac{f(x|I)}{1-F(x|I)}f_{Y_{K+1}}(x|I)}{\sum_{u=1}^I \frac{f(x|u)}{1-F(x|u)}f_{Y_{K+1}}(x|u)P(u)}}{\frac{f_{Y_{K+1}}(x|I)}{f_{Y_{K+1}}(x)}} \\ &= \frac{\sum_{u=1}^I \left( \frac{f(x|u)}{1-F(x|u)} - \frac{f(x|I)}{1-F(x|I)} \right) f_{Y_{K+1}}(x|u)P(u)}{\sum_{u=1}^I \frac{f(x|u)}{1-F(x|u)}f_{Y_{K+1}}(x|u)P(u)} \end{aligned}$$

where the second equality follows from

$$\begin{aligned} f_{Y_{K+1}, Y_K}(x, x|z) &= \frac{1}{2K} \frac{N!}{K!(N-K-1)!} F(x|z)^{N-K-1} (1-F(x|z))^{K-1} f(x|z)^2 \\ &= \frac{1}{2K} \frac{f(x|z)}{1-F(x|z)} f_{Y_{K+1}}(x|z) \end{aligned}$$

Lemma 13 implies that the inverse hazard rate  $\frac{f(x|z)}{1-F(x|z)}$  is strictly decreasing in  $z$  for  $x < H - 1$ , which shows that  $A(x)$ , and hence  $f_{Y_{K+1}}(x|I)A(x)$ , is strictly positive for  $x \in [0, H - 1)$ .

The remainder of Lemma 14 is proved as follows. Define  $z_{\max}(x)$  as

$$z_{\max}(x) \equiv \arg \max_{z \in \{0, \dots, I\}} \gamma(F(x|z))$$

In view of Lemma 12, we have that

$$\lim_{N \rightarrow \infty, N/K \rightarrow \beta} \frac{f_{Y_{K+1}}(x|z)}{f_{Y_{K+1}}(x|z_{\max})} = 0 \quad z \neq z_{\max} \quad (.4)$$

Also, since  $F^{1-\beta} (1 - F)^\beta$  is a concave function of  $F$ , and since  $F(x|z)$  is decreasing in  $z$ , we have that for any integer  $i$

$$\lim_{N \rightarrow \infty, N/K \rightarrow \beta} \frac{f_{Y_{K+1}}(x|z_{\max} - (i+1))}{f_{Y_{K+1}}(x|z_{\max} - i)} = 0 \quad (.5)$$

$$\lim_{N \rightarrow \infty, N/K \rightarrow \beta} \frac{f_{Y_{K+1}}(x|z_{\max} + (i+1))}{f_{Y_{K+1}}(x|z_{\max} + i)} = 0 \quad (.6)$$

We can now derive the limiting behavior of  $f_{Y_{K+1}}(x|I)A(x)$ :

$$\begin{aligned} & \lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \frac{f_{Y_{K+1}}(x|I)A(x)}{\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I))} \\ = & \lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \left( \frac{\sum_{u=0}^{I-1} \left( \frac{f(x|u)}{1-F(x|u)} - \frac{f(x|I)}{1-F(x|I)} \right) \frac{f_{Y_{K+1}}(x|u)}{f_{Y_{K+1}}(x|z_{\max})} \frac{f_{Y_{K+1}}(x|I)}{\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I))} P(u)}{\sum_{u=0}^I \frac{f(x|u)}{1-F(x|u)} \frac{f_{Y_{K+1}}(x|u)}{f_{Y_{K+1}}(x|z_{\max})} P(u)} \right) \\ = & \frac{\sum_{u=0}^{I-1} \left( \frac{f(x|u)}{1-F(x|u)} - \frac{f(x|I)}{1-F(x|I)} \right) \lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \left[ \frac{f_{Y_{K+1}}(x|u)}{f_{Y_{K+1}}(x|z_{\max})} \frac{f_{Y_{K+1}}(x|I)}{\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I))} \right] P(u)}{\sum_{u=0}^I \frac{f(x|u)}{1-F(x|u)} \lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \left[ \frac{f_{Y_{K+1}}(x|u)}{f_{Y_{K+1}}(x|z_{\max})} \right] P(u)} \end{aligned} \quad (.7)$$

The last step is valid provided that the denominator does not go to zero and provided both numerator and denominator are bounded, conditions that are verified now. First, from Lemma 12, the denominator goes to  $\frac{f(x|z_{\max})}{1-F(x|z_{\max})} > 0$ . Second, if  $z_{\max} \neq I$ , we have  $\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I)) = f_{Y_{K+1}}(x|I)$  from expression .6, so that the numerator goes to  $\frac{f(x|z_{\max})}{1-F(x|z_{\max})} - \frac{f(x|I)}{1-F(x|I)} > 0$ . For  $z_{\max} = I$ , we have

$$\begin{aligned} & \lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \frac{f_{Y_{K+1}}(x|u)}{f_{Y_{K+1}}(x|z_{\max})} \frac{f_{Y_{K+1}}(x|I)}{\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I))} \\ = & \lim_{N \rightarrow \infty, \frac{K}{N} \rightarrow \beta} \frac{f_{Y_{K+1}}(x|u)}{f_{Y_{K+1}}(x|I-1)} \\ = & \begin{cases} 1 & u = I-1 \\ 0 & u < I-1 \end{cases} \end{aligned}$$

where the first step follows from  $f_{Y_{K+1}}(x|I) = f_{Y_{K+1}}(x|z_{\max})$ , and the second from expression .5. Thus, the numerator of .7 goes to  $\frac{f(x|I-1)}{1-F(x|I-1)} - \frac{f(x|I)}{1-F(x|I)} > 0$ . It remains to show that  $\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I))$  does in fact go to zero with  $N$ . But this follows from Lemma 11, since  $F(x|I-1) \neq 1-\beta$  if  $\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I)) = f_{Y_{K+1}}(x|I-1)$



and  $F(x|I) \neq 1 - \beta$  if  $\min(f_{Y_{K+1}}(x|I-1), f_{Y_{K+1}}(x|I)) = f_{Y_{K+1}}(x|I)$ . ■

The convergence rate of  $U(a_2(Z_I))$  will be governed by the integrand in .3 with the slowest convergence.

From Lemma .7 and Lemma 12, it follows that the integrand with the slowest convergence in .3 is for the  $x$  that solves the problem

$$\max_{x \leq H-1} [\min(\gamma(F(x|I)), \gamma F(x|I-1))] \quad (.8)$$

I claim that the solution to problem .8 is given by  $x^{**}$  defined as

$$x^{**} \equiv \min(H-1, x^*) \quad (.9)$$

where  $x^*$  is defined implicitly by the solution to

$$\gamma(F(x^*|I)|I) = \gamma(F(x^*|I-1))$$

To see this, we first show that  $x^*$  is the solution to the relaxed problem

$$\max_x [\min(\gamma(F(x|I)), \gamma F(x|I-1))] \quad (.10)$$

To see this, notice that  $F(x|I) < F(x|I-1)$  for all  $0 < x < 1$  from MLRP. Then, since  $\gamma(F)$  is increasing for  $F < 1 - \beta$  and decreasing for  $F > 1 - \beta$ , it must be that  $F(x^*|I) < 1 - \beta < F(x^*|I-1)$ . Also,  $\gamma(F(x|I)) < \gamma(F(x|I-1))$  for  $0 < x < x^*$  and  $\gamma(F(x|I)) > \gamma(F(x|I-1))$  for  $x^* < x < H$ . Hence,  $\min(\gamma(F(x|I)), \gamma F(x|I-1))$  is maximized at  $x^*$ . This shows that  $x^*$  is also the solution to .8 if  $x^* \leq H-1$ . Now suppose  $x^* > H-1$ . Then, we have that  $F(x|I) < F(x^*|I) < 1 - \beta$  and  $\min(\gamma(F(x|I)), \gamma F(x|I-1)) = \gamma(F(x|I))$  for  $x \leq H-1$ . Since  $\gamma(F(x|I))$  is increasing in  $x$  for  $x < x^*$ , we then have  $x = H-1$  as the solution to .8. This shows that  $x^{**} = \min(H-1, x^*)$  is indeed the solution to .8.

I now show that the higher  $F(x^{**}|I)$ , the slower  $U(a_2(Z_I))$  goes to zero. Note that

$$\min(\gamma(F(x^{**}|I)), \gamma(F(x^{**}|I-1))) = \gamma(F(x^{**}|I))$$

Since the convergence rate of  $U(a_2(Z_I))$  is inversely proportional to  $\gamma(F(x^{**}|I))$ , the result

follows from the fact that  $F(x^{**}|I) < 1 - \beta$  so that  $\frac{\partial \gamma(F(x^{**}|I))}{\partial F(x^{**}|I)} > 0$ .<sup>5</sup>

To make the distinction between bundles of different size explicit, denote by  $F_I(x|z)$  the distribution function of  $X$  for a bundle with  $I$  assets. To finish the proof of the proposition, it is enough to show that  $F_I(x_I^{**}|I) > F_1(x_1^{**}|I)$  for  $I > 1$  (with the dependence of  $x^{**}$  on  $I$  made explicit).

From the definition of  $x^{**}$ , we have

$$\begin{aligned} F_I(x_I^{**}|I) &= \min(F_I(I-1|I), F_I(x_I^*|I)) \\ &= \min(1 - p^I, F_I(x_I^*|I)) \end{aligned}$$

Since  $1 - p^I$  is increasing in  $I$ , all that remains is to show that  $F_I(x_I^*|I) \geq F_1(x_1^*|1)$  for  $I > 1$ .

Define  $y(x)$  implicitly by

$$F_I(y(x)|I) \equiv F_1(x|1)$$

If we can show that

$$F_I(y(x)|I-1) \leq F_1(x|0) \quad \forall x \quad (.11)$$

we have that

$$\begin{aligned} \gamma(F_I(y(x_1^*)|I)) &= \gamma(F_1(x_1^*|1)) \\ &= \gamma(F_1(x_1^*|0)) < \gamma(F_I(y(x_1^*)|0)) \end{aligned}$$

The last inequality follows since  $F_I(y(x_1^*)|I-1) \in (F_1(x_1^*|1), F_1(x_1^*|0))$  and from the concavity of  $\gamma(F)$ . Hence,  $x_1^* > y(x_1^*)$ , and  $F_I(x_1^*|I) > F_1(x_1^*|1)$ , which proves the proposition.

Note that we have

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<sup>5</sup>More exactly, it is straightforward to show that

$$\frac{U(a_2(Z_I))}{\frac{N!}{K!(N-K-1)!} \gamma(F(x^{**}|I))^N} \rightarrow 0$$

while for all  $\varepsilon > 0$ ,

$$\frac{U(a_2(Z_I))}{\frac{N!}{K!(N-K-1)!} \gamma(F(x^{**}|I) - \varepsilon)^N} \rightarrow \infty$$

$$\begin{aligned}\frac{f_1(0|0)}{f_1(0|1)} &= \frac{1-q}{1-p} \\ \frac{f_1(1|0)}{f_1(1|1)} &= \frac{q}{p}\end{aligned}$$

and

$$\begin{aligned}\frac{f_I(x|I-1)}{f_I(x|I)} &= \frac{\frac{I-1!}{(J-1)!(I-J)!} (1-p)^{I-J} p^{J-1} q + \frac{I-1!}{J!(I-J-1)!} (1-p)^{I-J-1} p^J (1-q)}{\frac{I!}{J!(I-J)!} (1-p)^{I-J} p^J} \\ &= \frac{Jq}{I p} + \frac{I-J}{I} \frac{1-q}{1-p}\end{aligned}$$

where  $J$  is an integer such that  $J \leq x < J+1$ . Therefore, we have  $\frac{f_1(0|0)}{f_1(0|1)} \geq \frac{f_I(y|I-1)}{f_I(y|I)} \geq \frac{f_1(1|0)}{f_1(1|1)}$  for all  $y$ . This implies inequality .11, since for  $x < 1$ , we have

$$\begin{aligned}\frac{f_1(x|0)}{f_1(x|1)} &\geq \frac{f_I(y|I-1)}{f_I(y|I)} \Rightarrow \\ F_1(x|0) f_I(y|I) &\geq F_1(x|1) f_I(y|I-1) \Rightarrow \\ F_1(x|0) F_I(y|I) &\geq F_1(x|1) F_I(y|I-1) \Rightarrow \\ F_1(x|0) &\geq F_I(y(x)|I-1)\end{aligned}$$

Similarly, for  $x \geq 1$ ,

$$\begin{aligned}\frac{f(x|0)}{f(x|1)} &\leq \frac{f(y|I-1)}{f(y|I)} \Rightarrow \\ (1 - F_1(x|0)) f_I(y|I) &\leq (1 - F_1(x|1)) f_I(y|I-1) \Rightarrow \\ (1 - F_1(x|0)) (1 - F_I(y|I)) &\leq (1 - F_1(x|1)) (1 - F_I(y|I-1)) \Rightarrow \\ F_1(x|0) &\geq F_I(y(x)|I-1)\end{aligned}$$

This concludes the proof.

## .1. Ascending Price Auctions

$$U(a_2(Z_I)) = E(a_2(Z_I)) - E(\pi(a_2(Z_I)))$$

$$\begin{aligned}
&= \int_0^H \left( \frac{f_{Y_{K+1}}(x|I) P(I)}{f_{Y_{K+1}}(x)} \right) f_{Y_{K+1}}(x) dx \\
&\quad - \int_0^H \int_{Y_{K+2:N}} \frac{f_{Y_{K+2:N}, Y_{K+1}, Y_K}(Y, x, x|I) P(I)}{f_{Y_{K+2:N}, Y_{K+1}, Y_K}(Y, x, x)} f_{Y_{K+2:N}|Y_{K+1}}(Y_{K+2:N}) dY f_{Y_{K+1}}(x) dx \\
&\qquad\qquad\qquad = P(I) \int_0^H f_{Y_{K+1}}(x|I) dx \\
&\quad - P(I) \int_0^H \int_{Y_{K+2:N}} f_{Y_{K+2:N}, Y_{K+1}, Y_K}(Y, x, x|I) \frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)}{f_{Y_{K+2:N}, Y_{K+1}, Y_K}(Y, x, x)} dY dx \\
&\qquad\qquad\qquad = P(I) \int_0^H f_{Y_{K+1}}(x|I) dx \\
&\quad - P(I) \int_0^H \int_{Y_{K+2:N}} \frac{C}{D} \frac{f(x|I)}{1-F(x|I)} f_{Y_{K+2:N}, Y_{K+1}}(Y, x|I) \frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)}{f_{Y_{K+2:N}, Y_{K+1}, Y_K}(Y, x, x)} dY dx \\
&\qquad\qquad\qquad = P(I) \int_0^H f_{Y_{K+1}}(x|I) dx \\
&\quad - P(I) \int_0^H f_{Y_{K+1}}(x|I) \int_{Y_{K+2:N}} \frac{C}{D} \frac{f(x|I)}{1-F(x|I)} f_{Y_{K+2:N}}(Y|I, Y_{K+1}=x) \frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)}{f_{Y_{K+2:N}, Y_{K+1}, Y_K}(Y, x, x)} dY dx \\
&\qquad\qquad\qquad = P(I) \int_0^H f_{Y_{K+1}}(x|I) dx \\
&\quad - P(I) \int_0^H f_{Y_{K+1}}(x|I) \int_{Y_{K+2:N}} f_{Y_{K+2:N}}(Y|I, Y_{K+1}=x) \frac{\frac{C}{D} \frac{f(x|I)}{1-F(x|I)} f_{Y_{K+2:N}, Y_{K+1}}(Y, x)}{\sum_z \frac{C}{D} \frac{f(x|z)}{1-F(x|z)} f_{Y_{K+2:N}, Y_{K+1}}(Y, x|z) P(z)} dY dx \\
&\qquad\qquad\qquad = P(I) \int_0^H f_{Y_{K+1}}(x|I) dx \\
&\quad - P(I) \int_0^H f_{Y_{K+1}}(x|I) \int_{Y_{K+2:N}} \frac{\frac{f(x|I)}{1-F(x|I)}}{\sum_z \frac{f(x|z)}{1-F(x|z)} \frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x|z)}{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)}} f_{Y_{K+2:N}}(Y|I, Y_{K+1}=x) dY dx \\
&\qquad\qquad\qquad = P(I) \int_0^{H-1} f_{Y_{K+1}}(x|I) A(x) \\
&\quad - P(I) \int_0^H f_{Y_{K+1}}(x|I) \int_{Y_{K+2:N}} \frac{\frac{f(x|I)}{1-F(x|I)}}{\sum_z \frac{f(x|z)}{1-F(x|z)} \frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x|z)}{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)}} f_{Y_{K+2:N}}(Y|I, Y_{K+1}=x) dY dx \\
&\qquad\qquad\qquad = P(I) \int_0^{H-1} f_{Y_{K+1}}(x|I) A(x).
\end{aligned}$$

We need to show that for a cut off level  $P_1^*$  where  $A(x)$  converges for one asset, it does not

for many assets. Then, there is an  $x$  such that  $A(x)$  does not go to zero where  $\gamma(F(x|I)) > \gamma(P_1^*)$ , so that the bundle converges at a slower rate.

**Lemma 15.**  $A(x) \rightarrow 1 - \frac{\frac{f(x|I)}{1-F(x|I)}}{\frac{f(x|z_{\max})}{1-F(x|z_{\max})}}$  where  $z_{\max}$  solves

$$\max_z \left[ \prod_{i=1}^x P(i|z)^{\frac{P(i|I)}{F(x|I)}} \right]^{1-\beta} (1 - F(x|z))^\beta$$

Proof: It is enough to show that, with probability one under the measure  $f_{Y_{K+2:N}}(Y|I, Y_{K+1} = x)$ ,

$$\frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x|z) P(z)}{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)} \rightarrow \begin{cases} 1 & z = z_{\max} \\ 0 & z \neq z_{\max} \end{cases}$$

Denote by  $\#i$  the number of elements in  $Y$  that are equal to  $i$ . Then,

$$\frac{\#i}{N - K} \xrightarrow{a.s.} \frac{P(i|I)}{F(x|I)}$$

under the measure  $f_{Y_{K+2:N}}(Y | I, Y_{K+1} = x)$  for  $i \leq x$ . We have that

$$\begin{aligned}
& \frac{f_{Y_{K+2:N}, Y_{K+1}}(Y, x | z) P(z)}{f_{Y_{K+2:N}, Y_{K+1}}(Y, x)} \\
&= \frac{\left( \prod_{i=0}^x P(i | z)^{\frac{\#i}{N-K-1}} \right)^{N-K-1} P(x | z) (1 - F(x | z))^K P(z)}{\sum_u \left( \prod_{i=0}^x P(i | u)^{\frac{\#i}{N-K-1}} \right)^{N-K-1} P(x | u) (1 - F(x | u))^K P(u)} \\
&= \frac{\left( \frac{\prod_{i=0}^x P(i | z)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{N-K-1} P(x | z) \left( \frac{1 - F(x | z)}{1 - F(x | z_{\max})} \right)^K P(z)}{\sum_u \left( \frac{\prod_{i=0}^x P(i | u)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{N-K-1} P(x | u) \left( \frac{1 - F(x | u)}{1 - F(x | z_{\max})} \right)^K P(u)} \\
&= \frac{\left( \frac{\prod_{i=0}^x P(i | z)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{N((1-\beta) - \frac{1}{N})} \left( \frac{1 - F(x | z)}{1 - F(x | z_{\max})} \right)^{N\beta} P(x | z) P(z)}{\sum_u \left( \frac{\prod_{i=0}^x P(i | u)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{N((1-\beta) - \frac{1}{N})} \left( \frac{1 - F(x | u)}{1 - F(x | z_{\max})} \right)^{N\beta} P(x | u) P(u)} \\
&= \frac{\left[ \left( \frac{\prod_{i=0}^x P(i | z)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{(1-\beta) - \frac{1}{N}} \left( \frac{1 - F(x | z)}{1 - F(x | z_{\max})} \right)^\beta \right]^N P(x | z) P(z)}{\sum_u \left[ \left( \frac{\prod_{i=0}^x P(i | u)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{(1-\beta) - \frac{1}{N}} \left( \frac{1 - F(x | u)}{1 - F(x | z_{\max})} \right)^\beta \right]^N P(x | u) P(u)}
\end{aligned}$$

Here, for a set of measure one, if  $z \neq z_{\max}$ ,

$$\left( \frac{\prod_{i=0}^x P(i | z)^{\frac{\#i}{N-K-1}}}{\prod_{i=1}^x P(i | z_{\max})^{\frac{\#i}{N-K-1}} \frac{P(i | I)}{F(x | I)}} \right)^{(1-\beta) - \frac{1}{N}} \left( \frac{1 - F(x | z)}{1 - F(x | z_{\max})} \right)^\beta \rightarrow C < 1$$

And hence the result follows. Hence,  $A(x)$  goes to zero for  $x$  such that  $z_{\max} < I$ .

$$\left[ \prod_{i=1}^x f(y_i | I)^{\frac{f(y_i | I)}{F(x | I)}} \right]^{1-\beta} (1 - F(x | I))^\beta = \left[ \prod_{i=1}^x (f(y_i | I - 1))^{\frac{f(y_i | I)}{F(x | I)}} \right]^{1-\beta} (1 - F(x | I - 1))^\beta$$

We can rewrite this as

$$\left[ \prod_{i=1}^x \frac{f(y_i | I - 1)^{\frac{f(y_i | I)}{F(x | I)}}}{f(y_i | I)} \right]^{1-\beta} \left( \frac{1 - F(x | I - 1)}{1 - F(x | I)} \right)^\beta = 1$$

$$(1 - \beta) \sum_{J \leq x} \frac{f(J|I)}{F(x|I)} \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) + \beta \ln \sum_{J > x} \frac{f(J|I)}{1-F(x|I)} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) = 0$$

$$(1 - \beta) E \left( \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) | J \leq x, I \right) + \beta \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J > x, I \right) = 0$$

Denote by  $L(x)$  the function  $x \frac{q}{p} + (1-x) \frac{1-q}{1-p}$ . Then, we can rewrite this as

$$(1 - \beta) E \left( \ln \left( L \left( \frac{X}{I} \right) \right) | X \leq x, I \right) + \beta \ln E(L(J) | J > x, I) = 0$$

We know that

$$F(x|I) E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J \leq x, I \right) + (1 - F(x|I)) E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J > x, I \right) =$$

$$E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | I \right) =$$

$$\frac{\frac{q}{p} E(J) + \frac{1-q}{1-p} E(I-J)}{I} =$$

$$\frac{q}{p} p + \frac{1-q}{1-p} (1-p) = 1,$$

or

$$\ln \left[ F(x|I) E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J \leq x, I \right) + (1 - F(x|I)) E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J > x, I \right) \right] = 0.$$

Then, from Jensen's inequality, we have

$$F(x|I) \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J \leq x, I \right) +$$

$$+ (1 - F(x|I)) \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J > x, I \right) < 0$$

$$F(x|I) E \left( \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) | J \leq x, I \right) +$$

$$+ (1 - F(x|I)) \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} | J > x, I \right) < 0$$

Both these follow from Jensen's inequality:  $E(f(x)) < f(E(x))$  for  $f$  a concave function.

Also, we have

$$\begin{aligned} & E \left( \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \middle| J \leq x, I \right) \\ & > \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \middle| J > x, I \right) \end{aligned}$$

Hence, we have to have

$$1 - \beta > F(x|I)$$

Now, let  $I$  grow. Then,  $f(\frac{J}{I})$  goes to a peaked distribution around  $p$ . Since

$$\sqrt{I} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right) \rightarrow N(0, 1)$$

we have that

$$\sqrt{I} \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \rightarrow N(0, 1)$$

so that

$$\begin{aligned} & E \left( \sqrt{I} \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \middle| J \leq x, I \right) \\ \rightarrow & E \left( \sqrt{I} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right) \middle| J \leq x, I \right) \end{aligned}$$

and

$$\begin{aligned} & \sqrt{I} \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \middle| J > x, I \right) \\ \rightarrow & E \left( \sqrt{I} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right) \middle| J > x, I \right) \end{aligned}$$

$$\begin{aligned} & (1 - \beta) E \left( \sqrt{I} \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \middle| J \leq x, I \right) \\ & + \beta \sqrt{I} \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \middle| J > x, I \right) = 0 \end{aligned}$$

Show that

$$F(x|I) \rightarrow 1 - \beta$$



We have that

$$\sqrt{I} \left( \frac{J}{I} - p \right) \rightarrow N(0, p(1-p))$$

and

$$\sqrt{I} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right) \rightarrow N(0, \sigma^2)$$

Also,

$$\sqrt{I} \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \sim \sqrt{I} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right)$$

and

$$\begin{aligned} & E \left( \sqrt{I} \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \mid J \leq x, I \right) \\ \rightarrow & E \left( \sqrt{I} \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right) \mid J \leq x, I \right) \\ & \sqrt{I} \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \mid J > x, I \right) \\ \rightarrow & \sqrt{I} E \left( \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} - 1 \right) \mid J > x, I \right) \end{aligned}$$

Therefore, there exists  $I$  such that

which shows that the cut-off level is increasing in  $I$ . Hence, at the  $F(x|I)$  where we have convergence for one asset, there is no convergence of  $A(y(x))$  for many assets, and we can increase  $F(x|I)$  slightly and hence  $\gamma(F(x|I))$ .

Also, since

$$E \left( \ln \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \right) \mid J \leq x, I \right) < \ln E \left( \frac{Jq}{Ip} + \frac{I-J}{I} \frac{1-q}{1-p} \mid J \leq x, I \right)$$

It has to be that the cut off level for ascending price auctions is in general lower than for uniform price auctions. Hence, ascending price auctions converge faster.

If  $x^* > H - 1$ , the convergence is at  $F(H - 1|I)$  - the same for uniform and ascending price auctions. Hence, with a more concentrated allocation, the ascending price auction has less of an advantage. Also, with less bundling, it has less of an advantage?

■

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