Networks and inflation: A learning-based microfoundation for persistent cost-push shocks*

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Abstract

This paper presents a model of price setting wherein firms partially inform their decisions by watching price changes by other firms across an observation network. Within a context of imperfect common knowledge and for a wide range of plausible and commonly observed network structures, idiosyncratic shocks are shown to not "wash out" in aggregate prices. These aggregate effects are also shown to be persistent despite the underlying idiosyncratic shocks being entirely transitory. The model is therefore able to explain a variety of recently documented stylised facts regarding price setting, including the observation that short-lived price changes appear to contain macroeconomic content. The paper also presents a closed-form, readily implementable solution to Bayesian learning over a social network, with the aggregate network effects able to be simulated without the need to explicitly model the network.

JEL Classification: D21 (Firm Behavior), D83 (Search, Learning, and Information), E31 (Price Level; Inflation; Deflation)

Keywords: Network learning; Incomplete information; Inflation persistence; Aggregate volatility

1 Introduction

It is now broadly accepted that idiosyncratic shocks are an important aspect of firms' price-setting decision making. However, it remains almost universally assumed that these shocks "cancel out" in aggregation through some form of diversification argument. In such a setting, firm-specific shocks can only contribute to aggregate dynamics by causing sluggish responses in aggregate prices to aggregate shocks because firms take time to be sure that a given shock is truly common to all firms. This is inconsistent with stylised facts recently documented regarding price changes at the micro level, where price changes that appear most likely to be caused by idiosyncratic shocks do not cancel out and so appear to contain macroeconomic content.

To reconcile this puzzle, this paper develops a network learning-based microfoundation for aggregate cost-push shocks. That firms observe the prices of their competitors and adjust their own prices accordingly is widely recognised, but has hitherto been little used in macroeconomic modelling. By making use of an observation network in a setting of imperfect common knowledge (ICK), I illustrate that even with a continuum of firms, mean zero idiosyncratic shocks to firms' marginal costs need not wash out in aggregate prices. Furthermore, because of the recursive nature of agents' learning, these aggregate effects are persistent even though idiosyncratic shocks are entirely transitory. Because firms may choose to observe the prices of other firms with whom they are are not direct competitors, this also represents a novel transmission mechanism for inflation across industries or geographies independent of it's path along production chains.

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More generally, this paper develops a closed-form, readily implementable solution to Bayesian learning over a social network in a setting of repeated, simultaneous actions and a dynamic underlying state. Previous work on network learning has limited attention to sequential actions and an unchanging state, or relied on assumptions of bounded rationality, or characterised only the speed of convergence in social beliefs. The effects of the network on agents' learning are captured in a manner that permits researchers to simulate the effects of network learning without having to model the network explicitly, with results calibrated by a single additional parameter (beyond those typically found in ICK models) describing the degree of asymmetry within the network. This makes the model particularly amenable to nesting within broad general equilibrium models of the economy and may be of independent interest.

When firms exist in an observation network, it is necessary for them to estimate not only the average expectation (for reasons of strategic complementarity), but also the expectations of specific competitors (and their expectations of others again). As the number of agents in the network expands, this causes an explosion in the size of the state vector quite apart from the presence of higher-order expectations (see section 3.1 for more detail) and has typically been thought to prevent closed-form analysis in anything other than trivially small networks.

In contrast, this paper limits the state vector to growing in the number of higher orders of expecation only, even for networks with an infinite number of agents, by denying agents knowledge of the exact topology of the network (the network is *opaque*). Instead, agents are granted knowledge of the distribution from which observation targets are drawn and do not learn about the structure of the network over time. With unobserved aggregate variables following an AR(1) process, the full hierarchy of agents' expectations is shown to follow an ARMA(1,1) process, with current and lagged weighted sums of agents' idiosyncratic shocks entering at an aggregate level. For asymmetric networks – i.e. where some firms' prices are disproportionately observable – these weighted sums are shown to not converge to zero.

Within the literature on deriving aggregate volatility from firms' idiosyncratic shocks, this paper is most closely related to work by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Saleh (2011).[1] Examining the idea of firms operating within an intersectoral supply network, they show idiosyncratic productivity shocks leading to volatility in aggregate output and, for finite networks, derive an upper limit for the rate at which aggregate volatility declines as the number of firms increases. For sufficiently asymmetric trading networks, aggregate volatility need not vanish at all. In another vein, Gabaix (2011) demonstrates how aggregate volatility can emerge from idiosyncratic shocks when the distribution of firm *sizes* exhibits fat tails, even when those firms do not trade directly with each other. Each of these share with this paper an emphasis on unequal, or fat-tailed, distributions. In the model of Gabaix (2011), aggregate volatility arises because the largest firms contribute disproportionately to aggregate production. In that by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Saleh (2011), it emerges through those firms whose output is most extensively used as an intermediate good by other firms. In the current paper, with network-based learning, it derives from firms whose price changes are most readily observed.

A large literature also exists exploring network learning. To avoid the dimensionality problems mentioned above, a common approach in this literature has been to step away from fully Bayesian updating. DeMarzo, Vayanos, and Zwiebel (2003), for example, explore situations where where agents assume that signals they receive from observing each other contain *entirely new information*. Such a rule greatly simplifies analysis, but introduces what the authors label "persuasion bias" from the agents' failure to properly discount the repetition of information they receive. Somewhat more generally, Golub and Jackson (2010) study learning in a setting where agents "naïvely" update their beliefs by taking weighted averages of their neighbour's opinions and determine conditions under which social beliefs regarding a single, fixed state of the world converge to the truth. In examining Bayesian learning over a network, previous work has typically limited attention to settings with a fixed state of the world and with agents acting sequentially (and only once each). For example, Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) study the equilibrium of a sequential learning model over a general stochastic network, showing that there will be asymptotic learning when private beliefs are unbounded and characterising some settings under which asymptotic learning still emerges when private beliefs are bounded. Other work of note includes Calvo-Armengol and de Marti (2007), who characterise a method of calculating the welfare gains from a variety of network structures in communication networks that exhibit convergent learning.

[1] The work of this paper was first developed independently by Carvalho (2010) and Acemoglu, Ozdaglar, and Tahbaz-Saleh (2010) and later combined to the paper referenced in the text.

This paper falls broadly within and was initially inspired by the literature on imperfect common knowledge. The idea that real effects may arise from nominal disturbances through imperfect information dates to Lucas (1972) and, more recently, Woodford (2003a). The solution method developed by this paper builds upon that put forward by Nimark (2008, 2011a), who introduced dynamic pricing and idiosyncratic shocks in marginal costs to the Woodford (2003a) paper. Other recent work in this area includes Adam (2007), who looked at optimal monetary policy in the Woodford setting and Melosi (2011), who uses the Survey of Professonal Forecasters to estimate a DSGE model with price setters experiencing imperfect common knowledge.

The idea of firms' existing within observation networks need not only feed into a setting of imperfect common knowledge. It might also be readily applied to the the rational inattention work of Sims (2003) or the "sticky information" literature of Mankiw and Reis (2002) and Reis (2006). If one were to suppose that a full information update was costly and observing the price of a competitor less so, a natural incentive emerges to delay a full update and instead update one's price on the basis of those of one's competitors. However, as shown below, evidence from a variety of surveys of firms' price-setting behaviour suggest that the imperfect common knowledge setting may be the more likely reason for firms' observation of each others' prices.

The remainder of this paper is organised as follows. Section 2 provides evidence in support of the price-setting model described in this paper. Section 3 then examines a generalised definition of hierarchies of expectations and presents a context-free model of learning across opaque networks. This may be of independent interest. Section 4 applies this to a small model of price setting and considers the implications for inflation dynamics. Section 5 concludes.

2 Evidence

In this section we summarise a variety of facts regarding firms' price-setting behaviour garnered over recent decades and argue that they are strongly supportive of a model of price setting in which firms obtain information by observing the prices of other firms.

At some level, that firms operate within not just transactional but also observational networks is intuitive, or even self evident. An independent coffee shop will take note of the prices offered by their competitors, including other independent outlets nearby and larger chains like Starbucks. Firms might also observe the price movements of businesses that are not direct competitors in order to learn about the structure of their costs. When a book shop observes a price change at a Thai restaurant next door, or even a car mechanic around the corner, they obtain information about movements in average marginal costs, thereby improving their ability to ascertain that portion of their own cost changes that are idiosyncratic.

In further support of this, we here first describe evidence from a number of price-setting surveys conducted in the 1990s and 2000s and then explore a series of stylised facts identified from studies of recently available datasets of observed price changes.

2.1 Price-setting Surveys

Starting with the work of Blinder (1991) and Blinder, Canetti, Lebow, and Rudd (1998) in the United States and continuing through to the first half of the 2000s, a variety of surveys were been conducted in an attempt to shed light on precisely how firms set prices. These include work in the UK (Hall, Walsh, and Yates (1997)), Sweden (Apel, Friberg, and Hallsten (2005)), Japan (Nakagawa, Hattori, and Takagawa (2000)), Canada (Amirault, Kwan, and Wilkinson (2006)) and nine euro area countries (Fabiani, Druant, Hernando, Kwapil, Landau, Loupias, Martins, MathÃd', Sabbatini, Stahl, and Stokman (2005)).[2]

When looking at those firms following partially or completely state-based pricing, Canadian firms listed price changes by competitors as the most important cause in triggering an adjustment, as did those

[2] Countries included were: Austria (Kwapil, Baumgartner, and Scharler (2005)), Belgium (Aucremanne and Druant (2005)), France (Loupias and Ricart (2004)), Germany (Stahl (2005)), Luxembourg (Lunnemann and MathÃd' (2006)), the Netherlands (Hoeberichts and Stokman (2006)), Portugal (Martins (2005)) and Spain (ÃĄlvarez and Hernando (2005)).

in Sweden. 53% of Spanish firms reported that competitors' price movements were important factors in triggering their own price changes. In considering the *magnitude* of price changes, 25% of surveyed UK firms reported basing their prices on those of their competitors. This figure agreed with the 27% of surveyed eurozone firms reporting the same, although this ranged from 13% in Portugal to 38% in France. In the Netherlands, where the survey was unique in including very small firms among those polled, this figure was 21.6% overall but rose sharply to 34.1% for firms employing only one worker.

These responses are strongly supportive of the idea that firms observe each others' prices, and we can only assume that they do so as a result of some form of imperfect information: that they learn something from their observations. However, that firms observe each others' prices does not, in itself, speak to *why* they might do so. If, for example, firms experience significant costs in developing optimal price plans in the style of Mankiw and Reis (2002, 2006, 2007) and Reis (2006), then observing one's competitors may occur if by doing so a lower cost is incurred and a fair approximation of the optimal price achieved. Alternatively, if firms face strategic complementarity in their price-setting and there are unobservable aggregate state variables in the style of Woodford (2003a) and Nimark (2008), observing other firms' decisions may be used to inform businesses of the (average) actions or beliefs of their compeititors.

Fortunately, the surveys also queried firms as to their opinions regarding the reasons for price stickiness, from which four theories stand out as being significant: implicit contracts, explicit contracts, cost-based pricing and coordination failure. All of these were among the top five recognised reasons in all 14 surveys when they were included in the options put to surveyed firms. In stark contrast, menu costs and its more recent variant, information costs, were among the least supported ideas, being in the bottom three reasons for most European surveys and Canada. Only in America and Austria were these costs placed in the middle of the group, menu costs being cited as the sixth most proximate cause of price rigidity in the United States and seventh in Austria and information costs coming sixth in Austria.

The low importance attached to information costs suggests that while there may be imperfect information, it does not manifest in the form of infrequently updated information sets. On the contrary, the strong recognition of coordination concerns and cost-based pricing are supportive of this paper's underlying model: the former suggests that businesses are subject to some form of strategic complementarity in price-setting and the latter that (presumably marginal) costs drive movements in prices.

2.2 Stylised facts from observed price changes

Although early work suggested that most prices change around once per year,[3] the seminal work by Bils and Klenow (2004) observed that the median duration of prices in CPI data from the U.S. Bureau of Labor Statistics (BLS) was 4.3 months, a frequency almost three times higher than previously thought. This triggered a rush of further work exploring and broadly characterising microeconomic price changes. Klenow and Malin (2010) provide an excellent survey of this literature and provide a summary in the form of ten stylised facts. Among these are that:

- prices change at least once a year, twice in America;
- temporary price changes – both reductions and increases – around more rigid "reference prices" are common and do not cancel out in aggregation, suggesting that some macroeconomic content is present in the more frequent updates;
- price changes are typically larger than those needed to keep up up with inflation, suggesting that idiosyncratic factors weigh more heavily on a firm's price-setting decision than aggregate factors;
- changes in *relative* prices tend to be short lived, suggesting that idiosyncratic shocks are less persistent than aggregate disturbances; and
- price changes are generally linked to changes in marginal costs, particularly wages.

The first of these necessitates some form of structural, or *real* rigidity in addition to firms' nominal rigidities – a "contract multiplier", in the words of Taylor (1980) – to explain the sluggish responses

[3] See, for example, Taylor (1999).

observed aggregate price indicies.[4] The second and third points show that even if firms' idiosyncratic shocks have zero mean and "cancel out" when averaged, average temporary price changes (that are presumably based on them) do not cancel out.

The model presented in this paper is consistent with all of the above stylised facts and with observations of rigidity in aggregate prices. Because firms are able to observe the prices of any other firm, it also represents a framework for the transmission of inflation (and hence, its persistence) across industries or geographies and not simply along production chains.

3 A generalised model of learning across opaque networks

We here develop a generalised model of Bayesian learning across opaque networks. Although the current paper is focused primarily on price-setting, this section is presented largely free of context because the model may, in general, be applied to any setting in macroeconomics or finance where agents' expected payoffs depend on the (average) actions of their competitors and some unobserved aggregate state.[5]

A simple roadmap of how this section will proceed may be of some assistance. First, in subsection 3.1, we will provide a comprehensive characterisation of higher-order expectations (beyond the simple treatement found in other papers in the literature that focus only on hierarchies of average expectations), together with an explanation of how these have traditionally defeated attempts to model Bayesian learning over social networks. Next, in subsection 3.2, we will describe the agents' problem, the information available to them and how they make their decisions. Subsection 3.3 will characterise agents' average action and briefly describe the information assumptions used in previous research and how they differ to the current paper. The main result of this paper – a model of learning when agents observe the actions of individual competitors – is then developed, and its main consequences discussed, in subsection 3.4.

3.1 Higher-order expectations

Because agents observe the actions of individual competitors, the common description of higher-order expectations as only including *average* expectations is insufficient for our needs. We therefore first provide a generalised definition of a hierarchy of expectations.

Definition 1 *Let θ_t be an $(r \times 1)$ vector of random variables, $E\left[\theta_t | \mathcal{I}_t(i)\right]$ be the expectation of θ_t conditioned on the information set of agent i and $\mathcal{E}_t\left[\theta_t\right] \equiv \begin{bmatrix} E\left[\theta_t|\mathcal{I}_t(1)\right] & \cdots & E\left[\theta_t|\mathcal{I}_t(N)\right]\end{bmatrix}$ be the $(r \times N)$ matrix containing all agents' expectations of the same. Let $\mathbf{w}$ be an $(N \times 1)$ vector of weights across all agents such that $w_i \in [0,1]$ and $\sum_{i=1}^{N} w_i = 1$. We then define a compound expectation to be a weighted sum of all agents' expectations:*

$$E_{\mathbf{w},t}\left[\theta_t\right] \equiv \mathcal{E}_t\left[\theta_t\right]\mathbf{w} \tag{1}$$

Note that this nests both simple, or unweighted, average expectations (e.g. $\mathbf{w}_A = \begin{bmatrix}\frac{1}{N} & \cdots & \frac{1}{N}\end{bmatrix}'$) and individual expectations (e.g. $\mathbf{w}_B = \begin{bmatrix}\mathbf{0}' & 1 & \mathbf{0}'\end{bmatrix}'$).

[4] See, for example, Christiano, Eichenbaum, and Evans (1999) or Romer and Romer (2004) for the USA, or Peersman and Smets (2003) for the Euro area.

[5] Two examples may be of interest: First, when posting vacancies in a labour search model in the style of Mortensen and Pissarides (1994), firms' probability of finding a successful match is dependent on the number of vacancies that other firms post. When firms' productivity includes both aggregate and idiosyncratic components, observing the number of vacancies posted by their competitors allows firms to be able to predict the component of their productivity that is common to all and their expected gain from posting an additional vacancy themselves.

Alternatively, in the asset pricing model of Singleton (1987), traders' individual demand for a risky asset is dependent on their expectation of the next-period price, itself a function of all traders' actions and (unobserved in advance) shocks to the supply of the asset. Observing the actions of (some of) their competitors allows traders to learn about the (higher-order) expectations of other traders and adjust their responses accordingly.

Definition 2 *Let $W \equiv \begin{bmatrix} \mathbf{w}_A & \mathbf{w}_B & \cdots \end{bmatrix}$ be the $(N \times p)$ matrix formed of all weights of interest in a given problem and p be the number of those weights (i.e. the number of columns in W). We then define higher-order expectations as follows, using a blackboard-bold $\mathbb{E}^{(k)}$ to denote the vector containing all expectations of the k-th order:*

$$\mathbb{E}_t^{(0)}[\theta_t] \equiv \theta_t$$

$$\mathbb{E}_t^{(k)}[\theta] \equiv \begin{bmatrix} E_{\mathbf{w}_A,t}\left[\mathbb{E}_t^{(k-1)}[\theta_t]\right] \\ E_{\mathbf{w}_B,t}\left[\mathbb{E}_t^{(k-1)}[\theta_t]\right] \\ \vdots \end{bmatrix} = vec\left(\mathcal{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta_t]\right]' W\right) \quad \forall k \geq 1 \tag{2}$$

Note that if we are interested in p different compound expectations, there are p^k different permutations of k-th order expectations. For example, if θ is scalar and $p = 2$, then the vector describing the set of second-order expectations will be of size (4×1) and arranged in the following way:

$$\mathbb{E}_t^{(2)}[\theta] = \begin{bmatrix} E_{\mathbf{w}_A,t}\left[\mathbb{E}_t^{(1)}[\theta_t]\right] \\ E_{\mathbf{w}_B,t}\left[\mathbb{E}_t^{(1)}[\theta_t]\right] \end{bmatrix} = \begin{bmatrix} E_{\mathbf{w}_A,t}\begin{bmatrix} E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \\ E_{\mathbf{w}_B,t}\begin{bmatrix} E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \end{bmatrix}$$

Definition 3 *A hierarchy of expectations, from order 0 to k, is defined recursively as:*

$$\mathbb{E}_t^{(0:k)}[\theta_t] = \begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}\left[\mathbb{E}_t^{(0:k-1)}[\theta_t]\right] \\ E_{\mathbf{w}_B,t}\left[\mathbb{E}_t^{(0:k-1)}[\theta_t]\right] \\ \vdots \end{bmatrix} \tag{3}$$

Note that this is not simply the stacking each order of expectations on top of each other. For example, if θ is scalar and $p = 2$, the hierarchies $(0:1)$ and $(0:2)$ are given by:

$$\mathbb{E}_t^{(0:1)}[\theta_t] = \begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \qquad \mathbb{E}_t^{(0:2)}[\theta_t] = \begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}\begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \\ E_{\mathbf{w}_B,t}\begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \end{bmatrix}$$

The benefit of depicting hierarchies in this manner is that it becomes simple to extract sub-hierarchies comprised of a single compound expectation. For example, if $\mathbf{w}_A = \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix}'$ so that $E_{\mathbf{w}_A,t}[\theta_t] = \overline{E}_t[\theta_t]$ is the average expectation, the sub-hierarchy of $\overline{\theta}_t^{(0:k)} \equiv \left[\theta_t', \overline{E}_t[\theta_t'], \overline{E}_t\left[\overline{E}_t[\theta_t']\right], \cdots\right]'$ may be extracted as

$$\overline{\theta}_t^{(0:k)} = \begin{bmatrix} I & 0 \end{bmatrix} \mathbb{E}_t^{(0:k)}[\theta_t]$$

In solving our model of learning over an opaque network, $\mathbb{E}_t^{(0:k^*)}[\theta_t]$ will then represent the unknown state vector about which agents attempt to learn.

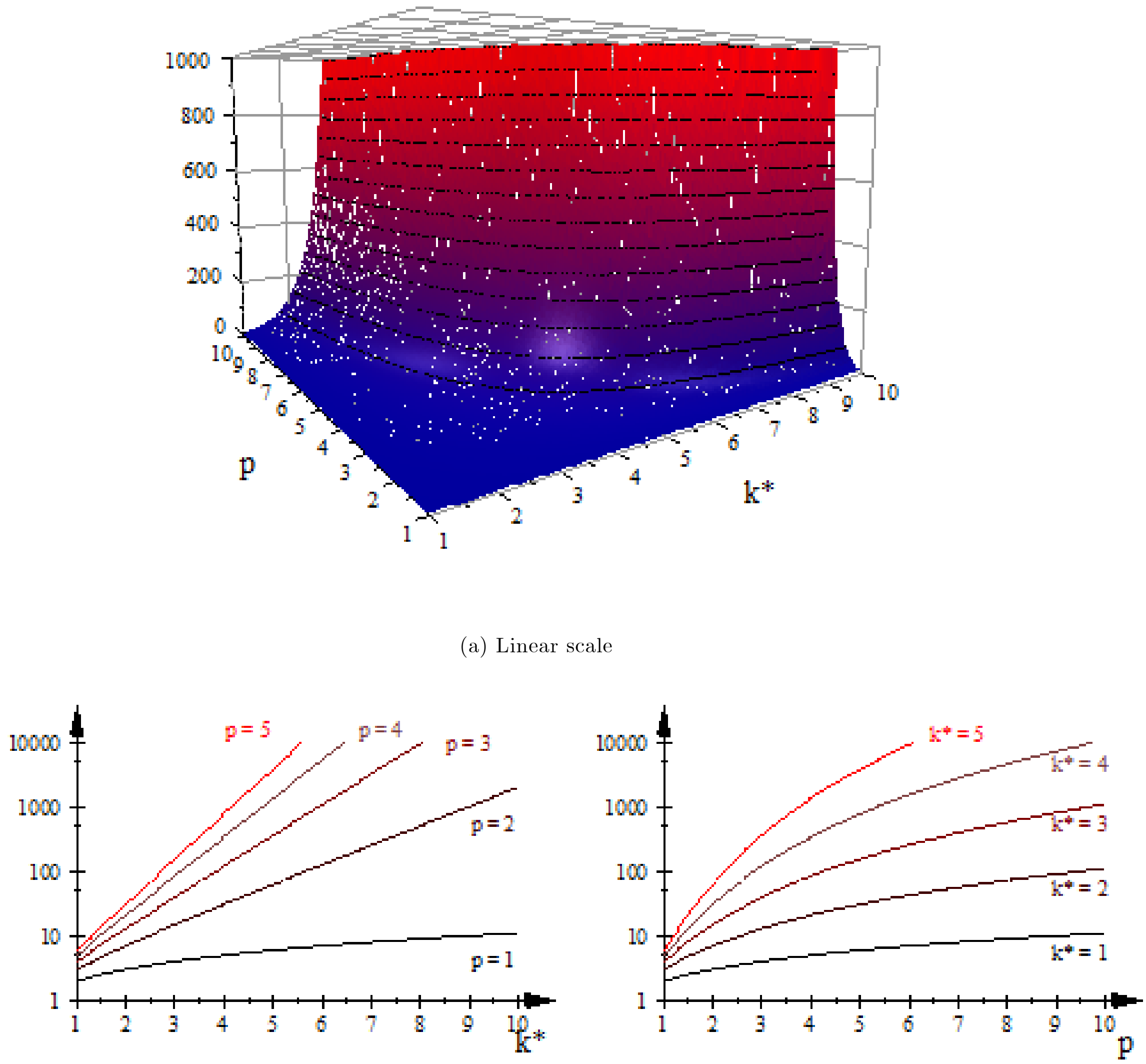
Temporarily dropping the time subscript, it is clear that if θ contains m elements, $\mathbb{E}^{(k)}[\theta]$ – the set of k-th order expectations – will contain mp^k distinct elements. However, it is worth emphasising that it does not in general follow that $\mathbb{E}^{(0:k^*)}[\theta]$ will contain $m\left(\sum_{k=0}^{k^*} p^k\right)$ unique elements. This is because if one of the compound expectations, say $E_{\mathbf{w}_B}[\cdot]$, is an individual expectation – i.e. formed from a single information set – then the law of iterated expectations implies that

$$E_{\mathbf{w}_B}\left[E_{\mathbf{w}_B}[\theta]\right] = E_{\mathbf{w}_B}[\theta]$$

In general, when $q \leq p$ is the number of individual expectations in W, the number of unique elements in the hierarchy $\mathbb{E}^{(0:k^*)}[\theta]$ will be given by:[6]

$$m\left(p^{k^*} + \sum_{k=0}^{k^*-1}\left(p^k - q\sum_{s=0}^{k} p^s\right)\right) < m\left(\sum_{k=0}^{k^*} p^k\right)$$

Nevertheless, even when $q = p$, the size of an expectation hierarchy explodes (goes to infinity) in both p and k^* (see figure 1).



(a) Linear scale

(b) Cross-section by p (log scale)

(c) Cross-section by k^* (log scale)

Figure 1: The number of elements in an expectation hierarchy ($q = 0$, θ scalar)

An infinite dimension state vector need not be a problem, *per se*, provided that the researcher is able to make a reasonable approximation of agents' actions by restricting attention to a finite subset of the

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$$m\left(\underbrace{[1]}_{\text{0-th order}} + \underbrace{[p]}_{\text{1-st order}} + \underbrace{\left[p^2 - q\right]}_{\text{2-nd order}} + \underbrace{\left[p*\left(p^2 - q\right) - q\right]}_{\text{3-rd order}} + \underbrace{\left[p*\left(p*\left(p^2 - q\right) - q\right) - q\right]}_{\text{4-th order}} + \cdots\right)$$

$$= m\left(\left(\sum_{k=0}^{k^*} p^k\right) - q\left(\sum_{k=0}^{k^*-1}\sum_{s=0}^{k} p^s\right)\right)$$, which rearranges to the equation in the text

state. In most models – including that of the current paper – imposing a finite upper limit, k^*, will be acceptable as in order to ensure stability in agent actions, decreasing weight is placed on higher order expectations.

Allowing the number of compound expectations to increase can be more problematic, however, as there is rarely an obvious reason for weighting them differently. Woodford (2003a) and Nimark (2008) avoid this difficulty by setting up their problems in a manner that implicitly assumes that $p = 1$. That is, that no matter the number of agents, they each only care about the *average* expectation of their competitors. This avenue is not available when considering learning via networks, where it is typically the case that p is given by the number of agents in the network.

3.2 The general setting

There is a countably infinite number of agents,[7] indexed in a continuum between zero and unity.[8]

The *underlying state* follows a vector autoregressive process:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + P\mathbf{u}_t \tag{4}$$

where $\mathbf{u}_t$ is a vector of shocks with mean zero, while A and P are appropriately dimensioned matricies of fixed and publicly known parameters.

Agents simultaneously determine their individual actions according to a common decision rule:

$$y_t(i) = \eta_s' \mathbf{s}_t(i) + \eta_x' E_t(i)[\mathbf{x}_t] + \eta_y E_t(i)[\overline{y}_t] \tag{5}$$

where $\mathbf{s}_t(i)$ is agent i's vector of observables (defined below), $E_t(i)[\cdot] \equiv E[\cdot|\mathcal{I}_t(i)]$ is agent i's (first-order) expectation of the element within the square brackets conditional on all information available to her in period t (defined below), $\overline{y}_t \equiv \int_o^1 y_t(i)\, di$ is the (simple, or unweighted) average action of all agents in period t, and η_s, η_x and η_y are vectors of parameters and a scalar parameter respectively, all fixed and publicly known. η_y may be thought of as a measure of agents' strategic complementarity ($\eta_y > 0$) or substitutability ($\eta_y < 0$) in actions.

The origin of the decision rule will be context-specific[9] and so it is left as general as possible for the moment. We impose only two constraints:

- that $|\eta_y| \in (0, 1)$ in order to ensure that agents place successively lower weight on higher-order expectations; and
- that only contemporaneous elements of agents' signals affect their actions directly, so that signal components obtained with a lag serve only an informational role (by helping agents construct their expectations).

Each agent's *signal vector* is made up of two, distinct components – a *private signal* based on the current-period underlying state and a *social signal* derived from observing competitors' actions with a one-period lag:

$$\mathbf{s}_t^p(i) = B\mathbf{x}_t + Q\mathbf{v}_t(i)$$
$$\mathbf{s}_t^s(i) = W_{t-1}(i)\,\mathbf{y}_{t-1}$$

It is assumed that private signals are noisy, with $\mathbf{v}_t(i)$ a vector of shocks specific to agent i in period t, drawn from independent and identical Gaussian distributions with mean zero and variance Σ_{vv}. The dimensions of B and Q are left unspecified, depending on the number of observables made available

[7] An infinite number of agents is assumed to allow an appeal to relevant laws of large numbers when considering simple averages of zero-mean shocks.

[8] The assumption of indexing agents from zero to one is innocuous and made only to simplify the calculation of averages.

[9] For example, in the context of price-setting to be explored in the next section, the underlying state will include aggregate shocks to marginal cost and demand, while agents' actions will be the price they choose and the "signal" they receive will be their private marginal cost and the previous-period price of a competitor.

from the underlying state. Social signals are assumed to be observed perfectly, where $\mathbf{y}_{t-1}$ is the $(\infty \times 1)$ vector of all agents' actions from the previous period and $W_{t-1}(i)$ is i's (potentially stochastic) observation matrix. For example, if in period t agent i observes the period $(t-1)$ actions of agents 1 and 2, then $W_{t-1}(i)$ would be given by:

$$W_{t-1}(i) = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \end{bmatrix}$$

We combine the two signals as a single signal vector thus:

$$\mathbf{s}_t(i) = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} \mathbf{0} \\ W_{t-1}(i) \end{bmatrix} \mathbf{y}_{t-1} + \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \mathbf{v}_t(i) \tag{6}$$

The assumption that observations based on the previous period serve only an informational role in the current period (by helping agents construct their expectations) and do not affect actions directly therefore gives the following form to η_s:

$$\eta_s \equiv \begin{bmatrix} \alpha \\ \mathbf{0} \end{bmatrix} \tag{7}$$

Note, in particular, that this implies that $\eta_s' \mathbf{s}_t(i) = \alpha' (B\mathbf{x}_t + Q\mathbf{v}_t(i)) \; \forall i, t$. Finally, we have that agent i's information set evolves as follows:

$$\begin{aligned} \mathcal{I}_0(i) &= \{A, P, B, Q, \Phi, W_0(i)\} \\ \mathcal{I}_t(i) &= \{\mathcal{I}_{t-1}(i), \mathbf{s}_t(i), W_t(i)\} \end{aligned} \tag{8}$$

That is, in each period, agent i is informed of their private signal, their social signal and the identity of the competitors whose actions they will receive in the next period. $\Phi : [0,1] \to [0,1]$ is the (cumulative) distribution from which observees are drawn, assumed to be identical and independent for every agent. $\Phi(j)$ is absolutely continuous over the range $[0,1]$ and has p.d.f. ϕ.

3.3 Average actions and imperfect common knowledge

At first glance, obtaining $E_t(i)[\overline{y}_t]$ in the agent's decision rule may appear implausible, as it would seem to require the formation of an expectation regarding every competitor's action ($E_t(i)[\overline{y}_t] = \int_0^1 E_t(i)[y_t(j)]\, dj$). However, we can greatly simplify matters by noting that the average action – i.e., the average of equation (5) – can be written as

$$\overline{y}_t = \eta_s' \overline{\mathbf{s}}_t + \eta_x' \overline{E}_t[\mathbf{x}_t] + \eta_y \overline{E}_t[\overline{y}_t]$$

Making use of the law of large numbers, we can next see that $\eta_s' \overline{\mathbf{s}}_t = \alpha' B \mathbf{x}_t$. Repeatedly substituting our expression for the average action back into itself, we therefore obtain

$$\overline{y}_t = \alpha' B \mathbf{x}_t + \sum_{k=1}^{\infty} \eta_y^{k-1} (\eta_x' + \eta_y \alpha' B) \overline{E}_t^{(k)}[\mathbf{x}_t]$$

where $\overline{E}_t^{(k)}[\cdot] \equiv \overline{E}_t\left[\overline{E}_t^{(k-1)}[\mathbf{x}_t]\right]$ is the (simple) average k-th order expectation of $\mathbf{x}_t$. Note that as $|\eta_y| \in (0,1)$, successively lower weight is placed on higher-order expectations. We can rewrite this more simply as:

$$\overline{y}_t = \alpha' B \mathbf{x}_t + \beta' \overline{\mathbf{x}}_{t|t}^{(1:\infty)} \tag{9}$$

where the vector β is given by:

$$\beta' = \begin{bmatrix} (\eta_x' + \eta_y \alpha' B) & \eta_y (\eta_x' + \eta_y \alpha' B) & \eta_y^2 (\eta_x' + \eta_y \alpha' B) & \cdots \end{bmatrix} \tag{10}$$

Substituting this into (5), we are then able to write agent i's decision rule as:

$$y_t(i) = \alpha' \underbrace{(B\mathbf{x}_t + Q\mathbf{v}_t(i))}_{\text{Private signal}} + \beta' E_t(i)\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right] \tag{11}$$

That is, each agent constructs their action as a linear combination of their private signal and their expectation of the entire hierarchy of *simple average* expectations of the underlying state. Agents therefore need to estimate an expanded *state vector of interest* that includes not just the underlying state, but also the higher-order average expectations of the same.

Because of the linearity of the underlying system, the best linear estimator – in the sense of minimising the mean squared error – will be a Kalman filter.[10] [11] Denoting the state vector to be estimated as Z_t (which will include $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ but may, depending on the model, have other components), the signal equation will be able to be rewritten in the following form

$$\mathbf{s}_t(i) = LZ_t + \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \mathbf{v}_t(i)$$

Supposing an AR(1) process for Z_t's law of motion

$$Z_t = MZ_{t-1} + N \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{bmatrix}$$

agent i's period-t expectation of Z_t will be formed recursively in the following manner

$$E_t(i)[Z_t] = K\mathbf{s}_t(i) + (M - KLM) E_{t-1}(i)[Z_{t-1}] \tag{12}$$

where K is a time-invariant Kalman gain matrix. As in other models of imperfect common knowledge, since Z_t includes $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$, we have that (a) the state vector to be estimated is of infinite dimension; and (b) the Kalman filter serves a dual role, both as estimator and as part of the law of motion for the state vector. Solving the system then requires finding the coefficients in K, M, N, L and V (the variance-covariance of the agents' estimates). This, in turn, depends on the exact signal structures faced by the agents.

Woodford (2003a), looking at firms' static price-setting decisions, supposed that agents receive only a private signal from the underlying state and no social signal from other agents ($W_t(i) = \mathbf{0}\ \forall i, t$). In such a setting, where $Z_t = \overline{\mathbf{x}}_{t|t}^{(0:\infty)}$, Woodford showed that M will be lower-triangular: each order of simple average expectations will be a linear combination of *lower* order expectations and current period shocks. Consequently, M may be constructed sequentially, first finding an expression for $E_t(i)[\mathbf{x}_t]$, then averaging it and repeating the process to find $E_t(i)\left[\overline{E}_t[\mathbf{x}_t]\right]$ and so forth.

Nimark (2008), who focused on a context of dynamic price-setting, extended Woodford's (2002) work to allow agents to observe the simple-average action from the previous period in addition to their private signals. In the nomenclature above, this amounts to

$$W_{t-1}(i)\,\mathbf{y}_{t-1} = \lim_{N\to\infty} \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} \mathbf{y}_{t-1} = \overline{y}_{t-1}\ \forall i$$

In this case, agents' signals are linear combinations of the entire hierarchy of previous-period expectations (since actions are based on the entire hierarchy and agents observe the average previous action) and as a result, the solution must be found for all higher-order expectations simultaneously. Note, too, that since the Kalman filter requires that agents form prior expecations about the signals they receive, this typically requires that the state vector also include $\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}$ so that $Z_t = \begin{bmatrix} \overline{\mathbf{x}}_{t|t}^{(0:\infty)\prime} & \overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)\prime} \end{bmatrix}'$.[12]

[10] If all shocks are drawn from Gaussian distributions, it will be the best such estimator, linear or otherwise.

[11] The derivation of the standard Kalman filter may be found in most texts on dynamic macroeconomics (e.g. Ljungqvist and Sargent (2004)) or timeseries analysis (e.g. Hamilton (1994)).

[12] An alternative to including $\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}$ in the state vector of interest is to use a slightly different specification of the signal vector:

$$\mathbf{s}_t(i) = L_1\overline{\mathbf{x}}_{t|t}^{(0:\infty)} + L_2\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)} + \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \mathbf{v}_t(i)$$

and correspondingly modify the Kalman filter:

$$E_t(i)\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right] = K\mathbf{s}_t(i) + (M - K(L_1M + L_2)) E_{t-1}(i)\left[\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}\right]$$

See Nimark (2011b) for more detail.

It is perhaps worth empahsising that the signal structures assumed by Woodford (2003a) and Nimark (2008) both result in agents only being concerned with the simple *average* expectation of their peers (or higher-order versions of the same). In the language of this paper, they have chosen signal structures that explicitly set $p = 1$, thereby having the infinite dimensionality of the state vector arising only from the presence of higher-order expectations. With the condition that $|\eta_y| \in (0, 1)$, successively lower weight is placed on higher-order expectations and so a finite system can be made arbitrarily accurate in approximating the full system by defining a threshold for the upper limit of orders of expectation.

3.4 Observing individual competitors' actions

Suppose that agents observe the previous-period actions of q competitors. We introduce the function $\delta_t : [0,1] \to [0,1]^q$ to map each agent to their observational target(s). For presentational simplicity, in what follows we will typically assume that $q = 1$ (i.e. that all agents observe a single competitor) and simply write $j = \delta_t(i)$ to mean that agent j's period-t action will be observed by agent i. To speak of the the observee of an observee, we write $\delta_s(\delta_t(i))$: the identity of the agent whose period-s action is observed by the agent whose period-t action is observed by agent i. The function $\delta_t(i)$ is related to $W_t(i)$ in the following way:

$$W_t(i) = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{\text{Column } j=\delta_t(i)} & 0 & \cdots & 0 \end{bmatrix}$$

In this setting, agent i's signal vector for period t will have two elements in it:

$$\mathbf{s}_t(i) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(i) \\ y_{t-1}(\delta_{t-1}(i)) \end{bmatrix}$$

Agent i's prior will therefore include $E_{t-1}(i)[y_{t-1}(\delta_{t-1}(i))]$ and, stepping forward one period, we have that agent i must form $E_t(i)[y_t(\delta_t(i))]$ in period t as part of her prior for period $t+1$.

To allow us to solve the general agent's problem, we make the following assumptions regarding the agents' observation network:

Assumption 1 *The network is stochastic and opaque, in that:*

- *all agents observe the same number of competitors;*
- *observees are drawn from identical, independent distributions with p.d.f. $\phi_N(i)$;*
- *agents know the identity of the other agents they observe;*
- *agents do not know who they are observed by; and*
- *agents do not learn about the network topology over time.*

To obtain this last point, we might either suppose that agents make a fresh draw of whom to observe every period,[13] in which case nothing *could* be learned about the network topology (since it changes every period), or allow the network to be drawn once and assume a form of bounded rationality in the agents, in that they focus only on the game in front of them and not the structure of the network. Purely for notational simplicity, the remainder of this paper shall suppose the latter scenario, thereby allowing us to drop any time subscripts from agents' observation mappings ($W_t(i) = W(i)$ and $\delta_t(i) = \delta(i)\ \forall t$). Proofs provided in the appendicies shall retain the time subscripts to allow for either scenario.

We are then in a position to assert the following:

Lemma 1 *Given assumption 1, agents' use of a linear estimator implies that all agents treat all other agents as though they observe a common, weighted average of previous-period actions, with the weights given by the distribution ϕ.*

[13] In such a setting, it may be better to imagine agents not operating in a network so much as a search model.

Proof. The proof may be found in appendix B. ■

From equation (11), we see that the weighted-average action, $\widetilde{y}_t$, is given by:

$$\widetilde{y}_t = \alpha' B \mathbf{x}_t + \alpha' Q \widetilde{\mathbf{v}}_t + \beta' \widetilde{E}_t \left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)} \right] \tag{13}$$

where $\widetilde{E}_t [\cdot] \equiv \int_0^1 E_t (j) [\cdot] \phi (j) \, dj$ is the weighted-average expectation and the last term is the weighted average of private expectations regarding the hierarchy containing only *unweighted* average expectations.

Note that we cannot, in general, make use of some law of large numbers to disregard the effect of idiosyncratic shocks in the weighted-average action – that is, we cannot assume that $\widetilde{\mathbf{v}}_t \equiv \int_0^1 \mathbf{v}_t (j) \phi (j) \, dj$ will be equal to zero – because the weights applied to each agent may not be sufficiently close to equal. As an extreme example, if all agents were to observe agent 1 and nobody else (i.e. $\phi(1) = 1$ and $\phi(i) = 0$ $\forall i \neq 1$), we would then have that $\widetilde{\mathbf{v}}_t = \mathbf{v}_t (1)$ which will, with probability one, be not equal to zero.

Identifying laws of large numbers for weighted sums of i.i.d. random variables (i.e. the limiting behaviour of $\sum_{i=1}^N a_{N,i} X_i$ when $E[X] = 0$) remains an area of active research. See, for example, Wu (1999), Sung (2001) or Cai (2006). However, it is not necessary for us to have an exact characterisation of the necessary conditions for the weighted sum to converge to zero, as there are a broad range of functions for the weights under which the weighted sum will *not* converge to zero. In particular, we make the following assumption:

Assumption 2 *The network is asymmetric. That is, denoting $g(N) \equiv \sum_{i=1}^N \phi_N (i)^2$, the p.d.f. $\phi_N (i)$ is such that:*

- $\lim_{N \to \infty} g(N) = g^*$ *where* $g^* \in (0, \infty)$.

This assumption then allows us to assert the following lemma regarding limiting properties of aggregate (random) variables derived from agents' idiosyncratic shocks:

Lemma 2 *Suppose that $\mathbf{v}_t (i) \sim i.i.d. N(\mathbf{0}, \Sigma_{vv})$ $\forall i, t$. For a finite number of agents (N), let $\widetilde{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^N \mathbf{v}_t (i) \phi_N (i)$ denote the weighted average of agents' own idiosyncratic shocks; $\ddot{\mathbf{v}}_{N,t} \equiv \frac{1}{N} \sum_{i=1}^N \mathbf{v}_t (\delta_t (i))$ denote the simple average of the idiosyncratic shocks of agents' observees; and $\widehat{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^N \mathbf{v}_t (\delta_t (i)) \phi_N (i)$ denote the weighted average of agents' observees' idiosyncratic shocks. Given assumption 2,we have the following results in the limit (as $N \to \infty$):*

- $\widetilde{\mathbf{v}}_{N,t} \longrightarrow^d \widetilde{\mathbf{v}}_t$ *where* $\widetilde{\mathbf{v}}_t \sim N(\mathbf{0}, g^* \Sigma_{vv})$
- $\ddot{\mathbf{v}}_{N,t} \longrightarrow^{\mathcal{L}^2} \widetilde{\mathbf{v}}_t$
- $\widehat{\mathbf{v}}_{N,t} \longrightarrow^d \widehat{\mathbf{v}}_t$ *where* $\widehat{\mathbf{v}}_t \sim N(\mathbf{0}, g^* (2 - g^*) \Sigma_{vv})$
- $Cov(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t) = g^* \Sigma_{vv}$

Proof. The proof may be found in appendix B. ■

The first result in this lemma shows that assumption 2 is sufficient to ensure that idiosyncratic shocks do not "wash out" in the weighted-average action. The set of distributions satisfying this assumption is quite broad. In particular, it is satisfied by the discrete power law distribution (the Zipf distribution)

$$\phi_N (i) = c_N i^{-\gamma}; \text{ where } c_N = \left(\sum_{i=1}^N i^{-\gamma} \right)^{-1} \quad \text{and } \gamma > 1$$

and it's equivalent for infinite N, the Zeta distribution. The shape parameter, $\gamma > 1$, governs the scaling of the distribution's tail: larger values of γ correspond to greater asymmetry in the distribution and, as such, the greater the variance that survives aggregation. Figure 2 plots the values of g^* for a range of values of γ for the Zeta distribution.

A great many observed networks – from webpages on the internet to established relationships in social networks – have been shown to have degree distributions well approximated by power law distributions (i.e. the networks are *scale free*). See, for example, Albert and BarabÃąsi (2002), Jackson and Rogers (2007) or Clauset, Shalizi, and Newman (2009).

Finally, we now present the following main result of this section:

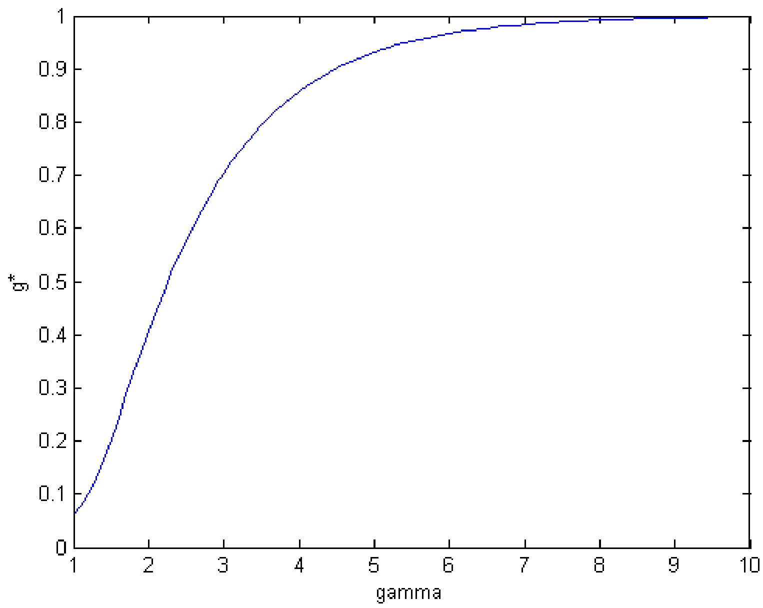


Figure 2: A plot of g^* for power law (Zeta) distributions with shape parameter γ

Proposition 1 *Given assumptions 1 and 2, and the generalised setting described above, agents' need only consider a hierarchy of $p = 3$ compound expectations and a solution may be obtained in two steps:*

- *First, for a given set of weights that are common knowledge, obtain a solution to a model where all agents observe the same weighted average of everybody's previous-period action; and*
- *Second, solve the individual problem supposing that information from any target comes from the setting in the first stage, where the weights used are the distribution from which agents' observation targets are drawn. The hierarchy of expectations of the underlying state will evolve according to the following ARMA(1,1) process:*

$$
\begin{aligned}
X_t &= \mathbb{E}_t^{(0:\infty)}\left[\mathbf{x}_t\right] \\
X_t &= F X_{t-1} + G_1 \mathbf{u}_t + G_2 \widetilde{\mathbf{v}}_t + G_3 \widetilde{\mathbf{v}}_{t-1} + G_4 \widehat{\mathbf{v}}_{t-1} \\
\overline{\mathbf{x}}_{t|t}^{(0:\infty)} &= \begin{bmatrix} I & \mathbf{0} \end{bmatrix} X_t
\end{aligned}
$$

Proof. The proof may be found in appendix B. ∎

An outline of agents' process of learning about the underlying state is provided in appendix A.

Three broad consequences of the model therefore emerge.

First, it is possible to simulate the effects of network learning without having to model the network explicitly: the results above represent, in essence, a sufficient statistic for the network's effects. This makes the model particularly amenable to nesting within broad General Equilibrium models of the economy.

Second, because the network has a distribution of links that is sufficiently far from uniform, mean zero idiosyncratic shocks do not wash out in aggregation, thereby leading to a network based origin for aggregate volatility. The size of this additional source of aggregate volatility depends on the asymmetry of the network, which is captured simply in a single parameter: g^*.

Third, the aggregate effects of idiosyncratic shocks are *persistent*. This comes about for two reasons: first, because agents observe competitors' prices with a one-period lag, the timing of their immediate impact will be delayed mechanically, and second, the aggregate effects, once present, become persistent because of the recursive nature of agents' learning. The degree of persistence will naturally vary with the parameterisation of the model, but broadly increases with the persistence of true aggregate shocks and with the degree of strategic complementarity in agents' actions.

4 Price setting within a network

We now present an application of the previous section's results to a simple model of firms' price setting behaviour. Firms are monopolistically competitive in the sense of Dixit and Stiglitz (1977) and, in the spirit of fairly high-frequency updates observed in the data, are free to update their prices every period. All firms will therefore have that their optimal price is a (common) markup over their marginal cost. Expressed in terms of log deviations from long-run trends, this is:

$$p_t(j)^* = p_t + mc_t(j) \tag{14}$$

where $p_t = \int p_t(j)\,dj$ is the aggregate price level of the economy. Firms do not know the aggregate price level or their own marginal costs at the time of setting prices and so instead set prices according to

$$p_t(j) = E_t(j)[p_t] + E_t(j)[mc_t(j)] \tag{15}$$

Marginal costs increase with aggregate output with constant elasticity and are subject to shocks

$$mc_t(j) = \psi y_t + \omega_t(j) \tag{16}$$

Firms observe $\omega_t(j)$ before setting their prices, but do not know the level of aggregate output until after all firms have set their prices. On the demand side of the economy, we simply suppose an exogenous, stochastic process – determined by a monetary authority – for the level of nominal GDP (NGDP).

$$n_t = y_t + p_t \tag{17}$$

Together, equations (15), (16) and (17) provide the following decision rule for firms' pricing decisions

$$p_t(j) = \omega_t(j) + \psi E_t(j)[n_t] + (1-\psi)E_t(j)[p_t] \tag{18}$$

so that, in setting their individual prices, firms need to form expectations regarding both the level of aggregate demand and the average price that will be set by all firms.

Shocks to firms' marginal costs have both an aggregate and an idiosyncratic component:

$$\omega_t(j) = \lambda_t + v_t(j) \tag{19}$$

Aggregate shocks to marginal cost are first-order autoregressive and affected, with a lag, by deviations in NGDP (n_t).

$$\begin{aligned} \lambda_t &= \rho_{\lambda\lambda}\lambda_{t-1} + \rho_{\lambda n} n_{t-1} + \varepsilon_t \\ \varepsilon_t &\sim N\left(0, \sigma_\varepsilon^2\right) \end{aligned} \tag{20}$$

where $\rho_{\lambda\lambda} \in (0,1)$ and $\rho_{\lambda n} \in (0,1)$. The parameter $\rho_{\lambda n}$ is designed to capture, in a simple fashion, the idea of increased aggregate demand leading to (lagged) increases in wage demands that are reflected in marginal costs. Idiosyncratic shocks to marginal costs are entirely transitory

$$v_t(j) \sim N\left(0, \sigma_v^2\right) \tag{21}$$

Deviations of NGDP from it's long term trend are assumed to follow a first-order autoregressive process,

$$\begin{aligned} n_t &= \rho_{nn} n_{t-1} + \xi_t \\ \xi_t &\sim N\left(0, \sigma_\xi^2\right) \end{aligned} \tag{22}$$

where $\rho_{nn} \in (0,1)$ parametises the persistence in deviations from that trend growth rate. The disturbance, ξ_t, may be thought of as representing a monetary shock.

To express this setup in the form of the previous section, define the underlying state vector, $\mathbf{x}_t \equiv \begin{bmatrix}\lambda_t & n_t\end{bmatrix}'$, and vector of aggregate shocks, $\mathbf{u}_t \equiv \begin{bmatrix}\varepsilon_t & \xi_t\end{bmatrix}'$, so the underlying system updates as

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + P\mathbf{u}_t$$
$$A = \begin{bmatrix}\rho_{\lambda\lambda} & \rho_{\lambda n} \\ 0 & \rho_{nn}\end{bmatrix} \quad , \quad P = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

Firms observe the combined shock to their marginal costs, $\omega_t(j)$, and the lagged price of a specific competitor:

$$\mathbf{s}_t(j) = \begin{bmatrix}\omega_t(j) \\ p_{t-1}(\delta_{t-1}(j))\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\mathbf{x}_t + \begin{bmatrix}\mathbf{0} \\ W_{t-1}(i)\end{bmatrix}\mathbf{p}_{t-1} + \begin{bmatrix}1 \\ 0\end{bmatrix} v_t(i) \tag{23}$$

In the notation of the previous section, we can then express firms' pricing decision rule as:

$$p_t(j) = \alpha'\left(\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}_t + v_t(i)\right) + \beta' E_t(j)\left[\overline{\mathbf{x}}_t^{(0:\infty)}\right] \tag{24}$$

where the vectors α and β will be given by:

$$\begin{aligned}\alpha' &= 1 \\ \beta' &= \left[\begin{bmatrix}(1-\psi) & \psi\end{bmatrix} \quad \begin{bmatrix}(1-\psi)^2 & \psi(1-\psi)\end{bmatrix} \quad \begin{bmatrix}(1-\psi)^3 & \psi(1-\psi)^2\end{bmatrix} \quad \cdots\right]\end{aligned} \tag{25}$$

Note that the weight attached to higher order expectations is governed by ψ (the elasticity of average marginal costs with respect to aggregate output), with $1-\psi$ controling the degree of strategic complementarity. Unless otherwise specified, the following baseline parameters are used to calibrate the simulation:

Parameter	Value
ψ	0.15
$\rho_{\lambda\lambda}$	0.7
$\rho_{\lambda n}$	0.1
$\rho_{\lambda n}$	0.7
g^*	0.1
$\sigma_\xi^2/\sigma_\varepsilon^2$	1
$\sigma_v^2/\sigma_\varepsilon^2$	4

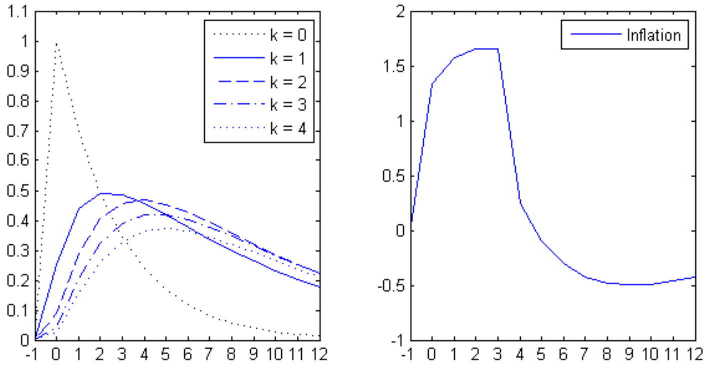
Table 1: Baseline parameterisation

Woodford (2003a,b) argues that a value of 0.15 for ψ is realistic for the U.S. economy and proposes a value of roughly 6 for $\sigma_v^2/\sigma_\varepsilon^2$ in order to have his formative version of imperfect common knowledge match the inflation dynamics predicted by a Calvo (1983) pricing model with one third of all prices revised in each quarter. The value of 0.1 for g^* corresponds to a power law distribution with shape parameter $\gamma = 1.2$.

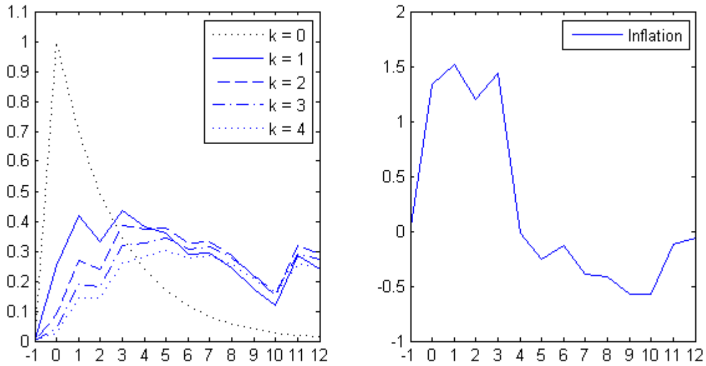
Figure 3 shows the hierarchy of simple average expectations regarding λ_t (the aggregate component of shocks to firms' marginal costs) and the path of inflation following a shock to aggregate marginal costs under the network learning model, both without and with an illustrative set of idiosyncratic shocks.[14] Expectations respond with a delay and exhibit considerable persistence above the shock itself.

Next, figure 4 illustrates the aggregate effects of a one-period shock to firms' idiosyncratic shocks. Note that these arise from two, correlated aggregate random variables derived from firms' idiosyncratic shocks. The first, from $\widetilde{v}_t = \int \mathbf{v}_t(i)\,\phi(i)\,di$, is the weighted average of all firms' shocks, while $\widehat{v}_t = \int \mathbf{v}_t(\delta(i))\,\phi(i)\,di$ may be thought of as a double-weighted average. The aggregate effects of $\widehat{v}_t$ are more muted, but enter with a greater lag and exhibit a more pronounced hump shape in their response.

[14] In these figures, inflation is defined as $p_t - p_{t-4}$, corresponding to the lagged 12 month inflation rate if each period is interpreted as a quarter.

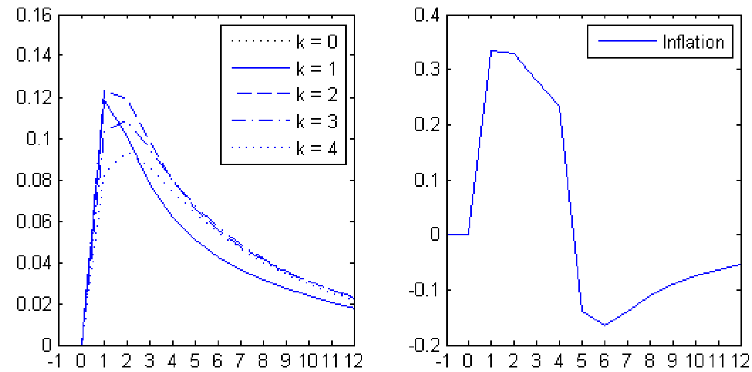


(a) Without idiosyncratic shocks

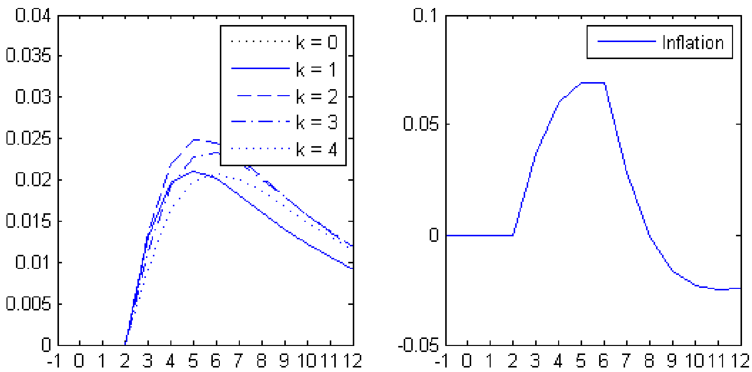


(b) With illustrative set of idiosyncratic shocks

Figure 3: Hierarchy of simple average expectations regarding λ_t and Inflation $(p_t - p_{t-4})$ following a shock to λ_t.



(a) Shock to $\widetilde{v}_t$



(b) Shock to $\widehat{v}_t$

Figure 4: Hierarchy of simple average expectations regarding λ_t and Inflation $(p_t - p_{t-4})$ following a one-period set of idiosyncratic shocks to firms' marginal costs.

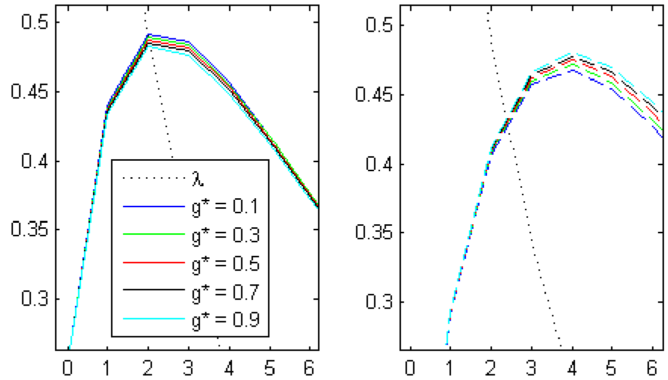


Figure 5: Comparison of first- and second-order simple average expectations for various values of g^*.

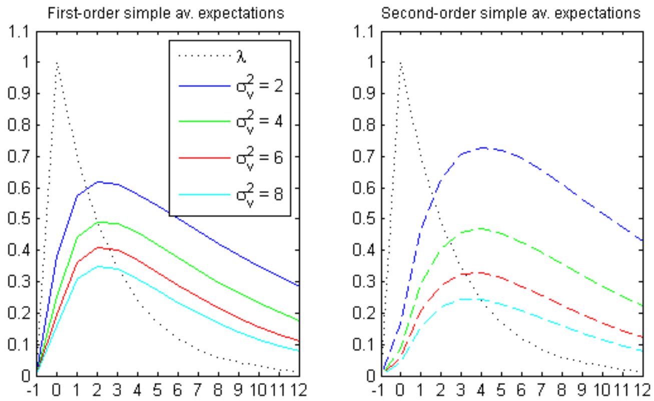


Figure 6: Comparison of first- and second-order simple average expectations for various values of $\sigma_v^2/\sigma_\varepsilon^2$.

The sensitivity of these impulse responses to changes in parameters is largely as might be expected. Figure 5 shows how the first-order simple average expectation of λ_t following an aggregate shock to marginal costs varies as the network becomes more asymmetric. Recall from above that low values of g^* correspond to relatively uniform network distributions and high values of g^* to highly asymmetric networks. As the network becomes more asymmetric, firms' higher-order expectations are shifted up, leading to a corresponding increase in the magnitude of the shock passed through to inflation.

Figure 6 illustrates the implications of varying the relative variance of idiosyncratic shocks $(\sigma_v^2/\sigma_\varepsilon^2)$ for the persistence of aggregate shocks. As one might expect, increased variance in idiosyncratic shocks relative to aggregate shocks causes agents' private signals to carry less information and the response of agents' expectations is correspondingly subdued.

Finally, in figure 7 and as is to be expected, we see that increasing ψ reduces the degree of strategic complementarity in the system and hence the persistence (although not the initial impact) of shocks' effects on the expectation hierarchy are reduced.

The contribution of idiosyncratic shocks to aggregate volatility may be examined by considering the variance of agents' simple-average action. As average actions are given by $\overline{y}_t = \begin{bmatrix} \alpha' B & \beta' \end{bmatrix} \overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ (see equation 9) and the hierarchy of simple-average expectations is extracted from the full hierarchy as $\begin{bmatrix} I & \mathbf{0} \end{bmatrix} X_t$, we can write the unconditional variance of the average action as

$$Var\left(\overline{y}_t\right) = \begin{bmatrix} \alpha' B & \beta' \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \Sigma \begin{bmatrix} I & \mathbf{0} \end{bmatrix}' \begin{bmatrix} \alpha' B & \beta' \end{bmatrix}'$$

where Σ is the time-invariant variance-covariance matrix of X_t obtained from the stage-two Kalman filter. Table 2 presents the implied variance of the aggregate price level for various values of $\sigma_v^2/\sigma_\varepsilon^2$ and g^*. The case of $g^* = 0$, corresponding to a scenario of a perfectly uniform network, is provided for reference. Note, from lemma 2, that the variance of $\widetilde{v}_t$ and $\widehat{v}_t$ are given by $g^* \sigma_v^2$ and $g^*(2 - g^*)\sigma_v^2$ respectively. Given that the latter quadratic is strictly increasing for $g^* \in (0, 1)$ and as we might therefore

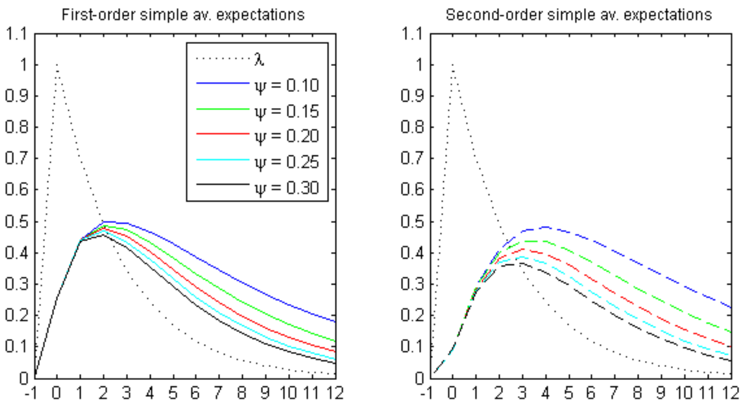


Figure 7: Comparison of first- and second-order simple average expectations for various values of ψ.

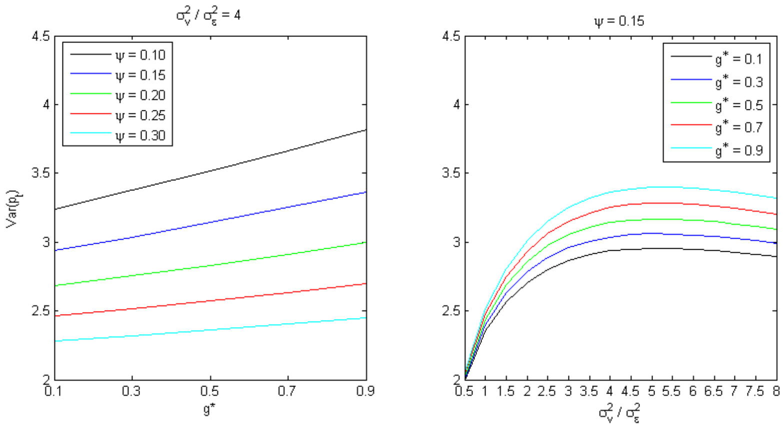


Figure 8: Unconditional variance of aggregate prices as a function of network asymmetry (g^*), strategic complementarity ($1 - \phi$) and the relative variance of idiosyncratic shocks ($\sigma_v^2/\sigma_\epsilon^2$).

expect, aggregate volatility is monatonically increasing in the asymmetry of the network. Curiously, however, aggregate volatility starts to decline for sufficiently high variance in idiosyncratic shocks relative to aggregate shocks.

	$g^* = 0.0$	$g^* = 0.1$	$g^* = 0.3$	$g^* = 0.5$	$g^* = 0.7$	$g^* = 0.9$
$\sigma_v^2/\sigma_\varepsilon^2 = 1$	2.34	2.36	2.40	2.44	2.48	2.52
$\sigma_v^2/\sigma_\varepsilon^2 = 2$	2.67	2.71	2.78	2.86	2.93	3.01
$\sigma_v^2/\sigma_\varepsilon^2 = 4$	2.89	2.94	3.04	3.14	3.25	3.36
$\sigma_v^2/\sigma_\varepsilon^2 = 6$	2.90	2.95	3.05	3.16	3.28	3.39
$\sigma_v^2/\sigma_\varepsilon^2 = 8$	2.85	2.90	2.99	3.09	3.20	3.32

Table 2: Variance of aggregate prices by network asymmetry (g^*) and the relative variance of idiosyncratic shocks (σ_v^2)

The reason for this result is not immediately clear, but would appear to represent a trade-off to agents' use of information gained from each others' prices. As σ_v^2 increases, the useful information available from another firm's price correspondingly decreases and, given the asymmetric nature of the network, increasingly comes to represent that distortion and not the underlying state. However, it not therefore apparent why this would persist for perfectly uniform networks ($g^* = 0$), too, and so remains to be explored in further research.

The interaction between the asymmetry of the network and the degree of strategic complementarity is shown in table 3. Increasing strategic complementarity (lower ψ) leads to greater amounts of aggregate volatility being passed through from idiosyncratic shocks, with this effect strengthening in the asymmetry of the network.

	$g^* = 0.0$	$g^* = 0.1$	$g^* = 0.3$	$g^* = 0.5$	$g^* = 0.7$	$g^* = 0.9$
$\psi = 0.10$	3.18	3.24	3.38	3.52	3.66	3.81
$\psi = 0.15$	2.89	2.94	3.04	3.14	3.25	3.36
$\psi = 0.20$	2.64	2.68	2.75	2.83	2.91	3.00
$\psi = 0.25$	2.44	2.46	2.52	2.58	2.64	2.70
$\psi = 0.30$	2.26	2.28	2.32	2.36	2.41	2.45

Table 3: Variance of aggregate prices by network asymmetry (g^*) and the elasticity of marginal costs to output (ψ)

5 Conclusion

This paper has argued that firms set their prices while operating in an observation network, making use of competitors' prices to learn about hidden aggregate states of the world. That firms operate in a network and that they do so in a model of imperfect common knowledge is motivated by the observation that when surveyed, a large fraction of firms across North America and Europe admit to looking to other firms in deciding both the timing and the magnitude of price changes and do so out of a desire to coordinate pricing changes with competitors.

The paper's central contribution is to present and solve a generalised linear model of social learning over a network. It is shown that when the network is opaque, in that agents do not know the full structure of the network but instead only the identity of their observees and the distribution from which observees are drawn, the social learning problem becomes analytically tractable and admits a closed-form solution. When the distribution of links within the network is sufficiently asymmetric, agents' mean zero idiosyncratic shocks will not "wash out" in aggregation and agents' expectations will follow an ARMA(1,1) process with current and lagged values of weighted sums of agents' idiosyncratic shocks entering at an aggregate level. The recursive nature of agents' learning then implies that the aggregate effects of idiosyncratic shocks will be persistent, despite the individual agents' shocks being entirely transitory, with this persistence increasing in the degree of strategic complementarity, the asymmetry of the network and the persistence of any aggregate shocks.

In the context of firms' price-setting decisions, these persistent aggregate effects therefore represent a learning-based microfoundation for cost push shocks, with inflation able to persistently deviate from it's long-run trend entirely in the absence of any aggregate shocks to the economy and despite firms being able to update prices in every period. Because firms may choose to observe the prices of other firms with whom they are are not direct competitors, this also represents a novel transmission mechanism for inflation across industries or geographies independent of it's path along production chains. It should be noted that recent work by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Saleh (2011) has previously illustrated the possibility of aggregate volatility emerging through this latter "real" network.

In contrast to the common assumption that idiosyncratic shocks cancel out in aggregation, the emergence of aggregate-level price changes based on short-lived idiosyncratic shocks is consistent with recent evidence garnered from a variety of observed panels of micro price changes. The level of aggregate volatility induced through network learning is increasing in the degree of strategic complementarity, the asymmetry of the network and, in general, the relative variance of idiosyncratic shocks. However, when the relative variance of idiosyncratic shocks is sufficiently large, the level of induced aggregate volatility begins to decline again. The reasons for this are not immediately apparent and clearly call for more research.

Future work might also seek to make use of the additional degree of freedom offered by this model of learning to better match observed data in aggregate price indicies and thence to including network learning in a broader Dynamic Stochastic General Equilibrium (DSGE) model of the economy. Further work to gather data on the exact structures of firms' observation networks is likewise clearly called for. Finally, an immediate extension of merit would be to consider a setting of dynamic price setting in a manner analogous to that achieved by Nimark (2008) of the original Woodford (2003a) paper.

Bibliography

Acemoglu, D., V. Carvalho, A. Ozdaglar, and A. Tahbaz-Saleh (2011): "The Network Origins of Aggregate Fluctuations," *Universitat Pompeu Fabra Economics Working Papers*, No. 1291.

Acemoglu, D., M. Dahleh, I. Lobel, and A. Ozdaglar (2011): "Bayesian Learning in Social Networks," *Review of Economic Studies*, 78(4).

Acemoglu, D., A. Ozdaglar, and A. Tahbaz-Saleh (2010): "Cascades in Networks and Aggregate Volatility," *NBER Working Papers*, 16516.

Adam, K. (2007): "Optimal monetary policy with imperfect common knowledge," *Journal of Monetary Economics*, 54(2), 267–301.

Albert, R., and A.-L. BarabÃąsi (2002): "Statistical mechanics of complex networks," *Reviews of Modern Physics*, 74(1), 47–97.

Amirault, D., C. Kwan, and G. Wilkinson (2006): "Survey of Price-Setting Behaviour of Canadian Companies," *Bank of Canada Working Paper*, No. 2006-35.

Apel, M., R. Friberg, and K. Hallsten (2005): "Microfoundations of Macroeconomic Price Adjustment: Survey Evidence from Swedish Firms," *Journal of Money, Credit, and Banking*, 37(2), 313–338.

Aucremanne, L., and M. Druant (2005): "Price-setting Behaviour in Belgium: What can be Learned from an Ad Hoc Survey?," *ECB Working Paper*, No. 448.

Bils, M., and P. Klenow (2004): "Some Evidence on the Importance of Sticky Prices," *Journal of Political Economy*, 112(5), 947–985.

Blinder, A. (1991): "Why are Prices Sticky? Preliminary Results from an Interview Survey," *American Economic Review*, 81(2), 89–96.

Blinder, A., E. Canetti, D. Lebow, and J. Rudd (1998): *Asking about prices: a new approach to understanding price stickiness.* Russell Sage Foundation, 1 edn.

Cai, G.-H. (2006): "Strong laws for weighted sums of i.i.d. random variables," *Electronic Research Announcements in Mathematical Sciences*, 12, 29–36.

Calvo, G. (1983): "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12(3), 383–398.

Calvo-Armengol, A., and J. de Marti (2007): "Communication Networks: Knowledge and Decisions," *American Economic Review*, 97(2), 86–91, âĂIJwe analyze how this decentralized information-sharing scheme shapes individual and organization decisions and outcomes when the network geometry varies and for arbitrary communication roundsâĂİ.

Carvalho, V. (2010): "Aggregate Fluctuations and the Network Structure of Intersectoral Trade," .

Christiano, L., M. Eichenbaum, and C. Evans (1999): "Monetary Policy Shocks: What Have We Learned and to What End?," in *Handbook of Macroeconomics*, ed. by J. Taylor, and M. Woodford, vol. 1A, chap. 2, pp. 65–148. Elsevier, New York.

Clauset, A., C. R. Shalizi, and M. E. J. Newman (2009): "Power-Law Distributions in Empirical Data," *SIAM Review*, 51(4), 661–703.

DeMarzo, P., D. Vayanos, and J. Zwiebel (2003): "Persuasion Bias, Social Influence and Unidimensional Opinions," *Quarterly Journal of Economics*, 118(3), 909–968.

Dixit, A., and J. Stiglitz (1977): "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, 67(3), 297–308.

Fabiani, S., M. Druant, I. Hernando, C. Kwapil, B. Landau, C. Loupias, F. Martins, T. MathÃď, R. Sabbatini, H. Stahl, and A. Stokman (2005): "The Pricing Behaviour of Firms in the Euro Area: New Survey Evidence," *ECB Working Paper*, No. 535.

GABAIX, X. (2011): "The Granular Origins of Aggregate Volatility," *Econometrica*, 79(3), 733–772.

GOLUB, B., AND M. JACKSON (2010): "NaÃ¯ve Learning in Social Networks and the Wisdom of Crowds," *American Economic Journal: Microeconomics*, 2(1), 112âĂŞ49.

HALL, S., M. WALSH, AND A. YATES (1997): "How do UK companies set prices?," *Bank of England Working Paper*, No. 67.

HAMILTON, J. (1994): *Time Series Analysis*. Princeton University Press, 1 edn.

HOEBERICHTS, M., AND A. STOKMAN (2006): "Price Setting Behaviour in the Netherlands: Results of a Survey," *ECB Working Paper*, No. 607.

JACKSON, M., AND B. ROGERS (2007): "Meeting Strangers and Friends of Friends: How Random Are Social Networks?," *American Economic Review*, 97(3), 890–915, Full Read: Oct 2007.

KLENOW, P., AND B. MALIN (2010): "Microeconomic Evidence on Price-Setting," in *Handbook of Monetary Economics*, ed. by B. Friedman, and M. Woodford, vol. 3, chap. 6, pp. 231–284. Elsevier.

KWAPIL, C., J. BAUMGARTNER, AND J. SCHARLER (2005): "The Price-Setting Behavior of Austrian Firms: Some Survey Evidence," *ECB Working Paper*, No. 464.

LJUNGQVIST, L., AND T. SARGENT (2004): *Recursive Macroeconomic Theory*. MIT Press, 2 edn.

LOUPIAS, C., AND R. RICART (2004): "Price Setting in France: New Evidence from Survey Data," *ECB Working Paper*, No. 423.

LUCAS, R. (1972): "Expectations and the Neutrality of Money," *Journal of Economic Theory*, 4, 103–124.

LUNNEMANN, P., AND T. MATHÃĎ (2006): "New Survey Evidence on the Pricing Behaviour of Luxembourg Firms," *ECB Working Paper*, No. 617.

MANKIW, G., AND R. REIS (2002): "Sticky information versus sticky prices: A proposal to replace the New Keynsian Phillips curve," *The Quarterly Journal of Economics*, 117(4), 1295–1328.

——— (2006): "Pervasive Stickiness," *American Economic Review*, 96(2), 164–169.

——— (2007): "Sticky Information in General Equilibrium," *Journal of the European Economic Association*, 5(2-3), 603–613, Brief read: Sep 2007.

MARTINS, F. (2005): "The Price Setting Behaviour of Portuguese Firms: Evidence from Survey Data," *ECB Working Paper*, No. 562.

MELOSI, L. (2011): "Public Inflation Expectations and Monetary Policy," .

MORTENSEN, D., AND C. PISSARIDES (1994): "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies*, 61(3), 397–415.

NAKAGAWA, S., R. HATTORI, AND I. TAKAGAWA (2000): "Price-Setting Behavior of Japanese Companies," *Bank of Japan Research Papers*, No. 2000-09.

NIMARK, K. (2008): "Dynamic pricing and imperfect common knowledge," *Journal of Monetary Economics*, 55(2), 365–382.

——— (2011a): "Dynamic Higher Order Expectations," Economics Working Papers.

——— (2011b): "A low dimensional Kalman Filter for systems with lagged observables," .

PEERSMAN, G., AND F. SMETS (2003): "The Monetary Transmission Mechanism in the Euro Area: More Evidence from VAR Analysis," in *Monetary Policy Transmission in the Euro Area*, ed. by I. Angeloni, A. Kashyap, and B. Mojon, chap. 2, pp. 36–55. Cambridge University Press.

REIS, R. (2006): "Inattentive Producers," *Review of Economic Studies*, 73(3), 793–821.

ROMER, D., AND C. ROMER (2004): "A New Measure of Monetary Shocks: Derivation and Implications," *American Economic Review*, 94, 1055–1084.

Sims, C. (2003): "Implications of Rational Inattention," *Journal of Monetary Economics*, 50(3), 665–690.

Singleton, K. (1987): "Asset prices in a time series model with disparately informed, competitive traders," in *New Approaches to Monetary Economics*, ed. by W. Burnett, and K. Singleton, vol. 1. Cambridge University Press.

Stahl, H. (2005): "Price Setting in German Manufacturing: New Evidence from New Survey Data," *ECB Working Paper*, No. 561.

Sung, S. H. (2001): "Strong laws for weighted sums of i.i.d. random variables," *Statistics and Probability Letters*, 52(4), 413–419.

Taylor, J. (1980): "Aggregated Dynamics and Staggered Contracts," *Journal of Political Economy*, 88(1), 1–23.

Woodford, M. (2003a): "Imperfect Common Knowledge and the Effects of Monetary Policy," in *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, ed. by P. Aghion, R. Frydman, J. Stiglitz, and M. Woodford. Princeton University Press.

——— (2003b): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

Wu, W. B. (1999): "On the strong convergence of a weighted sum," *Statistics and Probability Letters*, 44(1), 19–22.

ÃĄlvarez, L., and I. Hernando (2005): "The Price Setting Behaviour of Spanish Firms: Evidence from Survey Data," *ECB Working Paper*, No. 538.

Appendix A Network Learning Outline

This appendix provides an outline of Kalman Filter-based method for learning used by agents in each of the two stages.

A.1 Symbol Reference

The following reference table is provided to describe common symbols used in section 3.

A	Transition matrix for the underlying state
B	Mapping from the underlying state to a private signal
C	Mapping from the full state to individual signals in stage two
D	Mapping from the full state to individual signals in stage one
E	Expectation operator (always linear)
F	Transition matrix for the full state in stage two
G	Mapping from aggregate shocks to the full state in stage two
H	
I	The Identity matrix
J	Kalman Gain in stage two
K	Kalman Gain in stage one
L	
M	Transition matrix for the full state in stage one
N	Mapping from aggregate shocks to the full state in stage one
O	
P	Mapping from aggregate shocks to the underlying state
Q	Mapping from an idiosyncratic shock to a private signal
R	Mapping from shocks to individual signals in stage one
S	Mapping from shocks to individual signals in stage two
T	
U	Variance of the full state in stage two
V	Variance of the full state in stage one
W	Mapping from the previous-period actions to social signals
X	The full state in period two
Y	
Z	The full state in period one

A.1.1 Stage One: When all agents observe the same weighted-average action

In the first stage, we solve a model where all agents observe the same weighted average of previous-period actions, so that agents' signal vectors are given by:

$$\mathbf{s}_t(j) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(j) \\ \widetilde{y}_{t-1} \end{bmatrix}$$

Equation (13) shows us that agents will need to include weighted-average expectations in their state vectors of interest and so we define the vector of k-th order expectations in stage 1 as:

$$\mathbb{E}_t^{(0)}[\theta] = \theta$$
$$\mathbb{E}_t^{(k)}[\theta] = \begin{bmatrix} \overline{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta]\right] \\ \widetilde{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta]\right] \end{bmatrix} \quad \forall k \geq 1$$

We next observe that agents need to estimate $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ (to construct the period-t decision rule), but have signals derived from $\mathbf{x}_t$, $\mathbf{x}_{t-1}$and $\widetilde{E}_{t-1}\left[\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}\right]$. Define the state vector of interest, X_t, as the full hierarchy of expectations regarding the underlying state:

$$Z_t \equiv \mathbb{E}_t^{(0:\infty)}[\mathbf{x}_t] \tag{A.1}$$

Recalling the definition of a hierarchy of expectations (equation (3)), we can write Z_t as a vector of three components: first, the underlying state variables themselves; second, the simple-average expectation of the hierarchy; and third, the weighted-average expectation of the hierarchy. Using this, we can rewrite agents' observation vectors as:

$$\mathbf{s}_t(j) = D_1 Z_t + D_2 Z_{t-1} + R_1 \mathbf{v}_t(j) + R_2 \widetilde{\mathbf{v}}_{t-1} \tag{A.2a}$$

$$D_1 = \begin{bmatrix} \begin{bmatrix} B & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \mathbf{0} \end{bmatrix} \tag{A.2b}$$

$$D_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' \begin{bmatrix} B & \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \end{bmatrix} \tag{A.2c}$$

$$R_1 = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \tag{A.2d}$$

$$R_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' Q \end{bmatrix} \tag{A.2e}$$

where, within the description of D_2, the matrix $\begin{bmatrix} \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix}$ selects $\widetilde{E}_{t-1}\left[\overline{\mathbf{x}}^{(0:\infty)}_{t-1|t-1}\right]$ from Z_{t-1}. We conjecture (and verify below in appendix B) that the state vector may be written in the following ARMA(1,1) law of motion:

$$Z_t = M Z_{t-1} + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1} \tag{A.3}$$

with the matricies M, N_1, N_2 and N_3 broken down as:

$$M = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{bmatrix} \tag{A.4}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} & \mathbf{0} \\ \Psi_4 & \Psi_5 & \Psi_6 \\ \Upsilon_4 & \Upsilon_5 & \Upsilon_6 \end{bmatrix} \tag{A.5}$$

Which is to say that

$$\overline{E}_t[Z_t] = \Psi_1 \mathbf{x}_{t-1} + \Psi_2 \overline{E}_{t-1}[Z_{t-1}] + \Psi_3 \widetilde{E}_{t-1}[Z_{t-1}] + \Psi_4 \mathbf{u}_t + \Psi_5 \widetilde{\mathbf{v}}_t + \Psi_6 \widetilde{\mathbf{v}}_{t-1}$$
$$\widetilde{E}_t[Z_t] = \Upsilon_1 \mathbf{x}_{t-1} + \Upsilon_2 \overline{E}_{t-1}[Z_{t-1}] + \Upsilon_3 \widetilde{E}_{t-1}[Z_{t-1}] + \Upsilon_4 \mathbf{u}_t + \Upsilon_5 \widetilde{\mathbf{v}}_t + \Upsilon_6 \widetilde{\mathbf{v}}_{t-1}$$

There are two related complications in this system over a classic Kalman filtering problem. The first is that agents' signal vectors include observations available only with a lag, and the second, related to the first, is presence of lagged shocks (the MA(1) component). The most common approach is to stack the state vector with it's lag

$$\begin{bmatrix} Z_t \\ Z_{t-1} \end{bmatrix} = \begin{bmatrix} M & \mathbf{0} \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ Z_{t-2} \end{bmatrix} + \begin{bmatrix} N_1 & N_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_t \\ \widetilde{\mathbf{v}}_t \end{bmatrix} + \begin{bmatrix} \mathbf{0} & N_3 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{t-1} \\ \widetilde{\mathbf{v}}_{t-1} \end{bmatrix}$$

However, to do so doubles the size of the state vector, which may present problems when simulating the system with finite computing resources, and still requires accounting for the lagged disturbances. As such, the derivation of expressions for the Ψ_* and Υ_* coefficients in appendix B follows Nimark (2011b) in finding a modified Kalman filter that does not require the stacking of the system and explicitly allows for the presence of lagged shocks.

Finding the solution then involves finding a fixed point of the system for a chosen upper limit (k^*) on the number of higher-order expectations to include.

A.1.2 Stage Two: Solving the agents' problem in an opaque network

In the general problem for an agent that observes the previous period action of specific competitors,[15] agent i's social observation matrix will be given by:

[15] For the sake of brevity, we shall assume that agent i observes the previous-period action of only one competitor, $\delta_t(i)$, but how to increase that number should be readily apparant.

$$W_t(i) = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{\text{Column } j=\delta_t(i)} & 0 & \cdots & 0 \end{bmatrix}$$

so that her period-t social signal will be given by:

$$\mathbf{s}_t^s(i) = y_{t-1}(\delta_{t-1}(i)) = \alpha' B\mathbf{x}_{t-1} + \alpha' Q\mathbf{v}_{t-1}(\delta_{t-1}(i)) + \beta' E_{t-1}(\delta_{t-1}(i)) \left[\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}\right]$$

We can therefore see that agent i will need to keep track of three compound expectations: the simple-average ($\overline{E}_t[\cdot]$), the weighted-average ($\widetilde{E}_t[\cdot]$) and that of her observee ($E_t(\delta_t(i))[\cdot]$). However, since the observee is treated the same no matter who they are, we have that the expectations will update in a common manner for every agent and denote an expectation obtained from stage one as $E_t(\delta_t(i))[\cdot] = \mathring{E}_t[\cdot]\ \ \forall i$:

$$\mathbb{E}_t^{(k)}[\theta] = \begin{bmatrix} \overline{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta]\right] \\ \widetilde{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta]\right] \\ \mathring{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta]\right] \end{bmatrix} \quad \forall k \geq 1$$

The full, infinite-dimension state vector to be estimated will then be:

$$X_t \equiv \mathbb{E}_t^{(0:\infty)}[\mathbf{x}_t] \tag{A.6}$$

so that agent i's observation vector can be written as

$$\mathbf{s}_t(i) = C_1 X_t + C_2 X_{t-1} + S_1 \mathbf{v}_t(i) + S_2 \mathbf{v}_{t-1}(\delta_{t-1}(i)) \tag{A.7a}$$

$$C_1 = \begin{bmatrix} \begin{bmatrix} B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \mathbf{0} \end{bmatrix} \tag{A.7b}$$

$$C_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' \begin{bmatrix} B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \end{bmatrix} \tag{A.7c}$$

$$S_1 = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \tag{A.7d}$$

$$S_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' Q \end{bmatrix} \tag{A.7e}$$

We conjecture (and verify in appendix B) that the state vector may be written in the following law of motion:

$$X_t = FX_{t-1} + G_1 \mathbf{u}_t + G_2 \widetilde{\mathbf{v}}_t + G_3 \widetilde{\mathbf{v}}_{t-1} + G_4 \widehat{\mathbf{v}}_{t-1} \tag{A.8}$$

$$Z_t = MZ_{t-1} + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1} \tag{A.9}$$

with the matricies F, G_1,G_2 ,G_3 and G_4 given by:

$$F = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 \end{bmatrix} \tag{A.10}$$

$$\begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Theta_5 & \Theta_6 & \Theta_7 & \Theta_8 \\ \Omega_5 & \Omega_6 & \Omega_7 & \Omega_8 \\ \Gamma_5 & \Gamma_6 & \Gamma_7 & \Gamma_8 \end{bmatrix} \tag{A.11}$$

Which is to say that

$$\overline{E}_t[X_t] = \Theta_1 \mathbf{x}_{t-1} + \Theta_2 \overline{E}_{t-1}[X_{t-1}] + \Theta_3 \widetilde{E}_{t-1}[X_{t-1}] + \Theta_4 \mathring{E}_{t-1}[X_{t-1}] + \Theta_5 \mathbf{u}_t + \Theta_6 \widetilde{\mathbf{v}}_t + \Theta_7 \widetilde{\mathbf{v}}_{t-1} + \Theta_8 \widehat{\mathbf{v}}_{t-1}$$
$$\widetilde{E}_t[X_t] = \Omega_1 \mathbf{x}_{t-1} + \Omega_2 \overline{E}_{t-1}[X_{t-1}] + \Omega_3 \widetilde{E}_{t-1}[X_{t-1}] + \Omega_4 \mathring{E}_{t-1}[X_{t-1}] + \Omega_5 \mathbf{u}_t + \Omega_6 \widetilde{\mathbf{v}}_t + \Omega_7 \widetilde{\mathbf{v}}_{t-1} + \Omega_8 \widehat{\mathbf{v}}_{t-1}$$
$$\mathring{E}_t[X_t] = \Gamma_1 \mathbf{x}_{t-1} + \Gamma_2 \overline{E}_{t-1}[X_{t-1}] + \Gamma_3 \widetilde{E}_{t-1}[X_{t-1}] + \Gamma_4 \mathring{E}_{t-1}[X_{t-1}] + \Gamma_5 \mathbf{u}_t + \Gamma_6 \widetilde{\mathbf{v}}_t + \Gamma_7 \widetilde{\mathbf{v}}_{t-1} + \Gamma_8 \widehat{\mathbf{v}}_{t-1}$$

The derivation of the following expressions for these coeficients is provided in appendix B:

$\overline{E}_t[X_t]$	$\widetilde{E}_t[X_t]$	$\mathring{E}_t[X_t]$
$\Theta_1 = J_1 BA + J_2\alpha' B$	$\Omega_1 = J_1 BA + J_2\alpha' B$	$\Gamma_1 = K_1 BA + K_2\alpha' B$
$\Theta_2 = (F - (C_1 F + C_2))$	$\Omega_2 = \mathbf{0}$	$\Gamma_2 = \mathbf{0}$
$\Theta_3 = \mathbf{0}$	$\Omega_3 = (F - (C_1 F + C_2))$	$\Gamma_3 = K_2\beta' \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$
$\Theta_4 = J_2\beta' \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$	$\Omega_4 = J_2\beta' \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$	$\Gamma_4 = (M - (D_1 M + D_2))$
$\Theta_5 = J_1 BP$	$\Omega_5 = J_1 BP$	$\Gamma_5 = K_1 BP$
$\Theta_6 = \mathbf{0}$	$\Omega_6 = J_1 Q$	$\Gamma_6 = \mathbf{0}$
$\Theta_7 = J_2\alpha' Q$	$\Omega_7 = \mathbf{0}$	$\Gamma_7 = K_2\alpha' Q$
$\Theta_8 = \mathbf{0}$	$\Omega_8 = J_2\alpha' Q$	$\Gamma_8 = \mathbf{0}$

Finding the solution then involves finding a fixed point of the system for a chosen upper limit (k^*) on the number of higher-order expectations to include.

Appendix B Network Learning Proofs

B.1 Proof of lemma 1.

Using the equation for each agent's decision rule (11), we have that agent i will construct her prior expectation of her social signal as follows:

$$\begin{aligned} E_t(i)\left[y_t(\delta_t(i))\right] &= E_t(i)\left[\alpha'\left(B\mathbf{x}_t + Q\mathbf{v}_t(\delta_t(i))\right) + \beta' E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]\right] \\ &= \alpha' B E_t(i)\left[\mathbf{x}_t\right] + \beta' E_t(i)\left[E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]\right] \end{aligned}$$

where we have used the lack of persistence in idiosyncratic shocks to give $E_t(i)\left[\mathbf{v}_t(\delta_t(i))\right] = 0$. Supposing that $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ is the top-most portion of the common state vector of interest (Z_t), common knowledge of rationality then allows agent i to substitute in the Kalman filter (12) for agent $\delta_t(i)$'s expectation:

$$\begin{aligned} E_t(i)\left[E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]\right] &= \begin{bmatrix} I & 0 \end{bmatrix} E_t(i)\left[(M - KLM)E_{t-1}(\delta_t(i))\left[Z_{t-1}\right] + \begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(\delta_t(i)) \\ y_{t-1}(\delta_{t-1}(\delta_t(i))) \end{bmatrix}\right] \\ &= \begin{bmatrix} I & 0 \end{bmatrix}\left\{\begin{array}{l} (M - KLM)E_t(i)\left[E_{t-1}(\delta_t(i))\left[Z_{t-1}\right]\right] \\ +K_1 B E_t(i)\left[\mathbf{x}_t\right] \\ +K_2 E_t(i)\left[y_{t-1}(\delta_{t-1}(\delta_t(i)))\right] \end{array}\right\} \end{aligned}$$

The final term shows if agent i is going to observe the period-t action of agent $\delta_t(i)$, then in order to form her prior, she must also consider whomever agent $\delta_t(i)$ observed in period-$(t-1)$. This recursion of expectations (and expectations of expectations) across agents and backwards through time leads to an explosion in the dimensionality (this is the explosion of p) and typically prevents closed-form analysis in anything other than trivially small networks.

However, by denying agents knowledge of the full network and, instead, granting them knowledge of the distribution from which inbound links are drawn (ϕ) and using the assumption that this distribution is independent of other shocks, we can observe that:

$$\begin{aligned} E_t(i)\left[y_{t-1}(\delta_{t-1}(\delta_t(i)))\right] &= \int_0^1 E_t(i)\left[y_{t-1}(j)\right]\phi(j)\,dj \\ &= E_t(i)\left[\int_0^1 y_{t-1}(j)\phi(j)\,dj\right] \\ &= E_t(i)\left[\widetilde{y}_{t-1}\right] \end{aligned}$$

where the second line exploited the linearity of the expectation operator and $\widetilde{y}_t \equiv \int_0^1 y_t(j)\phi(j)\,dj$ is a *weighted* average of all agents' actions in period t using the observation p.d.f. as the weights. Substituting this back in above gives:

$$E_t(i)\left[E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]\right] = \begin{bmatrix} I & 0 \end{bmatrix} E_t(i) \left\{ \begin{array}{l} (M - KLM) E_{t-1}(\delta_t(i))[Z_{t-1}] \\ +K_1 B\mathbf{x}_t \\ +K_2 \widetilde{y}_{t-1} \end{array} \right\}$$

From agent i's perspective, their observee's own expectation updates recursively with no idiosyncratic information. This is, in effect, agent i treating agent $\delta_t(i)$ *as though* they receive a weighted average of everybody's period-$(t-1)$ actions. We can then replace $E_t(\delta_t(i))[\cdot]$ with a common expectation $\mathring{E}_t[\cdot]$ as *ex ante*, from i's perspective, all other agents are forming the same expectation. Agent i's prior expectation of their social signal can therefore be written as:

$$E_t(i)[y_t(\delta_t(i))] = \alpha' B E_t(i)[\mathbf{x}_t] + \beta' \begin{bmatrix} I & 0 \end{bmatrix} E_t(i)\left[\mathring{E}_t[Z_t]\right]$$

with

$$\mathring{E}_t[Z_t] = (M - KLM)\mathring{E}_{t-1}[Z_{t-1}] + K \begin{bmatrix} B\mathbf{x}_t \\ \widetilde{y}_{t-1} \end{bmatrix}$$

Note, too, that by identical logic we also have that when considering their observee's observee's observee, agent i will expect that:

$$E_t(i)[E_{t-1}(\delta_t(i))[y_{t-2}(\delta_{t-2}(\delta_{t-1}(\delta_t(i))))]] = E_t(i)[E_{t-1}(\delta_t(i))[\widetilde{y}_{t-2}]]$$

This, in turn, amounts to agent i treating agent $\delta_{t-1}(\delta_t(i))$ – that is, whoever $\delta_t(i)$ observed – as though they also received a weighted average of everybody's period-$(t-2)$ actions. The ongoing application backwards through time should be clear. So long as the weights used (the observation p.d.f.) are constant over time and common across agents – that is, so long as agents do not learn about the topology of the network – then we have that agent i's problem may be summarised as follows: observe the action of agent $\delta_t(i)$, but treat them as though they and and all information obtained through them comes from a setting in which all agents observe the weighted average action.

B.2 Proof of lemma 2.

Denoting $g(N) \equiv \sum_{i=1}^{N} \phi_N(i)^2$ and assuming that $\lim_{N\to\infty} g(N) = g^*$ where $g^* \in (0,\infty)$, we here demonstrate the following four results regarding agents' idiosyncratic shocks:

- $\widetilde{\mathbf{v}}_{N,t} \longrightarrow^d \widetilde{\mathbf{v}}_t$ where $\widetilde{\mathbf{v}}_t \sim N(\mathbf{0}, g^*\Sigma_{vv})$
- $\ddot{\mathbf{v}}_{N,t} \longrightarrow^{\mathcal{L}^2} \widetilde{\mathbf{v}}_t$
- $\widehat{\mathbf{v}}_{N,t} \longrightarrow^d \widehat{\mathbf{v}}_t$ where $\widehat{\mathbf{v}}_t \sim N(\mathbf{0}, g^*(2-g^*)\Sigma_{vv})$
- $Cov(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t) = g^*\Sigma_{vv}$

where the three weighted sums are defined as

$$\widetilde{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^{N} \mathbf{v}_t(i)\phi_N(i)$$
$$\ddot{\mathbf{v}}_{N,t} \equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{v}_t(\delta_t(i))$$
$$\widehat{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^{N} \mathbf{v}_t(\delta_t(i))\phi_N(i)$$

First, note that since the vector $\mathbf{v}_t(i)$ is drawn from independent and identical Gaussian distributions for each i and t, the sums $\widetilde{\mathbf{v}}_{N,t}$, $\ddot{\mathbf{v}}_{N,t}$ and $\widehat{\mathbf{v}}_{N,t} \equiv$ must all be distributed Normally and since $E[\mathbf{v}_t(i)] = 0$ $\forall i,t$ it must be that $E[\widetilde{\mathbf{v}}_{N,t}] = E[\ddot{\mathbf{v}}_{N,t}] = E[\widehat{\mathbf{v}}_{N,t}] = 0$ $\forall N,t$.

B.2.1 $\widetilde{\mathbf{v}}_{N,t} \longrightarrow^{d} \widetilde{\mathbf{v}}_t$

The variance of $\widetilde{\mathbf{v}}_{N,t}$ will then be given by

$$\begin{aligned}
Var\left[\widetilde{\mathbf{v}}_{N,t}\right] &= Var\left[\sum_{i=1}^{N} \mathbf{v}_t\left(i\right)\phi_N\left(i\right)di\right] \\
&= \sum_{i=1}^{N} Var\left[\mathbf{v}_t\left(i\right)\phi_N\left(i\right)\right]di \\
&= \sum_{i=1}^{N} \Sigma_{vv}\phi_N\left(i\right)^2 di \\
&= g\left(N\right)\Sigma_{vv}
\end{aligned}$$

where in moving to the second line we use the independence of each vector to ignore the covariance terms. The limiting variance as $N \to \infty$ is therefore $g^*\Sigma_{vv}$, giving the first result.

B.2.2 $\ddot{\mathbf{v}}_{N,t} \longrightarrow^{\mathcal{L}^2} \widetilde{\mathbf{v}}_t$

We next demonstrate that $\ddot{\mathbf{v}}_{N,t}$ converges to $\widetilde{\mathbf{v}}_t$ in mean square error (a stronger form of convergence than in probability) – i.e. that $\lim_{N\to\infty} E\left[\left(\ddot{\mathbf{v}}_{N,t} - \widetilde{\mathbf{v}}_t\right)^2\right] = 0$. First, see that

$$\begin{aligned}
E\left[\left(\ddot{\mathbf{v}}_{N,t} - \widetilde{\mathbf{v}}_t\right)^2\right] &= E\left[\left(\ddot{\mathbf{v}}_{N,t}\right)^2 - 2\ddot{\mathbf{v}}_{N,t}\widetilde{\mathbf{v}}_t + \left(\widetilde{\mathbf{v}}_t\right)^2\right] \\
&= Var\left[\ddot{\mathbf{v}}_{N,t}\right] - 2Cov\left[\ddot{\mathbf{v}}_{N,t}, \widetilde{\mathbf{v}}_t\right] + Var\left[\widetilde{\mathbf{v}}_t\right]
\end{aligned}$$

The third term was shown above to be given by $g^*\Sigma_{vv}$. We now consider the first and second terms in turn. The variance of $\ddot{\mathbf{v}}_{N,t}$ is given by:

$$\begin{aligned}
Var\left[\ddot{\mathbf{v}}_{N,t}\right] &= \frac{1}{N^2} Var\left[\mathbf{v}_t\left(\delta_t\left(1\right)\right) + \mathbf{v}_t\left(\delta_t\left(2\right)\right) + \cdots + \mathbf{v}_t\left(\delta_t\left(N\right)\right)\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \frac{1}{N^2}\left(N\Sigma_{vv} + \sum_{i=1}^{N}\sum_{j\neq i}^{N} E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right]\right)
\end{aligned}$$

However, when $i \neq j$, given the full independence of the distributions of agents' observees, it must be that

$$\begin{aligned}
E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] &= \sum_{k=1}^{N}\phi_N\left(k\right)E\left[\mathbf{v}_t\left(k\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \sum_{k=1}^{N}\phi_N\left(k\right)\left(\sum_{l=1}^{N}\phi_N\left(l\right)E\left[\mathbf{v}_t\left(k\right)\mathbf{v}_t\left(l\right)\right]\right) \\
&= \sum_{k=1}^{N}\phi_N\left(k\right)^2 E\left[\mathbf{v}_t\left(k\right)\mathbf{v}_t\left(k\right)\right] \\
&= g\left(N\right)\Sigma_{vv}
\end{aligned}$$

where in moving from the second line to the third we again make use of the independence of agents' idiosyncratic shocks. We therefore have that

$$
\begin{aligned}
Var\left[\ddot{\mathbf{v}}_{N,t}\right] &= \frac{1}{N^2}\left(N\Sigma_{vv} + \sum_{i=1}^{N}\sum_{j\neq i}^{N} g\left(N\right)\Sigma_{vv}\right) \\
&= \frac{1}{N^2}\left(N\Sigma_{vv} + \left(N^2 - N\right)g\left(N\right)\Sigma_{vv}\right) \\
&= \frac{1}{N}\Sigma_{vv} + \left(\frac{N-1}{N}\right)g\left(N\right)\Sigma_{vv}
\end{aligned}
$$

and thus, in the limit, that

$$
\lim_{N\to\infty} Var\left[\ddot{\mathbf{v}}_{N,t}\right] = g^{*}\Sigma_{vv}
$$

Next, we consider the covariance of $\widetilde{\mathbf{v}}_{M,t}$ and $\ddot{\mathbf{v}}_{N,t}$ for a general setting where $M > N$:

$$
\begin{aligned}
Cov\left[\widetilde{\mathbf{v}}_{M,t}, \ddot{\mathbf{v}}_{N,t}\right] &= E\left[\widetilde{\mathbf{v}}_{M,t}\ddot{\mathbf{v}}_{N,t}\right] \\
&= E\left[\left(\sum_{i=1}^{M}\mathbf{v}_t\left(i\right)\phi_N\left(i\right)\right)\left(\frac{1}{N}\sum_{j=1}^{N}\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right)\right] \\
&= \frac{1}{N}E\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)\left(\sum_{k=1}^{N}\phi_N\left(k\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(k\right)\right]\right) \\
&= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)\left(\phi_M\left(i\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(i\right)\right]\right) \\
&= \sum_{i=1}^{M}\phi_M\left(i\right)^2\frac{1}{N}\sum_{j=1}^{N}\Sigma_{vv} \\
&= g\left(M\right)\Sigma_{vv}
\end{aligned}
$$

where moving from the fifth to the sixth line again used the independence of each agents' idiosyncratic shocks. We can then note that

$$
\begin{aligned}
Cov\left[\widetilde{\mathbf{v}}_{t}, \ddot{\mathbf{v}}_{N,t}\right] &= \lim_{M\to\infty} Cov\left[\widetilde{\mathbf{v}}_{M,t}, \ddot{\mathbf{v}}_{N,t}\right] \\
&= \lim_{M\to\infty} g\left(M\right)\Sigma_{vv} \\
&= g^{*}\Sigma_{vv}
\end{aligned}
$$

We therefore have that

$$
\lim_{N\to\infty} E\left[\left(\ddot{\mathbf{v}}_{N,t} - \widetilde{\mathbf{v}}_t\right)^2\right] = g^{*}\Sigma_{vv} - 2g^{*}\Sigma_{vv} + g^{*}\Sigma_{vv} = 0
$$

as required for the second result.

B.2.3 $\widehat{\mathbf{v}}_{N,t} \longrightarrow^{d} \widehat{\mathbf{v}}_t$

Next, the variance of $\widehat{\mathbf{v}}_{N,t}$ is:

$$
\begin{aligned}
Var\left[\widehat{\mathbf{v}}_{N,t}\right] &= Var\left[\sum_{i=1}^{N} \phi_N\left(i\right) \mathbf{v}_t\left(\delta_t\left(i\right)\right)\right] \\
&= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N\left(i\right)\phi_N\left(j\right) \mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N\left(i\right)\phi_N\left(j\right) E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \sum_{i=1}^{N} \phi_N\left(i\right)^2 \Sigma_{vv} + \sum_{i=1}^{N}\sum_{j\neq i}^{N} \phi_N\left(i\right)\phi_N\left(j\right) E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right]
\end{aligned}
$$

From the analysis of $\ddot{\mathbf{v}}_{N,t}$ in the previous subsection, we have that when $i \neq j$,

$$E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] = g\left(N\right)\Sigma_{vv}$$

We therefore have that

$$Var\left[\widehat{\mathbf{v}}_{N,t}\right] = g\left(N\right)\Sigma_{vv} + g\left(N\right)\Sigma_{vv}\sum_{i=1}^{N}\sum_{j\neq i}^{N} \phi_N\left(i\right)\phi_N\left(j\right)$$

Next, consider that

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N\left(i\right)\phi_N\left(j\right) &= \sum_{i=1}^{N} \phi_N\left(i\right)\left(\sum_{j=1}^{N} \phi_N\left(j\right)\right) \\
&= \sum_{i=1}^{N} \phi_N\left(i\right) \\
&= 1
\end{aligned}
$$

as $\phi_N\left(i\right)$ and $\phi_N\left(j\right)$ are p.d.f's. We must therefore have that

$$\sum_{i=1}^{N}\sum_{j\neq i}^{N} \phi_N\left(i\right)\phi_N\left(j\right) = 1 - \sum_{i=1}^{N} \phi_N\left(i\right)^2 = 1 - g\left(N\right)$$

so that

$$Var\left[\widehat{\mathbf{v}}_{N,t}\right] = g\left(N\right)\Sigma_{vv}\left(1 + \left(1 - g\left(N\right)\right)\right)$$

and, in the limit,

$$\lim_{N\to\infty} Var\left[\widehat{\mathbf{v}}_{N,t}\right] = g^*\left(2 - g^*\right)\Sigma_{vv}$$

therefore giving the third result. Note that since $g^* \in (0,1)$, we also have that

$$1 > g^*\left(2 - g^*\right) > g^*$$

so that the variance of $\widehat{\mathbf{v}}_t$ is larger than that of $\widetilde{\mathbf{v}}_t$, but still smaller than that of $\mathbf{v}_t\left(i\right)$.

B.2.4 $Cov(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t) = g^* \Sigma_{vv}$

The covariance of $\widetilde{\mathbf{v}}_{N,t}$ and $\widehat{\mathbf{v}}_{N,t}$ is given by

$$\begin{aligned}
Cov\left[\widetilde{\mathbf{v}}_{N,t}, \widehat{\mathbf{v}}_{N,t}\right] &= E\left[\left(\sum_{i=1}^{N} \mathbf{v}_t(i)\,\phi_N(i)\right)\left(\sum_{j=1}^{N} \mathbf{v}_t(\delta_t(j))\,\phi_N(j)\right)\right] \\
&= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N(i)\,\phi_N(j)\,\mathbf{v}_t(i)\,\mathbf{v}_t(\delta_t(j))\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N(i)\,\phi_N(j)\,E\left[\mathbf{v}_t(i)\,\mathbf{v}_t(\delta_t(j))\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N(i)\,\phi_N(j)\left(\sum_{k=1}^{N} \phi_N(k)\,E\left[\mathbf{v}_t(i)\,\mathbf{v}_t(k)\right]\right) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N(i)\,\phi_N(j)\left(\phi_N(i)\,E\left[\mathbf{v}_t(i)\,\mathbf{v}_t(i)\right]\right) \\
&= \Sigma_{vv}\sum_{i=1}^{N}\sum_{j=1}^{N} \phi_N(i)^2\,\phi_N(j) \\
&= \Sigma_{vv}\sum_{i=1}^{N} \phi_N(i)^2\left(\sum_{j=1}^{N} \phi_N(j)\right) \\
&= \Sigma_{vv}\sum_{i=1}^{N} \phi_N(i)^2 \\
&= g(N)\,\Sigma_{vv}
\end{aligned}$$

so that, in the limit,

$$Cov(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t) = \lim_{N\to\infty} Cov(\widetilde{\mathbf{v}}_{N,t}, \widehat{\mathbf{v}}_{N,t}) = g^* \Sigma_{vv}$$

therefore giving the fourth result.

B.3 Proof of proposition 1.

Agent i's problem may therefore be solved in two stages:

- First, for a given set of weights that are common knowledge, obtain a solution to a model where all agents observe the same weighted average of everybody's previous-period action.
- Second, solve the individual problem supposing that information from any target comes from the setting in the first stage, where the weights used are the distribution from which agents' observation targets are drawn.

B.3.1 Solving stage one

We have the following state space system

$$Z_t = M Z_{t-1} + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1} \tag{B.1}$$
$$\mathbf{s}_t(j) = D_1 Z_t + D_2 Z_{t-1} + R_1 \mathbf{v}_t(j) + R_2 \widetilde{\mathbf{v}}_{t-1} \tag{B.2}$$

We will first develop a modified Kalman filter for agent j's estimation of Z_t and then turn to considering the evolution of Z_t itself (i.e. the coefficients of M and N).

The (modified) Kalman filter

The filter here closely follows that developed by Nimark (2011b) as a means of avoiding the doubling-up of the state vector more typical in the literature, thereby allowing more accurate simulation results when working with finite computing resources.

Denoting j's expectation formed with period-t information as $E_t(j)[\cdot] = E[\cdot|\mathcal{I}_t(j)]$, our goal is to find a mean square error minimising[16] formula for $E_t(j)[\mathbf{X}_t]$. To begin, we first substitute the state equation into the observation equation to get:

$$\begin{aligned}\mathbf{s}_t(j) &= D_1(MZ_{t-1} + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}) + D_2X_{t-1} + R_1\mathbf{v}_t(j) + R_2\widetilde{\mathbf{v}}_{t-1}\\ &= (D_1M + D_2)Z_{t-1} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}\end{aligned}$$

Next, note that the contemporaneous expectation of the weighted-average idiosyncratic shock must be zero:

$$\begin{aligned}E_t(j)[\widetilde{\mathbf{v}}_t] &= \lim_{N\to\infty}\sum_{i=1}^{N} E_t(j)[\mathbf{v}_t(i)]\,\phi(i,N)\\ &= \lim_{N\to\infty}\left(E_t(j)[\mathbf{v}_t(j)]\,\phi(j,N) + \sum_{i=1,i\neq j}^{N} E[\mathbf{v}_t(i)]\,\phi(i,N)\right)\\ &= \lim_{N\to\infty} E_t(j)[\mathbf{v}_t(j)]\,\phi(j,N)\\ &= 0\end{aligned}$$

where moving from the first line to the second uses the fact that idiosyncratic shocks are independent, so the best that agent j can do with respect to other agents' shocks is to use the unconditional expectation, and in moving from the third line to the fourth we have used the assumption that $\lim_{N\to\infty}\phi(j,N) = 0$. This then allows us to note that j's best estimate of $\mathbf{s}_t(j)$ given period-$(t-1)$ information is simply

$$E_{t-1}(j)[\mathbf{s}_t(j)] = (D_1M + D_2)\,E_{t-1}(j)[Z_{t-1}]$$

Define the innovation in $\mathbf{s}_t(j)$ – that is, the unexpected component – to be:

$$\overrightarrow{\mathbf{s}}_t(j) \equiv \mathbf{s}_t(j) - E_{t-1}(j)[\mathbf{s}_t(j)]$$

Since $\overrightarrow{\mathbf{s}}_t(j)$ contains only *new* information available in period t, it must be orthogonal to any of j's estimates based on information from earlier periods. We can therefore use the result that $E[w|y,z] = E[w|y] + E[w|z]$ when $y \perp z$, so that

$$\begin{aligned}E_t(j)[Z_t] &= E_{t-1}(j)[Z_t] + K_t\overrightarrow{\mathbf{s}}_t(j)\\ &= ME_{t-1}(j)[Z_{t-1}] + K_t\{\mathbf{s}_t(j) - (D_1M + D_2)\,E_{t-1}(j)[X_{t-1}]\}\\ &= (M - K_t(D_1M + D_2))\,E_{t-1}(j)[Z_{t-1}] + K_t\mathbf{s}_t(j)\end{aligned} \tag{B.3}$$

for some matrix, K_t (the Kalman gain). Note that K_t does not require an agent subscript as the problem is symmetric for all agents. For this to be the best linear estimator, we require K_t to be such that $\overrightarrow{\mathbf{s}}_t(j)$ is orthogonal to the projection error, $Z_t - K_t\overrightarrow{\mathbf{s}}_t(j)$. That is, we require that

$$E\left[\left(X_t - K_t\overrightarrow{\mathbf{s}}_t(j)\right)\overrightarrow{\mathbf{s}}_t(j)'\right] = 0 \tag{B.4}$$

Rearranging gives

$$K_t = E\left[Z_t\overrightarrow{\mathbf{s}}_t(j)'\right]\left(E\left[\overrightarrow{\mathbf{s}}_t(j)\,\overrightarrow{\mathbf{s}}_t(j)'\right]\right)^{-1} \tag{B.5}$$

Before evaluating this, note that we can rewrite the relevant vectors as:

[16] And hence, given that all shocks are mean zero, a variance-minimising estimator.

$$\overrightarrow{\mathbf{s}}_t(j) = \overbrace{(D_1M + D_2) Z_{t-1} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}}^{\mathbf{s}_t(j)} - \overbrace{(D_1M + D_2) E_{t-1}(j)[Z_{t-1}]}^{E_{t-1}(j)[\mathbf{s}_t(j)]}$$
$$= (D_1M + D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}$$

where $\widehat{Z_t(j)} \equiv Z_t - E_t(j)[Z_t]$ is j's contemporaneous error in estimating Z_t, and

$$Z_t = M\left(\widehat{Z_{t-1}(j)} + E_{t-1}(j)[Z_{t-1}]\right) + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}$$

The first term of equation (B.5) thus expands to:

$$E\left[Z_t\overrightarrow{\mathbf{s}}_t(j)'\right] = E\left[\begin{array}{l}\left(M\left(\widehat{Z_{t-1}(j)} + E_{t-1}(j)[Z_{t-1}]\right) + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}\right) \\ \times\left((D_1M + D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}\right)'\end{array}\right]$$
$$= MV_{t-1|t-1}(D_1M + D_2)' + N_1\Sigma_{uu}N_1'D_1' + g^*N_2\Sigma_{vv}N_2'D_1' + g^*N_3\Sigma_{vv}(D_1N_3 + R_2)'$$

where I have used $V_{t|t} \equiv E\left[\widehat{Z_t(j)}\widehat{Z_t(j)}'\right]$ as the variance-covariance matrix associated with $E_t(j)[Z_t]$. Given the symmetry of the problem across agents, although individual expectations may differ the variance of each estimate will be common. For the second term, we have that

$$E\left[\overrightarrow{\mathbf{s}}_t(j)\overrightarrow{\mathbf{s}}_t(j)'\right] = E\left[\begin{array}{l}\left((D_1M + D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}\right) \\ \times\left((D_1M + D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}\right)'\end{array}\right]$$
$$= (D_1M + D_2)V_{t-1|t-1}(D_1M + D_2) + D_1N_1\Sigma_{uu}N_1'D_1'$$
$$+ g^*D_1N_2\Sigma_{vv}N_1'D_2' + R_1\Sigma_{vv}R_1' + g^*(D_1N_3 + R_2)\Sigma_{vv}(D_1N_3 + R_2)'$$

so that, all together, the Kalman gain is given by

$$K_t = \left(MV_{t-1|t-1}(D_1M + D_2)' + N_1\Sigma_{uu}N_1'D_1' + g^*N_2\Sigma_{vv}N_2'D_1' + g^*N_3\Sigma_{vv}(D_1N_3 + R_2)'\right) \tag{B.6}$$
$$\times\left[\begin{array}{l}(D_1M + D_2)V_{t-1|t-1}(D_1M + D_2) + D_1N_1\Sigma_{uu}N_1'D_1' \\ +g^*D_1N_2\Sigma_{vv}N_2'D_1' + R_1\Sigma_{vv}R_1' + g^*(D_1N_3 + R_2)\Sigma_{vv}(D_1N_3 + R_2)'\end{array}\right]^{-1}$$

Evolution of the gain and variance matricies

First, since we can rewrite the state equation as

$$Z_t - E_{t-1}(j)[Z_t] = MZ_{t-1} + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1} - E_{t-1}(j)[Z_t]$$
$$= M(Z_{t-1} - E_{t-1}(j)[Z_{t-1}]) + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}$$

we have that

$$V_{t|t-1} = MV_{t-1|t-1}M' + N_1\Sigma_{uu}N_1' + g^*N_2\Sigma_{vv}N_2' + g^*N_3\Sigma_{vv}N_3' \tag{B.7}$$

Next, add Z_t to each side of equation (B.3) and rearrange to get

$$Z_t - E_t(j)[Z_t] = Z_t - E_{t-1}(j)[Z_t] - K_t\overrightarrow{\mathbf{s}}_t(j)$$

Since the innovation is orthogonal to both the prior error, $Z_t - E_{t-1}(j)[Z_t]$, and the posterior error, $Z_t - E_t(j)[Z_t]$, the variance of the left-hand side must equal the sum of the variances on the right-hand side, thereby giving

$$V_{t|t} = V_{t|t-1} - K_t Var\left(\left(D_1 M + D_2 \right) \widehat{Z_{t-1}\left(j \right)} + D_1 N_1 \mathbf{u}_t + D_1 N_2 \widetilde{\mathbf{v}}_t + R_1 \mathbf{v}_t\left(j \right) + \left(D_1 N_3 + R_2 \right) \widetilde{\mathbf{v}}_{t-1} \right) K_t' \tag{B.8}$$

$$= V_{t|t-1} - K_t \left[\begin{array}{l} \left(D_1 M + D_2 \right) V_{t-1|t-1} \left(D_1 M + D_2 \right)' + D_1 N_1 \Sigma_{uu} N_1' D_1' \\ + g^* D_1 N_2 \Sigma_{vv} N_2' D_1' + R_1 \Sigma_{vv} R_1' + g^* \left(D_1 N_3 + R_2 \right) \Sigma_{vv} \left(D_1 N_3 + R_2 \right)' \end{array} \right] K_t'$$

Provided that M represents a contraction, then there will exist steady state (i.e. time-invariant) Kalman gain and Variance matricies, found by iterating equations (B.6), (B.7) and (B.8) forward until convergence is achieved. The form of these matricies will be:

$$K = \left(MV \left(D_1 M + D_2 \right)' + N_1 \Sigma_{uu} N_1' D_1' + g^* N_2 \Sigma_{vv} N_2' D_1' + g^* N_3 \Sigma_{vv} \left(D_1 N_3 + R_2 \right)' \right)$$
$$\times \left[\begin{array}{l} \left(D_1 M + D_2 \right) V \left(D_1 M + D_2 \right) + D_1 N_1 \Sigma_{uu} N_1' D_1' \\ + g^* D_1 N_2 \Sigma_{vv} N_2' D_1' + R_1 \Sigma_{vv} R_1' + g^* \left(D_1 N_3 + R_2 \right) \Sigma_{vv} \left(D_1 N_3 + R_2 \right)' \end{array} \right]^{-1}$$

$$V = M \left(V - K \left[\begin{array}{l} \left(D_1 M + D_2 \right) V \left(D_1 M + D_2 \right)' + D_1 N_1 \Sigma_{uu} N_1' D_1' \\ + g^* D_1 N_2 \Sigma_{vv} N_1' D_2' + R_1 \Sigma_{vv} R_1' + g^* \left(D_1 N_3 + R_2 \right) \Sigma_{vv} \left(D_1 N_3 + R_2 \right)' \end{array} \right] K' \right) M'$$
$$+ N_1 \Sigma_{uu} N_1' + g^* N_2 \Sigma_{vv} N_2' + g^* N_3 \Sigma_{vv} N_3'$$

Identifying the state law of motion

We seek the coefficients Ψ_* and Υ_* in the matricies of the state vector's law of motion:

$$Z_t = M Z_{t-1} + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1}$$
$$M = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{bmatrix}$$
$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} & \mathbf{0} \\ \Psi_4 & \Psi_5 & \Psi_6 \\ \Upsilon_4 & \Upsilon_5 & \Upsilon_6 \end{bmatrix}$$

where the state vector is defined as:

$$Z_t \equiv \mathbb{E}_t^{(0:\infty)} \left[\mathbf{x}_t \right]$$
$$\mathbb{E}_t^{(0:k)} \left[\mathbf{x}_t \right] = \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t \left[\mathbb{E}_t^{(k-1)} \left[\mathbf{x}_t \right] \right] \\ \widetilde{E}_t \left[\mathbb{E}_t^{(k-1)} \left[\mathbf{x}_t \right] \right] \end{bmatrix}$$

Further, we have that agents update their estimates of the state vector according to

$$E_t\left(j \right) \left[Z_t \right] = \left(M - K \left(D_1 M + D_2 \right) \right) E_{t-1}\left(j \right) \left[Z_{t-1} \right] + K \mathbf{s}_t\left(j \right) \tag{B.9}$$

Partition the Kalman gain matrix (K) as:

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

so that the first column relates to the agent's private signal and the second column to her social signal. Given the structure of the agents' signals – see equation (A.2a) – we then expand equation (B.9) as

$$E_t\left(j \right) \left[Z_t \right] = \left(M - K \left(D_1 M + D_2 \right) \right) E_{t-1}\left(j \right) \left[Z_{t-1} \right] \tag{B.10}$$
$$+ \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} BA & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \alpha' \begin{bmatrix} B & \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \end{bmatrix} X_{t-1}$$
$$+ K_1 B P \mathbf{u}_t + K_1 Q \mathbf{v}_t\left(j \right) + K_2 \alpha' Q \widetilde{\mathbf{v}}_{t-1}$$

Taking the simple average of equation (B.10) gives

$$\begin{aligned}\overline{E}_t[Z_t] &= (M-(D_1M+D_2))\,\overline{E}_{t-1}[Z_{t-1}]\\ &+\begin{bmatrix}K_1 & K_2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}BA & \mathbf{0} & \mathbf{0}\end{bmatrix}\\ \alpha'\begin{bmatrix}B & \mathbf{0} & \mathbf{0}\end{bmatrix}+\beta'\begin{bmatrix}\mathbf{0} & \mathbf{0} & \begin{bmatrix}I & \mathbf{0}\end{bmatrix}\end{bmatrix}\end{bmatrix}X_{t-1}\\ &+K_1BP\mathbf{u}_t+K_2\alpha' Q\widetilde{\mathbf{v}}_{t-1}\end{aligned}$$

We can then immediately read that

$$\begin{aligned}\overline{E}_t[Z_t] &= \Psi_1\mathbf{x}_{t-1}+\Psi_2\overline{E}_{t-1}[Z_{t-1}]+\Psi_3\widetilde{E}_{t-1}[Z_{t-1}]+\Psi_4\mathbf{u}_t+\Psi_5\widetilde{\mathbf{v}}_t+\Psi_6\widetilde{\mathbf{v}}_{t-1}\\ \Psi_1 &= K_1BA+K_2\alpha' B\\ \Psi_2 &= (M-K(D_1M+D_2))\\ \Psi_3 &= K_2\beta'\begin{bmatrix}I & \mathbf{0}\end{bmatrix}\\ \Psi_4 &= K_1BP\\ \Psi_5 &= \mathbf{0}\\ \Psi_6 &= K_2\alpha' Q\end{aligned}$$

Taking the weighted average of equation (B.10) instead gives

$$\begin{aligned}\widetilde{E}_t[Z_t] &= (M-K(D_1M+D_2))\,\widetilde{E}_{t-1}[Z_{t-1}]\\ &+\begin{bmatrix}K_1 & K_2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}BA & \mathbf{0} & \mathbf{0}\end{bmatrix}\\ \alpha'\begin{bmatrix}B & \mathbf{0} & \mathbf{0}\end{bmatrix}+\beta'\begin{bmatrix}\mathbf{0} & \mathbf{0} & \begin{bmatrix}I & \mathbf{0}\end{bmatrix}\end{bmatrix}\end{bmatrix}X_{t-1}\\ &+K_1BP\mathbf{u}_t+K_1Q\widetilde{\mathbf{v}}_t+K_2\alpha' Q\widetilde{\mathbf{v}}_{t-1}\end{aligned}$$

From which we can immediately read that

$$\begin{aligned}\widetilde{E}_t[Z_t] &= \Upsilon_1\mathbf{x}_{t-1}+\Upsilon_2\overline{E}_{t-1}[Z_{t-1}]+\Upsilon_3\widetilde{E}_{t-1}[Z_{t-1}]+\Upsilon_4\mathbf{u}_t+\Upsilon_5\widetilde{\mathbf{v}}_t+\Upsilon_6\widetilde{\mathbf{v}}_{t-1}\\ \Upsilon_1 &= K_1BA+K_2\alpha' B\\ \Upsilon_2 &= \mathbf{0}\\ \Upsilon_3 &= (M-K(D_1M+D_2))+K_2\beta'\begin{bmatrix}I & \mathbf{0}\end{bmatrix}\\ \Upsilon_4 &= K_1BP\\ \Upsilon_5 &= K_1Q\\ \Upsilon_6 &= K_2\alpha' Q\end{aligned}$$

Finding the solution involves finding the fixed point of the system for a pre-chosen upper limit (k^*) on the number of orders of expectations to include.

B.3.2 Solving stage two

We have the following state space system

$$X_t = FX_{t-1}+G_1\mathbf{u}_t+G_2\widetilde{\mathbf{v}}_t+G_3\widetilde{\mathbf{v}}_{t-1}+G_4\widehat{\mathbf{v}}_{t-1} \tag{B.11}$$
$$\mathbf{s}_t(i) = C_1X_t+C_2X_{t-1}+S_1\mathbf{v}_t(i)+S_2\mathbf{v}_{t-1}(\delta_{t-1}(i)) \tag{B.12}$$

We will again first develop a modified Kalman filter for agent i's estimation of Y_t and then turn to considering the evolution of Y_t itself.

The (modified) Kalman filter

Substituting the state law of motion into the signal equation gives

$$\mathbf{s}_t\left(i\right) = \left(C_1F + C_2\right)X_{t-1} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t + S_1\mathbf{v}_t\left(i\right) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)$$

with a prior expectation of the signal given by

$$E_{t-1}\left(i\right)\left[\mathbf{s}_t\left(i\right)\right] = \left(C_1F + C_2\right)E_{t-1}\left(i\right)\left[X_{t-1}\right]$$

Denote the innovation in $\mathbf{s}_t\left(i\right)$ as

$$\begin{aligned}\overleftarrow{\mathbf{s}}_t\left(i\right) &\equiv \mathbf{s}_t\left(i\right) - E_{t-1}\left(i\right)\left[\mathbf{s}_t\left(i\right)\right] \\ &= \left(C_1F + C_2\right)\widehat{X_{t-1}\left(i\right)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t + S_1\mathbf{v}_t\left(i\right) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)\end{aligned}$$

The orthogonality of $\overleftarrow{\mathbf{s}}_t\left(i\right)$ gives us that

$$E_t\left(i\right)\left[X_t\right] = \left(F - J_t\left(C_1F + C_2\right)\right)E_{t-1}\left(i\right)\left[X_{t-1}\right] + J_t\mathbf{s}_t\left(i\right)$$

for some matrix, J_t (the Kalman gain), while optimality in the sense of minimising mean square error gives us that

$$J_t = E\left[X_t\overleftarrow{\mathbf{s}}_t\left(i\right)'\right]\left(E\left[\overleftarrow{\mathbf{s}}_t\left(i\right)\overleftarrow{\mathbf{s}}_t\left(i\right)'\right]\right)^{-1} \tag{B.13}$$

The first term in this expression expands as

$$X_t = F\left(\widehat{X_{t-1}\left(i\right)} + E_{t-1}\left(i\right)\left[X_{t-1}\right]\right) + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1}$$

$$\begin{aligned}E\left[X_t\overleftarrow{\mathbf{s}}_t\left(i\right)'\right] &= E\left[\begin{array}{c}\left(F\left(\widehat{X_{t-1}\left(i\right)} + E_t\left(i\right)\left[X_{t-1}\right]\right) + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1}\right) \\ \times\left(\begin{array}{l}\left(C_1F + C_2\right)\widehat{X_{t-1}\left(i\right)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t \\ +S_1\mathbf{v}_t\left(i\right) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)\end{array}\right)'\end{array}\right] \\ &= FU_{t-1|t-1}\left(C_1F + C_2\right)' + G_1\Sigma_{uu}G_1'C_1' + g^*G_2\Sigma_{vv}G_2'C_1' + g^*G_3\Sigma_{vv}G_3'C_1' + g^*\left(2 - g^*\right)G_4\Sigma_{vv}G_4'C_1' + 2g^*\end{aligned}$$

where $U_{t|t} \equiv E\left[\widehat{X_t\left(i\right)}\widehat{X_t\left(i\right)}'\right]$ as the variance-covariance matrix associated with $E_t\left(i\right)\left[X_t\right]$. The second term expands as:

$$\begin{aligned}E\left[\overleftarrow{\mathbf{s}}_t\left(i\right)\overleftarrow{\mathbf{s}}_t\left(i\right)'\right] &= E\left[\begin{array}{c}\left(\begin{array}{l}\left(C_1F + C_2\right)\widehat{X_{t-1}\left(i\right)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t \\ +S_1\mathbf{v}_t\left(i\right) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)\end{array}\right) \\ \times\left(\begin{array}{l}\left(C_1F + C_2\right)\widehat{X_{t-1}\left(i\right)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t \\ +S_1\mathbf{v}_t\left(i\right) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)\end{array}\right)'\end{array}\right] \\ &= \left(C_1F + C_2\right)U_{t-1|t-1}\left(C_1F + C_2\right)' + C_1G_1\Sigma_{uu}G_1'C_1' + g^*C_1G_2\Sigma_{vv}G_2'C_1' \\ &\quad + S_1\Sigma_{vv}S_1' + g^*C_1G_3\Sigma_{vv}G_3'C_1' + g^*\left(2 - g^*\right)C_1G_4\Sigma_{vv}G_4'C_1' + 2g^*C_1G_3\Sigma_{vv}G_4'C_1' + S_2\Sigma_{vv}S_2'\end{aligned}$$

Together, these give the Kalman gain for stage two as

$$\begin{aligned}J_t &= \left(FU_{t-1|t-1}\left(C_1F + C_2\right)' + G_1\Sigma_{uu}G_1'C_1' + g^*G_2\Sigma_{vv}G_2'C_1' + g^*G_3\Sigma_{vv}G_3'C_1' + g^*\left(2 - g^*\right)G_4\Sigma_{vv}G_4'C_1' + 2g^*G_3\Sigma_{vv}G_4'C\right. \\ &\quad\times\left[\begin{array}{c}\left(C_1F + C_2\right)U_{t-1|t-1}\left(C_1F + C_2\right)' + C_1G_1\Sigma_{uu}G_1'C_1' + g^*C_1G_2\Sigma_{vv}G_2'C_1' \\ +S_1\Sigma_{vv}S_1' + g^*C_1G_3\Sigma_{vv}G_3'C_1' + g^*\left(2 - g^*\right)C_1G_4\Sigma_{vv}G_4'C_1' + 2g^*C_1G_3\Sigma_{vv}G_4'C_1' + S_2\Sigma_{vv}S_2'\end{array}\right]^{-1}\end{aligned} \tag{B.14}$$

Evolution of the gain and variance matricies

The prior variance will be given by

$$U_{t|t-1} = FU_{t-1|t-1}F' + G_1\Sigma_{uu}G_1' + g^*G_2\Sigma_{vv}G_2' + g^*G_3\Sigma_{vv}G_3' + g^*\left(2-g^*\right)G_4\Sigma_{vv}G_4' + 2g^*G_3\Sigma_{vv}G_4' \tag{B.15}$$

And the posterior variance will update as

$$X_t - E_t\left(j\right)\left[X_t\right] = X_t - E_{t-1}\left(j\right)\left[X_t\right] - K_t\overrightarrow{\mathbf{s}}_t\left(j\right)$$

Since the innovation is orthogonal to both the prior error, $X_t - E_{t-1}\left(j\right)\left[X_t\right]$, and the posterior error, $X_t - E_t\left(j\right)\left[X_t\right]$, the variance of the left-hand side must equal the sum of the variances on the right-hand side, thereby giving

$$\begin{aligned} U_{t|t} &= U_{t|t-1} - J_t Var\left(\overleftarrow{\mathbf{s}}_t\left(j\right)\right)J_t' \qquad\qquad (\text{B.16})\\ &= U_{t|t-1} - J_t\left[\begin{array}{c}\left(C_1F+C_2\right)U_{t-1|t-1}\left(C_1F+C_2\right)' + C_1G_1\Sigma_{uu}G_1'C_1' + g^*C_1G_2\Sigma_{vv}G_2'C_1' \\ +S_1\Sigma_{vv}S_1' + g^*C_1G_3\Sigma_{vv}G_3'C_1' + g^*\left(2-g^*\right)C_1G_4\Sigma_{vv}G_4'C_1' + 2g^*C_1G_3\Sigma_{vv}G_4'C_1' + S_2\Sigma_{vv}S_2'\end{array}\right]J_t' \end{aligned}$$

Provided that F represents a contraction, then there will exist steady state (i.e. time-invariant) Kalman gain and variance matricies, found by iterating equations (B.14), (B.15) and (B.16) forward until convergence is achieved.

Identifying the state law of motion

We seek the coefficients Θ_*, Ω_* and Γ_* in the matricies of the state vector's law of motion:

$$X_t = FX_{t-1} + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1}$$

$$F = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 \end{bmatrix}$$

$$\begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Theta_5 & \Theta_6 & \Theta_7 & \Theta_8 \\ \Omega_5 & \Omega_6 & \Omega_7 & \Omega_8 \\ \Gamma_5 & \Gamma_6 & \Gamma_7 & \Gamma_8 \end{bmatrix}$$

where the state vector is defined as:

$$X_t \equiv \mathbb{E}_t^{(0:\infty)}\left[\mathbf{x}_t\right]$$

$$\mathbb{E}_t^{(0:k)}\left[\mathbf{x}_t\right] = \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t\left[\mathbb{E}_t^{(k-1)}\left[\mathbf{x}_t\right]\right] \\ \widetilde{E}_t\left[\mathbb{E}_t^{(k-1)}\left[\mathbf{x}_t\right]\right] \\ \mathring{E}_t\left[\mathbb{E}_t^{(k-1)}\left[\mathbf{x}_t\right]\right] \end{bmatrix}$$

Further, we have that agents update their estimates of the state vector according to

$$E_t\left(i\right)\left[X_t\right] = \left(F - J\left(C_1F+C_2\right)\right)E_{t-1}\left(i\right)\left[X_{t-1}\right] + J\mathbf{s}_t\left(i\right) \tag{B.17}$$

Partition the Kalman gain matrix (J) as:

$$J = \begin{bmatrix} J_1 & J_2 \end{bmatrix}$$

so that the first column relates to the agent's private signal and the second column to her social signal. Given the structure of the agents' signals – equation (A.7a) – we then expand equation (B.17) as

$$\begin{aligned} E_t(i)[X_t] &= (F - J(C_1F + C_2))\, E_{t-1}(i)[X_{t-1}] \\ &+ \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} BA & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \alpha' \begin{bmatrix} B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \end{bmatrix} X_{t-1} \\ &+ J_1BP\mathbf{u}_t + J_1Q\mathbf{v}_t(i) + J_2\alpha' Q\mathbf{v}_{t-1}(\delta_{t-1}(i)) \end{aligned} \tag{B.18}$$

Taking the simple average of equation (B.18) gives

$$\begin{aligned} \overline{E}_t[X_t] &= ((F - (C_1F + C_2))\, T_{sX} + J(C_1F + C_2))\, X_{t-1} \\ &+ J_1BP\mathbf{u}_t + J_1Q\mathbf{v}_t(i) + J_2\alpha' Q\mathbf{v}_{t-1}(\delta_{t-1}(i)) \end{aligned}$$

$$T_{sX}X_t = ((F - J(C_1F + C_2))\, T_{sX} + J(C_1F + C_2))\, X_{t-1} + J_1BP\mathbf{u}_t + J_1Q\mathbf{v}_t(i) + J_2\alpha' Q\mathbf{v}_{t-1}(\delta_{t-1}(i))$$

$$\begin{aligned} \overline{E}_t[X_t] &= (F - J(C_1F + C_2))\, \overline{E}_{t-1}[X_{t-1}] \\ &+ \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} BA & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \alpha' \begin{bmatrix} B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \end{bmatrix} X_{t-1} \\ &+ J_1BP\mathbf{u}_t + J_2\alpha' Q\widetilde{\mathbf{v}}_{t-1} \end{aligned}$$

where I have used lemma (2) to replace $\int_0^1 \mathbf{v}_{t-1}(\delta_{t-1}(i))\, di$ with $\widetilde{\mathbf{v}}_{t-1}$. We can then immediately read that

$$\overline{E}_t[X_t] = \Theta_1\mathbf{x}_{t-1} + \Theta_2\overline{E}_{t-1}[X_{t-1}] + \Theta_3\widetilde{E}_{t-1}[X_{t-1}] + \Theta_4\mathring{E}_{t-1}[X_{t-1}] + \Theta_5\mathbf{u}_t + \Theta_6\widetilde{\mathbf{v}}_t + \Theta_7\widetilde{\mathbf{v}}_{t-1} + \Theta_8\widehat{\mathbf{v}}_{t-1}$$

$$\begin{aligned} \Theta_1 &= J_1BA + J_2\alpha' B \\ \Theta_2 &= (F - (C_1F + C_2)) \\ \Theta_3 &= \mathbf{0} \\ \Theta_4 &= J_2\beta' \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \\ \Theta_5 &= J_1BP \\ \Theta_6 &= \mathbf{0} \\ \Theta_7 &= J_2\alpha' Q \\ \Theta_8 &= \mathbf{0} \end{aligned}$$

Taking the weighted average of equation (B.18) gives

$$\begin{aligned} \widetilde{E}_t[X_t] &= (F - J(C_1F + C_2))\, \widetilde{E}_t[X_{t-1}] \\ &+ \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} BA & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \alpha' \begin{bmatrix} B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \beta' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \end{bmatrix} X_{t-1} \\ &+ J_1BP\mathbf{u}_t + J_1Q\widetilde{\mathbf{v}}_t + J_2\alpha' Q\widehat{\mathbf{v}}_{t-1} \end{aligned}$$

where I have used lemma (2) to replace $\int_0^1 \mathbf{v}_{t-1}(\delta_{t-1}(i))\, \phi(i)\, di$ with $\widehat{\mathbf{v}}_{t-1}$. We can then immediately read that

$$\widetilde{E}_t[X_t] = \Omega_1\mathbf{x}_{t-1} + \Omega_2\overline{E}_{t-1}[X_{t-1}] + \Omega_3\widetilde{E}_{t-1}[X_{t-1}] + \Omega_4\mathring{E}_{t-1}[X_{t-1}] + \Omega_5\mathbf{u}_t + \Omega_6\widetilde{\mathbf{v}}_t + \Omega_7\widetilde{\mathbf{v}}_{t-1} + \Omega_8\widehat{\mathbf{v}}_{t-1}$$

$$\begin{aligned} \Omega_1 &= J_1BA + J_2\alpha' B \\ \Omega_2 &= \mathbf{0} \\ \Omega_3 &= (F - (C_1F + C_2)) \\ \Omega_4 &= J_2\beta' \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \\ \Omega_5 &= J_1BP \\ \Omega_6 &= J_1Q \\ \Omega_7 &= \mathbf{0} \\ \Omega_8 &= J_2\alpha' Q \end{aligned}$$

Finally, we use the expression for agent j's expectation from stage one – equation (B.10) – for $\mathring{E}_t[X_t]$:

$$\mathring{E}_t[X_t] = \Gamma_1 \mathbf{x}_{t-1} + \Gamma_2 \overline{E}_{t-1}[X_{t-1}] + \Gamma_3 \widetilde{E}_{t-1}[X_{t-1}] + \Gamma_4 \mathring{E}_{t-1}[X_{t-1}] + \Gamma_5 \mathbf{u}_t + \Gamma_6 \widetilde{\mathbf{v}}_t + \Gamma_7 \widetilde{\mathbf{v}}_{t-1} + \Gamma_8 \widehat{\mathbf{v}}_{t-1}$$

$$\begin{aligned}
\Gamma_1 &= K_1 BA + K_2 \alpha' B \\
\Gamma_2 &= \mathbf{0} \\
\Gamma_3 &= K_2 \beta' \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \\
\Gamma_4 &= (M - K(D_1 M + D_2)) \\
\Gamma_5 &= K_1 BP \\
\Gamma_6 &= \mathbf{0} \\
\Gamma_7 &= K_2 \alpha' Q \\
\Gamma_8 &= \mathbf{0}
\end{aligned}$$

where I have ignored the element $K_1 Q \mathbf{v}_t(j)$ from equation (B.10) as, from agent i's perspective, it is expected to be equal to zero.

Finding the solution involves finding the fixed point of the system for a pre-chosen upper limit (k^*) on the number of orders of expectations to include.