(*** INCOMPLETE ***) Networks and inflation: A learning-based microfoundation for persistent cost-push shocks[1]

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Abstract

This paper presents a model of price setting wherein firms partially inform their decisions by watching price changes by other firms across an observation network. Within a context of imperfect common knowledge and for a wide range of plausible and commonly observed network structures, idiosyncratic shocks are shown to not "wash out" in aggregate prices. These aggregate effects are also shown to be persistent despite the underlying idiosyncratic shocks being entirely transitory. The model is therefore able to explain a variety of recently documented stylised facts regarding price setting, including the observation that short-lived price changes appear to contain macroeconomic content. The paper also presents a general, readily implementable solution to Bayesian learning over an opaque social network, with the effects of network learning on aggregate expectations able to be simulated without the need to explicitly model the network.

JEL Classification: D21 (Firm Behavior), D83 (Search, Learning, and Information), E31 (Price Level; Inflation; Deflation)

Keywords: Network learning; Incomplete information; Inflation persistence; Aggregate volatility

Contents

1 Introduction

The UK appears to be experiencing a negative supply shock. Real GDP is slumping and inflation, originally predicted to fall quickly following the full pass-through of one-off factors like the VAT increase, remains stubbornly high. Although arguably to a lesser extent, the USA appears to be undergoing a similar episode, with inflation remaining higher than might be expected given the growth rate in real GDP.

Following experiences in the 1970s, it has come to be thought that stagflation persists in an economy through agents' expectations. In particular, most research has supposed that households, expecting high inflation in the coming period, demand higher wages in the current period. To the extent that firms use some version of cost-based pricing and are free to update their prices, this therefore causes high inflation in the current period. But in the UK the present period of stagflation has been accompanied with very low growth rates of nominal wages (and so sizable *contractions* in real wages), thereby ruling out a wage-price spiral as an explanation for its persistence.

In addition, the UK also appears to be experiencing a notable slump in estimates of TFP relative to its pre-financial-crisis trend.

This paper proposes a partial explanation of this phenomenon by generating persistent aggregate cost-push dynamics in the absense of aggregate shocks, the aggregate responses instead being driven by firms' idiosyncratic shocks. That idiosyncratic shocks are an important aspect of firms' price-setting decisions is now broadly accepted. However, it remains commonly assumed that since the shocks themselves must cancel out,[1] the *effects* of those shocks on firms' decisions must also wash out in aggregation. In such a setting, firm-specific shocks can only contribute to aggregate dynamics by causing sluggish responses to aggregate shocks, because firms take time to be sure that a given shock is truly common to all firms. In contrast to this, recently documented evidence from studies of micro-level price changes suggests that those price changes most likely to have been driven by idiosyncratic shocks do *not* cancel out and therefore appear to contain content of macroeconomic import.

To achieve the emergence of aggregate effects from idiosyncratic shocks, this paper develops a model in which firms learn about the state of the economy by observing each other's prices in a directed network and set those prices on the basis of their own marginal costs and their belief regarding the average price. If the network is sufficiently non-uniform, idiosyncratic shocks to firms' marginal costs will not wash out in aggregate prices. If a highly visible firm receives a positive shock to costs (e.g. a negative idiosyncratic productivity shock) and an obscure firm receives a negative cost shock such that the sum of them is zero, the average competitor will consequently believe that there has been a positive aggregate shock to costs. If actual aggregate shocks are persistent, then the average firm will therefore anticipate higher costs tomorrow and thus, if able, will raise their price today. Finally, as firms' learning is recursive, this aggregate effect will be persistent. To a macroeconomist focusing on aggregate data, this will be observationally equivalent to a persistent negative TFP shock.

[1] Over a continuum of agents, that mean-zero idiosyncratic shocks must sum to zero is true by definition; if they did not, they would necessarily include an aggregate component.

It may be noted that because firms may choose to observe the prices of other firms with whom they do not trade and are are not competitors (a perfectly reasonable action provided that their marginal costs are correlated), this model also represents a novel transmission mechanism for inflation across industries or geographies independent of it's path along production chains. However, the origin of an observation network remains largely outside the scope of the current paper, which takes the network as exogenously given.

More generally, this paper develops a general, readily implementable solution to Bayesian learning over an opaque social network in a setting of repeated, simultaneous actions and a dynamic underlying state. Previous work on network learning has typically limited attention to sequential actions and an unchanging state, or relied on assumptions of bounded rationality, or characterised only the speed of convergence in social beliefs. The effects of the network on agents' learning are here captured in a manner that permits researchers to simulate the effects of network learning without having to model the network explicitly, with results calibrated by a single additional parameter describing the degree of asymmetry within the network. This makes the model particularly amenable to nesting within broad general equilibrium models of the economy and may be of independent interest.

When firms exist in an observation network, it is necessary for them to estimate not only the average expectation (for reasons of strategic complementarity), but also the expectations of their observees (and, in turn, their observees' expectations of others again). As the number of agents in the network expands, this causes an explosion in the size of the state vector quite apart from the presence of higher-order expectations (see section 3.1 for more detail) and has typically been thought to prevent closed-form analysis in anything other than trivially small networks.

In contrast, this paper limits the state vector to growing in the number of higher orders of expecation only, even for networks with an infinite number of agents, by denying agents knowledge of the exact topology of the network (the network is *opaque*). Instead, agents are granted knowledge of the distribution from which observation targets are drawn and do not learn about the structure of the network over time. With unobserved aggregate variables following an AR(1) process, the full hierarchy of agents' expectations is shown to follow an ARMA(1,1) process, with current and lagged weighted sums of agents' idiosyncratic shocks entering at an aggregate level. For asymmetric networks – i.e. where some agents' actions are disproportionately observable – these weighted sums are shown to not converge to zero.

Within the literature on deriving aggregate volatility from firms' idiosyncratic shocks, this paper is most closely related to work by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Saleh (2011).[2] Examining the idea of firms operating within an intersectoral supply network, they show idiosyncratic productivity shocks leading to volatility in aggregate output and, for finite networks, derive an upper limit for the rate at which aggregate volatility declines as the number of firms increases. For sufficiently asymmetric trading networks, aggregate volatility need not vanish at all. In another vein, Gabaix (2011) demonstrates how aggregate volatility can emerge from idiosyncratic shocks when the distribution of firm *sizes* exhibits fat tails, even when those firms do not trade directly with each other. Each of these share with this paper an emphasis on unequal, or fat-tailed, distributions. In the model of Gabaix (2011), aggregate volatility arises because the largest firms contribute disproportionately to aggregate production. In that by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Saleh (2011), it emerges through those firms whose output is most extensively used as an intermediate good by other firms. In the current

[2]The work of this paper was first developed independently by Carvalho (2010) and Acemoglu, Ozdaglar, and Tahbaz-Saleh (2010) and later combined to the paper referenced in the text.

paper, with network-based learning, it derives from firms whose price changes are most readily observed.

A large literature also exists exploring network learning. To avoid the dimensionality problems mentioned above, a common approach in this literature has been to step away from fully Bayesian updating. DeMarzo, Vayanos, and Zwiebel (2003), for example, explore situations where where agents assume that signals they receive from observing each other contain *entirely new information*. Such a rule greatly simplifies analysis, but introduces what the authors label "persuasion bias" from the agents' failure to properly discount the repetition of information they receive. Somewhat more generally, Golub and Jackson (2010) study learning in a setting where agents "naïvely" update their beliefs by taking weighted averages of their neighbour's opinions and determine conditions under which social beliefs regarding a single, fixed state of the world converge to the truth. In examining Bayesian learning over a network, previous work has typically limited attention to settings with a fixed state of the world and with agents acting sequentially (and only once each). For example, Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) study the equilibrium of a sequential learning model over a general stochastic network, showing that there will be asymptotic learning when private beliefs are unbounded and characterising some settings under which asymptotic learning still emerges when private beliefs are bounded. Other work of note includes Calvo-Armengol and de Marti (2007), who characterise a method of calculating the welfare gains from a variety of network structures in communication networks that exhibit convergent learning.

This paper falls broadly within and was initially inspired by the literature on imperfect common knowledge or incomplete information. The idea that real effects may arise from nominal disturbances through imperfect information dates to Lucas (1972) and, more recently, Woodford (2003a). The solution method developed by this paper builds upon that put forward by Nimark (2008, 2011a), who introduced dynamic pricing and idiosyncratic shocks in marginal costs to the Woodford (2003a) paper. Other recent work in this area includes Adam (2007), who looked at optimal monetary policy in the Woodford setting and Melosi (2011), who uses the Survey of Professonal Forecasters to estimate a DSGE model with price setters experiencing imperfect common knowledge.

The idea of firms' existing within observation networks need not only feed into a setting of imperfect common knowledge. It might also be readily applied to the the rational inattention work of Sims (2003) or the "sticky information" literature of Mankiw and Reis (2002) and Reis (2006). If one were to suppose that a full information update was costly and observing the price of a competitor less so, a natural incentive emerges to delay a full update and instead update one's price on the basis of those of one's competitors. However, as shown below, evidence from a variety of surveys of firms' price-setting behaviour suggest that the imperfect common knowledge setting may be the more likely reason for firms' observation of each others' prices.

The remainder of this paper is organised as follows. Section 2 provides evidence in support of the price-setting model described in this paper. Section 3 then examines a generalised definition of hierarchies of expectations and presents a context-free model of learning across opaque networks. This may be of independent interest. Section **??** applies this to a small model of price setting and considers the implications for inflation dynamics. Section 5 concludes.

2 Evidence

In this section we summarise a variety of facts regarding firms' price-setting behaviour garnered over recent decades and argue that they are strongly supportive of a model of price setting in which firms obtain information by observing the prices of other firms.

At some level, that firms operate within not just transactional but also observational networks is intuitive, or even self evident. An independent coffee shop will take note of the prices offered by their competitors, including other independent outlets nearby and larger chains like Starbucks. Firms might also observe the price movements of businesses that are not direct competitors in order to learn about the structure of their costs. When a book shop observes a price change at a Thai restaurant next door, or even a car mechanic around the corner, they obtain information about movements in average marginal costs, thereby improving their ability to ascertain that portion of their own cost changes that are idiosyncratic.

In further support of this, we here first describe evidence from a number of price-setting surveys conducted in the 1990s and 2000s and then explore a series of stylised facts identified from studies of recently available datasets of observed price changes.

2.1 Price-setting Surveys

Starting with the work of Blinder (1991) and Blinder, Canetti, Lebow, and Rudd (1998) in the United States and continuing through to the first half of the 2000s, a variety of surveys were been conducted in an attempt to shed light on precisely how firms set prices. These include work in the UK (Hall, Walsh, and Yates (1997)), Sweden (Apel, Friberg, and Hallsten (2005)), Japan (Nakagawa, Hattori, and Takagawa (2000)), Canada (Amirault, Kwan, and Wilkinson (2006)) and nine euro area countries (Fabiani, Druant, Hernando, Kwapil, Landau, Loupias, Martins, MathÃď, Sabbatini, Stahl, and Stokman (2005)).[3]

When looking at those firms following partially or completely state-based pricing, Canadian firms listed price changes by competitors as the most important cause in triggering an adjustment, as did those in Sweden. 53% of Spanish firms reported that competitors' price movements were important factors in triggering their own price changes. In considering the *magnitude* of price changes, 25% of surveyed UK firms reported basing their prices on those of their competitors. This figure agreed with the 27% of surveyed eurozone firms reporting the same, although this ranged from 13% in Portugal to 38% in France. In the Netherlands, where the survey was unique in including very small firms among those polled, this figure was 21.6% overall but rose sharply to 34.1% for firms employing only one worker.

These responses are strongly supportive of the idea that firms observe each others' prices, and we can only assume that they do so as a result of some form of imperfect information: that they learn something from their observations. However, that firms observe each others' prices does not, in itself, speak to *why* they might do so. If, for example, firms experience significant costs in developing optimal price plans in the style of Mankiw and Reis (2002, 2006, 2007) and Reis (2006), then observing one's competitors may occur if by doing so a lower cost is incurred and a fair approximation of the optimal price achieved. Alternatively, if firms face strategic complementarity in their price-setting and there are unobservable aggregate state variables in the style of Woodford (2003a) and Nimark (2008), observing other firms' decisions may be used to inform businesses of the (average) actions or beliefs of their compeititors.

Fortunately, the surveys also queried firms as to their opinions regarding the reasons for price stickiness, from which four theories stand out as being significant: implicit contracts, explicit contracts, cost-based pricing and coordination failure. All of these were among the top five recognised reasons in all 14 surveys when they were included in the options put to surveyed

[3] Countries included were: Austria (Kwapil, Baumgartner, and Scharler (2005)), Belgium (Aucremanne and Druant (2005)), France (Loupias and Ricart (2004)), Germany (Stahl (2005)), Luxembourg (Lunnemann and MathÃď (2006)), the Netherlands (Hoeberichts and Stokman (2006)), Portugal (Martins (2005)) and Spain (ÃĄlvarez and Hernando (2005)).

firms. In stark contrast, menu costs and its more recent variant, information costs, were among the least supported ideas, being in the bottom three reasons for most European surveys and Canada. Only in America and Austria were these costs placed in the middle of the group, menu costs being cited as the sixth most proximate cause of price rigidity in the United States and seventh in Austria and information costs coming sixth in Austria.

The low importance attached to information costs suggests that while there may be imperfect information, it does not manifest in the form of infrequently updated information sets. On the contrary, the strong recognition of coordination concerns and cost-based pricing are supportive of this paper's underlying model: the former suggests that businesses are subject to some form of strategic complementarity in price-setting and the latter that (presumably marginal) costs drive movements in prices.

2.2 Stylised facts from observed price changes

Although early work suggested that most prices change around once per year,[4] the seminal work by Bils and Klenow (2004) observed that the median duration of prices in CPI data from the U.S. Bureau of Labor Statistics (BLS) was 4.3 months, a frequency almost three times higher than previously thought. This triggered a rush of further work exploring and broadly characterising microeconomic price changes. Klenow and Malin (2010) provide an excellent survey of this literature and provide a summary in the form of ten stylised facts. Among these are that:

- prices change at least once a year, twice in America;
- temporary price changes – both reductions and increases – around more rigid "reference prices" are common and do not cancel out in aggregation, suggesting that some macroeconomic content is present in the more frequent updates;
- price changes are typically larger than those needed to keep up up with inflation, suggesting that idiosyncratic factors weigh more heavily on a firm's price-setting decision than aggregate factors;
- changes in *relative* prices tend to be short lived, suggesting that idiosyncratic shocks are less persistent than aggregate disturbances; and
- price changes are generally linked to changes in marginal costs, particularly wages.

The first of these necessitates some form of structural, or *real* rigidity in addition to firms' nominal rigidities – a "contract multiplier", in the words of Taylor (1980) – to explain the sluggish responses observed aggregate price indicies.[5] The second and third points show that even if firms' idiosyncratic shocks have zero mean and "cancel out" when averaged, average temporary price changes (that are presumably based on them) do not cancel out.

The model presented in this paper is consistent with all of the above stylised facts and with observations of rigidity in aggregate prices. Because firms are able to observe the prices of any other firm, it also represents a framework for the transmission of inflation (and hence, its persistence) across industries or geographies and not simply along production chains.

[4] See, for example, Taylor (1999).

[5] See, for example, Christiano, Eichenbaum, and Evans (1999) or Romer and Romer (2004) for the USA, or Peersman and Smets (2003) for the Euro area.

3 A generalised model of learning across opaque networks

We here develop a generalised model of Bayesian learning across opaque networks. Although the current paper is focused primarily on price-setting, this section is presented largely free of context because the model may, in general, be applied to any setting in macroeconomics or finance where agents' expected payoffs depend on the (average) actions of their competitors and some unobserved aggregate state.[6]

A simple roadmap of how this section will proceed may be of some assistance. First, in subsection 3.1, we will provide a comprehensive characterisation of higher-order expectations (beyond the simple treatement found in other papers in the literature that focus only on hierarchies of average expectations), together with an explanation of how these have traditionally defeated attempts to model Bayesian learning over social networks. Next, in subsection 3.2, we will describe the agents' problem, the information available to them and how they make their decisions. Subsection 3.3 will characterise agents' average action and briefly describe the informational assumptions used in previous research and how they differ to the current paper. The main result of this paper – a model of learning when agents observe the actions of individual competitors – is then developed, and its main consequences discussed, in subsection 3.4. Finally, subsection 3.6 demonstrates that the model may be readily extended to dynamic settings where agents decision rules include consideration of past or future variables.

3.1 Higher-order expectations

Because agents observe the actions of individual competitors, the common description of higher-order expectations as only including *average* expectations is insufficient for our needs. We therefore first provide a generalised definition of a hierarchy of expectations.

Definition 1 *Let θ_t be an $(m \times 1)$ vector of random variables, $E[\theta_t|\mathcal{I}_t(i)]$ be the expectation of θ_t conditioned on the information set of agent i and $\mathcal{E}_t[\theta_t] \equiv \begin{bmatrix} E[\theta_t|\mathcal{I}_t(1)] & \cdots & E[\theta_t|\mathcal{I}_t(N)] \end{bmatrix}$ be the $(m \times N)$ matrix containing all agents' expectations of the same. Let $\mathbf{w}$ be an $(N \times 1)$ vector of weights across all agents such that $w_i \in [0,1]$ and $\sum_{i=1}^{N} w_i = 1$. We then define a compound expectation to be a weighted sum of all agents' expectations:*

$$E_{\mathbf{w},t}[\theta_t] \equiv \mathcal{E}_t[\theta_t]\,\mathbf{w} \tag{1}$$

Note that this nests both simple, or unweighted, average expectations (e.g. $\mathbf{w}_A = \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix}'$) and individual expectations (e.g. $\mathbf{w}_B = \begin{bmatrix} \mathbf{0}' & 1 & \mathbf{0}' \end{bmatrix}'$).

Definition 2 *Let $W \equiv \begin{bmatrix} \mathbf{w}_A & \mathbf{w}_B & \cdots \end{bmatrix}$ be the $(N \times p)$ matrix formed of all weights of interest in a given problem and p be the number of those weights (i.e. the number of columns in W).*

[6] Two examples may be of interest: First, when posting vacancies in a labour search model in the style of Mortensen and Pissarides (1994), firms' probability of finding a successful match is dependent on the number of vacancies that other firms post. When firms' productivity includes both aggregate and idiosyncratic components, observing the number of vacancies posted by their competitors allows firms to be able to predict the component of their productivity that is common to all and their expected gain from posting an additional vacancy themselves.

Alternatively, in the asset pricing model of Singleton (1987), traders' individual demand for a risky asset is dependent on their expectation of the next-period price, itself a function of all traders' actions and (unobserved in advance) shocks to the supply of the asset. Observing the actions of (some of) their competitors allows traders to learn about the (higher-order) expectations of other traders and adjust their responses accordingly.

We then define higher-order expectations as follows, using a blackboard-bold $\mathbb{E}^{(k)}$ to denote the vector containing all expectations of the k-th order:

$$\mathbb{E}_t^{(0)}[\theta_t] \equiv \theta_t$$

$$\mathbb{E}_t^{(k)}[\theta_t] \equiv \begin{bmatrix} E_{\mathbf{w}_A,t}\left[\mathbb{E}_t^{(k-1)}[\theta_t]\right] \\ E_{\mathbf{w}_B,t}\left[\mathbb{E}_t^{(k-1)}[\theta_t]\right] \\ \vdots \end{bmatrix} = vec\left(\mathcal{E}_t\left[\mathbb{E}_t^{(k-1)}[\theta_t]\right]' W\right) \ \forall k \geq 1 \tag{2}$$

Note that if we are interested in p different compound expectations, there are p^k different permutations of k-th order expectations. For example, if θ_t is scalar and $p = 2$, then the vector describing the set of second-order expectations will be of size (4×1) and arranged in the following way:

$$\mathbb{E}_t^{(2)}[\theta_t] = \begin{bmatrix} E_{\mathbf{w}_A,t}\left[\mathbb{E}_t^{(1)}[\theta_t]\right] \\ E_{\mathbf{w}_B,t}\left[\mathbb{E}_t^{(1)}[\theta_t]\right] \end{bmatrix} = \begin{bmatrix} E_{\mathbf{w}_A,t}\begin{bmatrix} E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \\ E_{\mathbf{w}_B,t}\begin{bmatrix} E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \end{bmatrix}$$

Definition 3 *A hierarchy of expectations, from order* 0 *to* k*, is defined recursively as:*

$$\mathbb{E}_t^{(0:k)}[\theta_t] = \begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}\left[\mathbb{E}_t^{(0:k-1)}[\theta_t]\right] \\ E_{\mathbf{w}_B,t}\left[\mathbb{E}_t^{(0:k-1)}[\theta_t]\right] \\ \vdots \end{bmatrix} \tag{3}$$

Note that this is not simply the stacking each order of expectations on top of each other. For example, if θ_t is scalar and $p = 2$, the hierarchies $(0:1)$ and $(0:2)$ are given by:

$$\mathbb{E}_t^{(0:1)}[\theta_t] = \begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \quad \mathbb{E}_t^{(0:2)}[\theta_t] = \begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}\begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \\ E_{\mathbf{w}_B,t}\begin{bmatrix} \theta_t \\ E_{\mathbf{w}_A,t}[\theta_t] \\ E_{\mathbf{w}_B,t}[\theta_t] \end{bmatrix} \end{bmatrix}$$

The benefit of depicting hierarchies in this manner is that it becomes simple to extract sub-hierarchies comprised of a single compound expectation. For example, if $\mathbf{w}_A = \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix}'$ so that $E_{\mathbf{w}_A,t}[\theta_t] = \overline{E}_t[\theta_t]$ is the average expectation, the sub-hierarchy of $\overline{\theta}_t^{(0:k)} \equiv \left[\theta_t', \overline{E}_t[\theta_t'], \overline{E}_t\left[\overline{E}_t[\theta_t']\right], \cdots\right]'$ may be extracted as

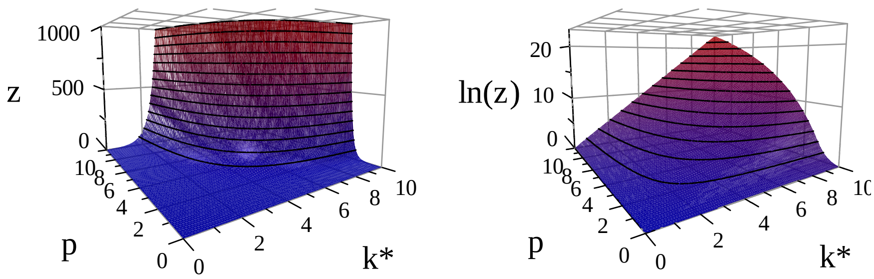
$$\overline{\theta}_t^{(0:k)} = \begin{bmatrix} I & 0 \end{bmatrix} \mathbb{E}_t^{(0:k)}[\theta_t]$$

In solving our model of learning over an opaque network, $\mathbb{E}_t^{(0:k^*)}[\theta_t]$ will represent the unknown state vector about which agents attempt to learn.

Temporarily dropping the time subscript, it is clear that if θ contains m elements, $\mathbb{E}^{(k)}[\theta]$ – the set of k-th order expectations – will contain mp^k distinct elements. However, it is worth emphasising that it does not in general follow that $\mathbb{E}^{(0:k^*)}[\theta]$ will contain $m\left(\sum_{k=0}^{k^*} p^k\right)$ unique elements. This is because if one of the compound expectations, say $E_{\mathbf{w}_B}[\cdot]$, is an individual expectation – i.e. formed from a single information set – then the law of iterated expectations implies that $E_{\mathbf{w}_B}[E_{\mathbf{w}_B}[\theta]] = E_{\mathbf{w}_B}[\theta]$. In general, when $q \leq p$ is the number of individual expectations in W, the number of unique elements in the hierarchy $\mathbb{E}^{(0:k^*)}[\theta]$ will be given by:[7]

$$m\left(p^{k^*} + \sum_{k=0}^{k^*-1}\left(p^k - q\sum_{s=0}^{k} p^s\right)\right) < m\left(\sum_{k=0}^{k^*} p^k\right)$$

Nevertheless, even when $q = p$, the size of an expectation hierarchy explodes (goes to infinity) in both p and k^* (see figure 1).



An infinite dimension state vector need not be a problem, *per se*, provided that the researcher is able to make a reasonable approximation of agents' actions by restricting attention to a finite subset of the state. In most models – including that of the current paper – imposing a finite upper limit, k^*, on the number of orders of expectation will be acceptable as in order to ensure stability in agent actions, decreasing weight is placed on higher order expectations.

Allowing the number of compound expectations to increase can be more problematic, however, as there is rarely an obvious reason for weighting them differently. Previous literature has generally avoided this difficulty by setting up their problems in a manner that implicitly assumes that $p = 1$. That is, that no matter the number of agents, they each only care about the *average* expectation of their competitors. However, this avenue is not available when considering learning via networks, where it is typically the case that p is given by the number of agents in the network.

3.2 The general setting

There is a countably infinite number of agents,[8] indexed in a continuum between zero and unity.[9]

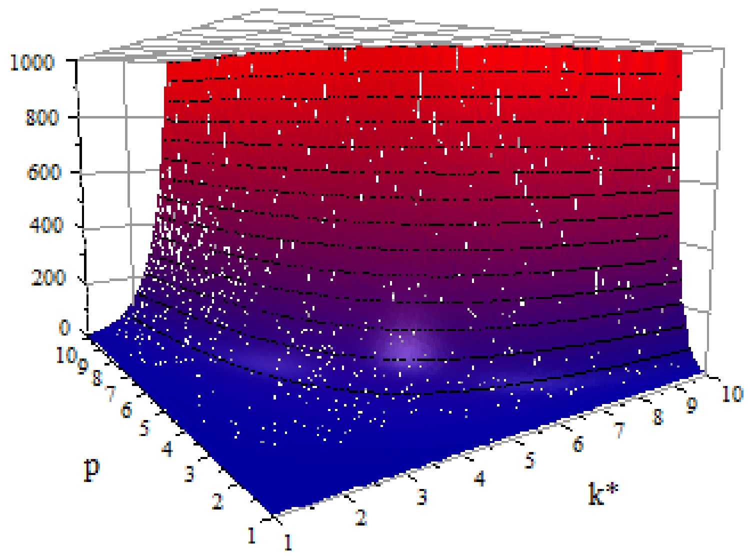
[7]

$$m\left(\underbrace{[1]}_{\text{0-th order}} + \underbrace{[p]}_{\text{1-st order}} + \underbrace{[p^2 - q]}_{\text{2-nd order}} + \underbrace{[p*(p^2-q)-q]}_{\text{3-rd order}} + \underbrace{[p*(p*(p^2-q)-q)-q]}_{\text{4-th order}} + \cdots\right)$$

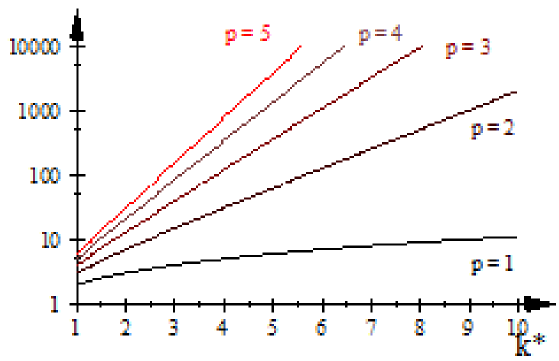
$$= m\left(\left(\sum_{k=0}^{k^*} p^k\right) - q\left(\sum_{k=0}^{k^*-1}\sum_{s=0}^{k} p^s\right)\right), \text{ which rearranges to the equation in the text}$$

[8] An infinite number of agents is assumed to allow an appeal to relevant laws of large numbers when considering simple averages of zero-mean shocks.

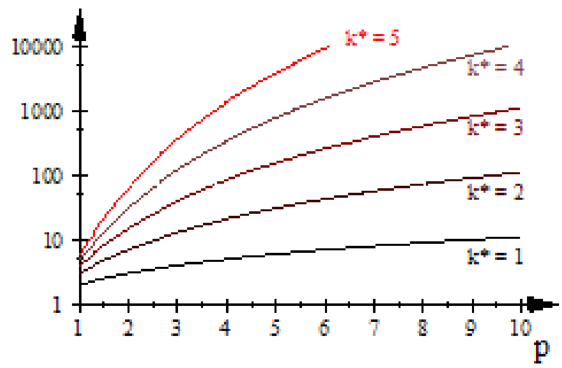
[9] The assumption of indexing agents from zero to one is innocuous and made only to simplify the calculation of averages.



(a) Linear scale



(b) Cross-section by p (log scale)



(c) Cross-section by k^* (log scale)

Figure 1: The number of elements in an expectation hierarchy ($q = 0$, θ scalar)

The *underlying state* follows a vector autoregressive process:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + P\mathbf{u}_t \tag{4}$$

where $\mathbf{u}_t$ is a vector of shocks with mean zero, while A and P are appropriately dimensioned matricies of fixed and publicly known parameters.

Agents simultaneously determine their individual actions according to a common decision rule:[10]

$$g_t(i) = \eta_s'\mathbf{s}_t(i) + \eta_x' E_t(i)[\mathbf{x}_t] + \eta_y E_t(i)[\overline{g}_t] \tag{5}$$

[10]The derivation of the decision rule will invariably be context-specific. For example, in the context of price-setting to be explored in the next section, the underlying state will include aggregate shocks to marginal cost and demand, while agents' actions will be the price they choose and the "signal" they receive will include their private marginal cost and the previous-period price of a competitor.

where $\mathbf{s}_t(i)$ is agent i's vector of observables (defined below), $E_t(i)[\cdot] \equiv E[\cdot|\mathcal{I}_t(i)]$ is agent i's (first-order) expectation of the element within the square brackets conditional on all information available to her in period t (defined below), $\overline{g}_t \equiv \int_o^1 g_t(i)\,di$ is the (simple, or unweighted) average action of all agents in period t, and η_s, η_x and η_y are vectors of parameters and a scalar parameter respectively, all fixed and publicly known. η_y may be thought of as a measure of agents' strategic complementarity ($\eta_y > 0$) or substitutability ($\eta_y < 0$) in actions.

Note that more general decision rules can be accomodated (see section 3.6), but we proceed with that specified in (5) for the sake of clarity. We impose only two constraints:

- that $|\eta_y| \in (0,1)$ in order to ensure that agents place successively lower weight on higher-order expectations; and
- that only contemporaneous elements of agents' signals affect their actions directly, so that signal components obtained with a lag serve only an informational role (by helping agents construct their expectations).

Each agent's *signal vector* is made up of two, distinct components – a *private signal* based on the current-period underlying state and a *social signal* derived from observing competitors' actions with a one-period lag:

$$\begin{aligned}\mathbf{s}_t(i) &= \begin{bmatrix}\mathbf{s}_t^p(i)\\ \mathbf{s}_t^s(i)\end{bmatrix} \\ \mathbf{s}_t^p(i) &= B\mathbf{x}_t + Q\mathbf{v}_t(i) \\ \mathbf{s}_t^s(i) &= W_{t-1}(i)\,\mathbf{g}_{t-1}\end{aligned} \tag{6}$$

It is assumed that private signals are noisy, with $\mathbf{v}_t(i)$ a vector of shocks specific to agent i in period t, drawn from independent and identical Gaussian distributions with mean zero and variance Σ_{vv}. The dimensions of B and Q are left unspecified, depending on the number of observables made available from the underlying state. Social signals are assumed to be observed perfectly, where $\mathbf{g}_{t-1}$ is the $(\infty \times 1)$ vector of all agents' actions from the previous period and $W_{t-1}(i)$ is i's (potentially stochastic) observation matrix. For example, if in period t agent i observes the period $(t-1)$ actions of agents 1 and 2, then $W_{t-1}(i)$ would be given by:

$$W_{t-1}(i) = \begin{bmatrix}1 & 0 & 0 & \cdots\\ 0 & 1 & 0 & \cdots\end{bmatrix}$$

The assumption that observations based on the previous period serve only an informational role in the current period (by helping agents construct their expectations) and do not directly affect actions therefore gives the following form to η_s:

$$\eta_s' \equiv \begin{bmatrix}\alpha' & 0\end{bmatrix} \tag{7}$$

Note, in particular, that this implies that $\eta_s' \mathbf{s}_t(i) = \alpha'(B\mathbf{x}_t + Q\mathbf{v}_t(i))\ \forall i, t$. Finally, we have that agent i's information set evolves as follows:

$$\begin{aligned}\mathcal{I}_0(i) &= \{A, P, B, Q, \Phi, W_0(i)\} \\ \mathcal{I}_t(i) &= \{\mathcal{I}_{t-1}(i), \mathbf{s}_t(i), W_t(i)\}\end{aligned} \tag{8}$$

That is, in each period, agent i is informed of their private signal, their social signal and the identity of the competitors whose actions they will receive in the next period. $\Phi : [0,1] \to [0,1]$ is the (cumulative) distribution from which observees are drawn, assumed to be identical and independent for every agent. $\Phi(j)$ is absolutely continuous over the range $[0,1]$ and has p.d.f. ϕ.

3.3 Average actions and imperfect common knowledge

At first glance, obtaining $E_t(i)[\overline{g}_t]$ in the agent's decision rule may appear implausible, as it would seem to require the formation of an expectation regarding the action of every competitor. However, we can greatly simplify matters by noting that the average action – i.e., the average of equation (5) – can be written as

$$\overline{g}_t = \eta_s' \overline{\mathbf{s}}_t + \eta_x' \overline{E}_t[\mathbf{x}_t] + \eta_y \overline{E}_t[\overline{g}_t]$$

Making use of the law of large numbers, we can next see that $\eta_s' \overline{\mathbf{s}}_t = \alpha' B \mathbf{x}_t$. Repeatedly substituting our expression for the average action back into itself, we therefore obtain

$$\overline{g}_t = \alpha' B \mathbf{x}_t + \beta' \overline{\mathbf{x}}_{t|t}^{(1:\infty)} \tag{9}$$

where $\overline{\mathbf{x}}_{t|t}^{(1:\infty)}$ is a vector containing the hierarchy of simple-average expectations regarding $\mathbf{x}_t$ and the vector β is given by:

$$\beta' = \begin{bmatrix} (\eta_x' + \eta_y \alpha' B) & \eta_y (\eta_x' + \eta_y \alpha' B) & \eta_y^2 (\eta_x' + \eta_y \alpha' B) & \cdots \end{bmatrix} \tag{10}$$

Note that as $|\eta_y| \in (0,1)$, successively lower weight is placed on higher-order expectations. Substituting this into (5), we are then able to write agent i's decision rule as:

$$g_t(i) = \alpha' \underbrace{(B\mathbf{x}_t + Q\mathbf{v}_t(i))}_{\text{Private signal}} + \beta' E_t(i) \left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)} \right] \tag{11}$$

That is, each agent constructs their action as a linear combination of their private signal and their expectation of the entire hierarchy of *simple average* expectations of the underlying state. Agents therefore need to estimate an expanded *state vector of interest* that includes not just the underlying state, but also the higher-order average expectations of the same.

Because of the linearity of the underlying system, the best linear estimator – in the sense of minimising the mean squared error – will be a Kalman filter.[11] [12] Denoting the state vector to be estimated as Z_t (which will include $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ but may, depending on the model, have other components), the signal equation will be able to be rewritten in the following form

$$\mathbf{s}_t(i) = L Z_t + \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \mathbf{v}_t(i)$$

Supposing an AR(1) process for Z_t's law of motion

[11] If all shocks are drawn from Gaussian distributions, it will be the best such estimator, linear or otherwise.

[12] The derivation of the standard Kalman filter may be found in most texts on dynamic macroeconomics (e.g. Ljungqvist and Sargent (2004)) or timeseries analysis (e.g. Hamilton (1994)).

$$Z_t = MZ_{t-1} + N \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{bmatrix}$$

agent i's period-t expectation of Z_t will be formed recursively in the following manner

$$E_t(i)[Z_t] = K\mathbf{s}_t(i) + (M - KLM) E_{t-1}(i)[Z_{t-1}] \tag{12}$$

where K is a time-invariant Kalman gain matrix. As in other models of imperfect common knowledge, since Z_t includes $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$, we have that (a) the state vector to be estimated is of infinite dimension; and (b) the Kalman filter serves a dual role, both as estimator and as part of the law of motion for the state vector. Solving the system then requires finding the coefficients in K, M, N, L and V (the variance-covariance of the agents' estimates). This, in turn, depends on the exact signal structures faced by the agents.

Woodford (2003a), looking at firms' static price-setting decisions, supposed that agents receive only a private signal from the underlying state and no social signal from other agents ($W_t(i) = \mathbf{0}\ \forall i, t$). In such a setting, where $Z_t = \overline{\mathbf{x}}_{t|t}^{(0:\infty)}$, Woodford showed that M will be lower-triangular: each order of simple average expectations will be a linear combination of *lower* order expectations and current period shocks. Consequently, M may be constructed sequentially, first finding an expression for $E_t(i)[\mathbf{x}_t]$, then averaging it and repeating the process to find $E_t(i)\left[\overline{E}_t[\mathbf{x}_t]\right]$ and so forth.

Nimark (2008), who focused on a context of dynamic price-setting, extended Woodford's (2002) work to allow agents to observe the simple-average action from the previous period in addition to their private signals. In the nomenclature above, this amounts to

$$W_{t-1}(i)\,\mathbf{g}_{t-1} = \lim_{N\to\infty} \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} \mathbf{g}_{t-1} = \overline{g}_{t-1}\ \forall i$$

In this case, agents' signals are linear combinations of the entire hierarchy of previous-period expectations (since actions are based on the entire hierarchy and agents observe the average previous action) and as a result, the solution must be found for all higher-order expectations simultaneously. Note, too, that since the Kalman filter requires that agents form prior expecations about the signals they receive, this typically requires that the state vector also include $\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}$ so that $Z_t = \begin{bmatrix} \overline{\mathbf{x}}_{t|t}^{(0:\infty)\prime} & \overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)\prime} \end{bmatrix}^{\prime}$.[13]

It is perhaps worth empahsising that the signal structures assumed by Woodford (2003a) and Nimark (2008) both result in agents only being concerned with the simple *average* expectation of their peers (or higher-order versions of the same). In the language of this paper, they have

[13] An alternative to including $\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}$ in the state vector of interest is to use a slightly different specification of the signal vector:

$$\mathbf{s}_t(i) = L_1 \overline{\mathbf{x}}_{t|t}^{(0:\infty)} + L_2 \overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)} + \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \mathbf{v}_t(i)$$

and correspondingly modify the Kalman filter:

$$E_t(i)\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right] = K\mathbf{s}_t(i) + (M - K(L_1 M + L_2)) E_{t-1}(i)\left[\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}\right]$$

See Nimark (2011b) for more detail.

chosen signal structures that explicitly set $p = 1$, thereby having the infinite dimensionality of the state vector arising only from the presence of higher-order expectations. With the condition that $|\eta_y| \in (0,1)$, successively lower weight is placed on higher-order expectations and so a finite system can be made arbitrarily accurate in approximating the full system by defining a threshold for the upper limit of orders of expectation.

3.4 Observing individual competitors' actions

Suppose that agents observe the previous-period actions of q competitors. We introduce the function $\delta_t : [0,1] \to [0,1]^q$ to map each agent to their observational target(s). For presentational simplicity, in what follows we will typically assume that $q = 1$ (i.e. that all agents observe a single competitor) and simply write $j = \delta_t(i)$ to mean that agent j's period-t action will be observed by agent i. To speak of the the observee of an observee, we write $\delta_s(\delta_t(i))$: the identity of the agent whose period-s action is observed by the agent whose period-t action is observed by agent i. The function $\delta_t(i)$ is related to $W_t(i)$ in the following way:

$$W_t(i) = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{\text{Column } j=\delta_t(i)} & 0 & \cdots & 0 \end{bmatrix}$$

In this setting, agent i's signal vector for period t will have two elements in it:

$$\mathbf{s}_t(i) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(i) \\ g_{t-1}(\delta_{t-1}(i)) \end{bmatrix}$$

Agent i's prior will therefore include $E_{t-1}(i)[g_{t-1}(\delta_{t-1}(i))]$ and, stepping forward one period, we have that agent i must form $E_t(i)[g_t(\delta_t(i))]$ as part of her prior for period $t+1$.

To allow us to solve the general agent's problem, we make the following assumptions regarding the agents' observation network:

Assumption 1 *The network is stochastic and opaque, in that:*

- *all agents observe the same number of competitors;*
- *observees are drawn from identical, independent distributions with p.d.f. $\phi(i)$;*
- *agents know the identity of the other agents they observe;*
- *agents do not know who they are observed by; and*
- *agents do not learn about the network topology over time.*

To obtain this last point, we might either suppose that agents make a fresh draw of whom to observe every period,[14] in which case nothing *could* be learned about the network topology (since it changes every period), or allow the network to be drawn once and assume a form of bounded rationality in the agents, in that they focus only on the game in front of them and not the structure of the network. We are then in a position to assert the following:

Lemma 1 *Given assumption 1, agents' use of a linear estimator implies that all agents treat all other agents as though they observe a common, weighted average of previous-period actions, with the weights given by the distribution ϕ.*

[14] In such a setting, it may be better to imagine agents not operating in a network so much as a search model.

Proof. The proof may be found in appendix A.2. ■

From equation (11), we see that the weighted-average action, $\widetilde{g}_t$, is given by:

$$\widetilde{g}_t = \alpha' B \mathbf{x}_t + \alpha' Q \widetilde{\mathbf{v}}_t + \beta' \widetilde{E}_t \left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)} \right] \tag{13}$$

where $\widetilde{E}_t [\cdot] \equiv \int_0^1 E_t (j) [\cdot] \phi (j) \, dj$ is the weighted-average expectation and the last term is the weighted average of private expectations regarding the hierarchy containing only *unweighted* average expectations.

Note that we cannot, in general, make use of some law of large numbers to disregard the effect of idiosyncratic shocks in the weighted-average action – that is, we cannot assume that $\widetilde{\mathbf{v}}_t \equiv \int_0^1 \mathbf{v}_t (j) \phi (j) \, dj$ will be equal to zero – because the weights applied to each agent may not be sufficiently close to equal. As an extreme example, if all agents were to observe agent 1 and nobody else (i.e. $\phi (1) = 1$ and $\phi (i) = 0 \ \forall i \neq 1$), we would then have that $\widetilde{\mathbf{v}}_t = \mathbf{v}_t (1)$ which will, with probability one, be not equal to zero.

Identifying laws of large numbers for weighted sums of i.i.d. random variables (i.e. the limiting behaviour of $\sum_{i=1}^N a_{N,i} X_i$ when $E[X] = 0$) remains an area of active research. See, for example, Wu (1999), Sung (2001) or Cai (2006). However, it is not necessary for us to have an exact characterisation of the necessary conditions for the weighted sum to converge to zero, as there are a broad range of functions for the weights under which the weighted sum will *not* converge to zero. In particular, we make the following assumption:

Assumption 2 *The network is asymmetric. That is, defining $\zeta (N) \equiv \sum_{i=1}^N \phi_N (i)^2$, the p.d.f. $\phi_N (i)$ is such that:*

- $\lim_{N \to \infty} \phi_N (i) = 0 \quad \forall i$
- $\lim_{N \to \infty} \zeta (N) = \zeta^*$ *where* $\zeta^* \in (0, \infty)$.

This assumption then allows us to assert the following lemma regarding limiting properties of aggregate (random) variables derived from agents' idiosyncratic shocks:

Lemma 2 *Suppose that $\mathbf{v}_t (i) \sim i.i.d. N (\mathbf{0}, \Sigma_{vv}) \ \forall i, t$. For a finite number of agents (N), let $\widetilde{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^N \mathbf{v}_t (i) \phi_N (i)$ denote the weighted average of agents' own idiosyncratic shocks; $\ddot{\mathbf{v}}_{N,t} \equiv \frac{1}{N} \sum_{i=1}^N \mathbf{v}_t (\delta_t (i))$ denote the simple average of the idiosyncratic shocks of agents' observees; and $\widehat{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^N \mathbf{v}_t (\delta_t (i)) \phi_N (i)$ denote the weighted average of agents' observees' idiosyncratic shocks. Given assumption 2, we have the following results in the limit (as $N \to \infty$):*

- $\widetilde{\mathbf{v}}_{N,t} \xrightarrow{d} \widetilde{\mathbf{v}}_t$ *where* $\widetilde{\mathbf{v}}_t \sim N (\mathbf{0}, \zeta^* \Sigma_{vv})$
- $\ddot{\mathbf{v}}_{N,t} \xrightarrow{\mathcal{L}^2} \widetilde{\mathbf{v}}_t$
- $\widehat{\mathbf{v}}_{N,t} \xrightarrow{d} \widehat{\mathbf{v}}_t$ *where* $\widehat{\mathbf{v}}_t \sim N (\mathbf{0}, \zeta^* (2 - \zeta^*) \Sigma_{vv})$
- $Cov (\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t) = \zeta^* \Sigma_{vv}$

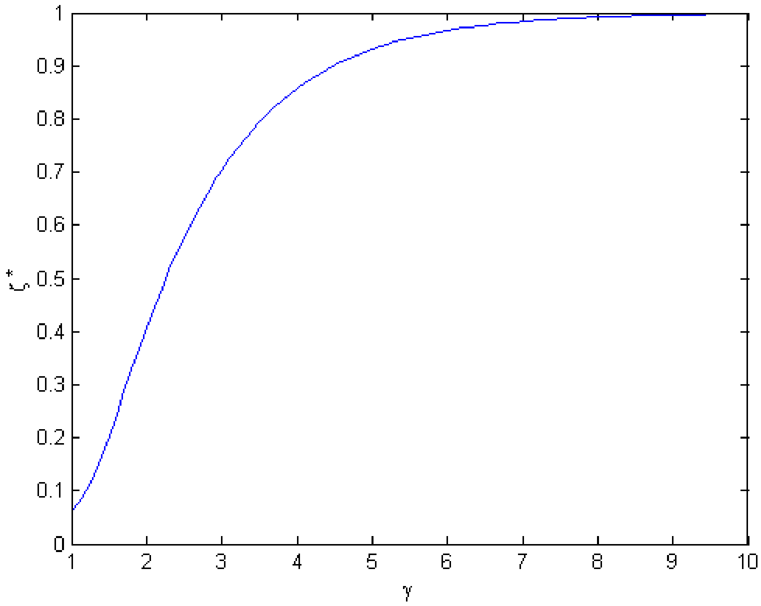


Figure 2: A plot of ζ^* for power law (Zeta) distributions with shape parameter γ

Proof. The proof may be found in appendix A.3. ■

The first result in this lemma shows that assumption 2 is sufficient to ensure that idiosyncratic shocks do not "wash out" in the weighted-average action. The set of distributions satisfying this assumption is quite broad. In particular, it is satisfied by the discrete power law distribution (the Zipf distribution)

$$\phi_N(i) = c_N i^{-\gamma}; \text{ where } c_N = \left(\sum_{i=1}^{N} i^{-\gamma}\right)^{-1} \text{ and } \gamma > 1$$

and it's equivalent for infinite N, the Zeta distribution. The shape parameter, $\gamma > 1$, governs the scaling of the distribution's tail: larger values of γ correspond to greater asymmetry in the distribution and, as such, the greater the variance that survives aggregation. Figure 2 plots the values of ζ^* for a range of values of γ for the Zeta distribution.

A great many observed networks, from webpages on the internet to established relationships in social networks, have been shown to have degree distributions well approximated by power law distributions (i.e. the networks are *scale free*). See, for example, Albert and BarabÃąsi (2002), Jackson and Rogers (2007) or Clauset, Shalizi, and Newman (2009).

Finally, we now present the following main result of this section:

Proposition 1 *Given assumptions 1 and 2, and the generalised setting described above, agents' need only consider a hierarchy of $p = 3$ compound expectations and a solution may be obtained in two steps:*

- *First, for a given set of weights that are common knowledge, obtain a solution to a model where all agents observe the same weighted average of everybody's previous-period action; and*

- *Second, solve the individual problem supposing that information from any target comes from the setting in the first stage, where the weights used are the distribution from which agents' observation targets are drawn. The hierarchy of expectations of the underlying state will evolve according to the following ARMA(1,1) process:*

$$
\begin{aligned}
X_t &\equiv \mathbb{E}_t^{(0:\infty)}\left[\mathbf{x}_t\right] \\
X_t &= F X_{t-1} + G_1 \mathbf{u}_t + G_2 \widetilde{\mathbf{v}}_t + G_3 \widetilde{\mathbf{v}}_{t-1} + G_4 \widehat{\mathbf{v}}_{t-1} \\
\overline{\mathbf{x}}_{t|t}^{(0:\infty)} &= \begin{bmatrix} I & \mathbf{0} \end{bmatrix} X_t
\end{aligned}
$$

Proof. Although the complete derivation is provided in appendix A.4, an outline of the agents' learning process may be of expositional interest. ■

3.4.1 Stage One: When all agents observe the same weighted-average action

In the first stage, we solve a model where all agents observe the same weighted average of previous-period actions, so that agents' signal vectors are given by:

$$
\mathbf{s}_t\left(j\right) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t\left(j\right) \\ \widetilde{g}_{t-1} \end{bmatrix}
$$

Equation (13) shows us that agents will therefore need to consider weighted-average expectations in addition to simple-average expectations. We define the state vector of interest, Z_t, as the full hierarchy of expectations regarding the underlying state, expressed recursively as:

$$
Z_t \equiv \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t\left[Z_t\right] \\ \widetilde{E}_t\left[Z_t\right] \end{bmatrix} \tag{14}
$$

We also define S, T_s and T_w as the matricies that select $\mathbf{x}_t$, $\overline{E}_t\left[Z_t\right]$ and $\widetilde{E}_t\left[Z_t\right]$ from Z_t respectively. Using this, we can rewrite agents' observation vectors as:

$$
\mathbf{s}_t\left(j\right) = D_1 Z_t + D_2 Z_{t-1} + R_1 \mathbf{v}_t\left(j\right) + R_2 \widetilde{\mathbf{v}}_{t-1} \tag{15a}
$$

In the context of the model presented above, the parameters here are given by:

$$
D_1 = \begin{bmatrix} BS \\ \mathbf{0} \end{bmatrix} \tag{16a}
$$

$$
D_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' BS + \beta' T_w \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \tag{16b}
$$

$$
R_1 = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \tag{16c}
$$

$$
R_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' Q \end{bmatrix} \tag{16d}
$$

where, within the description of D_2, the matrix $T_w \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$ selects $\widetilde{E}_{t-1}\left[\overline{\mathbf{x}}_{t-1|t-1}^{(0:\infty)}\right]$ from Z_{t-1}. We conjecture (and verify below in appendix A) that the state vector may be written in the following ARMA(1,1) law of motion:

$$
Z_t = M Z_{t-1} + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1} \tag{17}
$$

There are two complications in this system over a classic Kalman filtering problem. The first is that agents' signal vectors include observations available only with a lag, and the second, related to the first, is presence of lagged shocks (the MA(1) component). The most common approach is to stack the state vector with it's lag

$$\begin{bmatrix} Z_t \\ Z_{t-1} \end{bmatrix} = \begin{bmatrix} M & \mathbf{0} \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ Z_{t-2} \end{bmatrix} + \begin{bmatrix} N_1 & N_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_t \\ \widetilde{\mathbf{v}}_t \end{bmatrix} + \begin{bmatrix} \mathbf{0} & N_3 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{t-1} \\ \widetilde{\mathbf{v}}_{t-1} \end{bmatrix}$$

However, to do so doubles the size of the state vector – which may present problems when simulating the system with finite computing resources – and still requires accounting for the lagged disturbances. Instead, the derivation of expressions for the M and N_* coefficients in appendix A follows Nimark (2011b) in finding a modified Kalman filter that does not require the stacking of the system and explicitly allows for the presence of lagged shocks. That is, starting from an expression of agent j's filtering problem in recursive form

$$E_t(j)[Z_t] = E_{t-1}(j)[Z_t] + K_t\{\mathbf{s}_t(j) - E_{t-1}(j)[\mathbf{s}_t(j)]\}$$

we substitute in the state law of motion (17) and the signal (15a); find the optimal Kalman gain (K_t) and its time-invariant form; and then take averages to obtain expressions for $\overline{E}_t[Z_t]$ and $\widetilde{E}_t[Z_t]$, which in turn allows us to identify the elements of M and N_*.

3.4.2 Stage Two: Solving the agents' problem in an opaque network

In the general problem for an agent that observes the previous period action of specific competitors,[15] agent i's social observation matrix will be given by:

$$W_t(i) = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{\text{Column } j=\delta_t(i)} & 0 & \cdots & 0 \end{bmatrix}$$

so that her period-t social signal will be given by:

$$\mathbf{s}^s_t(i) = g_{t-1}(\delta_{t-1}(i)) = \alpha' B\mathbf{x}_{t-1} + \alpha' Q\mathbf{v}_{t-1}(\delta_{t-1}(i)) + \beta' E_{t-1}(\delta_{t-1}(i))\left[\overline{\mathbf{x}}^{(0:\infty)}_{t-1|t-1}\right]$$

We can therefore see that agent i will need to keep track of three compound expectations: the simple-average $(\overline{E}_t[\cdot])$, the weighted-average $(\widetilde{E}_t[\cdot])$ and that of her observee $(E_t(\delta_t(i))[\cdot])$. However, since the observee is treated the same no matter who they are, we have that the expectations will update in a common manner for every agent and denote an expectation obtained from stage one as $E_t(\delta_t(i))[\cdot] = \mathring{E}_t[\cdot]\ \ \forall i$:

$$X_t \equiv \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t[X_t] \\ \widetilde{E}_t[X_t] \\ \mathring{E}_t[X_t] \end{bmatrix} \tag{18}$$

We can then write agent i's observation vector as

$$\mathbf{s}_t(i) = C_1 X_t + C_2 X_{t-1} + S_1\mathbf{v}_t(i) + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i)) \tag{19}$$

[15] For the sake of brevity, we shall assume that agent i observes the previous-period action of only one competitor, $\delta_t(i)$, but how to increase that number should be readily apparent.

In the context of the model presented above, these coefficients are given by:

$$C_1 = \begin{bmatrix} BS \\ \mathbf{0} \end{bmatrix} \tag{20a}$$

$$C_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' BS + \beta' T_o \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \end{bmatrix} \tag{20b}$$

$$S_1 = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \tag{20c}$$

$$S_2 = \begin{bmatrix} \mathbf{0} \\ \alpha' Q \end{bmatrix} \tag{20d}$$

where, within the description of C_2, the matrix $T_o \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$ selects $\mathring{E}_{t-1} \left[\overline{\mathbf{x}}^{(0:\infty)}_{t-1|t-1} \right]$ from X_{t-1}. We conjecture (and verify in appendix A) that the state vector may be written in the following law of motion:

$$X_t = F X_{t-1} + G_1 \mathbf{u}_t + G_2 \widetilde{\mathbf{v}}_t + G_3 \widetilde{\mathbf{v}}_{t-1} + G_4 \widehat{\mathbf{v}}_{t-1} \tag{21}$$

The solution is found in the same manner as in stage 1, with the exception that the coefficients for the expression of $\mathring{E}_t [X_t]$ are taken from stage 1.[16]

3.4.3 Finding the solution

For a given set of D_*, R_*, C_*, S_* matricies and initial guesses for M, N_*, F and G_*, the solution is found via the following algorithm:

1. In stage one
   (a) Find the time-invariant Kalman Gain (K) and Variance (V) matricies by iterating equations (A.6), (A.7) and (A.8) until convergence is achieved
   (b) Update the M and N_* matricies
   (c) Repeat steps 1a and 1b until convergence is achieved
2. In stage two
   (a) Find the time-invariant Kalman Gain (J) and Variance (U) matricies by iterating equations (A.16), (A.17) and (A.18) until convergence is achieved
   (b) Update the F and G_* matricies, including use of the individual agent results from stage one for $\mathring{E}_t [\cdot]$
   (c) Repeat steps 2a and 2b until convergence is achieved

[16] More strictly, the first stage is expanded to also include $\mathring{E}_t [Z_t]$ (the expectation of any generic competitor) in its state vector. Although it is not referenced by the stage-one expressions for $\overline{E}_t [Z_t]$ and $\widetilde{E}_t [Z_t]$, this is necessary to ensure conformability between the two stages.

3.5 An initial discussion

The presence of $\widetilde{\mathbf{v}}_t$ and $\widehat{\mathbf{v}}_t$ in agents' learning represent a network-based origin of volatility in aggregate beliefs, independent of "true" aggregate shocks. The first of these represents a first-order effect: the idiosyncratic shocks of the most frequently observed agents have outsized effects on average beliefs and, hence, average actions. The second captures a second-order effect: observed agents also make observations themselves and so *transmit* volatility in addition to creating it. That we need not consider effects beyond these two emerges directly from the assumed opacity of the network. Three broad consequences of the model are immediately apparent from proposition 1.

First, it is possible to simulate the effects of network learning without having to model the network explicitly: the shocks $\widetilde{\mathbf{v}}_t$ and $\widehat{\mathbf{v}}_t$ together represent a sufficient statistic for the effect of the network on agents' aggregate beliefs. This makes the model particularly amenable to nesting within broad General Equilibrium models of the economy.

Second, because the network has a distribution of links that is sufficiently far from uniform, mean zero idiosyncratic shocks do not wash out in aggregation, thereby leading to a network-based source of aggregate volatility. The scale of this additional volatility depends on the asymmetry of the network, which is captured simply in a single parameter: ζ^*.

Third, the aggregate effects of idiosyncratic shocks are *persistent*. This comes about for two reasons. First, because agents observe competitors' actions with a one-period lag, the timing of their immediate impact will be delayed mechanically. Second, the aggregate effects, once present, become persistent because of the recursive nature of agents' learning. The degree of persistence will naturally vary with the parameterisation of the model, but broadly increases with the persistence of true aggregate shocks and with the degree of strategic complementarity in agents' actions.

3.6 A generalisation and extension to dynamic actions

Extending the model of the previous section to consideration of dynamic actions is relatively straightforward. In the static setting, the agents' decision rule was combined with the expression for their signals to obtain the following:

$$g_t(i) = \alpha' \underbrace{(B\mathbf{x}_t + Q\mathbf{v}_t(i))}_{\text{Private signal}} + \beta' E_t(i)\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]$$

It should be clear that this may be written more generally as

$$g_t(i) = \gamma_1' w_{t-1} + \gamma_2' X_t + \gamma_3' E_t(i)[X_t] + \gamma_4' \mathbf{v}_t(i) \tag{22}$$

where w_{t-1} is any commonly and perfectly observed variable from period $t-1$. Note that X_t includes $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ by construction, which itself includes $\mathbf{x}_t$. Any setting that can be expressed in this reduced form can be applied to the network learning environment of the previous section. The solution developed in appendix A.4 to prove proposition 1 remains unchanged, although the coefficients used in each stage's signal vectors are slightly different. In the first stage, they will be:

$$\mathbf{s}_t(j) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(j) \\ \widetilde{g}_{t-1} \end{bmatrix} = D_1 Z_t + D_2 Z_{t-1} + D_3 w_{t-2} + R_1 \mathbf{v}_t(j) + R_2 \widetilde{\mathbf{v}}_{t-1} \tag{23a}$$

$$D_1 = \begin{bmatrix} BS \\ \mathbf{0} \end{bmatrix} \qquad D_2 = \begin{bmatrix} \mathbf{0} \\ \gamma_2' + \gamma_3' T_w \end{bmatrix} \qquad D_3 = \begin{bmatrix} \mathbf{0} \\ \gamma_1' \end{bmatrix} \tag{23b}$$

$$R_1 = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \qquad R_2 = \begin{bmatrix} \mathbf{0} \\ \gamma_4' \end{bmatrix} \tag{23c}$$

while in the second stage, they will be:

$$\mathbf{s}_t(i) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(j) \\ g_{t-1}(\delta_{t-1}(i)) \end{bmatrix} = C_1 X_t + C_2 X_{t-1} + S_1 \mathbf{v}_t(i) + S_2 \mathbf{v}_{t-1}(\delta_{t-1}(i)) \tag{24a}$$

$$C_1 = \begin{bmatrix} BS \\ \mathbf{0} \end{bmatrix} \qquad C_2 = \begin{bmatrix} \mathbf{0} \\ \gamma_2' + \gamma_3' T_o \end{bmatrix} \qquad C_3 = \begin{bmatrix} \mathbf{0} \\ \gamma_1' \end{bmatrix} \tag{24b}$$

$$S_1 = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} \qquad S_2 = \begin{bmatrix} \mathbf{0} \\ \gamma_4' \end{bmatrix} \tag{24c}$$

Note that because the variable w_{t-2} is commonly and perfectly observed, it is known with certainy in period $t-1$ when agents are forming their priors in preparation for period t. As such, it will drop out in the Kalman filter, which updates beliefs on the basis of unexpected information in the signal.

Because of agents' joint rationality, equation (22) includes a wide array possible of forward- and backward-looking decision rules. For example, suppose that individual decisions are made according to the following rule:

$$g_t(i) = \eta_s' \mathbf{s}_t(i) + \eta_x' E_t(i)[X_t] + \eta_y E_t(i)[\overline{g}_t] + \eta_z E_t(i)[\overline{g}_{t+1}]$$

Note that the previous section's setup is nested within this by imposing that (a) η_x' have non-zero elements against $\mathbf{x}_t$ and zero against all other components of X_t; and (b) $\eta_z = 0$. It is shown in appendix A.5 that this may be expressed as

$$g_t(i) = \underbrace{\alpha' BS}_{\gamma_2'} X_t + \underbrace{\left(\eta_x' + \eta_y \mathbf{a}' + \eta_z \mathbf{a}' F\right)}_{\gamma_3'} E_t(i)[X_t] + \underbrace{\alpha' Q}_{\gamma_4'} \mathbf{v}_t(i)$$

where $\gamma_1' = 0$ and

$$\mathbf{a}' \equiv \left(\alpha' BS + \eta_x' T_s\right)(I - \eta_y T_s)^{-1} \left(I - \eta_z F T_s (I - \eta_y T_s)^{-1}\right)^{-1}$$

while S selects $\mathbf{x}_t$ from X_t and T_s selects the simple-average expectation of X_t from X_t (i.e. $\overline{E}_t^{(1)}[X_t] = T_s X_t$). This is, of course, by no means the only dynamic setting that may be modelled here. The dynamic price-setting model explored in section **??** considers an environment with an infinite sum of forward-looking variables in the individual firm's decision rule.

4 Price setting with network learning

In this section we construct and analyse a dynamic, stochastic, general equilibrium (DSGE) model in which firms make use of network learning in their pricing decisions. The real economy is presented here in an entirely standard model with no capital. A representative household purchases differentiated goods via a Dixit-Stiglitz aggregator and supplies labour to firms. Monopolistic firms produce the goods and sell them to the household, but are restricted to using Calvo (1983) pricing. Aggregate shocks occur within the household's preferences, the central bank's interest rate policy and economy-wide TFP, while firms also face idiosyncratic shocks to their TFP. Only the main results are presented here; readers interested in the full derivation are referred to appendix B. In what follows, lower-case letters are used to denote (natural) log deviations from the long-run steady state values of the corresponding upper-case letters.

The model presented here deviates from the basic New Keynesian framework in two key respects. The first is that the presence of imperfect common knowledge gives rise to higher-order expectations within the Phillips Curve and prevents the Phillips Curve from being expressed in recursive form. The second is that network learning gives rise to persistent, aggregate effects from firms' idiosyncratic shocks. These aspects are therefore presented first, before the bulk of the model.

4.1 The New Keynesian Phillips Curve under Imperfect Common Knowledge

In a linearised setting, firms operating under Calvo (1983) pricing will, when able, choose their reset price, $g_t(j)$, according to:

$$g_t(j) = (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s E_t(j)\left[p_{t+s} + \upsilon y_{t+s} + \omega_{t+s}(j)\right] \tag{25}$$

where β is the discount factor used by households in their objective function; θ is the probability of *not* being able to update their price each period (so θ^s is the probability of a price still being in effect s periods hence); $E_t(j)[\cdot]$ is the mathematical expectation conditional on firm j's information in period t; p_t and y_t are aggregate (i.e. average) prices and aggregate output respectively; $\upsilon y_t + \omega_t(j)$ is the marginal cost that firm j would incur if they were to produce the average quantity of output (i.e. if $y_t(j) = y_t$); and $\omega_t(j) \equiv \lambda_t + Qv_t(j)$ is a shock to firm j's marginal cost comprised of a persistent aggregate component and a transitory idiosyncratic component, both with unconditional means of zero.

We suppose that firms observe $\omega_t(j)$ and some combination of signals based on previous-period prices (detailed below). In this case, the following Phillips Curve arises:

$$\begin{aligned}\pi_t = &\left(\theta\sum_{k=0}^{\infty}(1-\theta)^k\,\overline{E}_t^{(k)}[p_{t-1}]\right) - p_{t-1} \\ &+ (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k\,\overline{E}_t^{(k)}\left[\upsilon\overline{E}_t^{(1)}[y_t] + \lambda_t\right] \\ &+ \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}(\beta\theta)^s(1-\theta)^k\,\overline{E}_t^{(k)}\left[\pi_{t+s} + (1-\beta\theta)\,\overline{mc}_{t+s}\right]\end{aligned} \tag{26}$$

where $\pi_t \equiv p_t - p_{t-1}$ is the net rate of inflation and $\overline{mc}_t = \upsilon y_t + \lambda_t$ is the marginal cost any firm would experience if they produced the average quantity of output *and* experienced the average TFP.

The first line of this expression represents any contribution to current-period inflation from uncertainty regarding p_{t-1}. If p_{t-1} is observed perfectly by all firms in period t, then the first line cancels out exactly. The second line represents the hierarchy of simple-average expectations of period t marginal costs for the average firm. Note that the additional order of expectation regarding y_t emerges because although firms observe $\omega_t(j)$ before resetting their prices, they do not know y_t ahead of time and so must estimate it. The average marginal cost for the purpose of determining reset prices is therefore $\upsilon \overline{E}_t^{(1)}[y_t] + \lambda_t$. The third line represents the period-t hierarchy of simple-average expectations regarding future inflation and future average marginal costs. Note that we cannot express the Phillips Curve in a recursive form here because each firm has their own information set so that $\overline{E}_t\left[\overline{E}_{t+1}[\cdot]\right] \neq \overline{E}_t\left[\overline{E}_t[\cdot]\right]$ in general.

It may be observed that this expression of the NKPC differs from that derived by Nimark (2008) in two respects. First, we here generalise slightly to a setting where p_{t-1} is not observed perfectly. Second, we correct an error in Nimark (2008) that effectively pre-supposed that in period t, agents expect on average that period $t+1$ expectations will be accurate (see the appendix for more detail). Although our preferred modelling setup is to suppose that agents do *not* observe p_{t-1} perfectly,[17] to make use of this is to build in inflation persistence by assumption. As the focus of the present paper is to highlight learning-based persistence, we will turn this channel off by assuming that p_{t-1} is indeed observed without noise.

4.2 Firms' network learning

We gather all aggregate-sourced shocks in the vector, $\mathbf{x}_t$, so that

$$\lambda_t = B\mathbf{x}_t \tag{27}$$

We suppose that $\mathbf{x}_t$ follows an AR(1) process:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + P\mathbf{u}_t$$

where $\mathbf{u}_t$ is a vector of period-t innovations identically and independently distributed as $N(0, I)$ and A and P are matricies of fixed and common-knowledge parameters. Idiosyncratic shocks to firms' marginal costs, $v_t(j)$, are entirely transitory, independent and distributed identically as $N(0, 1)$.

Each period, firms observe $\omega_t(j)$ (but not its constituent components), the previous period's aggregate price and a competitor's reset price from the previous period. That is, it is assumed that the network is destroyed and redrawn each period and that the distribution of observees is over the subset of firms that updated their price in the previous period. Firm j's information set therefore evolves as

$$\begin{aligned} \mathcal{I}_0(j) &= \{A, P, B, Q, \Phi\} \\ \mathcal{I}_t(j) &= \{\mathcal{I}_{t-1}(j), \mathbf{s}_t(j), \delta_{t-1}(j)\} \end{aligned} \tag{28}$$

where

$$\mathbf{s}_t(j) = \begin{bmatrix} B\mathbf{x}_t + Qv_t(j) \\ p_{t-1} \\ p_{t-1}(\delta_{t-1}(j)) \end{bmatrix} \tag{29}$$

[17] Any public signal of aggregate prices is, after all, necessarily compiled from an incomplete sample and subject to ongoing revision as statistical agencies gather more data and update their methodologies.

The hierarchy of firms' expectations of $\mathbf{x}_t$ includes three compound expectations: the simple-average ($\overline{E}_t[\cdot]$), the weighted-average ($\widetilde{E}_t[\cdot]$) and that of a generic observee ($\mathring{E}_t[\cdot]$):

$$X_t \equiv \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t[X_t] \\ \widetilde{E}_t[X_t] \\ \mathring{E}_t[X_t] \end{bmatrix} \tag{30}$$

It is shown in appendix B.3 that in this setting, firm j's decision rule for their reset prices can be expressed as:

$$\begin{aligned} g_t(j) = {} & p_{t-1} \\ & + \underbrace{(1-\beta\theta)BS}_{\gamma_2'}X_t \\ & + \underbrace{\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)BS\right)(1-(1-\theta)T_s)^{-1}}_{\gamma_3'}E_t(j)[X_t] \\ & + \underbrace{(1-\beta\theta)Q}_{\gamma_4'}v_t(j) \end{aligned} \tag{31}$$

where

$$\mathbf{f}' = (1-\beta\theta)\upsilon\mathbf{a}' + \left(\mathbf{c}' + (1-\beta\theta)\left(\upsilon\mathbf{a}' + BS\right)\right)\beta\theta F(I-\beta\theta F)^{-1} \tag{32}$$

and $\mathbf{a}'$ and $\mathbf{c}'$ are from the reduced form expressions of y_t and π_t respectively (see below). As this is in the generalised form of section (3.6), proposition 1 applies and firms' hierarchy of expectations will evolve according to the ARMA(1,1) process:

$$X_t = FX_{t-1} + G_1\mathbf{u}_t + G_2\widetilde{v}_t + G_3\widetilde{v}_{t-1} + G_4\widehat{v}_{t-1} \tag{33}$$

where the idiosyncratic-derived aggregate innovations are given by

$$\widetilde{v}_t = \int_0^1 v_t(j)\,\phi(j)\,dj \ ; \ \widehat{v}_t = \int_0^1 v_t(\delta_t(j))\,\phi(j)\,dj \tag{34}$$

The vector X_t may therefore be considered the state vector of the entire system.

4.3 The model

4.3.1 The household

Each period, the representative household maximises

$$E_t^{HH}\left[\sum_{s=0}^{\infty}\beta^s\left\{e^{\epsilon_{Ct+s}}\frac{C_{t+s}^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} - e^{\epsilon_{Ht+s}}\frac{H_{t+s}^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}}\right\}\right] \tag{35}$$

subject to a standard budget constraint and where $E_t^{HH}[\cdot]$ is the mathematical expectation conditional on the household's information set in period t (defined below); C_t is aggregate consumption; H_t is the aggregate labour supply; σ is the elasticity of intertemporal substitution; ψ is the Frisch elasticity of labour supply; and ϵ_{Ct} and ϵ_{Ht} are persistent, mean zero shocks (specified below) to the utility of consumption and the disutility of labour respectively. The shock to the disutility of labour may be considered a reduced-form way of capturing broad shocks to the labour supply, such as a temporary impairment to labour mobility. Aggregate consumption is given by the Dixit and Stiglitz (1977) aggregator over individual consumption goods:

$$C_t = \left(\int C_t\left(j\right)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{36}$$

where ε is the elasticity of substitution. The household's subsequent first-order conditions are:

$$\frac{W_t}{P_t} e^{\epsilon_{Ct}} C_t^{-\frac{1}{\sigma}} = e^{\epsilon_{Ht}} H_t^{\frac{1}{\psi}} \tag{37}$$

$$e^{\epsilon_{Ct}} C_t^{-\frac{1}{\sigma}} = \beta \left(1 + i_t\right) E_t^{HH} \left[e^{\epsilon_{Ct+1}} C_{t+1}^{-\frac{1}{\sigma}} \frac{1}{\Pi_{t+1}} \right] \tag{38}$$

where W_t/P_t is the real wage; i_t is the net nominal interest rate; and $\Pi_t \equiv P_t/P_{t-1}$ is the gross rate of inflation. It can also be shown that household demand for good j is given by:

$$C_t\left(j\right) = \left(\frac{P_t\left(j\right)}{P_t} \right)^{-\varepsilon} C_t \tag{39}$$

and the aggregate price level by:

$$P_t = \left(\int P_t\left(j\right)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} \tag{40}$$

4.3.2 Firms

Each good is produced by a single firm according to a common production function that deploys labour with decreasing marginal productivity:

$$Y_t\left(j\right) = A_t\left(j\right) H_t\left(j\right)^{1-\alpha} \tag{41}$$

Each firm's productivity, $A_t\left(j\right)$, is given by

$$\ln\left(A_t\left(j\right)\right) = \epsilon_{At} + v_t\left(j\right)$$

where ϵ_{At} is a persistent aggregate shock and $v_t\left(j\right)$ is a transitory idiosyncratic shock (each specified below) to the firm's productivity, broadly defined. Firm j's real marginal cost is then

$$MC_t\left(j\right) = \left(1 + \eta\right) \frac{W_t}{P_t} \frac{1}{A_t\left(j\right)} \left(\frac{Y_t\left(j\right)}{A_t\left(j\right)} \right)^{\eta} \tag{42}$$

where $\eta \equiv \frac{\alpha}{1-\alpha}$ is the elasticity of marginal cost w.r.t. output. Shocks to $A_t\left(j\right)$ may therefore be considered a reduced-form means of capturing shocks to firms' marginal costs other than those that act through demand or the (real) wage. Following Calvo (1983), each firm faces a constant, common and exogenous probability, $1-\theta$, of being able to adjust its price in any given period. When they are able, firms' choose a new price, $G_t(j)$, to maximise their discounted flow of real profits

$$E_t(j)\left[\sum_{s=0}^{\infty}\theta^s\left(\beta^s Q_{t+s|t}\right)\left\{\frac{G_t(j)}{P_{t+s}}Y_{t+s}(j)-\frac{W_{t+s}}{P_{t+s}}\left(\frac{Y_{t+s}(j)}{A_{t+s}(j)}\right)^{1+\eta}\right\}\right] \tag{43}$$

subject to demand (39) and where $E_t(j)[\cdot]\equiv E[\cdot|\mathcal{I}_t(j)]$ is the mathematical expectation conditional on firm j's information set in period t (defined below); θ^s is the probability that $G_t(j)$ will still be in effect s periods hence; and $\beta^s Q_{t+s|t}$ is the stochastic discount factor imposed by the shareholder (the household). The firm's corresponding first-order condition may be written as

$$G_t(j)\,E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s Q_{t+s|t}Y_{t+s}P_{t+s}^{\varepsilon-1}\right]=\left(\frac{\varepsilon}{\varepsilon-1}\right)E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s Q_{t+s|t}MC_t(j)\,Y_{t+s}P_{t+s}^{\varepsilon}\right] \tag{44}$$

4.3.3 Market clearing

Markets clear each period, so that

$$\begin{array}{ll} Y_t(j)=C_t(j) & \forall t,j \\ H_t=\int H_t(j)\,dj & \forall t \end{array} \tag{45}$$

This implies that aggregate output is given by

$$Y_t=Z_tH_t^{1-\alpha} \tag{46}$$

where aggregate TFP, Z_t, combines individual firm productivities and a distortion from relative prices

$$Z_t\equiv\left(\int A_t(j)^{-(1+\eta)}\left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon(1+\eta)}dj\right)^{\frac{1}{1+\eta}} \tag{47}$$

4.3.4 The central bank

To close the model, we assume that the central bank sets nominal interest rates according to the policy function

$$i_t=\kappa_y E_t^{CB}[y_t]+\kappa_{\pi 0}E_t^{CB}[\pi_t]+\kappa_{\pi 1}E_t^{CB}[\pi_{t+1}]+\epsilon_{Mt} \tag{48}$$

where $E_t^{CB}[\cdot]$ is the mathematical expectation conditional on the central bank's information set in period t (defined below) and ϵ_{Mt} is a persistent, mean zero shock to monetary policy (specified below).

4.4 Solving the model

The underlying state of the economy, $\mathbf{x}_t$, therefore contains four aggregate shocks:

$$\mathbf{x}_t' \equiv \begin{bmatrix} \epsilon_{At} & \epsilon_{Ct} & \epsilon_{Ht} & \epsilon_{Mt} \end{bmatrix} \tag{49}$$

Of these, the shocks to productivity (ϵ_{At}) and the disutility of labour (ϵ_{Ht}) are pure supply shocks as they enter the model only through firms' marginal costs (although note that the latter acts via higher real wages); the shock to monetary poliy (ϵ_{Mt}) is a pure demand shock as it only enters through the IS relation; and the shock to the utility of consumption (ϵ_{Ct}) has both supply and demand aspects in that it affects both the spending/saving decision and the labour supply.

Combining the household intratemporal constraint (37) with market clearly conditions (45) - (47) and firms' reset prices (44), we obtain the linearised expression for firms' reset prices (25) presented in section 4.1 above, with the parameters υ, B and Q given by:

$$\begin{aligned}
\upsilon &= \frac{1}{1+\varepsilon\eta}\left(\frac{1}{\sigma} + \frac{1+\eta}{\psi} + \eta\right) \\
B &= \frac{1}{1+\varepsilon\eta}\begin{bmatrix} -\left(\frac{1+\eta}{\psi} + 1 + \eta\right) & -1 & 1 & 0 \end{bmatrix} \\
Q &= -\left(\frac{1+\eta}{1+\varepsilon\eta}\right)
\end{aligned}$$

Households and the central bank are assumed to have complete information[18] and to update their beliefs rationally, so that

$$E_t^{CB}\left[X_{t+s}\right] = E_t^{HH}\left[X_{t+s}\right] = E_t\left[X_{t+s}\right] = \begin{cases} X_t & \text{when } s = 0 \\ F^{s-1}\left(FX_t + G_3\widetilde{v}_t + G_4\widehat{v}_t\right) & \text{when } s \geq 1 \end{cases}$$

We therefore have that the economy is described by the Phillips curve (26) with perfect knowledge of previous-period prices:

$$\begin{aligned}
\pi_t &= (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k\left(\upsilon\overline{E}_t^{(k+1)}\left[y_t\right] + \overline{E}_t^{(k)}\left[\lambda_t\right]\right) \\
&+ \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}(\beta\theta)^s(1-\theta)^k\,\overline{E}_t^{(k)}\left[\pi_{t+s} + (1-\beta\theta)\,\overline{mc}_{t+s}\right]
\end{aligned}$$

and, making use of the fact that both have complete information, the household's Eular equation (B.4) and the central bank's policy function (48):

$$\begin{aligned}
y_t &= E_t\left[y_{t+1}\right] - \sigma E_t\left[i_t - \pi_{t+1}\right] + \sigma\left(\epsilon_{Ct} - E_t\left[\epsilon_{Ct+1}\right]\right) \\
i_t &= \kappa_y y_t + \kappa_{\pi 0}\pi_t + \kappa_{\pi 1}E_t\left[\pi_{t+1}\right] + \epsilon_{Mt}
\end{aligned}$$

We posit a solution of the following form:

[18] An arguably more plausible (and certainly more interesting) scenario would be to restrict the household and central bank to less than complete information so that dynamics might arise from their higher-order expectations of each others' and firms' beliefs. Exploration of such a setting is held for future research.

$$y_t = \mathbf{a}' X_t + \mathbf{b}' \begin{bmatrix} \widetilde{v}_t \\ \widehat{v}_t \end{bmatrix} \tag{50}$$

$$\pi_t = \mathbf{c}' X_t \tag{51}$$

so that the dynamics of inflation and output are characterised entirely by equations (33), (50) and (51). It is shown in appendix B.4 that the following conditions must hold for this solution to apply:

$$\mathbf{a}' = \mathbf{a}' F - \sigma \left(\kappa_y \mathbf{a}' + \kappa_{\pi 0} \mathbf{c}' + (\kappa_{\pi 1} - 1) \mathbf{c}' F \right) + \mathbf{d}' \tag{52a}$$

$$\mathbf{b}' = \mathbf{a}' \begin{bmatrix} G_3 & G_4 \end{bmatrix} - \sigma \kappa_y \mathbf{b}' \tag{52b}$$

$$\mathbf{c}' = (1-\theta) \left\{ \begin{array}{c} (1-\beta\theta)(\upsilon \mathbf{a}' T_s + BS) \\ +\beta\theta \left(\mathbf{c}' + (1-\beta\theta)(\upsilon \mathbf{a}' + BS) \right) F (I - \beta\theta F)^{-1} T_s \end{array} \right\} (I - (1-\theta) T_s)^{-1} \tag{52c}$$

$$\mathbf{d}' = \sigma \left(\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} & 0_{1\times\infty} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} & 0_{1\times\infty} \end{bmatrix} F \right) \tag{52d}$$

where S is such that $SX_t = \mathbf{x}_t$ and T_s is such that $T_s X_t = \overline{E}_t [X_t]$.

4.4.1 Finding the solution

Finding the true solution to the model requires working with expectations of infinite order, which cannot be handled in practice. However, as the model places decreasing weight on higher order expectations (a weight of $(1-\theta)^k$ is applied to the average k-th order expectation),[19] an arbitrarily accurate approximation of the solution may be found by truncating firms' expectation hierarchy at an upper limit, k^*, of the number of orders to include.

For a given set of parameters and an upper limit for the number of orders of higher-order expectations to include, k^*, the solution is obtained by finding the fixed point of the system (33), (52):

1. Start with initial guesses for **a**, **b**, **c** and **d**
2. Find the corresponding coefficients for C_1, C_2, C_3, R_1, R_2, D_1, D_2, D_3, S_1 and S_2
3. Solve the agents' network learning problem (i.e. find F, G_1, G_2, G_3 and G_4 and the corresponding Kalman gain and Variance matricies) via the technique outlined in section 3.4.3
4. Update **a**, **b**, **c** and **d** via equation (52)
5. Repeat steps 2-4 until convergence is achieved

4.5 Simulation

Unless otherwise specified, the following baseline parameters are used to calibrate the simulation:

[19] For example, if $\mathbf{x}_t$ were only a single variable and $\mathbb{E}_t^{(k)}[\mathbf{x}_t]$ contained only simple-average expectations so that $T_s = \begin{bmatrix} 0_{\infty\times 1} & I_\infty \end{bmatrix}$, then $(I - (1-\theta) T_s)^{-1}$ would be given by

$$(I - (1-\theta) T_s)^{-1} = \begin{bmatrix} 1 & (1-\theta) & (1-\theta)^2 & \cdots \\ 0 & 1 & (1-\theta) & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

Parameter	Value	Description
β	0.99	The household's discount factor
ε	4	The elasticity of substitution in the household's consumption aggregator
σ	0.25	The household's elasticity of intertemporal substitution
ψ	1.1	The Frisch elasticity of labour supply
κ_y	0.5	Coefficient against the output gap in the central bank's policy function
$\kappa_{\pi 0}$	1.1	Coefficient against current inflation in the central bank's policy function
$\kappa_{\pi 1}$	0.4	Coefficient against expected next-period inflation in the central bank's policy function
η	0.54	The elasticity of marginal cost ($\equiv \frac{\alpha}{1-\alpha}$, corresponds to $\alpha = 0.35$)
θ	0.5	The probability of a firm *not* updating their price in a given period (corresponds to 2 periods)
ζ^*	0.2	The degree of asymmetry in the distribution from which agents' observees are drawn (0 = uniform, 1 = degenerate)
A	$0.9I_4$	The (AR(1)) transition matrix for the underlying state
P	I_4	The mapping from aggregate innovations to the underlying state
Σ_u	I_4	The variance-covariance matrix for aggregate shocks
σ_v^2	4	The variance of idiosyncratic shocks

Table 1: Baseline parameterisation

Figure 3 illustrates the primary result of this paper: the aggregate effects of a one-period shock to firms' idiosyncratic shocks in the absense of any aggregate shocks. Note that these arise from two, correlated aggregate variables derived from firms' idiosyncratic shocks. The first, from $\widetilde{v}_t = \int \mathbf{v}_t(i)\,\phi(i)\,di$, is the weighted average of all firms' shocks, while $\widehat{v}_t = \int \mathbf{v}_t(\delta(i))\,\phi(i)\,di$ may be thought of as a double-weighted average that captures second order effects (observees of observees). The impulse responses presented are for a one standard deviation shock to $\widetilde{v}_t$ and the corresponding conditional expected value of $\widehat{v}_t$. The top row plots the hierarchy of simple-average expectations regarding aggregate TFP and the aggregate shock to the disutility of labour. Despite their true values ($k = 0$) remaining at zero, firms mistakenly attribute the observed increases in visible firms' prices to a combination of these aggregate shocks. The bottom row plots the impulse responses for real GDP and inflation. The aggregate effect of the idiosyncratic shocks is larger on inflation than on real GDP, but is equally persistent across the two. Note that there is an impact response on GDP when the more visible firms experience the shock to costs, but inflation responds only with a one-period lag as firms only observe each others' prices with a one-period lag.

Next, figure 4 displayes the equivalent plot following a shock to aggregate productivity (ϵ_A). Firms are unable to differentiate between a shock to aggregate TFP and a shock to the disutility of labour, so these remain in fixed proportion throughout the response window. The impulse responses of real GDP and inflation are notably hump-shaped, a consequence of the shock's innate persistence (recall that ϵ_A has an AR(1) coefficient of 0.9). In order to illustrate the additional volatility created by the presence of idiosyncratic shocks, the dashed lines in the aggregate IRFs represent the two-standard deviation boundaries for idiosyncratic shocks around the aggregate IRF.[20] It is important to realise that these do not represent confidence intervals, but rather bounds on the *distribution* of impulse responses. Instead, were a researcher to simulate the economy described here by giving a persistent shock to aggregate TFP (ϵ_A), holding all other aggregate shocks at zero and having a full gamut of idiosyncratic shocks to

[20] To find these boundaries, we assume that all period $t-1$ variables are zero; that the aggregate variable has a unit shock in period t and then stays at zero forever; and that starting in period t, there is a full gamut of idiosyncratic shocks. We denote this scenario by Ω. The variance of X_{t+s} conditional on Ω is be given by

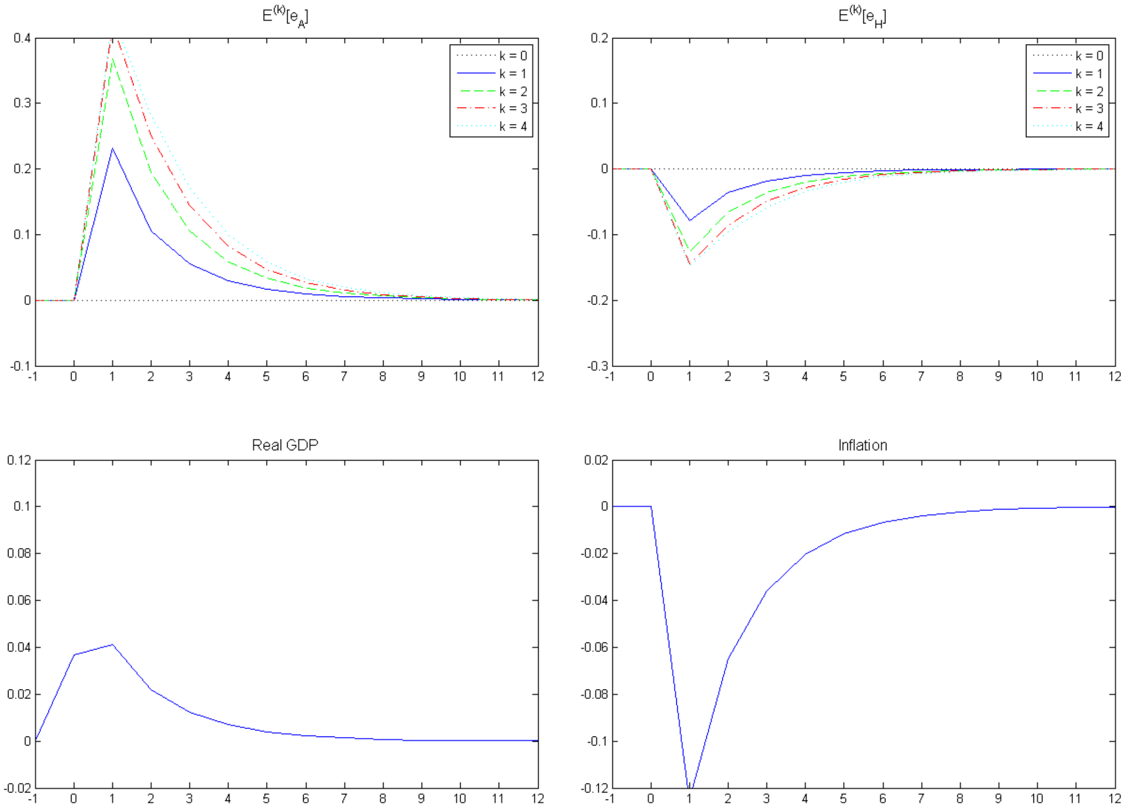


Figure 3: Hierarchy of simple average expectations regarding ϵ_A and ϵ_H, and impulse responses of Real GDP and Inflation following a one-period set of idiosyncratic shocks to firms' marginal costs.

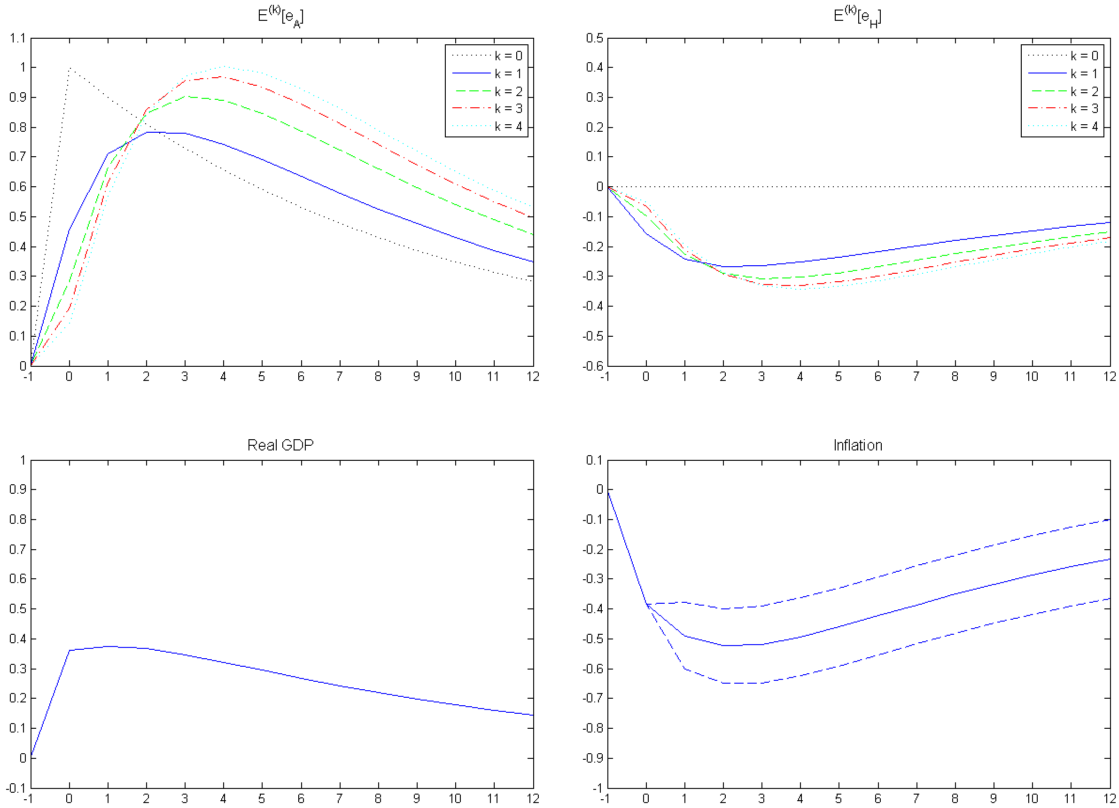


Figure 4: Hierarchy of simple average expectations regarding ϵ_A and ϵ_H, and impulse responses of Real GDP and Inflation following a shock to ϵ_A.

TFP each period, the subsequent impulse responses would fall within the dashed lines 95% of the time.

Returning to the effects of idiosyncratic shocks, figure 5 illustrates how the aggregate effects of such shocks varies with the rigidity of firms' prices. As θ increases, the average duration of any given price increases and, as might be expected, the magnitude and persistence of the responses to idiosyncratic shocks correspondingly rises.

Recall that the variances of $\widetilde{v}_t$ and $\widehat{v}_t$ are given by $\zeta^* \sigma_v^2$ and $\zeta^* (2 - \zeta^*) \sigma_v^2$ respectively, with low values of ζ^* (close to 0) corresponding to relatively uniform network distributions and high values of ζ^* (close to 1) to highly asymmetric networks. As might therefore be expected, increasing either of these causes the aggregate responses of real GDP and inflation to be larger and more persistent in entirely similar ways. Figure 6 plots these impulse responses.

However, varying ζ^* and σ_v^2 produces notably different results in the impulse responses following a shock to *aggregate* TFP, as shown in figure 7. Increasing the variance of idiosyncratic shocks reduces the information that may be gleaned from both firms' private signals and

---

$$Var\left(X_t|\Omega\right) = \zeta^* \sigma_v^2 G_2 G_2'$$

$$Var\left(X_{t+s}|\Omega\right) = \zeta^* \sigma_v^2 G_2 G_2' + \zeta^* \sigma_v^2 \sum_{i=1}^{s} F^{s-1} \begin{pmatrix} \left(FG_2 + G_3\right)\left(FG_2 + G_3\right)' \\ + \left(2 - \zeta^*\right) G_4 G_4' \\ + 2G_4 \left(FG_2 + G_3\right)' \end{pmatrix} F^{s-1'} \; \forall s \geq 1$$

The conditional variance of y_t and π_t then immediately follows from the reduced-form solution.

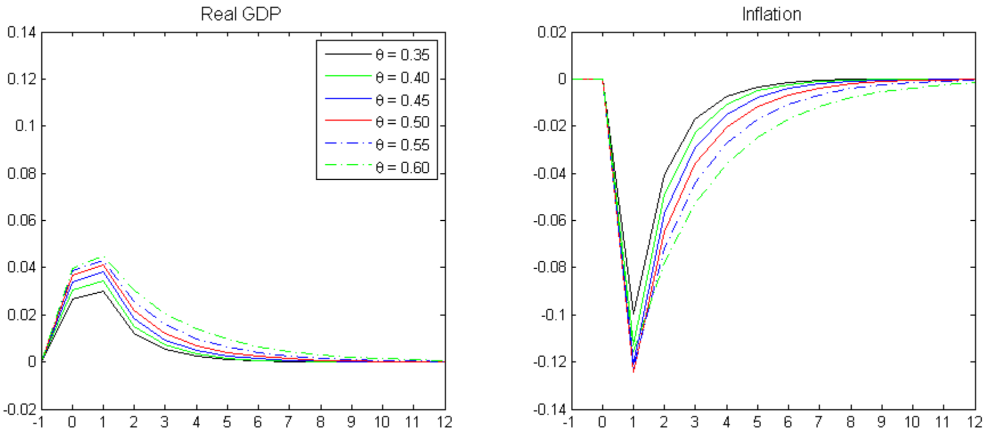
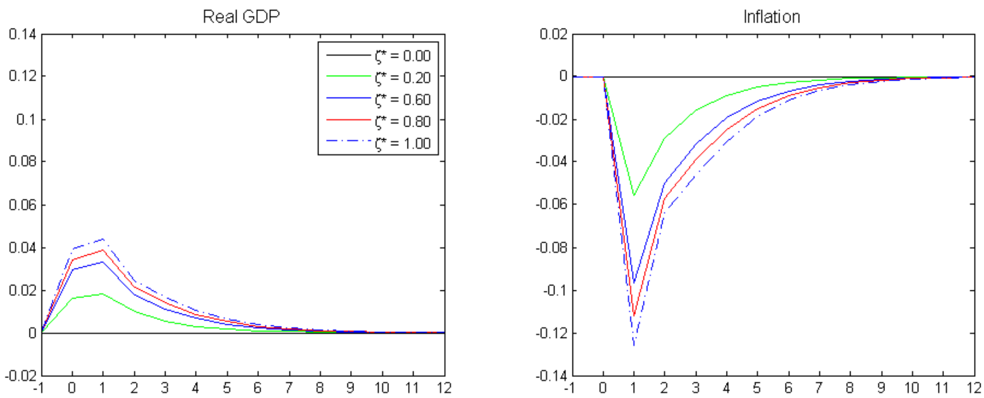
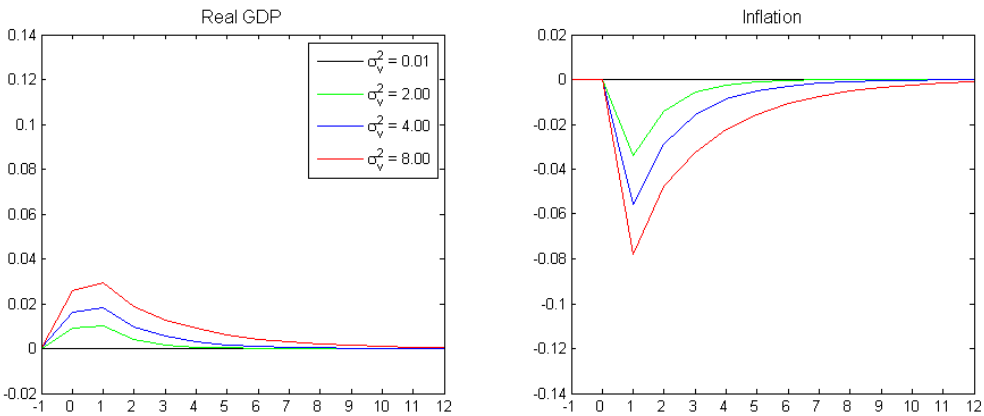


Figure 5: Impulse responses for Real GDP and Inflation following a one-standard deviation shock to $[\widetilde{v}_t \quad \widehat{v}_t]'$ for various measures of nominal price rigidity (θ).

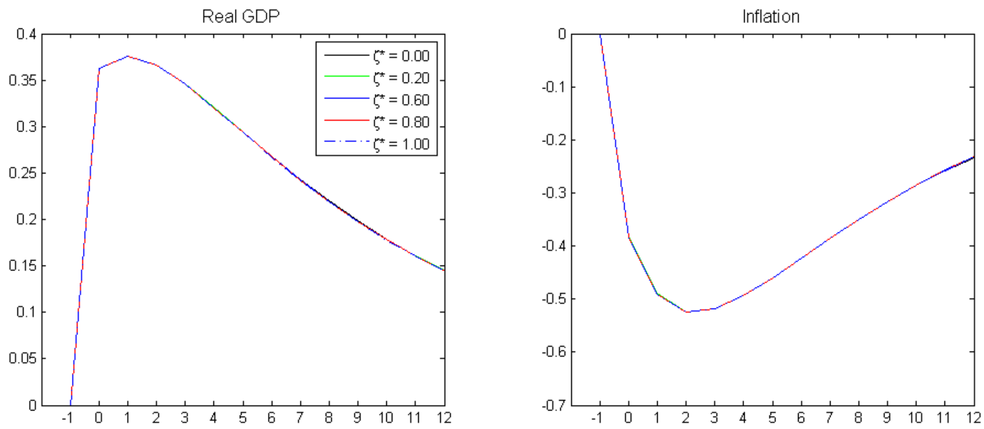


(a) Varying network asymmetry (ζ^*)

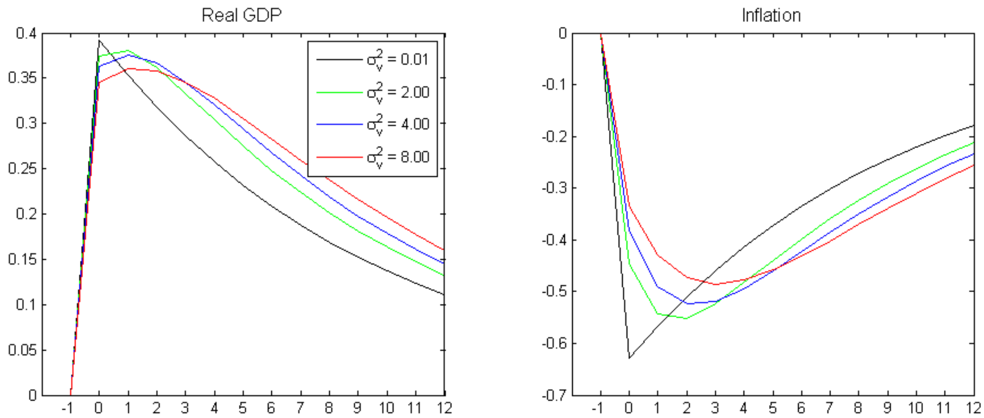


(b) Varying idiosyncratic variance (σ_v^2)

Figure 6: Impulse responses for Real GDP and Inflation following a one-standard deviation shock to $[\widetilde{v}_t \quad \widehat{v}_t]'$.



(a) Varying network asymmetry (ζ^*)



(b) Varying idiosyncratic variance (σ_v^2)

Figure 7: Impulse responses for Real GDP and Inflation following a one-standard deviation shock to aggregate TFP (ϵ_A).

the prices of their competitors, thereby lengthening the time required for them to learn that the shock they experience is indeed aggregate in nature. In marked contrast, increasing the asymmetry of the network has very little effect on learning about an aggregate shock. This is because, in the absense of $\left[\widetilde{v}_t \quad \widehat{v}_t\right]$ shocks, the observation of any one competitor is as good as another when estimating underlying aggregate shocks.

5 Conclusion

This paper has argued that firms set their prices while operating in an observation network, making use of competitors' prices to learn about hidden aggregate states of the world. That firms operate in a network and that they do so in a model of imperfect common knowledge is motivated by the observation that when surveyed, a large fraction of firms across North America and Europe admit to looking to other firms in deciding both the timing and the magnitude of price changes and do so out of a desire to coordinate pricing changes with competitors.

The paper's central contribution is to present and solve a generalised linear model of social learning over a network. It is shown that when the network is opaque, in that agents do not know the full structure of the network but instead only the identity of their observees and the distribution from which observees are drawn, the social learning problem becomes computationally tractable and admits simulation to an arbitrary degree of accuracy. When the distribution of links within the network is sufficiently asymmetric, the effects of agents' mean zero idiosyncratic shocks will not "wash out" in aggregation and agents' expectations will follow an ARMA(1,1) process with current and lagged values of weighted sums of agents' idiosyncratic shocks entering at an aggregate level. The recursive nature of agents' learning then implies that the aggregate effects of idiosyncratic shocks will be persistent, despite the individual agents' shocks being entirely transitory, with this persistence increasing in the degree of strategic complementarity, the asymmetry of the network and the persistence of any aggregate shocks.

In the context of firms' price-setting decisions, these persistent aggregate effects therefore represent a learning-based microfoundation for cost push shocks, with inflation able to persistently deviate from it's long-run trend entirely in the absence of any aggregate shocks to the economy. Because firms may choose to observe the prices of other firms with whom they are are not direct competitors, this also represents a novel transmission mechanism for inflation across industries or geographies independent of it's path along production chains. It should be noted that recent work by Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Saleh (2011) has previously illustrated the possibility of aggregate volatility emerging through this latter "real" network.

In contrast to the common assumption that idiosyncratic shocks cancel out in aggregation, the emergence of aggregate-level price changes based on short-lived idiosyncratic shocks is consistent with recent evidence garnered from a variety of observed panels of micro price changes. The level of aggregate volatility induced through network learning is increasing in the degree of strategic complementarity, the asymmetry of the network and the relative variance of idiosyncratic shocks.

This model clearly calls for future work to estimate the parameters of the model – particularly ζ^* and σ_v^2. While the obvious choice in this would be to pursue data on a panel of firms, the differential responses of aggregate variables predicted here following aggregate and idiosyncratic shocks may permit such an estimation even in the absence of individual firm data. The implications for monetary policy are a second area of research that warrants further work. In the model presented here, firms were able to observe the combined supply-side shock (and so could differentiate it from any monetary policy shock). Future work would grant firms somewhat less knowledge of, say, only the nominal wage and examine the real effects of shocks to monetary policy. There are also potential implications for optimal monetary policy. Just has previous work has suggested that monetary authorities focus their attention on the "stickiest" prices, it may also be necessary to focus on the most *visible* prices in the economy. Finally, further research into the origins of firms' observation networks would seem prudent.

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Appendix A Network Learning Proofs

A.1 Symbol Reference

Table 2 provides a description of common symbols used in section 3.

A.2 Proof of lemma 1.

Using the equation for each agent's decision rule (11), we have that agent i will construct her prior expectation of her social signal as follows:

$$\begin{aligned} E_t(i)\left[g_t(\delta_t(i))\right] &= E_t(i)\left[\alpha'\left(B\mathbf{x}_t + Q\mathbf{v}_t(\delta_t(i))\right) + \beta' E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t\mid t}^{(0:\infty)}\right]\right] \\ &= \alpha' B E_t(i)\left[\mathbf{x}_t\right] + \beta' E_t(i)\left[E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t\mid t}^{(0:\infty)}\right]\right] \end{aligned}$$

Symbol	Description
A	Transition matrix for the underlying state
B	Mapping from the underlying state to a private signal
C	Mapping from the full state to individual signals in stage two
D	Mapping from the full state to individual signals in stage one
E	Expectation operator (always linear)
F	Transition matrix for the full state in stage two
G	Mapping from aggregate shocks to the full state in stage two
I	The Identity matrix
J	Kalman Gain in stage two
K	Kalman Gain in stage one
M	Transition matrix for the full state in stage one
N	Mapping from aggregate shocks to the full state in stage one
P	Mapping from aggregate shocks to the underlying state
Q	Mapping from an idiosyncratic shock to a private signal
R	Mapping from shocks to individual signals in stage one
S	Mapping from shocks to individual signals in stage two
T_*	Selects $*$ from the full state (T_s selects $\overline{E}_t[X_t]$, T_w selects $\widetilde{E}_t[X_t]$)
U	Variance of the full state in stage two
V	Variance of the full state in stage one
W	Mapping from the previous-period actions to social signals
X	The full state in period two
Z	The full state in period one
α	Coefficients against private signals in agents' decision rule; the exponent on labour in production
β	Coefficients against the hierarchy of expectations in agents' decision rule; the household's discount factor
γ	Coefficients in the agents' generalised decision rule
δ	Mapping from one agent to their observee(s)
ε	The elasticity substitution in the household's consumption aggregator
ϵ	Aggregate shocks in the dynamic price-setting model
ζ^*	Degree of asymmetry in the distribution of observees across the network
η	Coefficients in agents' decision rule; the elasticity of marginal cost
θ	The probability of a firm *not* updating their price in a given period
κ	The parameters in the central bank's policy function
λ	The aggregate component of the shock to firms' marginal costs
π	Inflation in the dynamic price-setting model
σ	The household's elasticity of intertemporal substitution
ϕ	The p.d.f. from which agents' observees are drawn;
ω	The combined shock (aggregate and idiosyncratic) to firms' marginal costs

Table 2: Symbol reference

where we have used the lack of persistence in idiosyncratic shocks to give $E_t(i)[\mathbf{v}_t(\delta_t(i))] = 0$. Supposing that $\overline{\mathbf{x}}_{t|t}^{(0:\infty)}$ is the top-most portion of the common state vector of interest (Z_t), common knowledge of rationality then allows agent i to substitute in the Kalman filter (12) for agent $\delta_t(i)$'s expectation:

$$E_t(i)\left[E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]\right] = \begin{bmatrix} I & 0 \end{bmatrix} E_t(i)\left[\begin{array}{l} (M-KLM)\,E_{t-1}(\delta_t(i))[Z_{t-1}] \\ +\begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t(\delta_t(i)) \\ g_{t-1}(\delta_{t-1}(\delta_t(i))) \end{bmatrix} \end{array}\right]$$
$$= \begin{bmatrix} I & 0 \end{bmatrix}\left\{\begin{array}{l} (M-KLM)\,E_t(i)[E_{t-1}(\delta_t(i))[Z_{t-1}]] \\ +K_1 B E_t(i)[\mathbf{x}_t] \\ +K_2 E_t(i)[g_{t-1}(\delta_{t-1}(\delta_t(i)))] \end{array}\right\}$$

The final term shows if agent i is going to observe the period-t action of agent $\delta_t(i)$, then in order to form her prior, she must also consider whomever agent $\delta_t(i)$ observed in period-$(t-1)$. This recursion of expectations (and expectations of expectations) across agents and backwards through time leads to an explosion in the dimensionality (this is the explosion of p) and typically prevents closed-form analysis in anything other than trivially small networks.

However, by denying agents knowledge of the full network and, instead, granting them knowledge of the distribution from which inbound links are drawn (ϕ) and using the assumption that this distribution is independent of other shocks, we can observe that:

$$E_t(i)[g_{t-1}(\delta_{t-1}(\delta_t(i)))] = \int_0^1 E_t(i)[g_{t-1}(j)]\,\phi(j)\,dj$$
$$= E_t(i)\left[\int_0^1 g_{t-1}(j)\,\phi(j)\,dj\right]$$
$$= E_t(i)[\widetilde{g}_{t-1}]$$

where the second line exploited the linearity of the expectation operator and $\widetilde{g}_t \equiv \int_0^1 g_t(j)\,\phi(j)\,dj$ is a *weighted* average of all agents' actions in period t using the observation p.d.f. as the weights. Substituting this back in above gives:

$$E_t(i)\left[E_t(\delta_t(i))\left[\overline{\mathbf{x}}_{t|t}^{(0:\infty)}\right]\right] = \begin{bmatrix} I & 0 \end{bmatrix} E_t(i)\left\{\begin{array}{l} (M-KLM)\,E_{t-1}(\delta_t(i))[Z_{t-1}] \\ +K_1 B\mathbf{x}_t \\ +K_2\widetilde{g}_{t-1} \end{array}\right\}$$

From agent i's perspective, their observee's own expectation updates recursively with no idiosyncratic information. This is, in effect, agent i treating agent $\delta_t(i)$ *as though* they receive a weighted average of everybody's period-$(t-1)$ actions. We can then replace $E_t(\delta_t(i))[\cdot]$ with a common expectation $\mathring{E}_t[\cdot]$ as *ex ante*, from i's perspective, all other agents are forming the same expectation. Agent i's prior expectation of their social signal can therefore be written as:

$$E_t(i)[g_t(\delta_t(i))] = \alpha' B E_t(i)[\mathbf{x}_t] + \beta' \begin{bmatrix} I & 0 \end{bmatrix} E_t(i)\left[\mathring{E}_t[Z_t]\right]$$

with

$$\mathring{E}_t[Z_t] = (M-KLM)\,\mathring{E}_{t-1}[Z_{t-1}] + K\begin{bmatrix} B\mathbf{x}_t \\ \widetilde{g}_{t-1} \end{bmatrix}$$

Note, too, that by identical logic we also have that when considering their observee's observee's observee, agent i will expect that:

$$E_t(i)\left[E_{t-1}(\delta_t(i))\left[g_{t-2}(\delta_{t-2}(\delta_{t-1}(\delta_t(i))))\right]\right] = E_t(i)\left[E_{t-1}(\delta_t(i))\left[\widetilde{g}_{t-2}\right]\right]$$

This, in turn, amounts to agent i treating agent $\delta_{t-1}(\delta_t(i))$ – that is, whoever $\delta_t(i)$ observed – as though they also received a weighted average of everybody's period-$(t-2)$ actions. The ongoing application backwards through time should be clear. So long as the weights used (the observation p.d.f.) are constant over time and common across agents – that is, so long as agents do not learn about the topology of the network – then we have that agent i's problem may be summarised as follows: observe the action of agent $\delta_t(i)$, but treat them as though they and and all information obtained through them comes from a setting in which all agents observe the weighted average action.

A.3 Proof of lemma 2.

Denoting $\zeta(N) \equiv \sum_{i=1}^{N} \phi_N(i)^2$ and assuming that $\lim_{N\to\infty} \zeta(N) = \psi^*$ where $\zeta^* \in (0,\infty)$, we here demonstrate the following four results regarding agents' idiosyncratic shocks:

- $\widetilde{\mathbf{v}}_{N,t} \xrightarrow{d} \widetilde{\mathbf{v}}_t$ where $\widetilde{\mathbf{v}}_t \sim N(\mathbf{0}, \zeta^* \Sigma_{vv})$
- $\ddot{\mathbf{v}}_{N,t} \xrightarrow{\mathcal{L}^2} \widetilde{\mathbf{v}}_t$
- $\widehat{\mathbf{v}}_{N,t} \xrightarrow{d} \widehat{\mathbf{v}}_t$ where $\widehat{\mathbf{v}}_t \sim N(\mathbf{0}, \zeta^*(2-\zeta^*)\Sigma_{vv})$
- $Cov(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t) = \zeta^* \Sigma_{vv}$

where the three weighted sums are defined as

$$\widetilde{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^{N} \mathbf{v}_t(i)\,\phi_N(i)$$
$$\ddot{\mathbf{v}}_{N,t} \equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{v}_t(\delta_t(i))$$
$$\widehat{\mathbf{v}}_{N,t} \equiv \sum_{i=1}^{N} \mathbf{v}_t(\delta_t(i))\,\phi_N(i)$$

First, note that since the vector $\mathbf{v}_t(i)$ is drawn from independent and identical Gaussian distributions for each i and t, the sums $\widetilde{\mathbf{v}}_{N,t}$, $\ddot{\mathbf{v}}_{N,t}$ and $\widehat{\mathbf{v}}_{N,t}$ must all be distributed Normally and since $E[\mathbf{v}_t(i)] = 0\ \forall i,t$ it must be that $E[\widetilde{\mathbf{v}}_{N,t}] = E[\ddot{\mathbf{v}}_{N,t}] = E[\widehat{\mathbf{v}}_{N,t}] = 0\ \forall N,t$.

A.3.1 $\widetilde{\mathbf{v}}_{N,t} \xrightarrow{d} \widetilde{\mathbf{v}}_t$

The variance of $\widetilde{\mathbf{v}}_{N,t}$ will then be given by

$$
\begin{aligned}
Var\left[\widetilde{\mathbf{v}}_{N,t}\right] &= Var\left[\sum_{i=1}^{N} \mathbf{v}_t\left(i\right)\phi_N\left(i\right)di\right] \\
&= \sum_{i=1}^{N} Var\left[\mathbf{v}_t\left(i\right)\phi_N\left(i\right)\right]di \\
&= \sum_{i=1}^{N} \Sigma_{vv}\phi_N\left(i\right)^2 di \\
&= \zeta\left(N\right)\Sigma_{vv}
\end{aligned}
$$

where in moving to the second line we use the independence of each vector to ignore the covariance terms. The limiting variance as $N \to \infty$ is therefore $\zeta^*\Sigma_{vv}$, giving the first result.

A.3.2 $\ddot{\mathbf{v}}_{N,t} \xrightarrow{\mathcal{L}^2} \widetilde{\mathbf{v}}_t$

We next demonstrate that $\ddot{\mathbf{v}}_{N,t}$ converges to $\widetilde{\mathbf{v}}_t$ in mean square error (a stronger form of convergence than in probability). That is, we show that $\lim_{N\to\infty} E\left[\left(\ddot{\mathbf{v}}_{N,t} - \widetilde{\mathbf{v}}_t\right)^2\right] = 0$. First, see that

$$
\begin{aligned}
E\left[\left(\ddot{\mathbf{v}}_{N,t} - \widetilde{\mathbf{v}}_t\right)^2\right] &= E\left[\left(\ddot{\mathbf{v}}_{N,t}\right)^2 - 2\ddot{\mathbf{v}}_{N,t}\widetilde{\mathbf{v}}_t + \left(\widetilde{\mathbf{v}}_t\right)^2\right] \\
&= Var\left[\ddot{\mathbf{v}}_{N,t}\right] - 2Cov\left[\ddot{\mathbf{v}}_{N,t}, \widetilde{\mathbf{v}}_t\right] + Var\left[\widetilde{\mathbf{v}}_t\right]
\end{aligned}
$$

The third term was shown above to be given by $\zeta^*\Sigma_{vv}$. We now consider the first and second terms in turn. The variance of $\ddot{\mathbf{v}}_{N,t}$ is given by:

$$
\begin{aligned}
Var\left[\ddot{\mathbf{v}}_{N,t}\right] &= \frac{1}{N^2} Var\left[\mathbf{v}_t\left(\delta_t\left(1\right)\right) + \mathbf{v}_t\left(\delta_t\left(2\right)\right) + \cdots + \mathbf{v}_t\left(\delta_t\left(N\right)\right)\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \frac{1}{N^2}\left(N\Sigma_{vv} + \sum_{i=1}^{N}\sum_{j\neq i}^{N} E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right]\right)
\end{aligned}
$$

However, when $i \neq j$, given the full independence of the distributions of agents' observees, it must be that

$$
\begin{aligned}
E\left[\mathbf{v}_t\left(\delta_t\left(i\right)\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] &= \sum_{k=1}^{N}\phi_N\left(k\right)E\left[\mathbf{v}_t\left(k\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \sum_{k=1}^{N}\phi_N\left(k\right)\left(\sum_{l=1}^{N}\phi_N\left(l\right)E\left[\mathbf{v}_t\left(k\right)\mathbf{v}_t\left(l\right)\right]\right) \\
&= \sum_{k=1}^{N}\phi_N\left(k\right)^2 E\left[\mathbf{v}_t\left(k\right)\mathbf{v}_t\left(k\right)\right] \\
&= \zeta\left(N\right)\Sigma_{vv}
\end{aligned}
$$

where in moving from the second line to the third we again make use of the independence of agents' idiosyncratic shocks. We therefore have that

$$\begin{aligned}
Var\left[\ddot{\mathbf{v}}_{N,t}\right] &= \frac{1}{N^2}\left(N\Sigma_{vv} + \sum_{i=1}^{N}\sum_{j\neq i}^{N}\psi\left(N\right)\Sigma_{vv}\right) \\
&= \frac{1}{N^2}\left(N\Sigma_{vv} + \left(N^2 - N\right)\zeta\left(N\right)\Sigma_{vv}\right) \\
&= \frac{1}{N}\Sigma_{vv} + \left(\frac{N-1}{N}\right)\zeta\left(N\right)\Sigma_{vv}
\end{aligned}$$

and thus, in the limit, that

$$\lim_{N\to\infty} Var\left[\ddot{\mathbf{v}}_{N,t}\right] = \zeta^*\Sigma_{vv}$$

Next, we consider the covariance of $\widetilde{\mathbf{v}}_{M,t}$ and $\ddot{\mathbf{v}}_{N,t}$ for a general setting where $M > N$:

$$\begin{aligned}
Cov\left[\widetilde{\mathbf{v}}_{M,t}, \ddot{\mathbf{v}}_{N,t}\right] &= E\left[\widetilde{\mathbf{v}}_{M,t}\ddot{\mathbf{v}}_{N,t}\right] \\
&= E\left[\left(\sum_{i=1}^{M}\mathbf{v}_t\left(i\right)\phi_N\left(i\right)\right)\left(\frac{1}{N}\sum_{j=1}^{N}\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right)\right] \\
&= \frac{1}{N}E\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)\left(\sum_{k=1}^{N}\phi_N\left(k\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(k\right)\right]\right) \\
&= \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N}\phi_M\left(i\right)\left(\phi_M\left(i\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(i\right)\right]\right) \\
&= \sum_{i=1}^{M}\phi_M\left(i\right)^2\frac{1}{N}\sum_{j=1}^{N}\Sigma_{vv} \\
&= \zeta\left(M\right)\Sigma_{vv}
\end{aligned}$$

where moving from the fifth to the sixth line again used the independence of each agents' idiosyncratic shocks. We can then note that

$$\begin{aligned}
Cov\left[\widetilde{\mathbf{v}}_t, \ddot{\mathbf{v}}_{N,t}\right] &= \lim_{M\to\infty} Cov\left[\widetilde{\mathbf{v}}_{M,t}, \ddot{\mathbf{v}}_{N,t}\right] \\
&= \lim_{M\to\infty}\zeta\left(M\right)\Sigma_{vv} \\
&= \zeta^*\Sigma_{vv}
\end{aligned}$$

We therefore have that

$$\begin{aligned}
\lim_{N\to\infty} E\left[(\ddot{\mathbf{v}}_{N,t}-\widetilde{\mathbf{v}}_t)^2\right] &= Var\left[\ddot{\mathbf{v}}_{N,t}\right]-2Cov\left[\ddot{\mathbf{v}}_{N,t},\widetilde{\mathbf{v}}_t\right]+Var\left[\widetilde{\mathbf{v}}_t\right]\\
&= \zeta^*\Sigma_{vv}-2\zeta^*\Sigma_{vv}+\zeta^*\Sigma_{vv}\\
&= 0
\end{aligned}$$

as required for the second result.

A.3.3 $\widehat{\mathbf{v}}_{N,t} \xrightarrow{d} \widehat{\mathbf{v}}_t$

Next, the variance of $\widehat{\mathbf{v}}_{N,t}$ is:

$$\begin{aligned}
Var\left[\widehat{\mathbf{v}}_{N,t}\right] &= Var\left[\sum_{i=1}^{N}\phi_N(i)\,\mathbf{v}_t(\delta_t(i))\right]\\
&= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N(i)\,\phi_N(j)\,\mathbf{v}_t(\delta_t(i))\,\mathbf{v}_t(\delta_t(j))\right]\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N(i)\,\phi_N(j)\,E\left[\mathbf{v}_t(\delta_t(i))\,\mathbf{v}_t(\delta_t(j))\right]\\
&= \sum_{i=1}^{N}\phi_N(i)^2\,\Sigma_{vv}+\sum_{i=1}^{N}\sum_{j\neq i}^{N}\phi_N(i)\,\phi_N(j)\,E\left[\mathbf{v}_t(\delta_t(i))\,\mathbf{v}_t(\delta_t(j))\right]
\end{aligned}$$

From the analysis of $\ddot{\mathbf{v}}_{N,t}$ in the previous subsection, we have that when $i\neq j$,

$$E\left[\mathbf{v}_t(\delta_t(i))\,\mathbf{v}_t(\delta_t(j))\right]=\zeta(N)\,\Sigma_{vv}$$

We therefore have that

$$Var\left[\widehat{\mathbf{v}}_{N,t}\right]=\zeta(N)\,\Sigma_{vv}+\zeta(N)\,\Sigma_{vv}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\phi_N(i)\,\phi_N(j)$$

Next, consider that

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N(i)\,\phi_N(j) &= \sum_{i=1}^{N}\phi_N(i)\left(\sum_{j=1}^{N}\phi_N(j)\right)\\
&= \sum_{i=1}^{N}\phi_N(i)\\
&= 1
\end{aligned}$$

as $\phi_N(i)$ and $\phi_N(j)$ are p.d.f's. We must therefore have that

$$\sum_{i=1}^{N}\sum_{j\neq i}^{N}\phi_N(i)\,\phi_N(j)=1-\sum_{i=1}^{N}\phi_N(i)^2=1-\zeta(N)$$

so that

$$Var\left[\widehat{\mathbf{v}}_{N,t}\right] = \zeta\left(N\right)\Sigma_{vv}\left(1 + \left(1 - \zeta\left(N\right)\right)\right)$$

and, in the limit,

$$\lim_{N\to\infty} Var\left[\widehat{\mathbf{v}}_{N,t}\right] = \zeta^*\left(2 - \zeta^*\right)\Sigma_{vv}$$

therefore giving the third result. Note that since $\zeta^* \in (0,1)$, we also have that

$$1 > \zeta^*\left(2 - \zeta^*\right) > \zeta^*$$

so that the variance of $\widehat{\mathbf{v}}_t$ is larger than that of $\widetilde{\mathbf{v}}_t$, but still smaller than that of $\mathbf{v}_t\left(i\right)$.

A.3.4 $Cov\left(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t\right) = \zeta^*\Sigma_{vv}$

The covariance of $\widetilde{\mathbf{v}}_{N,t}$ and $\widehat{\mathbf{v}}_{N,t}$ is given by

$$\begin{aligned}
Cov\left[\widetilde{\mathbf{v}}_{N,t}, \widehat{\mathbf{v}}_{N,t}\right] &= E\left[\left(\sum_{i=1}^{N}\mathbf{v}_t\left(i\right)\phi_N\left(i\right)\right)\left(\sum_{j=1}^{N}\mathbf{v}_t\left(\delta_t\left(j\right)\right)\phi_N\left(j\right)\right)\right] \\
&= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N\left(i\right)\phi_N\left(j\right)\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N\left(i\right)\phi_N\left(j\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(\delta_t\left(j\right)\right)\right] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N\left(i\right)\phi_N\left(j\right)\left(\sum_{k=1}^{N}\phi_N\left(k\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(k\right)\right]\right) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N\left(i\right)\phi_N\left(j\right)\left(\phi_N\left(i\right)E\left[\mathbf{v}_t\left(i\right)\mathbf{v}_t\left(i\right)\right]\right) \\
&= \Sigma_{vv}\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_N\left(i\right)^2\phi_N\left(j\right) \\
&= \Sigma_{vv}\sum_{i=1}^{N}\phi_N\left(i\right)^2\left(\sum_{j=1}^{N}\phi_N\left(j\right)\right) \\
&= \Sigma_{vv}\sum_{i=1}^{N}\phi_N\left(i\right)^2 \\
&= \zeta\left(N\right)\Sigma_{vv}
\end{aligned}$$

so that, in the limit,

$$Cov\left(\widetilde{\mathbf{v}}_t, \widehat{\mathbf{v}}_t\right) = \lim_{N\to\infty} Cov\left(\widetilde{\mathbf{v}}_{N,t}, \widehat{\mathbf{v}}_{N,t}\right) = \zeta^*\Sigma_{vv}$$

therefore giving the fourth result.

A.4 Proof of proposition 1.

That the general agent's problem may be solved in the two stages described in the proposition follows immediately from lemma 1. We here provide a full derivation of the learning in each stage.

A.4.1 Solving stage one

We have the following state space system

$$Z_t = MZ_{t-1} + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1} \tag{A.1}$$
$$\mathbf{s}_t(j) = D_1 Z_t + D_2 Z_{t-1} + D_3\mathbf{w}_{t-2} + R_1\mathbf{v}_t(j) + R_2\widetilde{\mathbf{v}}_{t-1} \tag{A.2}$$

where $\mathbf{w}_{t-2}$ is a vector of observables known to the agent in period $t-1$. We will first develop a modified Kalman filter for agent j's estimation of Z_t and then turn to considering the evolution of Z_t itself (i.e. the coefficients of M and N).

The (modified) Kalman filter The filter here closely follows that developed by Nimark (2011b) as a means of avoiding the doubling-up of the state vector more typical in the literature, thereby allowing more accurate simulation results when working with finite computing resources.

Denoting j's expectation formed with period-t information as $E_t(j)[\cdot] = E[\cdot|\mathcal{I}_t(j)]$, our goal is to find a mean square error minimising[21] formula for $E_t(j)[X_t]$. To begin, we first substitute the state equation into the observation equation to get:

$$\begin{aligned}\mathbf{s}_t(j) &= D_1(MZ_{t-1} + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}) + D_2X_{t-1} + D_3\mathbf{w}_{t-2} + R_1\mathbf{v}_t(j) + R_2\widetilde{\mathbf{v}}_{t-1}\\ &= (D_1M + D_2)Z_{t-1} + D_3\mathbf{w}_{t-2} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}\end{aligned}$$

Next, note that the contemporaneous expectation of the weighted-average idiosyncratic shock must be zero:

$$\begin{aligned}E_t(j)[\widetilde{\mathbf{v}}_t] &= \lim_{N\to\infty}\sum_{i=1}^{N} E_t(j)[\mathbf{v}_t(i)]\,\phi_N(i)\\ &= \lim_{N\to\infty}\left(E_t(j)[\mathbf{v}_t(j)]\,\phi_N(j) + \sum_{i=1,i\neq j}^{N} E[\mathbf{v}_t(i)]\,\phi_N(i)\right)\\ &= \lim_{N\to\infty} E_t(j)[\mathbf{v}_t(j)]\,\phi_N(j)\\ &= 0\end{aligned}$$

where moving from the first line to the second uses the fact that idiosyncratic shocks are independent, so the best that agent j can do with respect to other agents' shocks is to use the unconditional expectation, and in moving from the third line to the fourth we have used the assumption that $\lim_{N\to\infty}\phi_N(j) = 0$. This then allows us to note that j's best estimate of $\mathbf{s}_t(j)$ given period-$(t-1)$ information is simply

[21] And hence, given that all shocks are mean zero, a variance-minimising estimator.

$$E_{t-1}(j)[\mathbf{s}_t(j)] = (D_1 M + D_2) E_{t-1}(j)[Z_{t-1}] + D_3 \mathbf{w}_{t-2}$$

Note that there is no expectation around the term in $\mathbf{w}_{t-2}$ as it is observed directly in period $t-1$. Define the innovation in $\mathbf{s}_t(j)$ – that is, the unexpected component – to be:

$$\overrightarrow{\mathbf{s}}_t(j) \equiv \mathbf{s}_t(j) - E_{t-1}(j)[\mathbf{s}_t(j)]$$

Since $\overrightarrow{\mathbf{s}}_t(j)$ contains only *new* information available in period t, it must be orthogonal to any of j's estimates based on information from earlier periods. We can therefore use the result that $E[w|y,z] = E[w|y] + E[w|z]$ when $y \perp z$, so that

$$\begin{aligned} E_t(j)[Z_t] &= E_{t-1}(j)[Z_t] + K_t \overrightarrow{\mathbf{s}}_t(j) \\ &= M E_{t-1}(j)[Z_{t-1}] + K_t \{\mathbf{s}_t(j) - (D_1 M + D_2) E_{t-1}(j)[X_{t-1}]\} \\ &= (M - K_t(D_1 M + D_2)) E_{t-1}(j)[Z_{t-1}] + K_t \mathbf{s}_t(j) \end{aligned} \tag{A.3}$$

for some matrix, K_t (the Kalman gain). Note that K_t does not require an agent subscript as the problem is symmetric for all agents. For this to be the best linear estimator, we require K_t to be such that $\overrightarrow{\mathbf{s}}_t(j)$ is orthogonal to the projection error, $Z_t - K_t \overrightarrow{\mathbf{s}}_t(j)$. That is, we require that

$$E\left[\left(X_t - K_t \overrightarrow{\mathbf{s}}_t(j)\right) \overrightarrow{\mathbf{s}}_t(j)'\right] = 0 \tag{A.4}$$

Rearranging gives

$$K_t = E\left[Z_t \overrightarrow{\mathbf{s}}_t(j)'\right] \left(E\left[\overrightarrow{\mathbf{s}}_t(j) \overrightarrow{\mathbf{s}}_t(j)'\right]\right)^{-1} \tag{A.5}$$

Before evaluating this, note that we can rewrite the relevant vectors as:

$$\begin{aligned} \overrightarrow{\mathbf{s}}_t(j) &= \overbrace{(D_1 M + D_2) Z_{t-1} + D_3 \mathbf{w}_{t-2} + D_1 N_1 \mathbf{u}_t + D_1 N_2 \widetilde{\mathbf{v}}_t + R_1 \mathbf{v}_t(j) + (D_1 N_3 + R_2) \widetilde{\mathbf{v}}_{t-1}}^{\mathbf{s}_t(j)} \\ &\quad - \overbrace{(D_1 M + D_2) E_{t-1}(j)[Z_{t-1}] - D_3 \mathbf{w}_{t-2}}^{E_{t-1}(j)[\mathbf{s}_t(j)]} \\ &= (D_1 M + D_2) \widehat{Z_{t-1}(j)} + D_1 N_1 \mathbf{u}_t + D_1 N_2 \widetilde{\mathbf{v}}_t + R_1 \mathbf{v}_t(j) + (D_1 N_3 + R_2) \widetilde{\mathbf{v}}_{t-1} \end{aligned}$$

where $\widehat{Z_t(j)} \equiv Z_t - E_t(j)[Z_t]$ is j's contemporaneous error in estimating Z_t. Note that the terms in $\mathbf{w}_{t-2}$ have dropped out as they are known with certainty in period $t-1$. We can also rewrite the law of motion for the hidden state as

$$Z_t = M\left(\widehat{Z_{t-1}(j)} + E_{t-1}(j)[Z_{t-1}]\right) + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1}$$

The first term of equation (A.5) thus expands to:

$$\begin{aligned} E\left[Z_t \overrightarrow{\mathbf{s}}_t(j)'\right] &= E\left[\begin{array}{l} \left(M\left(\widehat{Z_{t-1}(j)} + E_{t-1}(j)[Z_{t-1}]\right) + N_1 \mathbf{u}_t + N_2 \widetilde{\mathbf{v}}_t + N_3 \widetilde{\mathbf{v}}_{t-1}\right) \\ \times \left((D_1 M + D_2) \widehat{Z_{t-1}(j)} + D_1 N_1 \mathbf{u}_t + D_1 N_2 \widetilde{\mathbf{v}}_t + R_1 \mathbf{v}_t(j) + (D_1 N_3 + R_2) \widetilde{\mathbf{v}}_{t-1}\right)' \end{array}\right] \\ &= M V_{t-1|t-1} (D_1 M + D_2)' + N_1 \Sigma_{uu} N_1' D_1' + \zeta^* N_2 \Sigma_{vv} N_2' D_1' + \zeta^* N_3 \Sigma_{vv} (D_1 N_3 + R_2)' \end{aligned}$$

where I have used $V_{t|t} \equiv E\left[\widehat{Z_t(j)}\widehat{Z_t(j)}'\right]$ as the variance-covariance matrix associated with $E_t(j)[Z_t]$. Given the symmetry of the problem across agents, although individual expectations may differ the variance of each estimate will be common. For the second term, we have that

$$
\begin{aligned}
E\left[\overrightarrow{\mathbf{s}}_t(j)\,\overrightarrow{\mathbf{s}}_t(j)'\right] &= E\left[\begin{array}{c} \left((D_1M+D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3+R_2)\widetilde{\mathbf{v}}_{t-1}\right) \\ \times\left((D_1M+D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3+R_2)\widetilde{\mathbf{v}}_{t-1}\right) \end{array}\right. \\
&= (D_1M+D_2)V_{t-1|t-1}(D_1M+D_2) + D_1N_1\Sigma_{uu}N_1'D_1' \\
&\quad + \zeta^* D_1N_2\Sigma_{vv}N_1'D_2' + R_1\Sigma_{vv}R_1' + \zeta^*(D_1N_3+R_2)\Sigma_{vv}(D_1N_3+R_2)'
\end{aligned}
$$

so that, all together, the Kalman gain is given by

$$
\begin{aligned}
K_t &= \left(MV_{t-1|t-1}(D_1M+D_2)' + N_1\Sigma_{uu}N_1'D_1' + g^*N_2\Sigma_{vv}N_2'D_1' + g^*N_3\Sigma_{vv}(D_1N_3+R_2)'\right) \\
&\quad \times\left[\begin{array}{l} (D_1M+D_2)V_{t-1|t-1}(D_1M+D_2) + D_1N_1\Sigma_{uu}N_1'D_1' \\ +\zeta^* D_1N_2\Sigma_{vv}N_2'D_1' + R_1\Sigma_{vv}R_1' + \zeta^*(D_1N_3+R_2)\Sigma_{vv}(D_1N_3+R_2)' \end{array}\right]^{-1}
\end{aligned} \tag{A.6}
$$

Evolution of the gain and variance matricies First, since we can rewrite the state equation as

$$
\begin{aligned}
Z_t - E_{t-1}(j)[Z_t] &= MZ_{t-1} + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1} - E_{t-1}(j)[Z_t] \\
&= M(Z_{t-1} - E_{t-1}(j)[Z_{t-1}]) + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}
\end{aligned}
$$

we have that

$$
V_{t|t-1} = MV_{t-1|t-1}M' + N_1\Sigma_{uu}N_1' + \zeta^*N_2\Sigma_{vv}N_2' + \zeta^*N_3\Sigma_{vv}N_3' \tag{A.7}
$$

Next, add Z_t to each side of equation (A.3) and rearrange to get

$$
Z_t - E_t(j)[Z_t] = Z_t - E_{t-1}(j)[Z_t] - K_t\overrightarrow{\mathbf{s}}_t(j)
$$

Since the innovation is orthogonal to both the prior error, $Z_t - E_{t-1}(j)[Z_t]$, and the posterior error, $Z_t - E_t(j)[Z_t]$, the variance of the left-hand side must equal the sum of the variances on the right-hand side, thereby giving

$$
\begin{aligned}
V_{t|t} &= V_{t|t-1} - K_t Var\left((D_1M+D_2)\widehat{Z_{t-1}(j)} + D_1N_1\mathbf{u}_t + D_1N_2\widetilde{\mathbf{v}}_t + R_1\mathbf{v}_t(j) + (D_1N_3+R_2)\widetilde{\mathbf{v}}_{t-1}\right)K_t' \\
&= V_{t|t-1} - K_t\left[\begin{array}{l} (D_1M+D_2)V_{t-1|t-1}(D_1M+D_2)' + D_1N_1\Sigma_{uu}N_1'D_1' \\ +\zeta^* D_1N_2\Sigma_{vv}N_2'D_1' + R_1\Sigma_{vv}R_1' + \zeta^*(D_1N_3+R_2)\Sigma_{vv}(D_1N_3+R_2)' \end{array}\right]K_t'
\end{aligned} \tag{A.8}
$$

Provided that M represents a contraction, then there will exist steady state (i.e. time-invariant) Kalman gain and Variance matricies, found by iterating equations (A.6), (A.7) and (A.8) forward until convergence is achieved. The form of these matricies will be:

$$K = \left(MV\left(D_1M + D_2\right)' + N_1\Sigma_{uu}N_1'D_1' + \zeta^* N_2\Sigma_{vv}N_2'D_1' + \zeta^* N_3\Sigma_{vv}\left(D_1N_3 + R_2\right)'\right)$$
$$\times \left[\begin{array}{l} \left(D_1M + D_2\right)V\left(D_1M + D_2\right) + D_1N_1\Sigma_{uu}N_1'D_1' \\ +\zeta^* D_1N_2\Sigma_{vv}N_2'D_1' + R_1\Sigma_{vv}R_1' + \zeta^*\left(D_1N_3 + R_2\right)\Sigma_{vv}\left(D_1N_3 + R_2\right)' \end{array}\right]^{-1}$$

$$V = M\left(V - K\left[\begin{array}{l} \left(D_1M + D_2\right)V\left(D_1M + D_2\right)' + D_1N_1\Sigma_{uu}N_1'D_1' \\ +\zeta^* D_1N_2\Sigma_{vv}N_1'D_2' + R_1\Sigma_{vv}R_1' + \zeta^*\left(D_1N_3 + R_2\right)\Sigma_{vv}\left(D_1N_3 + R_2\right)' \end{array}\right]K'\right)M'$$
$$+ N_1\Sigma_{uu}N_1' + \zeta^* N_2\Sigma_{vv}N_2' + \zeta^* N_3\Sigma_{vv}N_3'$$

Identifying the state law of motion We seek the coefficients of the state vector's law of motion:

$$Z_t = MZ_{t-1} + N_1\mathbf{u}_t + N_2\widetilde{\mathbf{v}}_t + N_3\widetilde{\mathbf{v}}_{t-1}$$

First note that the state vector is defined recursively as

$$Z_t \equiv \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t\left[Z_t\right] \\ \widetilde{E}_t\left[Z_t\right] \end{bmatrix}$$

and define the matricies T_s and T_w as extracting the simple-average and weighted-average expectations of Z_t from Z_t. That is, $T_sZ_t = \overline{E}_t\left[Z_t\right]$ and $T_wZ_t = \widetilde{E}_t\left[Z_t\right]$. Next, note that we can break the M and N_* matricies down as

$$M = \begin{bmatrix} \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \Psi_1 \\ \Upsilon_1 \end{bmatrix} \qquad \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} & \mathbf{0} \\ \Psi_2 & \Psi_3 & \Psi_4 \\ \Upsilon_2 & \Upsilon_3 & \Upsilon_4 \end{bmatrix}$$

Further, we have that agents update their estimates of the state vector according to

$$E_t\left(j\right)\left[Z_t\right] = \left(M - K\left(D_1M + D_2\right)\right)E_{t-1}\left(j\right)\left[Z_{t-1}\right] + K\mathbf{s}_t\left(j\right) \tag{A.9}$$

with the Kalman gain matrix (K) defined above. Signals are comprised as

$$\mathbf{s}_t\left(j\right) = D_1Z_t + D_2Z_{t-1} + R_1\mathbf{v}_t\left(j\right) + R_2\widetilde{\mathbf{v}}_{t-1}$$

Combining these two equations gives

$$E_t\left(j\right)\left[Z_t\right] = \left(M - K\left(D_1M + D_2\right)\right)E_{t-1}\left(j\right)\left[Z_{t-1}\right] + K\left(D_1Z_t + D_2Z_{t-1} + R_1\mathbf{v}_t\left(j\right) + R_2\widetilde{\mathbf{v}}_{t-1}\right)$$

Expanding the Z_t on the right hand side and gathering like terms then gives

$$
\begin{aligned}
E_t(j)[Z_t] = {} & K(D_1M + D_2)Z_{t-1} \\
& + (M - K(D_1M + D_2))E_{t-1}(j)[Z_{t-1}] \\
& + KD_1N_1\mathbf{u}_t \\
& + KD_1N_2\widetilde{\mathbf{v}}_t \\
& + K(D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1} \\
& + KR_1\mathbf{v}_t(j)
\end{aligned} \tag{A.10}
$$

Taking the simple average of equation (A.10) gives

$$
\begin{aligned}
\overline{E}_t[Z_t] = {} & \{MT_s + K(D_1M + D_2)(I - T_s)\}Z_{t-1} \\
& + KD_1N_1\mathbf{u}_t \\
& + KD_1N_2\widetilde{\mathbf{v}}_t \\
& + K(D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}
\end{aligned} \tag{A.11}
$$

Similarly, the weighted average of equation (A.10) is:

$$
\begin{aligned}
\widetilde{E}_t[Z_t] = {} & \{MT_w + K(D_1M + D_2)(I - T_w)\}Z_{t-1} \\
& + KD_1N_1\mathbf{u}_t \\
& + K(D_1N_2 + R_1)\widetilde{\mathbf{v}}_t \\
& + K(D_1N_3 + R_2)\widetilde{\mathbf{v}}_{t-1}
\end{aligned} \tag{A.12}
$$

From these we may immediately read that

$$
\begin{aligned}
\Psi_1 &= MT_s + K(D_1M + D_2)(I - T_s) \\
\Psi_2 &= KD_1N_1 \\
\Psi_3 &= KD_1N_2 \\
\Psi_4 &= K(D_1N_3 + R_2)
\end{aligned}
$$

and that

$$
\begin{aligned}
\Upsilon_1 &= MT_w + K(D_1M + D_2)(I - T_w) \\
\Upsilon_2 &= KD_1N_1 \\
\Upsilon_3 &= K(D_1N_2 + R_1) \\
\Upsilon_4 &= K(D_1N_3 + R_2)
\end{aligned}
$$

Finding the solution involves finding the fixed point of the system for a pre-chosen upper limit (k^*) on the number of orders of expectations to include.

A.4.2 Solving stage two

We have the following state space system

$$
X_t = FX_{t-1} + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1} \tag{A.13}
$$

$$
\mathbf{s}_t(i) = C_1X_t + C_2X_{t-1} + C_3\mathbf{w}_{t-2} + S_1\mathbf{v}_t(i) + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i)) \tag{A.14}
$$

where, as in stage one, $\mathbf{w}_{t-2}$ is a vector containing information relating to period $t-2$ and earlier that is observed in period $t-1$ (or earlier). We will again first develop a modified Kalman filter for agent i's estimation of Y_t and then turn to considering the evolution of Y_t itself.

The (modified) Kalman filter Substituting the state law of motion into the signal equation gives

$$\mathbf{s}_t(i) = (C_1F + C_2)X_{t-1} + C_3\mathbf{w}_{t-2} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t + S_1\mathbf{v}_t(i) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i))$$

with a prior expectation of the signal given by

$$E_{t-1}(i)[\mathbf{s}_t(i)] = (C_1F + C_2)E_{t-1}(i)[X_{t-1}] + C_3\mathbf{w}_{t-2}$$

Denote the innovation in $\mathbf{s}_t(i)$ as

$$\begin{aligned}\overleftarrow{\mathbf{s}}_t(i) &\equiv \mathbf{s}_t(i) - E_{t-1}(i)[\mathbf{s}_t(i)] \\ &= (C_1F + C_2)\widehat{X_{t-1}(i)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t + S_1\mathbf{v}_t(i) \\ &+ C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i))\end{aligned}$$

The orthogonality of $\overleftarrow{\mathbf{s}}_t(i)$ gives us that

$$E_t(i)[X_t] = (F - J_t(C_1F + C_2))E_{t-1}(i)[X_{t-1}] + J_t\mathbf{s}_t(i)$$

for some matrix, J_t (the Kalman gain), while optimality in the sense of minimising mean square error gives us that

$$J_t = E\left[X_t\overleftarrow{\mathbf{s}}_t(i)'\right]\left(E\left[\overleftarrow{\mathbf{s}}_t(i)\,\overleftarrow{\mathbf{s}}_t(i)'\right]\right)^{-1} \tag{A.15}$$

The first term in this expression expands as

$$X_t = F\left(\widehat{X_{t-1}(i)} + E_{t-1}(i)[X_{t-1}]\right) + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1}$$

$$\begin{aligned}E\left[X_t\overleftarrow{\mathbf{s}}_t(i)'\right] &= E\left[\begin{array}{c}\left(F\left(\widehat{X_{t-1}(i)} + E_t(i)[X_{t-1}]\right) + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1}\right) \\ \times\left(\begin{array}{c}(C_1F + C_2)\widehat{X_{t-1}(i)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t \\ +S_1\mathbf{v}_t(i) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i))\end{array}\right)'\end{array}\right] \\ &= FU_{t-1|t-1}(C_1F + C_2)' + G_1\Sigma_{uu}G_1'C_1' + \zeta^*G_2\Sigma_{vv}G_2'C_1' \\ &+ \zeta^*G_3\Sigma_{vv}G_3'C_1' + \zeta^*(2 - \psi^*)G_4\Sigma_{vv}G_4'C_1' + 2\zeta^*G_3\Sigma_{vv}G_4'C_1'\end{aligned}$$

where $U_{t|t} \equiv E\left[\widehat{X_t(i)}\widehat{X_t(i)}'\right]$ as the variance-covariance matrix associated with $E_t(i)[X_t]$. The second term expands as:

$$E\left[\overleftarrow{\mathbf{s}}_t(i)\,\overleftarrow{\mathbf{s}}_t(i)'\right] = E\left[\begin{array}{c}\left(\begin{array}{c}(C_1F+C_2)\,\widehat{X_{t-1}(i)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t \\ +S_1\mathbf{v}_t(i) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i))\end{array}\right)' \\ \times\left(\begin{array}{c}(C_1F+C_2)\,\widehat{X_{t-1}(i)} + C_1G_1\mathbf{u}_t + C_1G_2\widetilde{\mathbf{v}}_t \\ +S_1\mathbf{v}_t(i) + C_1G_3\widetilde{\mathbf{v}}_{t-1} + C_1G_4\widehat{\mathbf{v}}_{t-1} + S_2\mathbf{v}_{t-1}(\delta_{t-1}(i))\end{array}\right)\end{array}\right]$$

$$= (C_1F+C_2)\,U_{t-1|t-1}\,(C_1F+C_2)' + C_1G_1\Sigma_{uu}G_1'C_1' + \zeta^*C_1G_2\Sigma_{vv}G_2'C_1'$$
$$+ S_1\Sigma_{vv}S_1' + \zeta^*C_1G_3\Sigma_{vv}G_3'C_1' + \zeta^*(2-\zeta^*)\,C_1G_4\Sigma_{vv}G_4'C_1'$$
$$+ 2\zeta^*C_1G_3\Sigma_{vv}G_4'C_1' + S_2\Sigma_{vv}S_2'$$

Together, these give the Kalman gain for stage two as

$$J_t = \left(\begin{array}{c}FU_{t-1|t-1}(C_1F+C_2)' + G_1\Sigma_{uu}G_1'C_1' + \zeta^*G_2\Sigma_{vv}G_2'C_1' \\ +\zeta^*G_3\Sigma_{vv}G_3'C_1' + \zeta^*(2-\zeta^*)\,G_4\Sigma_{vv}G_4'C_1' + 2\zeta^*G_3\Sigma_{vv}G_4'C_1'\end{array}\right) \tag{A.16}$$
$$\times\left[\begin{array}{c}(C_1F+C_2)\,U_{t-1|t-1}\,(C_1F+C_2)' \\ +C_1G_1\Sigma_{uu}G_1'C_1' \\ +\zeta^*C_1G_2\Sigma_{vv}G_2'C_1' \\ +S_1\Sigma_{vv}S_1' \\ +\zeta^*C_1G_3\Sigma_{vv}G_3'C_1' \\ +\zeta^*(2-\zeta^*)\,C_1G_4\Sigma_{vv}G_4'C_1' \\ +2\zeta^*C_1G_3\Sigma_{vv}G_4'C_1' \\ +S_2\Sigma_{vv}S_2'\end{array}\right]^{-1}$$

Evolution of the gain and variance matricies The prior variance will be given by

$$U_{t|t-1} = FU_{t-1|t-1}F' + G_1\Sigma_{uu}G_1' + \zeta^*G_2\Sigma_{vv}G_2' + \zeta^*G_3\Sigma_{vv}G_3' + \zeta^*(2-\zeta^*)\,G_4\Sigma_{vv}G_4' + 2\zeta^*G_3\Sigma_{vv}G_4' \tag{A.17}$$

And the posterior variance will update as

$$X_t - E_t(j)[X_t] = X_t - E_{t-1}(j)[X_t] - K_t\overrightarrow{\mathbf{s}}_t(j)$$

Since the innovation is orthogonal to both the prior error, $X_t - E_{t-1}(j)[X_t]$, and the posterior error, $X_t - E_t(j)[X_t]$, the variance of the left-hand side must equal the sum of the variances on the right-hand side, thereby giving

$$U_{t|t} = U_{t|t-1} - J_t Var\left(\overleftarrow{\mathbf{s}}_t(j)\right)J_t' \tag{A.18}$$
$$= U_{t|t-1} - J_t\left[\begin{array}{c}(C_1F+C_2)\,U_{t-1|t-1}\,(C_1F+C_2)' \\ +C_1G_1\Sigma_{uu}G_1'C_1' \\ +\zeta^*C_1G_2\Sigma_{vv}G_2'C_1' \\ +S_1\Sigma_{vv}S_1' \\ +\zeta^*C_1G_3\Sigma_{vv}G_3'C_1' \\ +\zeta^*(2-\zeta^*)\,C_1G_4\Sigma_{vv}G_4'C_1' \\ +2\zeta^*C_1G_3\Sigma_{vv}G_4'C_1' \\ +S_2\Sigma_{vv}S_2'\end{array}\right]J_t'$$

Provided that F represents a contraction, then there will exist steady state (i.e. time-invariant) Kalman gain and variance matricies, found by iterating equations (A.16), (A.17) and (A.18) forward until convergence is achieved.

Identifying the state law of motion We seek the coefficients of the state vector's law of motion:

$$X_t = FX_{t-1} + G_1\mathbf{u}_t + G_2\widetilde{\mathbf{v}}_t + G_3\widetilde{\mathbf{v}}_{t-1} + G_4\widehat{\mathbf{v}}_{t-1}$$

First note that the state vector is defined recursively as

$$X_t \equiv \begin{bmatrix} \mathbf{x}_t \\ \overline{E}_t\left[X_t\right] \\ \widetilde{E}_t\left[X_t\right] \\ \mathring{E}_t\left[X_t\right] \end{bmatrix}$$

and so break the F and G_* matrices down as

$$F = \begin{bmatrix} \begin{bmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \Theta_1 \\ \Omega_1 \\ \Gamma_1 \end{bmatrix} \qquad \begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix} = \begin{bmatrix} P & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Theta_2 & \Theta_3 & \Theta_4 & \Theta_5 \\ \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 \\ \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 \end{bmatrix}$$

and define the matricies T_s, T_w and T_o as such that $T_sX_t = \overline{E}_t\left[X_t\right]$ and $T_wX_t = \widetilde{E}_t\left[X_t\right]$ and $T_oX_t = \mathring{E}_t\left[X_t\right]$. Next, note that we have that agents update their estimates of the state vector according to

$$E_t\left(i\right)\left[X_t\right] = \left(F - J\left(C_1F + C_2\right)\right)E_{t-1}\left(i\right)\left[X_{t-1}\right] + J\mathbf{s}_t\left(i\right) \tag{A.19}$$

and individual signals are comprised as

$$\mathbf{s}_t\left(i\right) = C_1X_t + C_2X_{t-1} + S_1\mathbf{v}_t\left(i\right) + S_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)$$

Combining these last two equations with the state vector's law of motion gives

$$\begin{aligned} E_t\left(i\right)\left[X_t\right] = {} & J\left(C_1F + C_2\right)X_{t-1} \\ & + \left(F - J\left(C_1F + C_2\right)\right)E_{t-1}\left(i\right)\left[X_{t-1}\right] \\ & + JC_1G_1\mathbf{u}_t \\ & + JC_1G_2\widetilde{\mathbf{v}}_t \\ & + JC_1G_3\widetilde{\mathbf{v}}_{t-1} \\ & + JC_1G_4\widehat{\mathbf{v}}_{t-1} \\ & + JS_1\mathbf{v}_t\left(i\right) \\ & + JS_2\mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right) \end{aligned} \tag{A.20}$$

Taking the simple average of equation (A.20) gives

$$\overline{E}_t\left[X_t\right] = \left\{FT_s + J\left(C_1F + C_2\right)\left(I - T_s\right)\right\}X_{t-1} \\ + JC_1G_1\mathbf{u}_t \\ + JC_1G_2\widetilde{\mathbf{v}}_t \\ + J\left(C_1G_3 + S_2\right)\widetilde{\mathbf{v}}_{t-1} \\ + JC_1G_4\widehat{\mathbf{v}}_{t-1} \tag{A.21}$$

where I have used lemma (2) to replace $\int_0^1 \mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)di$ with $\widetilde{\mathbf{v}}_{t-1}$. Taking the weighted average of equation (A.20) gives

$$E_t\left(i\right)\left[X_t\right] = \left\{FT_w + J\left(C_1F + C_2\right)\left(I - T_w\right)\right\}X_{t-1} \\ + JC_1G_1\mathbf{u}_t \\ + J\left(C_1G_2 + S_1\right)\widetilde{\mathbf{v}}_t \\ + JC_1G_3\widetilde{\mathbf{v}}_{t-1} \\ + J\left(C_1G_4 + S_2\right)\widehat{\mathbf{v}}_{t-1} \tag{A.22}$$

where I have used lemma (2) to replace $\int_0^1 \mathbf{v}_{t-1}\left(\delta_{t-1}\left(i\right)\right)\phi\left(i\right)di$ with $\widehat{\mathbf{v}}_{t-1}$. We can then immediately read that

$$\Theta_1 = FT_s + J\left(C_1F + C_2\right)\left(I - T_s\right) \\ \Theta_2 = JC_1G_1 \\ \Theta_3 = JC_1G_2 \\ \Theta_4 = J\left(C_1G_3 + S_2\right) \\ \Theta_5 = JC_1G_4$$

and

$$\Omega_1 = FT_w + J\left(C_1F + C_2\right)\left(I - T_w\right) \\ \Omega_2 = JC_1G_1 \\ \Omega_3 = J\left(C_1G_2 + S_1\right) \\ \Omega_4 = JC_1G_3 \\ \Omega_5 = J\left(C_1G_4 + S_2\right)$$

and

$$\Gamma_1 = MT_o + K\left(D_1M + D_2\right)\left(I - T_o\right) \\ \Gamma_2 = KD_1N_1 \\ \Gamma_3 = KD_1N_2 \\ \Gamma_4 = K\left(D_1N_3 + R_2\right) \\ \Gamma_5 = 0$$

where I have ignored the element $K_1Q\mathbf{v}_t\left(j\right)$ from equation (A.10) as, from agent i's perspective, it is expected to be equal to zero.

Finding the solution involves finding the fixed point of the system for a pre-chosen upper limit (k^*) on the number of orders of expectations to include.

A.5 Extending the model to a dynamic setting

We here consider an important extension to the basic model of the previous section: consideration of dynamic actions. In particular, we here allow agents' decision rules to be somewhat more general, including a consideration of the future average action. We still have that the underlying state follows an AR(1) process:

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + P\mathbf{u}_t$$

and still suppose that the full hierarchy of expectations regarding the underlying state is given by:

$$X_t = \mathbb{E}_t^{(0:\infty)}\left[\mathbf{x}_t\right]$$

We also leave individual signals unchanged as:

$$\mathbf{s}_t\left(i\right) = \begin{bmatrix} B\mathbf{x}_t + Q\mathbf{v}_t\left(i\right) \\ g_{t-1}\left(\delta_{t-1}\left(i\right)\right) \end{bmatrix}$$

We then suppose that individual decisions are made according to the following rule:

$$g_t\left(i\right) = \eta_s'\mathbf{s}_t\left(i\right) + \eta_x' E_t\left(i\right)\left[X_t\right] + \eta_y E_t\left(i\right)\left[\overline{g}_t\right] + \eta_z E_t\left(i\right)\left[\overline{g}_{t+1}\right] \tag{A.23}$$

and maintain the assumption that $\eta_s' = \begin{bmatrix}\alpha' & 0\end{bmatrix}$. It is required to show that $g_t\left(i\right)$ may be expressed in the general form

$$g_t\left(i\right) = \gamma_1' w_{t-1} + \gamma_2' X_t + \gamma_3' E_t\left(i\right)\left[X_t\right] + \gamma_4'\mathbf{v}_t\left(i\right)$$

To do this, we start by taking the simple average of equation (A.23) to give:

$$\overline{g}_t = \eta_s'\overline{\mathbf{s}}_t + \eta_x'\overline{E}_t\left[X_t\right] + \eta_y\overline{E}_t\left[\overline{g}_t\right] + \eta_z\overline{E}_t\left[\overline{g}_{t+1}\right]$$

To keep the notation clean, define $\theta_t \equiv \eta_s'\overline{\mathbf{s}}_t + \eta_x'\overline{E}_t\left[X_t\right]$ so that

$$\overline{g}_t = \theta_t + \eta_y\overline{E}_t\left[\overline{g}_t\right] + \eta_z\overline{E}_t\left[\overline{g}_{t+1}\right]$$

We now substitute this equation back into itself in the second element $(\eta_y\overline{E}_t\left[\overline{g}_t\right])$:

$$\overline{g}_t = \theta_t + \eta_y\overline{E}_t\left[\theta_t\right] + \eta_y^2\overline{E}_t^{(2)}\left[\overline{g}_t\right] + \eta_z\overline{E}_t\left[\overline{g}_{t+1}\right] + \eta_y\eta_z\overline{E}_t^{(2)}\left[\overline{g}_{t+1}\right]$$

Repeating this process, in the limit (and using the fact that $\eta_y \in (0,1)$ and assuming that average expectations don't explode), this becomes:

$$\overline{g}_t = \left(\sum_{k=0}^{\infty}\eta_y^k\overline{E}_t^{(k)}\left[\theta_t\right]\right) + \left(\eta_z\sum_{k=1}^{\infty}\eta_y^{k-1}\overline{E}_t^{(k)}\left[\overline{g}_{t+1}\right]\right)$$

Now briefly consider θ_t and simple-average expectations of θ_t. First note that $\overline{\mathbf{s}}'_t = \begin{bmatrix} (B\mathbf{x}_t)' & \widetilde{g}_{t-1} \end{bmatrix}$ and we maintain our assumption that $\eta'_s = \begin{bmatrix} \alpha' & 0 \end{bmatrix}$ so that

$$\eta'_s \overline{\mathbf{s}}_t = \alpha' B \mathbf{x}_t$$

Therefore, we can write that:

$$\begin{aligned}
\theta_t &= \alpha' B \mathbf{x}_t + \eta'_x \overline{E}^{(1)}_t [X_t] \\
\overline{E}^{(1)}_t [\theta_t] &= \alpha' B \overline{E}^{(1)}_t [\mathbf{x}_t] + \eta'_x \overline{E}^{(2)}_t [X_t] \\
\overline{E}^{(2)}_t [\theta_t] &= \alpha' B \overline{E}^{(2)}_t [\mathbf{x}_t] + \eta'_x \overline{E}^{(3)}_t [X_t] \\
&\cdots
\end{aligned}$$

Next, suppose that the matrix T_s selects the simple-average expectation of X_t from X_t:

$$\overline{E}^{(1)}_t [X_t] = T_s X_t$$

and that the matrix S selects $\mathbf{x}_t$ from X_t (obviously $S = \begin{bmatrix} I_l & 0_{l \times \infty} \end{bmatrix}$ where l is the number of elements in $\mathbf{x}_t$):

$$\mathbf{x}_t = S X_t$$

Then we can write:

$$\begin{aligned}
\theta_t &= \left(\alpha' B S + \eta'_x T_s \right) X_t \\
\overline{E}^{(1)}_t [\theta_t] &= \left(\alpha' B S + \eta'_x T_s \right) T_s X_t \\
\overline{E}^{(2)}_t [\theta_t] &= \left(\alpha' B S + \eta'_x T_s \right) T_s^2 X_t \\
&\cdots
\end{aligned}$$

or, in general,

$$\overline{E}^{(k)}_t [\theta_t] = \left(\alpha' B S + \eta'_x T_s \right) T_s^k X_t$$

The average period-t action can therefore be written as

$$\begin{aligned}
\overline{g}_t &= \left(\alpha' B S + \eta'_x T_s \right) \left(\sum_{k=0}^{\infty} \left(\eta_y T_s \right)^k \right) X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}^{(k)}_t \left[\overline{g}_{t+1} \right] \\
&= \left(\alpha' B S + \eta'_x T_s \right) \left(I - \eta_y T_s \right)^{-1} X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}^{(k)}_t \left[\overline{g}_{t+1} \right] \\
&= \beta' X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}^{(k)}_t \left[\overline{g}_{t+1} \right]
\end{aligned}$$

where $\beta' \equiv \left(\alpha' B S + \eta'_x T_s \right) \left(I - \eta_y T_s \right)^{-1}$. Next, substitute this back into itself for the next-period average action:

$$
\begin{aligned}
\overline{g}_t &= \beta' X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\beta' X_{t+1} + \eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\overline{g}_{t+2} \right] \right] \\
&= \beta' X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \beta' \overline{E}_t^{(k)} \left[X_{t+1} \right] + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\overline{g}_{t+2} \right] \right]
\end{aligned}
$$

Next, we use the following conjectured aspect of the law of motion for X_t:

$$
E_t(i) \left[X_{t+1} \right] = E_t(i) \left[F X_t \right]
$$

for some matrix of parameters F. This implies that

$$
\overline{E}_t^{(k)} \left[X_{t+1} \right] = F \overline{E}_t^{(k)} \left[X_t \right]
$$

and hence that

$$
\begin{aligned}
\overline{g}_t &= \beta' X_t + \eta_z \beta' F \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[X_t \right] + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\overline{g}_{t+2} \right] \right] \\
&= \beta' X_t + \eta_z \beta' F \left(\sum_{k=1}^{\infty} \eta_y^{k-1} T_s^k \right) X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\overline{g}_{t+2} \right] \right] \\
&= \beta' X_t + \eta_z \beta' F T_s \left(I - \eta_y T_s \right)^{-1} X_t + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\overline{g}_{t+2} \right] \right]
\end{aligned}
$$

Next, expand the $\overline{g}_{t+2}$ term to give

$$
\begin{aligned}
\overline{g}_t &= \beta' X_t + \eta_z \beta' F T_s \left(I - \eta_y T_s \right)^{-1} X_t \\
&\quad + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\beta' X_{t+2} + \eta_z \sum_{m=1}^{\infty} \eta_y^{m-1} \overline{E}_{t+2}^{(m)} \left[\overline{g}_{t+3} \right] \right] \right] \\
&= \beta' X_t \\
&\quad + \eta_z \beta' F T_s \left(I - \eta_y T_s \right)^{-1} X_t \\
&\quad + \beta' \left(\eta_z F T_s \left(I - \eta_y T_s \right)^{-1} \right)^2 X_t \\
&\quad + \eta_z \sum_{k=1}^{\infty} \eta_y^{k-1} \overline{E}_t^{(k)} \left[\eta_z \sum_{l=1}^{\infty} \eta_y^{l-1} \overline{E}_{t+1}^{(l)} \left[\eta_z \sum_{m=1}^{\infty} \eta_y^{m-1} \overline{E}_{t+2}^{(m)} \left[\overline{g}_{t+3} \right] \right] \right]
\end{aligned}
$$

Continued substitution then arrives at:

$$
\overline{g}_t = \beta' \sum_{j=0}^{\infty} \left(\eta_z F T_s \left(I - \eta_y T_s \right)^{-1} \right)^j X_t
$$

which, in turn, becomes

$$
\overline{g}_t = \underbrace{\left(\alpha' B S + \eta_x' T_s \right) \left(I - \eta_y T_s \right)^{-1} \left(I - \eta_z F T_s \left(I - \eta_y T_s \right)^{-1} \right)^{-1}}_{\mathbf{a}'} X_t
$$

Using this simple expression of $\overline{g}_t = \mathbf{a}' X_t$, we can substitute it back into the agents' individual decision rule to obtain

$$\begin{aligned} g_t(i) &= \alpha'(B\mathbf{x}_t + Q\mathbf{v}_t(i)) + (\eta_x' + \eta_y \mathbf{a}' + \eta_z \mathbf{a}' F) E_t(i)[X_t] \\ &= \underbrace{\alpha' BS}_{\gamma_2'} X_t + \underbrace{(\eta_x' + \eta_y \mathbf{a}' + \eta_z \mathbf{a}' F)}_{\gamma_3'} E_t(i)[X_t] + \underbrace{\alpha' Q}_{\gamma_4'} \mathbf{v}_t(i) \end{aligned}$$

which is now in the necessary form. As an aside, taking a simple average of this gives

$$\overline{g}_t = \alpha' BS X_t + (\eta_x' + \eta_y \mathbf{a}' + \eta_z \mathbf{a}' F) \overline{E}_t[X_t]$$

which implies the following constraint on the coefficients of the decision rule $(\alpha, \eta_x, \eta_y, \eta_z)$ and the expectation transition matrix (F):

$$\mathbf{a}' = \alpha' BS + (\eta_x' + \eta_y \mathbf{a}' + \eta_z \mathbf{a}' F) T_s$$

Appendix B Dynamic price setting with network learning

This appendix provides a full derivation of the model of dynamic price setting with network learning presented in the text.

B.1 Getting to $mc_t(j) = \upsilon y_t + \omega_t(j)$

The treatment of the household is entirely standard and omitted. For firms' price-setting problem, note that (40) implies that the aggregate price must evolve as

$$P_t = \left(\theta \int P_{t-1}(j)^{1-\varepsilon} dj + (1-\theta) \int G_t(j)^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$$

which, in linearised form, is

$$p_t = \theta p_{t-1} + (1-\theta)\overline{g}_t \tag{B.1}$$

Substituting the demand function (39) into the firm's objective function (43), a generic firm's problem when able to update their price in period t is given by:

$$\max_{G_t(j)} E_t(j)\left[\sum_{s=0}^{\infty} (\beta\theta)^s Q_{t+s|t} \left\{\left(\frac{G_t(j)}{P_{t+s}}\right)^{1-\varepsilon} Y_{t+s} - \frac{W_{t+s}}{P_{t+s}} \frac{1}{A_{t+s}(j)^{1+\eta}} \left(\frac{G_t(j)}{P_{t+s}}\right)^{-\varepsilon(1+\eta)} Y_{t+s}^{1+\eta}\right\}\right]$$

The first-order condition, after some tweaking, may be expressed as:

$$E_t(j)\left[\sum_{s=0}^{\infty} (\beta\theta)^s Q_{t+s|t} \left(\frac{G_t(j)}{P_{t+s}}\right)^{-\varepsilon} Y_{t+s} \left\{\frac{G_t(j)}{P_{t+s}} - \left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{G_t(j)}{P_{t+s}}\right)^{-\varepsilon\eta} (1+\eta) \frac{W_{t+s}}{P_{t+s}} \frac{1}{A_{t+s}(j)^{1+\eta}} Y_{t+s}^{\eta}\right\}\right] = 0$$

Further use of (39) and the definition of marginal cost (42), this can be written as

$$G_t(j)\,E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\,Q_{t+s|t}Y_{t+s}P_{t+s}^{\varepsilon-1}\right]=\left(\frac{\varepsilon}{\varepsilon-1}\right)E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\,Q_{t+s|t}MC_t(j)\,Y_{t+s}P_{t+s}^{\varepsilon}\right]$$

In steady state, there are no shocks, technology and real output are constant, prices are constant and all firms make the same decisions:

$$\begin{aligned}
Y_{t+s}(j) &= Y^{ss}\\
A_{t+s}(j) &= A^{ss}\\
\Pi_t &= \Pi^{ss}=1\\
W_{t+s} &= W_{t+s}^{ss}=W^{ss}\\
G_t(j) &= G^{ss}=P^{ss}\\
MC_t(j) &= MC^{ss}\\
Q_{t+s|t} &= Q^{ss}=1
\end{aligned}$$

We normalise $P^{ss}=1$. Denoting lower-case letters as log-deviations from the steady-state ($x_t\equiv\ln(X_t)-\ln(X_t^*)$) then (44) can be written as:

$$Y^{ss}e^{g_t(j)}E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\,e^{q_{t+s}+y_{t+s}+(\varepsilon-1)p_{t+s}}\right]=\left(\frac{\varepsilon}{\varepsilon-1}\right)MC^{ss}Y^{ss}E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\,e^{q_{t+s}+mc_t(j)+y_{t+s}+\varepsilon p_{t+s}}\right]$$

In steady state, we therefore have that

$$1=\left(\frac{\varepsilon}{\varepsilon-1}\right)MC^{ss}$$

A first-order approximation of (44) can then be written as:

$$E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\left(g_t(j)+q_{t+s}+y_{t+s}+(\varepsilon-1)\,p_{t+s}\right)\right]=E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\left(q_{t+s}+mc_t(j)+y_{t+s}+\varepsilon p_{t+s}\right)\right]$$

Which rearranges to the standard:

$$g_t(j)=(1-\beta\theta)\,E_t(j)\left[\sum_{s=0}^{\infty}(\beta\theta)^s\left(p_{t+s}+mc_{t+s}(j)\right)\right]$$

The firm's linearised marginal cost is given by:

$$mc_t(j)=w_t-p_t+\eta y_t-\varepsilon\eta\left(p_t(j)-p_t\right)-(1+\eta)\,a_t(j)$$

If we had linear production technology ($\alpha=0$) then we would have that $\eta=0$ and that would be the end of it. However, because we want to speak of decreasing marginal productivity, we also need to linearise the firms' demand function (39):

$$y_t(j)=y_t-\varepsilon\left(p_t(j)-p_t\right)$$

So that

$$mc_t(j) = w_t - p_t + \eta y_t - \varepsilon\eta (p_t(j) - p_t) - (1+\eta) a_t(j)$$

Subsituting this in and collecting terms in $g_t(j)$ then gives:

$$g_t(j) = (1-\beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s E_t(j) \left[p_{t+s} + \frac{1}{1+\varepsilon\eta} \left((w_{t+s} - p_{t+s}) + \eta y_{t+s} - (1+\eta) a_{t+s}(j) \right) \right] \tag{B.2}$$

To obtain the aggregate production function, substitute the firm's production function (41) into the labour market clearing condition (45) to obtain

$$H_t = \int \left(\frac{Y_t(j)}{A_t(j)} \right)^{1+\eta} dj$$

Further substituting in the firm's demand function (39) gives

$$\begin{aligned} H_t &= \int \left(\frac{\left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} Y_t}{A_t(j)} \right)^{1+\eta} dj \\ &= Y_t^{1+\eta} \underbrace{\int A_t(j)^{-(1+\eta)} \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon(1+\eta)} dj}_{\equiv Z_t^{1+\eta}} \end{aligned}$$

Rearranging (recall that $1+\eta = \frac{1}{1-\alpha}$), we arrive at

$$Y_t = Z_t H_t^{1-\alpha}$$

Substituting the market-clearing requirements into the household's labour supply and Euler equation gives:

$$\begin{aligned} \frac{W_t}{P_t} &= e^{\epsilon_{Ht} - \epsilon_{Ct}} Y_t^{\frac{1}{\sigma} + \frac{1+\eta}{\psi}} Z_t^{\frac{1+\eta}{\psi}} \\ e^{\epsilon_{Ct}} Y_t^{-\frac{1}{\sigma}} &= \beta (1+i_t) E_t^{HH} \left[e^{\epsilon_{Ct+1}} Y_{t+1}^{-\frac{1}{\sigma}} \frac{1}{\Pi_{t+1}} \right] \end{aligned}$$

Linearising these gives:

$$w_t - p_t = \left(\frac{1}{\sigma} + \frac{1+\eta}{\psi} \right) y_t + \frac{1+\eta}{\psi} z_t + \epsilon_{Ht} - \epsilon_{Ct} \tag{B.3}$$

$$y_t = E_t^{HH}[y_{t+1}] - \sigma E_t^{HH}[i_t - \pi_{t+1}] + \sigma \left(\epsilon_{Ct} - E_t^{HH}[\epsilon_{Ct+1}] \right) \tag{B.4}$$

While the aggregate TFP linearises as:

$$(1+\eta) z_t = \int -(1+\eta) a_t(j) - \varepsilon(1+\eta)(p_t(j) - p_t)\, dj$$

But since $p_t = \int p_t(j)\, dj$ in a linear approximation, this is just

$$z_t = -\int a_t(j)\, dj = -\epsilon_{At}$$

so that the equilibrium real wage in period t is given by:

$$w_t - p_t = \left(\frac{1}{\sigma} + \frac{1+\eta}{\psi}\right) y_t + \epsilon_{Ht} - \epsilon_{Ct} - \frac{1+\eta}{\psi}\epsilon_{At} \tag{B.5}$$

For reference, recall that σ is the elasticity of intertemporal substitution, ψ is the Frisch elasticity of labour supply and η is the elasticity of marginal cost.[22] Next, we can substitute the equilibrium real wage (B.5) into the firms' reset price (B.2) to give:

$$g_t(j) = (1-\beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s E_t(j) \left[p_{t+s} + \upsilon y_{t+s} + \omega_{t+s}(j)\right]$$

where $\upsilon = \left(\frac{1}{\sigma} + \frac{1+\eta}{\psi} + \eta\right)$, which is equation (25) in the main text, with $\omega_t(j)$ given by

$$\begin{aligned}
\omega_t(j) &= \lambda_t + Q v_t(j) \\
\lambda_t &= \frac{1}{1+\varepsilon\eta}\left(\epsilon_{Ht+s} - \epsilon_{Ct+s} - \left(\frac{1+\eta}{\psi} + 1 + \eta\right)\epsilon_{At+s}\right) \\
Q &= -\left(\frac{1+\eta}{1+\varepsilon\eta}\right)
\end{aligned}$$

B.2 Deriving the Phillips Curve

Given that firms observe $\omega_t(j)$ each period and the idiosyncratic shocks are entirely transitory, we can write:

$$g_t(j) = (1-\beta\theta)\left\{E_t(j)\left[p_t + \upsilon y_t\right] + \omega_t(j)\right\} + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s E_t(j)\left[p_{t+s} + \overline{mc}_{t+s}\right] \tag{B.6}$$

where $\overline{mc}_{t+s} = \upsilon y_{t+s} + \lambda_{t+s}$ represents the marginal cost that any generic firm would experience if they produced the average quantity of goods *and* had the average level of TFP. Take the (simple) average of this expression:

$$\overline{g}_t = (1-\beta\theta)\left\{\overline{E}_t^{(1)}\left[p_t + \upsilon y_t\right] + \lambda_t\right\} + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s \overline{E}_t^{(1)}\left[p_{t+s} + \overline{mc}_{t+s}\right] \tag{B.7}$$

[22] As an aside, if we instead supposed that wages were negotiated prior to prices being set or markets clearing, then we would be best to rearrange this to describe aggregate demand in period t as a function of the wage, the aggregate price level and period-t shocks:

$$y_t = \left(\frac{\sigma\psi}{\psi + \sigma(1+\eta)}\right)\left(w_t - p_t + \frac{1+\eta}{\psi}\epsilon_{At} + \epsilon_{Ct} - \epsilon_{Ht}\right)$$

Next, denote by g_t^* the average reset price that would prevail if all firms had perfect foresight:

$$g_t^* \equiv (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s\left(p_{t+s}+\overline{mc}_{t+s}\right) \tag{B.8}$$

It is important to recognise that $\overline{g}_t \neq \overline{E}_t^{(1)}\left[g_t^*\right]$ in general. Indeed, we can combine the previous two equations to express $\overline{g}_t$ as a function of g_t^*:

$$\overline{g}_t = \overline{E}_t^{(1)}\left[g_t^*\right] + (1-\beta\theta)\left(\lambda_t - \overline{E}_t^{(1)}\left[\lambda_t\right]\right) \tag{B.9}$$

Alternatively, we might reverse the process and write g_t^* as a function of $\overline{g}_t$:

$$g_t^* = \overline{g}_t + (1-\beta\theta)\left\{\left(p_t - \overline{E}_t^{(1)}\left[p_t\right]\right) + \upsilon\left(y_t - \overline{E}_t^{(1)}\left[y_t\right]\right)\right\} + (\beta\theta)\left(g_{t+1}^* - \overline{E}_t^{(1)}\left[g_{t+1}^*\right]\right) \tag{B.10}$$

Nimark (2008) inadvertantly made the mistake of replacing $\overline{E}_t^{(k+1)}\left[g_{t+1}^*\right]$ with $\overline{E}_t^{(k+1)}\left[\overline{g}_{t+1}\right]$ (which can, in turn, be replaced by $\overline{E}_t^{(k+1)}\left[p_t + \frac{1}{1+\theta}\pi_{t+1}\right]$). In essence, this is to pre-suppose that in period t, agents expect on average that period $t+1$ expectations will be accurate.

To see this, consider agent j's period-t expectation regarding $y_{t+1} - \overline{E}_{t+1}^{(1)}\left[y_{t+1}\right]$. If, for some state vector X_t, we have that $y_t = a'X_t$ and $\overline{E}_t^{(1)}\left[X_t\right] = T_s X_t$ and $E_t\left(j\right)\left[X_{t+1}\right] = FE_t\left(j\right)\left[X_t\right]$ then:

$$\begin{aligned}
E_t\left(j\right)\left[y_{t+1} - \overline{E}_{t+1}^{(1)}\left[y_{t+1}\right]\right] &= E_t\left(j\right)\left[a'X_{t+1} - \overline{E}_{t+1}^{(1)}\left[a'X_{t+1}\right]\right] \\
&= a'E_t\left(j\right)\left[X_{t+1} - T_s X_{t+1}\right] \\
&= a'\left(I - T_s\right)FE_t\left(j\right)\left[X_t\right]
\end{aligned}$$

In the setting of Nimark (2008), $T_s = \begin{bmatrix}0 & I\end{bmatrix}$ (in the current paper it is $T_s = \begin{bmatrix}0 & I & 0 & 0\end{bmatrix}$), so that $I - T_s \neq 0$.

Instead, we proceed by first converting equation (B.8) into a sum of future marginal costs and inflation:

$$g_t^* \equiv p_t + \sum_{s=1}^{\infty}(\beta\theta)^s\pi_{t+s} + (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s\overline{mc}_{t+s}$$

Substituting this into (B.9) gives:

$$\overline{g}_t = (1-\beta\theta)\left(\lambda_t - \overline{E}_t^{(1)}\left[\lambda_t\right]\right) + \sum_{s=1}^{\infty}(\beta\theta)^s\overline{E}_t^{(1)}\left[\pi_{t+s}\right] + (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s\overline{E}_t^{(1)}\left[\overline{mc}_{t+s}\right] + \overline{E}_t^{(1)}\left[p_t\right]$$

Combining this with the evolution of the aggregate price level (B.1) then gives:

$$\begin{aligned}
p_t &= \theta p_{t-1} + (1-\theta)(1-\beta\theta)\left(\lambda_t - \overline{E}_t^{(1)}\left[\lambda_t\right]\right) \\
&\quad + (1-\theta)\sum_{s=1}^{\infty}(\beta\theta)^s\overline{E}_t^{(1)}\left[\pi_{t+s}\right] + (1-\theta)(1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s\overline{E}_t^{(1)}\left[\overline{mc}_{t+s}\right] + (1-\theta)\overline{E}_t^{(1)}\left[p_t\right]
\end{aligned}$$

Repeated substituting this back into itself then gives:

$$p_t = \sum_{k=0}^{\infty} (1-\theta)^k \overline{E}_t^{(k)} \left[\begin{array}{c} \theta p_{t-1} + (1-\theta)(1-\beta\theta)\left(\lambda_t - \overline{E}_t^{(1)}[\lambda_t]\right) \\ +(1-\theta)\sum_{s=1}^{\infty}(\beta\theta)^s \overline{E}_t^{(1)}[\pi_{t+s}] + (1-\theta)(1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s \overline{E}_t^{(1)}[\overline{mc}_{t+s}] \end{array} \right]$$

Subtracting p_{t-1} from both sides and rearranging then gives

$$\begin{aligned}\pi_t &= \left(\theta \sum_{k=0}^{\infty}(1-\theta)^k \overline{E}_t^{(k)}[p_{t-1}]\right) - p_{t-1} \\ &+ (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k \left(\upsilon \overline{E}_t^{(k+1)}[y_t] + \overline{E}_t^{(k)}[\lambda_t]\right) \\ &+ \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}(\beta\theta)^s (1-\theta)^k \overline{E}_t^{(k)}[\pi_{t+s} + (1-\beta\theta)\overline{mc}_{t+s}]\end{aligned}$$

which is the New Keysian Phillips Curve under Imperfect Common Knowledge (equation (26) in the main text).

B.3 Firms' learning and the evolution of X_t

Combining equation (B.6) and the definition of g_t^* (B.8) gives

$$g_t(j) = (1-\beta\theta)\{E_t(j)[p_t + \upsilon y_t] + \omega_t(j)\} + \beta\theta E_t(j)\left[g_{t+1}^*\right]$$

But since g_{t+1}^* can be written as

$$g_{t+1}^* \equiv p_t + \sum_{s=1}^{\infty}(\beta\theta)^{s-1}\pi_{t+s} + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s-1}\overline{mc}_{t+s}$$

we combine these two with the evolution of the aggregate price level to get

$$\begin{aligned}g_t(j) &= \theta p_{t-1} + (1-\beta\theta)\omega_t(j) + (1-\beta\theta)\upsilon E_t(j)[y_t] + (1-\theta)E_t(j)[\overline{g}_t] \\ &+ E_t(j)\left[\sum_{s=1}^{\infty}(\beta\theta)^s \pi_{t+s} + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s \overline{mc}_{t+s}\right]\end{aligned}$$

Using the conjectured solution (50) - (51), this becomes

$$\begin{aligned}g_t(j) &= \theta p_{t-1} + (1-\beta\theta)\omega_t(j) + (1-\beta\theta)\upsilon \mathbf{a}' E_t(j)[X_t] + (1-\theta)E_t(j)[\overline{g}_t] \\ &+ E_t(j)\left[\sum_{s=1}^{\infty}(\beta\theta)^s \mathbf{c}' X_{t+s} + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s \left(\upsilon \mathbf{a}' X_t + BSX_{t+s}\right)\right]\end{aligned}$$

where I have made use of the fact that $E_t(j)[\widetilde{v}_t] = E_t(j)[\widehat{v}_t] = 0$. Using the conjectured law of motion for the state, we obtain

$$
\begin{aligned}
g_t(j) &= \theta p_{t-1} + (1-\beta\theta)\,\omega_t(j) + (1-\beta\theta)\,\upsilon\mathbf{a}' E_t(j)[X_t] + (1-\theta)\,E_t(j)[\overline{g}_t] \\
&\quad + E_t(j)\left[\mathbf{c}'\sum_{s=1}^{\infty}(\beta\theta)^s F^s X_t + (1-\beta\theta)\left(\upsilon\mathbf{a}' + BS\right)\sum_{s=1}^{\infty}(\beta\theta)^s F^s X_t\right]
\end{aligned}
$$

Collecting terms, we get

$$
\begin{aligned}
g_t(j) &= \theta p_{t-1} + (1-\beta\theta)\,\omega_t(j) \\
&\quad + \underbrace{\left\{(1-\beta\theta)\,\upsilon\mathbf{a}' + \left(\mathbf{c}' + (1-\beta\theta)\left(\upsilon\mathbf{a}' + BS\right)\right)\beta\theta F\,(I-\beta\theta F)^{-1}\right\}}_{\mathbf{f}'} E_t(j)[X_t] \\
&\quad + (1-\theta)\,E_t(j)[\overline{g}_t]
\end{aligned}
\tag{B.11}
$$

Taking the simple average gives

$$
\overline{g}_t = \theta p_{t-1} + (1-\beta\theta)\,BSX_t + \mathbf{f}'\overline{E}_t^{(1)}[X_t] + (1-\theta)\,E_t^{(1)}[\overline{g}_t]
$$

where I have used the law of large numbers to write $\int \omega_t(j)\,dj = \lambda_t + Q\int \upsilon_t(j)\,dj = B\mathbf{x}_t = BSX_t$, where S is a matrix that selects $\mathbf{x}_t$ from X_t. Repeatedly substituting this equation back into itself gives

$$
\begin{aligned}
\overline{g}_t &= \theta p_{t-1} + (1-\beta\theta)\,BSX_t + \mathbf{f}'\overline{E}_t^{(1)}[X_t] \\
&\quad + (1-\theta)\,E_t^{(1)}\left[\theta p_{t-1} + (1-\beta\theta)\,BSX_t + \mathbf{f}'\overline{E}_t^{(1)}[X_t]\right] \\
&\quad + (1-\theta)^2\,E_t^{(2)}\left[\theta p_{t-1} + (1-\beta\theta)\,BSX_t + \mathbf{f}'\overline{E}_t^{(1)}[X_t]\right] \\
&\quad + \cdots
\end{aligned}
$$

In the limit, this becomes

$$
\begin{aligned}
\overline{g}_t &= \frac{\theta}{1-(1-\theta)}p_{t-1} \\
&\quad + (1-\beta\theta)\,BSX_t \\
&\quad + \left(\mathbf{f}' + (1-\theta)(1-\beta\theta)\,BS\right)\overline{E}_t^{(1)}[X_t] \\
&\quad + (1-\theta)\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)\,BS\right)\overline{E}_t^{(2)}[X_t] \\
&\quad + (1-\theta)^2\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)\,BS\right)\overline{E}_t^{(3)}[X_t] \\
&\quad + \cdots
\end{aligned}
$$

Substituting this back into (B.11) gives

$$
g_t(j) = \theta p_{t-1} + (1-\beta\theta)\,\omega_t(j) + \mathbf{f}'E_t(j)[X_t] + (1-\theta)\,E_t(j)\left[\begin{array}{c}
\frac{\theta}{1-(1-\theta)}p_{t-1} \\
+(1-\beta\theta)\,BSX_t \\
+\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)\,BS\right)\overline{E}_t^{(1)}[X_t] \\
+(1-\theta)\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)\,BS\right)\overline{E}_t^{(2)}[X_t] \\
+(1-\theta)^2\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)\,BS\right)\overline{E}_t^{(3)}[X_t] \\
+\cdots
\end{array}\right]
$$

(B.12)

Expanding $\omega_t(j)$ and rearranging, this gives

$$g_t(j) = p_{t-1} + (1-\beta\theta)(BSX_t + Qv_t(j)) + E_t(j)\left[\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)BS\right)\sum_{k=0}^{\infty}(1-\theta)^k \overline{E}_t^{(k)}[X_t]\right] \tag{B.13}$$

and, eventually,

$$g_t(j) = p_{t-1} + \underbrace{(1-\beta\theta)BS}_{\gamma_2'}X_t + \underbrace{\left(\mathbf{f}' + (1-\theta)(1-\beta\theta)BS\right)(1-(1-\theta)T_s)^{-1}}_{\gamma_3'}E_t(j)[X_t] + \underbrace{(1-\beta\theta)Q}_{\gamma_4'}v_t(j) \tag{B.14}$$

which is in the generalised format detailed in section (3.6) of the main text. Proposition 1 therefore applies and firms' hierarchy of expectations will evolve according to:

$$X_t = FX_{t-1} + G_1\mathbf{u}_t + G_2\widetilde{v}_t + G_3\widetilde{v}_{t-1} + G_4\widehat{v}_{t-1}$$

As an aside, we can take the average of (B.14) and plug it into the linearised expression for the evolution of the price level to obtain an alternative expression for inflation as a function of the current state:

$$\begin{aligned}
p_t &= \theta p_{t-1} + (1-\theta)\overline{g}_t \\
&= \theta p_{t-1} + (1-\theta)\left(p_{t-1} + \gamma_2' X_t + \gamma_3'\overline{E}_t[X_t]\right) \\
&= p_{t-1} + \underbrace{(1-\theta)\left(\gamma_2' + \gamma_3' T_s\right)X_t}_{\pi_t}
\end{aligned}$$

B.4 Solving the model

At this point, we have that the economy is described by the Phillips curve (26) with perfect knowledge of previous-period prices:

$$\begin{aligned}
\pi_t &= (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k\left(\upsilon\overline{E}_t^{(k+1)}[y_t] + \overline{E}_t^{(k)}[\lambda_t]\right) \\
&+ \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}(\beta\theta)^s(1-\theta)^k\,\overline{E}_t^{(k)}\left[\pi_{t+s} + (1-\beta\theta)\overline{mc}_{t+s}\right]
\end{aligned}$$

and, making use of the fact that both have complete information, the household's Eular equation (B.4) and the central bank's policy function (48):

$$\begin{aligned}
y_t &= E_t[y_{t+1}] - \sigma E_t[i_t - \pi_{t+1}] + \sigma(\epsilon_{Ct} - E_t[\epsilon_{Ct+1}]) \\
i_t &= \kappa_y y_t + \kappa_{\pi 0}\pi_t + \kappa_{\pi 1}E_t[\pi_{t+1}] + \epsilon_{Mt}
\end{aligned}$$

In addition, we have that endogenous variables are linear functions of the state ((50) - (51)):

$$y_t = \mathbf{a}' X_t + \mathbf{b}' \begin{bmatrix} \widetilde{v}_t \\ \widehat{v}_t \end{bmatrix} \tag{B.15}$$

$$\pi_t = \mathbf{c}' X_t \tag{B.16}$$

and that the state follows an ARMA(1,1) process (33):

$$X_t = F X_{t-1} + G_1 \mathbf{u}_t + G_2 \widetilde{v}_t + G_3 \widetilde{v}_{t-1} + G_4 \widehat{v}_{t-1}$$

with $\widetilde{v}_t$ and $\widehat{v}_t$ defined in equation (34).

To obtain the necessary conditions on the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, start by substituting the central bank policy into the household Eular equation

$$y_t = E_t [y_{t+1}] - \sigma \left\{ (\kappa_y y_t + \kappa_{\pi 0} \pi_t + (\kappa_{\pi 1} - 1) E_t [\pi_{t+1}]) \right\} + \sigma (\epsilon_{Ct} - E_t [\epsilon_{Ct+1}] - \epsilon_{Mt})$$

Substituting in the conjectured solutions for y_t and π_t on the right hand side and gathering the underlying shocks together into $\mathbf{x}_t$ then gives

$$\begin{aligned} y_t = {} & \mathbf{a}' E_t [X_{t+1}] - \sigma \left\{ \left(\kappa_y \left(\mathbf{a}' X_t + \mathbf{b}' \begin{bmatrix} \widetilde{v}_t \\ \widehat{v}_t \end{bmatrix} \right) + \kappa_{\pi_0} \mathbf{c}' X_t + (\kappa_{\pi_1} - 1) \mathbf{c}' E_t [X_{t+1}] \right) \right\} \\ & + \sigma \left(\begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} A \right) \mathbf{x}_t \end{aligned}$$

Finally, making use of the definition of X_t and its law of motion, we have

$$\begin{aligned} y_t = {} & \mathbf{a}' (F X_t + G_3 \widetilde{v}_t + G_4 \widehat{v}_t) \\ & - \sigma \left\{ \left(\kappa_y \left(\mathbf{a}' X_t + \mathbf{b}' \begin{bmatrix} \widetilde{v}_t \\ \widehat{v}_t \end{bmatrix} \right) + \kappa_{\pi_0} \mathbf{c}' X_t + (\kappa_{\pi_1} - 1) \mathbf{c}' (F X_t + G_3 \widetilde{v}_t + G_4 \widehat{v}_t) \right) \right\} \\ & + \mathbf{d}' X_t \end{aligned}$$

where

$$\mathbf{d}' = \sigma \left(\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} & 0_{1 \times \infty} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} & 0_{1 \times \infty} \end{bmatrix} F \right)$$

This then simplifies to

$$\begin{aligned} y_t = {} & \left\{ \mathbf{a}' F - \sigma \left(\kappa_y \mathbf{a}' + \kappa_{\pi_0} \mathbf{c}' + (\kappa_{\pi 1} - 1) \mathbf{c}' F \right) + \mathbf{d}' \right\} X_t \\ & + \left\{ \mathbf{a}' \begin{bmatrix} G_3 & G_4 \end{bmatrix} - \sigma \kappa_y \mathbf{b}' \right\} \begin{bmatrix} \widetilde{v}_t \\ \widehat{v}_t \end{bmatrix} \end{aligned}$$

from which we can immediately read off that $\mathbf{a}$ and $\mathbf{b}$ must satisfy:

$$\begin{aligned} \mathbf{a}' &= \mathbf{a}' F - \sigma \left(\kappa_y \mathbf{a}' + \kappa_{\pi_0} \mathbf{c}' + (\kappa_{\pi 1} - 1) \mathbf{c}' F \right) + \mathbf{d}' \\ \mathbf{b}' &= \mathbf{a}' \begin{bmatrix} G_3 & G_4 \end{bmatrix} - \sigma \kappa_y \mathbf{b}' \end{aligned}$$

Repeating the process with the Phillips Curve, we substitute in the conjectured solutions for y_t and π_t on the right hand side

$$\pi_t = (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k\left(\upsilon\overline{E}_t^{(k+1)}\left[\mathbf{a}'X_t + \mathbf{b}'\begin{bmatrix}\widetilde{\upsilon}_t\\ \widehat{\upsilon}_t\end{bmatrix}\right] + \overline{E}_t^{(k)}[BSX_t]\right)$$
$$+\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}(\beta\theta)^s(1-\theta)^k\overline{E}_t^{(k)}\left[\mathbf{c}'X_{t+s} + (1-\beta\theta)\left(\upsilon\left(\mathbf{a}'X_{t+s} + \mathbf{b}'\begin{bmatrix}\widetilde{\upsilon}_{t+s}\\ \widehat{\upsilon}_{t+s}\end{bmatrix}\right) + BSX_{t+s}\right)\right]$$

where the matrix S selects $\mathbf{x}_t$ from X_t. Crucially, note that the expectations within the Phillips Curve are the average of those formed by firms. As such, we have that $E_t(j)[\widetilde{\upsilon}_{t+s}] = E_t(j)[\widehat{\upsilon}_{t+s}] = 0\ \forall j$ and $\forall s \geq 0$,[23] so that

$$\pi_t = (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1-\theta)^k\left(\upsilon\mathbf{a}'\overline{E}_t^{(k+1)}[X_t] + BS\overline{E}_t^{(k)}[X_t]\right)$$
$$+\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}(\beta\theta)^s(1-\theta)^k\left\{\mathbf{c}'F^s + (1-\beta\theta)\left(\upsilon\mathbf{a}'F^s + BSF^s\right)\right\}\overline{E}_t^{(k)}[X_t]$$

Next, using T_s as the matrix that selects $\overline{E}_t^{(1)}[X_t]$ from X_t, this becomes

$$\pi_t = (1-\theta)(1-\beta\theta)\left(\sum_{k=0}^{\infty}(1-\theta)^k\left(\upsilon\mathbf{a}'T_s^{k+1} + BST_s^k\right)\right)X_t$$
$$+\left(\mathbf{c} + (1-\beta\theta)\left(\upsilon\mathbf{a}' + BS\right)\right)\beta\theta F(I-\beta\theta F)^{-1}\left(\sum_{k=1}^{\infty}(1-\theta)^k T_s^k\right)X_t$$

Further simplification and equalisation of coefficients then eventually yields the following constraint

$$\mathbf{c}' = (1-\theta)\left\{\begin{array}{c}(1-\beta\theta)(\upsilon\mathbf{a}'T_s + BS)\\ +\beta\theta(\mathbf{c}' + (1-\beta\theta)(\upsilon\mathbf{a}' + BS))F(I-\beta\theta F)^{-1}T_s\end{array}\right\}(I-(1-\theta)T_s)^{-1}$$

[23]Recall that $E_t(j)[\widetilde{\upsilon}_t] = E_t(j)[\widehat{\upsilon}_t] = 0$ because, despite having $E_t(j)[\upsilon_t(j)] \neq 0$, in the limit each firm receives a weight of zero ($\lim_{N\to\infty}\phi_N(j) = 0$) and they have no information about other firms' shocks.