Appendix to: “Monetary Policy Rules and the Exchange Rate”

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(details of the model for the referees)

In this appendix we describe the model in details.

Preferences

The world economy is populated by a continuum of agents on the interval $[0,1]$. The population on the segment $[0,n)$ belongs to the country H, while the segment $[n,1]$ belongs to F. A generic agent j belonging to the world economy is both producer and consumer: a producer of a single differentiated product and a consumer of all the goods produced in both countries H and F. All goods are traded. Preferences of the generic household j are given by

$$U_t^j = \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left[U(C_s^j) + L\left(\frac{M_s^j}{P_s^i}, \xi^i\right) - V(y_s^j, z_s^i) \right],$$

where the upper index j denotes a variable that is specific to agent j, while the upper index i denotes a variable that is specific to country i. We have that $i = H$ if $j \in [0,n)$, while $i = F$ if $j \in [n,1]$. E_t denotes the expectation conditional on the information set at date t, while β is the intertemporal discount factor, with $0 < \beta < 1$.

Agents obtain utility from consumption and from the liquidity services of holding money, while they receive disutility from producing goods.[1] We have that U is an increasing concave function of the index C^j defined as

$$C^j \equiv \frac{(C_H^j)^n (C_F^j)^{1-n}}{n^n (1-n)^{1-n}} \tag{1}$$

and C_H^j and C_F^j are indexes of consumption across the continuum of differentiated goods produced respectively in country H and F. Specifically,

$$C_H^j \equiv \left[\left(\frac{1}{n}\right)^{\frac{1}{\sigma}} \int_o^n c^j(h)^{\frac{\sigma-1}{\sigma}} dh \right]^{\frac{\sigma}{\sigma-1}}, \qquad C_F^j \equiv \left[\left(\frac{1}{1-n}\right)^{\frac{1}{\sigma}} \int_n^1 c^j(f)^{\frac{\sigma-1}{\sigma}} df \right]^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $\sigma > 1$ is the elasticity of substitution across goods produced within a country; the elasticity of substitution between the bundles C_H and C_F is 1. [2]The parameter n denotes both the population size and the 'economic' size of country H, where the 'economic size' is the share of the bundle of goods produced within that country in the consumption index.

L is an increasing concave function of the real money balances, while ξ^i is a country-specific shock to the liquidity preference; we will interpret it as an

[1] We have assumed that the utility function is separable in these three factors.

[2] This model can be easily generalized to the case in which the elasticity of intratemporal substitution is different from 1.

exogenous disturbance to money demand. Agents derive utility from the real money balances, $\frac{M_t^j}{P^i}$ where P^i is defined as[3]

$$P^i \equiv (P_H^i)^n (P_F^i)^{1-n},$$

$$P_H^i \equiv \left[\left(\frac{1}{n}\right) \int_o^n p^i(h)^{1-\sigma} dh \right]^{\frac{1}{1-\sigma}}, \qquad P_F^i \equiv \left[\left(\frac{1}{1-n}\right) \int_n^1 p^i(f)^{1-\sigma} df \right]^{\frac{1}{1-\sigma}},$$

where $p^i(h)$ is the price of good h sold in country i. We assume that prices are set in the producer currency and that the law of one price holds: $p^H(h)/S = p^F(h)$ and $p^H(f) = S \cdot p^F(f)$, where S is the nominal exchange rate (the price of F currency in terms of H currency). Given these assumptions and the structure of the preferences, purchasing power parity holds, i.e. $P^H = SP^F$.

We define the terms of trade T of country F as the ratio of the price of the bundle of goods produced in country F relative to the price of the bundle imported from country H. We have then $T \equiv P_F^F / P_H^F = P_F^H / P_H^H$.

Finally V is an increasing convex function of agent j's supply of its product y^j. Assuming that agents have disutility of working $g(N^j)$, where N^j is the number of hours worked by agent j, and that the production function is $y^j = f(N^j)$, we can interpret $V(y^j)$ as being equal to $g(f^{-1}(y^j))$. z^i is a country-specific stochastic disturbances and can be interpreted as a productivity shock.

For a given C^j, household j chooses C_H^j and C_F^j by minimizing the total expenditure $P^i C^j$ under the constraint given by (1). Similarly, for given C_H^j and C_F^j, household j allocates the expenditure among the differentiated goods by minimizing $P_H^i C_H^j$ and $P_F^i C_F^j$ under the constraints given by (2). The demands of the generic good h, produced in country H, and of the generic good f, produced in country F are

$$c^j(h) = \left(\frac{p(h)}{P_H}\right)^{-\sigma} T^{1-n} C^j, \qquad c^j(f) = \left(\frac{p(f)}{P_F}\right)^{-\sigma} T^{-n} C^j, \tag{3}$$

where we have suppressed the upper-script in the relative prices' argument given that relative prices are the same independently of the currency in which they are expressed.

We assume that each fiscal authority allocates a level of public expenditure only among domestics goods. The public expenditure production functions are given by

$$G^H = \left[\frac{1}{n} \int_0^n g(h)^{\frac{\sigma-1}{\sigma}} dh \right]^{\frac{\sigma}{\sigma-1}}, \qquad G^F = \left[\frac{1}{1-n} \int_n^1 g(f)^{\frac{\sigma-1}{\sigma}} df \right]^{\frac{\sigma}{\sigma-1}};$$

and they imply the following demands for the generic goods h and f

$$g(h) = \left(\frac{p(h)}{P_H}\right)^{-\sigma} G^H, \qquad g(f) = \left(\frac{p(f)}{P_F}\right)^{-\sigma} G^F. \tag{4}$$

[3]The price index P^i is properly defined as the minimum expenditure in country i required to purchase goods resulting in the consumption index of C^j, such that $C^j = 1$. Similar definitions are given for P_H^i and P_F^i.

Combining (3) with (4) we can write total demand of good h and f as

$$y^d(h) = \left(\frac{p(h)}{P_H}\right)^{-\sigma} [T^{1-n}C^W + G^H], \qquad y^d(f) = \left(\frac{p(f)}{P_F}\right)^{-\sigma} [T^{-n}C^W + G^F] \tag{5}$$

where world consumption C^W is defined as

$$C^W \equiv \int_0^1 C^j dj.$$

Assets Market

In each period t the economy faces one of the finitely many events $\theta_t \in \Upsilon$ (where Υ is the set of finitely many states). We denote by κ_t the history of events up through and including period t. Looking ahead from period t the conditional probability of occurrence of state θ_{t+1} is $\mu(\theta_{t+1} \mid \kappa_t)$. The initial realization θ_{-1} is given.

There are complete markets in this economy both at the domestic and international levels. We represent the asset structure by having complete contingent one-period nominal bonds denominated in the Home currency (see Chari et al. 1998). We let $B^j_{H,t}(\theta_{t+1})$ denote the consumer's holdings at time t of this bond, which pays one unit of the Home currency if state θ_{t+1} occurs (at time $t+1$) and 0 otherwise, and we let $q_t(\theta_{t+1} \mid \kappa_t)$ denote the price of one unit of such a bond at date t and state θ_t in units of the Home currency. The Home consumer maximizes utility subject to the sequence of budget constraints

$$\sum_{\theta^t} \frac{q_t(\theta_{t+1} \mid .) B^j_{H,t}(\theta_{t+1})}{P^H_t} + \frac{M^j_t}{P^H_t} \leq W^j_{t-1} + (1-\tau^i)\frac{p^H_t(j) y_t(j)}{P^H_t} - C^j_t + \frac{TR^{H,j}_t}{P^H_t},$$

with

$$W^j_{t-1} \equiv \frac{B^j_{H,t-1}}{P^H_t} + \frac{M^j_{t-1}}{P^H_t}.$$

For the Foreign consumer we have the following sequence of budget constraints

$$\frac{1}{S_t P^F_t}\left[\sum_{\theta^t} q_t(\theta_{t+1} \mid .) B^j_{H,t}(\theta_{t+1})\right] + \frac{M^j_t}{P^F_t} \leq W^j_{t-1} + (1-\tau^i)\frac{p^F_t(j) y_t(j)}{P^F_t} - C^j_t + \frac{TR^{F,j}_t}{P^F_t},$$

with

$$W^j_{t-1} \equiv \frac{B^j_{H,t-1}}{S_t P^F_t} + \frac{M^j_{t-1}}{P^F_t},$$

$TR^{i,j}_t$ are nominal lump sum transfers from the fiscal authority of country i in which j resides to the household j, while τ^i is a country-specific proportional tax on nominal income.

The budget constraint at date t of the fiscal authority of country i for $i = H$ or F is

$$\tau^i \int_{j\in i} p_t(j) y_t(j) dj = \int_{j\in i} M^j_t - \int_{j\in i} M^j_{t-1} + G^i_t + \int_{j\in i} TR^{i,j}_t,$$

We set the initial conditions $B^j_{H,-1} = 0 \; \forall j \in [0,1]$.

Consumer Optimization

Given the sequences of prices and incomes and the initial conditions, the allocation of consumption is completely characterized by the utility function and the resource constraint.[4]

Since households have identical preferences, the law of one price holds and markets are complete, the assumption that the initial wealth is identical among all the agents implies that there is perfect risk sharing of consumption within and across countries.

The Euler equation between any state at date t and the state θ_{t+1} at date $t+1$, for any state θ_{t+1} at time $t+1$ and for any agents $j \in [0,1]$, is

$$\frac{q_t(\theta_{t+1} \mid .)}{P_t^H} U_C(C_t^j) = \beta \mu(\theta_{t+1} \mid .) \frac{U_C(C_t^j(\theta_{t+1}))}{P_{t+1}^H(\theta_{t+1})}.$$

From here onward, we suppress the upper index j in describing the allocation of the general consumption index.

The risk-free nominal interest rate is the inverse of the price of a bond that delivers one unit of the currency in which is denominated independently of the state. We have that

$$\begin{aligned} \frac{1}{1+i_t^H} &= \sum_{\theta_{t+1}} q_t(\theta_{t+1} \mid .), \\ \frac{1}{1+i_t^F} &= \frac{\sum_{\theta_{t+1}} q_t(\theta_{t+1} \mid .) S_{t+1}(\theta_{t+1})}{S_t}, \end{aligned}$$

where i^i denote the risk-free nominal interest rate on a the bond denominated in currency i. We can then obtain the Euler Equation in terms of the risk-free nominal bonds

$$U_C(C_t) = (1+i_t^i)\beta \mathrm{E}_t \left\{ U_C(C_{t+1}) \frac{P_t^i}{P_{t+1}^i} \right\}. \tag{6}$$

for each country i. A no arbitrage implication of (6) is the uncovered interest parity

$$(1+i_t^H)\mathrm{E}_t \left[\frac{U_C(C_{t+1})}{U_C(C_t)} \frac{P_t^H}{P_{t+1}^H} \right] = (1+i_t^F)\mathrm{E}_t \left[\frac{U_C(C_{t+1})}{U_C(C_t)} \frac{P_t^F}{P_{t+1}^F} \right]. \tag{7}$$

Households' maximization problem is completed by the money demand equations

$$L_{M/P}\left(\frac{M_t^i}{P_t}, \xi^i \right) = \frac{i_t^i}{1+i_t^i} U_C(C_t) \tag{8}$$

$\forall t$ and for each $i = H$ or F.[5] From (8), the marginal rate of substitution between real money balances and consumption is equated to the user cost in

[4] The latter is derived by combining an appropriate borrowing limit with the budget constraint of the households.

[5] $L_{M/P}$ denotes the derivative of L with respect to the real money balance.

terms of the consumption good index of holding an extra unit of real money balances for one period.

Finally at the optimum the resource constraint holds with equality at each date t and in every history κ_t.

The demand side of the economy is completed by computing aggregate demands in both countries. Using the appropriate Dixit-Stiglitz aggregators related to (2) we define

$$Y^H \equiv \left[\left(\frac{1}{n}\right)\int_o^n y^d(h)^{\frac{\sigma-1}{\sigma}}dh\right]^{\frac{\sigma}{\sigma-1}}, \qquad Y^F \equiv \left[\left(\frac{1}{1-n}\right)\int_n^1 y^d(f)^{\frac{\sigma-1}{\sigma}}df\right]^{\frac{\sigma}{\sigma-1}}. \tag{9}$$

Applying (9) to (5), we obtain

$$Y^H = T^{1-n}C + G^H, \qquad Y^F = T^{-n}C + G^F. \tag{10}$$

While consumption is completely insured, aggregate production can vary between countries. From (10), it follows that changes in the terms of trade determine divergences in output across countries.

Interest Rate Rules

The model is closed by identifying the instrument of monetary policy. The central banks set their instruments in terms of the one-period risk free nominal interest rates on the nominal bond denominated in their currency. In this work, we consider feedback rules of the form

$$1 + i_t^H = \Phi^H(\Sigma_t), \tag{11}$$
$$1 + i_t^F = \Phi^F(\Sigma_t), \tag{12}$$

where Σ_t is the information set at time t and Φ^i are generic functions. An example of a rule of this kind is a feedback rule in which the nominal interest rate reacts to the domestic producer inflation as

$$1 + i_t^H = \Phi^H\left(\frac{P_{H,t}^H}{P_{H,t-1}^H}\right),$$
$$1 + i_t^F = \Phi^F\left(\frac{P_{F,t}^F}{P_{F,t-1}^F}\right).$$

Firms and Price Setting

Each producer of a single differentiated good acts in a monopolistic competitive market. The demand for the differentiated good, (5), is affected by the pricing decision, $p(j)$. [6]

The firms' price setting behavior is modelled through a Calvo-type contract. Under the Calvo pricing assumption, each firm has the opportunity to adjust its price at stochastic intervals. In each period a firm can set a new price with a

[6] Producers are small with respect to the overall market and they take as given the indexes P^i, P_H^i, P_F^i, T and C, with $i = H, F$.

fixed probability $1-\alpha$ which is the same for all firms and is independent from the amount of time elapsed since it last changed price. However, we allow this probability to be different between the two countries; then $1-\alpha^i$ denotes this probability with $i = H, F$. When a firm has an opportunity to set a new price at period t, it does so in order to maximize the expected discounted value of its net profits. The price setting decision at t determines the net profits at $t+k$ only in states of nature in which the producer does not change the price from $t+1$ to $t+k$ inclusive: this occurs with probability $(\alpha^i)^k$.

It is important to note that all the sellers that belong to the same country and that can modify their price at a certain time will face the same discounted future demands and future marginal costs under the assumption that the new price is maintained. Thus they will set the same price.

The objective for a firm is to maximize the expected discounted value of profits:

$$\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^i \beta)^k \left[\zeta^i_{t+k}(1-\tau^i)\widetilde{p}_t(j)\widetilde{y}_{t,t+k}(j) - V(\widetilde{y}_{t,t+k}(j), z^i_{t+k})\right], \qquad (13)$$

where we have denoted with $\widetilde{p}_t(j)$ the price of the good j chosen at date t and with $\widetilde{y}_{t,t+k}(j)$ the total demand of good j at time $t+k$ conditional on the fact that the price $\widetilde{p}_t(j)$ has not changed:[7]

$$\widetilde{y}_{t,t+k}(h) = \left(\frac{\widetilde{p}_t(h)}{P_{H,t+k}}\right)^{-\sigma} \left[T^{1-n}_{t+k} C_{t+k} + G^H_{t+k}\right], \qquad (14)$$

$$\widetilde{y}_{t,t+k}(f) = \left(\frac{\widetilde{p}_t(f)}{P_{F,t+k}}\right)^{-\sigma} \left[T^{-n}_{t+k} C_{t+k} + G^F_{t+k}\right]. \qquad (15)$$

Revenues are evaluated using the marginal utility of nominal income $\zeta^i_{t+k} = U_C(C_{t+k})/P^i_{t+k}$ which is the same for all consumers belonging to country i, because of the complete-market assumption.

The seller maximizes (13) with respect to $\widetilde{p}_t(j)$ taking as given the sequences $\{P^i_{H,t}, P^i_{F,t}, P^i_t, C_t, G^i_t\}$, the optimal choice of $\widetilde{p}_t(j)$ is

$$\widetilde{p}_t(j) = \frac{\sigma}{(\sigma-1)(1-\tau^i)} \frac{\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^i\beta)^k V_y(\widetilde{y}_{t,t+k}(j), z^i_{t+k})\widetilde{y}_{t,t+k}(j)}{\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^i\beta)^k \zeta^i_{t+k}\widetilde{y}_{t,t+k}(j)}. \qquad (16)$$

Calvo-price setting implies the following state equation for $P^H_{H,t}$ and $P^F_{F,t}$

$$(P^H_{H,t})^{1-\sigma} = \alpha^H (P^H_{H,t-1})^{1-\sigma} + (1-\alpha^H)\widetilde{p}_t(h)^{1-\sigma}, \qquad (17)$$

$$(P^F_{F,t})^{1-\sigma} = \alpha^F (P^F_{F,t-1})^{1-\sigma} + (1-\alpha^F)\widetilde{p}_t(f)^{1-\sigma} \qquad (18)$$

Equilibrium

Our model is not solvable in a closed-form solution. We focus on equilibria where the state variables follow paths that are close to a deterministic stationary equilibrium, in which the producer inflation rates and the exchange rate

[7] We have that $i = H$ if $j = h$ and $i = F$ if $j = f$.

depreciation are zero. In this steady state, we interpret the stochastic shocks $\{\xi_t^i, G_t^i, z_t^i\}$ for $i = H$ or F as zero at all dates.

In this steady state, the instrument of monetary policy, the one period nominal interest rate, is equal to the inverse of the intertemporal discount factor:

$$1 + i^H = 1 + i^F = \frac{1}{\beta}.$$

From the pricing decision in country H, we obtain[8]

$$(1-\tau^H) U_C(\overline{C}) = \frac{\sigma}{\sigma - 1} \overline{T}^{1-n} V_y \left(\overline{T}^{1-n} \overline{C}, 0 \right), \tag{19}$$

while in country F we have

$$(1-\tau^F) U_C(\overline{C}) = \frac{\sigma}{\sigma - 1} \overline{T}^{-n} V_y \left(\overline{T}^{-n} \overline{C}, 0 \right). \tag{20}$$

When $\tau^H = \tau^F$, it follows that $\overline{T} = 1$ and that $\overline{Y}^H = \overline{Y}^F = \overline{C}$.

Given the initial conditions $P_{H,-1}^H$, $P_{F,-1}^F$, S_{-1}, the sequences of stochastic shocks $\{\xi_t^i, G_t^i, z_t^i\}$ for $i = H,\ F$, the sticky price equilibrium is the sequence $\{C_t, P_{H,t}^H, \widetilde{p}_t(h), P_{F,t}^F, \widetilde{p}_t(f), T_t, S_t, i_t^H, i_t^F\}$ such that:

(i) interest rates follow the rules given by (11) and (12);

(ii) consumer optimization conditions, (6), (7), hold;

(iii) producer optimization condition, (16), hold;

(iv) aggregate demand is given by (10), and prices follow (17) and (18);

(v) home and foreign transversality conditions hold.

It is worth noting that with appropriate interest rate rules the money-market equilibrium determines only the level of money in each country. Moreover shocks to the liquidity preference affect only the path of money.

[8] As it is common in models with monopolistic competition, the marginal utility of consumption is not equated to the marginal disutility of producing output.

Log-Linear Model

Here we describe the stochastic equilibrium which arises from perturbations around the deterministic equilibrium identified above.

AD Block

From the log-linearization of (6), we obtain

$$E_t\widehat{C}_{t+1} - \widehat{C}_t = \rho^{-1}(\widehat{i}_t^H - E_t\pi_{t+1}) = \rho^{-1}(\widehat{i}_t^F - E_t\pi_{t+1}^*), \tag{21}$$

where $\widehat{C}$ is the consumption index, $\widehat{i}^H$ and $\widehat{i}^F$ are the nominal interest rates in the Home and Foreign countries, π and π^* are the respective CPI inflation rates (where $\pi_t = \ln P_t^H/P_{t-1}^H$ and $\pi_t^* = \ln P_t^F/P_{t-1}^F$); ρ is the inverse of the intertemporal elasticity of substitution in consumption. The log-linearized uncovered interest parity is obtained from (7)

$$\mathrm{E}_t\Delta S_{t+1} = \widehat{i}_t^H - \widehat{i}_t^F, \tag{22}$$

where $\Delta S = \ln S_t/S_{t-1}$ is the exchange rate depreciation. Noting that

$$\pi_t = \Delta S_t + \pi_t^* = n\pi_t^H + (1-n)(\Delta S_t + \pi_t^F)$$

where $\pi_t^H = \ln P_{H,t}^H/P_{H,t-1}^H$ and $\pi_t^F = \ln P_{F,t}^F/P_{F,t-1}^F$, we can rewrite (21) using (22) as

$$E_t\widehat{C}_{t+1} = \widehat{C}_t + \rho^{-1}n(\widehat{i}_t^H - E_t\pi_{t+1}^H) + \rho^{-1}(1-n)(\widehat{i}_t^F - E_t\pi_{t+1}^F), \tag{23}$$

where $\rho \equiv -U_{CC}(\overline{C})\overline{C}/U_C(\overline{C})$. The terms of trade identity can be written in a log-linear form as

$$\widehat{T}_t = \widehat{T}_{t-1} + \Delta S_t + \pi_t^F - \pi_t^H. \tag{24}$$

AS Block

The log-linearization of the aggregate supply side of the model implies the following equations:

$$\begin{aligned}\pi_t^H &= \lambda^H[(1-n)(1+\eta)(\widehat{T}_t - \widetilde{T}_t) + (\rho+\eta)(\widehat{C}_t - \widetilde{C}_t)] + \beta E_t\pi_{t+1}^H \\ &= (1-n)\,k_T^H(\widehat{T}_t - \widetilde{T}_t) + k_C^H(\widehat{C}_t - \widetilde{C}_t) + \beta E_t\pi_{t+1}^H,\end{aligned} \tag{25}$$

$$\begin{aligned}\pi_t^F &= \lambda^F[-n(1+\eta)(\widehat{T}_t - \widetilde{T}_t) + (\rho+\eta)(\widehat{C}_t - \widetilde{C}_t)] + \beta E_t\pi_{t+1}^F \\ &= -nk_T^F(\widehat{T}_t - \widetilde{T}_t) + k_C^F(\widehat{C}_t - \widetilde{C}_t) + \beta E_t\pi_{t+1}^F.\end{aligned} \tag{26}$$

where $\widetilde{T}_t$ and $\widetilde{C}_t$ are the flexible-price allocation; $\eta \equiv V_{yy}(\overline{C},0)\overline{C}/V_y(\overline{C},0)$ and $\lambda^i \equiv [(1-\alpha^i\beta)(1-\alpha^i)/\alpha^i]\cdot[1/(1+\sigma\eta)]$ for $i = H$ or F.

Equations (21), (22), (24), (25) and (26) and correspond to equations (1), (4), (5), (6) and (7) in Benigno and Benigno (2000).

Details on the log-linearization of the AS equations

We present the details of the derivation of the AS equation (25) for country H. The derivation of the country F's supply side follows in a specular way. The optimal paths of prices $\{\widetilde{p}_t(h), P_{H,t}^H\}$ is described by equations (16), (14) and (17).

We can write (16) as

$$\begin{aligned} 0 &= \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \{[(1-\sigma)(1-\tau^H)\zeta_{t+k}^H \widetilde{p}_t(h) + \\ &\quad +\sigma V_y(\widetilde{y}_{t,t+k}(h), z_{t+k}^H)]\widetilde{y}_{t,t+k}^d(h)\}, \end{aligned}$$

and after substituting the expression for ζ_{t+k}^H

$$\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \left\{ \left[\begin{array}{c} (1-\sigma)(1-\tau^H)U_C(C_{t+k})\frac{\widetilde{p}_t(h)}{P_{t+k}^H} + \\ +\sigma V_y(\widetilde{y}_{t,t+k}(h), z_{t+k}^H) \end{array} \right] \right\} \widetilde{y}_{t,t+k}(h) = 0,$$

or

$$\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \left\{ \left[\begin{array}{c} (1-\sigma)(1-\tau^H)U_C(C_{t+k})\frac{\widetilde{p}_t(h)}{P_{H,t+k}^H} T_{t+k}^{n-1} + \\ +\sigma V_y(\widetilde{y}_{t,t+k}(h), z_{t+k}^H) \end{array} \right] \widetilde{y}_{t,t+k}(h) \right\} = 0, \tag{27}$$

where $T_{t+k} = P_{F,t+k}^H / P_{H,t+k}^H$. We take a log-linear approximation of this equilibrium condition around a steady state in which $C_t = \overline{C}$, $T_t = 1$, $\widetilde{p}_t(h)/P_{H,t}^H = 1$, $G_t^H = 0$, $z_t^H = 0$ and $(1-\tau^H)U_C(\overline{C}) = \frac{\sigma}{\sigma-1} V_y(\overline{C}, 0)$ at all times, obtaining

$$\begin{aligned} 0 &= \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \{(1-\sigma)(1-\tau^H)U_C(\overline{C})\widehat{p}_{t,t+k} + \\ &\quad +(1-\sigma)(1-\tau^H)U_C(\overline{C})[-(1-n)\widehat{T}_{t+k}] \\ &\quad +(1-\sigma)(1-\tau^H)U_{CC}(\overline{C})\overline{C}\widehat{C}_{t+k} + \sigma\overline{C}V_{yy}(\overline{C},0)[-\sigma\widehat{p}_{t,t+k} + \\ &\quad +(1-n)\widehat{T}_{t+k} + \widehat{C}_{t+k} + g_{t+k}^H] + \sigma V_{yz}(\overline{C},0)\widehat{z}_{t+k}^H\} \end{aligned}$$

where $\widehat{p}_{t,t+k} = \ln(\widetilde{p}_t(h)/P_{H,t+k}^H)$. We can further simplify the equation above to

$$\begin{aligned} 0 &= \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \{(\widehat{p}_{t,t+k} - (1-n)\widehat{T}_{t+k} - \rho\widehat{C}_{t+k} - \eta[-\sigma\widehat{p}_{t,t+k} + (1-n)\widehat{T}_{t+k} \\ &\quad +\widehat{C}_{t+k} + g_{t+k}^H - \overline{Y}_t^H]\}, \end{aligned}$$

where $\rho \equiv -U_{CC}(\overline{C})\overline{C}/U_C(\overline{C})$ and $\eta \equiv V_{yy}(\overline{C},0)\overline{C}/V_y(\overline{C},0)$, while we have define $\overline{Y}_t^H$ such that $V_{yz}(\overline{C},0)\widehat{z}_{t+k}^H \equiv -\overline{C}V_{yy}(\overline{C},0)\overline{Y}_t^H$. We note that

$$\widehat{p}_{t,t+k} = \widehat{p}_{t,t} - \sum_{s=1}^{k} \pi_{H,t+s}$$

we can then simplify to

$$\frac{\widehat{p}_{t,t}}{1-\alpha^H\beta} = \mathrm{E}_t\sum_{k=0}^{\infty}(\alpha^H\beta)^k[\frac{1+\eta}{1+\sigma\eta}(1-n)\widehat{T}_{t+k}+\frac{\rho+\eta}{1+\sigma\eta}\widehat{C}_{t+k} + \frac{\eta}{1+\sigma\eta}(g^H_{t+k}-\overline{Y}^H_{t+k})]+\mathrm{E}_t\sum_{k=0}^{\infty}(\alpha^H\beta)^k\left[\sum_{s=1}^{k}\pi_{H,t+s}\right]. \quad (28)$$

Log-linearizing (17), we obtain

$$\widehat{p}_{t,t}=\frac{\alpha^H}{1-\alpha^H}\pi^H_t$$

Thus we can simplify (28) further to

$$\frac{\pi^H_t}{1-\alpha^H\beta}\frac{\alpha^H}{1-\alpha^H} = \mathrm{E}_t\sum_{k=0}^{\infty}(\alpha^H\beta)^k[\frac{1+\eta}{1+\sigma\eta}(1-n)\widehat{T}_{t+k}+\frac{\rho+\eta}{1+\sigma\eta}\widehat{C}_{t+k}+ + \frac{\eta}{1+\sigma\eta}(g^H_{t+k}-\overline{Y}^H_{t+k})]+\mathrm{E}_t\sum_{k=1}^{\infty}(\alpha^H\beta)^k\frac{\pi^H_{t+k}}{1-\alpha\beta}$$

We obtain

$$\pi^H_t = (1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{1+\eta}{1+\sigma\eta}(1-n)\widehat{T}_t+(1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{\rho+\eta}{1+\sigma\eta}\widehat{C}_t +(1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{\eta}{1+\sigma\eta}(g_t-\overline{Y}_t)+\beta\mathrm{E}_t\pi^H_{t+1} \quad (29)$$

noting that the natural rate of world consumption and of the terms of trade, which will arise when prices are flexible, are defined as

$$\widetilde{C}_t \equiv \frac{\eta}{\rho+\eta}(\overline{Y}^W_t-g^W_t),$$
$$\widetilde{T}_t \equiv \frac{\eta}{1+\eta}(g^R_t-\overline{Y}^R_t).$$

we can simplify the equation above to

$$\pi^H_t=(1-n)k^H_T(\widehat{T}_t-\widetilde{T}_t)+k^H_C(\widehat{C}_t-\widetilde{C}_t)+\beta E_t\pi^H_{t+1}, \quad (30)$$

Making use of the log-linearized versions of (10) we get equation (25) (note that $\widehat{C}_t=\widehat{C}^W_t$) where

$$k^H_C \equiv (1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{\rho+\eta}{1+\sigma\eta}\equiv\lambda^H(\rho+\eta)$$
$$k^H_T \equiv (1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{1+\eta}{1+\sigma\eta}\equiv k^H_C\left(\frac{1+\eta}{\rho+\eta}\right).$$

The model is closed by the log-linear approximation of the interest rate rules.

Welfare criterion

The centralized welfare criterion is the discounted value of a weighted average of the average utility flows of the countries,

$$W = \mathrm{E}_0 \sum_{j=0}^{\infty} \beta^j (n w_{t+j}^H + (1-n) w_{t+j}^F), \tag{31}$$

where the average utility flow among all the households belonging to country H is

$$w_t^H = U(C_t) - \frac{\int_0^n v(y_t(h), z_t^H) dh}{n}, \tag{32}$$

while that of country F is

$$w_t^F = U(C_t) - \frac{\int_{1-n}^1 v(y_t(f), z_t^F) df}{1-n}. \tag{33}$$

Through a second-order expansion of (31), we obtain

$$W_t = -\Omega \sum_{j=0}^{\infty} \beta^j L_{t+j} \tag{34}$$

where

$$\begin{aligned} L_{t+j} \quad = \quad & \Lambda[c_{t+j}^W - \overline{c}^W]^2 + n(1-n)\Gamma[\widehat{T}_{t+j} - \widetilde{T}_{t+j}]^2 + \\ & \gamma(\pi_{t+j}^H)^2 + (1-\gamma)(\pi_{t+j}^F)^2 + \text{t.i.p.} + o(\|\xi\|^3), \end{aligned}$$

Here we show the details of the derivations. We follow Rotemberg and Woodford (1997,1998) and Woodford (1999a).

We take a Taylor expansion of each term of the utility function. Taking a second-order linear expansion of $U(C_t)$ around the steady state value $\overline{C}$ defined by equation (19), we obtain

$$U(C_t) = U(\overline{C}) + U_C(C_t - \overline{C}) + \frac{1}{2} U_{CC}(C_t - \overline{C})^2 + o(\|\xi\|^3), \tag{35}$$

where in $o(\|\xi\|^3)$ we group all the terms that are of third or higher order in the deviations of the various variables from their steady-state values. Furthermore expanding C_t with a second-order Taylor approximation we obtain

$$C_t = \overline{C}(1 + \widehat{C}_t + \frac{1}{2}\widehat{C}_t^2) + o(\|\xi\|^3), \tag{36}$$

where $\widehat{C}_t = \ln(C_t/\overline{C})$. Substituting (36) into (35) we obtain

$$U(C_t) = U_C \overline{C} \widehat{C}_t + \frac{1}{2}(U_C \overline{C} + U_{CC} \overline{C}^2) \widehat{C}_t^2 + \text{t.i.p.} + o(\|\xi\|^3), \tag{37}$$

which can be written as

$$U(C_t) = U_C \overline{C}[\widehat{C}_t + \frac{1}{2}(1-\rho)\widehat{C}_t^2] + \text{t.i.p.} + o(\|\xi\|^3),$$

where we have defined $\rho \equiv -U_{CC}\overline{C}/U_C$ and where in t.i.p. we include all the terms that are independent of monetary policy. Similarly we take a second-order Taylor expansion of $v(y_t(h), z_t^H)$ around a steady state where $y_t(h) = \overline{Y}^H$ for each h, and at each date t, and where $z_t^H = 0$ at each date t. We obtain

$$\begin{aligned} v(y_t(h), z_t^H) &= v(\overline{Y}^H, 0) + v_y(y_t(h) - \overline{Y}^H) + v_z z_t^H + \frac{1}{2} v_{yy}(y_t(h) - \overline{Y}^H)^2 \\ &\quad + v_{yz}(y_t(h) - \overline{Y}^H) z_t^H + \frac{1}{2} v_{zz}(z_t^H)^2 + o(\|\xi\|^3), \end{aligned} \tag{38}$$

where $\widehat{y}_t(h) = \ln(y_t(h)/\overline{Y}^H)$. Here we recall that

$$y(h) = \left(\frac{p(h)}{P_H}\right)^{-\sigma} \left[(T)^{1-n} C^W + G^H\right],$$

which can be rewritten as

$$y(h) = y^d(h) + y^g(h),$$

where we have defined

$$\begin{aligned} y^d(h) &\equiv \left(\frac{p(h)}{P_H}\right)^{-\sigma} (T)^{1-n} C^W, \\ y^g(h) &\equiv \left(\frac{p(h)}{P_H}\right)^{-\sigma} G^H. \end{aligned}$$

Here we take a second order Taylor expansion of $y_t^d(h)$ and $y_t^g(h)$ obtaining

$$\begin{aligned} y_t^d(h) &= \overline{Y}^H \cdot (1 + \widehat{y}_t^d(h) + \frac{1}{2} \cdot [\widehat{y}_t^d(h)]^2) + o(\|\xi\|^3), \\ y_t^g(h) &= \overline{Y}^H \cdot (\widehat{y}_t^g(h) + \frac{1}{2} \cdot [\widehat{y}_t^g(h)]^2) + o(\|\xi\|^3). \end{aligned}$$

We note that $y_t^g(h)$ can be neglected because in its expansion, the term of order less than $o(\|\xi\|^3)$ are independent of monetary policy, being the shock G^H equal to zero in the steady state. We can simplify (38) to

$$\begin{aligned} v(y_t(h), z_t^H) &= v_y \overline{Y}^H \cdot [\widehat{y}_t^d(h) + \frac{1}{2} \cdot \widehat{y}_t^d(h)^2 + \frac{\eta}{2} \cdot \widehat{y}_t(h)^2 \\ &\quad - \eta \cdot \widehat{y}_t(h) \overline{Y}_t^H] + \text{ t.i.p.} + o(\|\xi\|^3), \end{aligned} \tag{39}$$

where $\overline{Y}_t^H$ has been defined by the relation $v_{yz} z_t^H \equiv -v_{yy} \overline{Y}^H \overline{Y}_t^H$ and we have that $\eta \equiv V_{yy}(\overline{Y}^H, 0)\overline{Y}^H / V_y(\overline{Y}^H, 0)$. We can rewrite the steady-state conditions as

$$(1 - \Phi^H) U_C(\overline{C}) = \overline{T}^{1-n} V_y\left(\overline{T}^{1-n}\overline{C}, 0\right), \tag{40}$$

$$(1-\Phi^F)U_C(\overline{C}) = \overline{T}^{-n} V_y\left(\overline{T}^{-n}\overline{C}, 0\right), \tag{41}$$

having defined

$$\begin{aligned}
(1-\Phi^H) &\equiv (1-\tau^H)\frac{\sigma-1}{\sigma}, \\
(1-\Phi^F) &\equiv (1-\tau^F)\frac{\sigma-1}{\sigma}.
\end{aligned}$$

In the efficient equilibrium, we have that $\Phi^H = \Phi^F = 0$. As outlined in Woodford (1999), we have to restrict our attention on steady state in which the deviations of Φ^H and Φ^F are of order at least $o(\|\xi\|)$. We also restrict the analysis to the case in which $\Phi^H = \Phi^F$. In this case we have that $\overline{Y}^H = \overline{Y}^F = \overline{C}$. In the neighbor of the efficient level of production and consumption we can write the steady state term of trade and consumption, by using conditions (40) and (41), as

$$\begin{aligned}
\overline{T} &= 1, \\
\ln\overline{C}/C^* &= -\frac{n\Phi^H + (1-n)\Phi^F}{\rho+\eta},
\end{aligned} \tag{42}$$

where C^* is the efficient level of consumption. By using (40) we can write (39) as

$$\begin{aligned}
v(y_t(h), z_t^H) &= U_C\overline{C}\cdot[(1-\Phi)\cdot\widehat{y}_t^d(h) + \frac{1}{2}\cdot\widehat{y}_t^d(h)^2 + \frac{\eta}{2}\cdot\widehat{y}_t(h)^2 \\
&\quad -\eta\cdot\widehat{y}_t(h)\overline{Y}_t^H] + \text{ t.i.p.} + o(\|\xi\|^3).
\end{aligned} \tag{43}$$

Here we integrate (43) across the households belonging to country H, obtaining

$$\begin{aligned}
\frac{\int_0^n v(y_t(h), z_t^H)dh}{n} &= U_C\overline{C}\cdot\{(1-\Phi)\cdot E_h\widehat{y}_t^d(h) + \frac{1}{2}\cdot[\text{var}_h\widehat{y}_t^d(h) + [E_h\widehat{y}_t^d(h)]^2] \\
&\quad + \frac{\eta}{2}\cdot[\text{var}_h\widehat{y}_t(h) + [E_h\widehat{y}_t(h)]^2] - \eta E_h\widehat{y}_t(h)\overline{Y}_t^H\} \\
&\quad + \text{t.i.p.} + o(\|\xi\|^3).
\end{aligned} \tag{44}$$

Using the aggregator (9) we can write

$$Y_{H,t} = Y_{H,t}^d + Y_{H,t}^g,$$

where

$$\begin{aligned}
Y_{H,t}^d &= T_t^{1-n}C_t^W, \\
Y_{H,t}^g &= G^H.
\end{aligned}$$

We take a second-order approximation of the aggregators obtaining

$$\begin{aligned}
\widehat{Y}_{H,t} &= E_h\widehat{y}_t(h) + \frac{1}{2}\left(\frac{\sigma-1}{\sigma}\right)\text{var}_h\widehat{y}_t(h) + o(\|\xi\|^3), \\
\widehat{Y}_{H,t}^d &= E_h\widehat{y}_t^d(h) + \frac{1}{2}\left(\frac{\sigma-1}{\sigma}\right)\text{var}_h\widehat{y}_t^d(h) + o(\|\xi\|^3).
\end{aligned} \tag{45}$$

Finally substituting (45) into (44) we obtain

$$\begin{aligned}\frac{\int_0^n v(y_t(h), z_t)}{n} &= U_C\overline{C}\cdot[(1-\Phi^H)\cdot\widehat{Y}^d_{H,t}+\frac{1}{2}\cdot[\widehat{Y}^d_{H,t}]^2+\frac{\eta}{2}\cdot[\widehat{Y}_{H,t}]^2 \\ &\quad +\frac{1}{2}(\sigma^{-1}+\eta)\cdot\mathrm{var}_h\widehat{y}_t(h)-\eta\widehat{Y}^d_{H,t}\overline{Y}^H_t] \\ &\quad +\text{t.i.p.}+o(\|\xi\|^3) \qquad (46)\end{aligned}$$

where we have used the fact that $\mathrm{var}_h\widehat{y}_t(h) = \mathrm{var}_h\widehat{y}^d_t(h)$.

Combining (46) and (37) into (32), we obtain

$$\begin{aligned}w^H_t &= U_C\overline{C}[\widehat{C}_t+\frac{1}{2}(1-\rho)\widehat{C}^2_t-(1-\Phi^H)\cdot\widehat{Y}^d_{H,t}-\frac{1}{2}\cdot[\widehat{Y}^d_{H,t}]^2-\frac{\eta}{2}\cdot[\widehat{Y}_{H,t}]^2 \\ &\quad -\frac{1}{2}(\sigma^{-1}+\eta)\cdot\mathrm{var}_h\widehat{y}_t(h)+\eta\widehat{Y}^d_{H,t}\overline{Y}^H_t] \\ &\quad +\text{t.i.p.}+o(\|\xi\|^3), \qquad (47)\end{aligned}$$

while for country F we have

$$\begin{aligned}w^F_t &= U_C\overline{C}[\widehat{C}_t+\frac{1}{2}(1-\rho)\widehat{C}^2_t-(1-\Phi^F)\cdot\widehat{Y}^d_{F,t}-\frac{1}{2}\cdot[\widehat{Y}^d_{F,t}]^2-\frac{\eta}{2}\cdot[\widehat{Y}_{F,t}]^2 \\ &\quad -\frac{1}{2}(\sigma^{-1}+\eta)\cdot\mathrm{var}_f\widehat{y}_t(f)+\eta\widehat{Y}^d_{F,t}\overline{Y}^F_t] \\ &\quad +\text{t.i.p.}+o(\|\xi\|^3). \qquad (48)\end{aligned}$$

Taking a linear combination of (47) and (48) with weight n, we obtain

$$\begin{aligned}w_t &= U_C\overline{C}\{\widehat{C}_t\cdot[n\Phi^H+(1-n)\Phi^F]+\frac{1}{2}(1-\rho)\widehat{C}^2_t \\ &\quad -\frac{1}{2}\cdot[n(\widehat{Y}^d_{H,t})^2+(1-n)(\widehat{Y}^d_{F,t})^2]-\frac{1}{2}\eta\cdot[n\widehat{Y}^2_{H,t}+(1-n)\widehat{Y}^2_{F,t}] \\ &\quad +\eta\cdot[n\widehat{Y}_{H,t}\overline{Y}^H_t+(1-n)\widehat{Y}_{F,t}\overline{Y}^F_t]+ \\ &\quad -\frac{1}{2}(\sigma^{-1}+\eta)\cdot[n\mathrm{var}_h\widehat{y}_t(h)+(1-n)\mathrm{var}_f\widehat{y}_t(f)]\} \\ &\quad +\text{t.i.p.}+o(\|\xi\|^3), \qquad (49)\end{aligned}$$

and after substituting the expressions for $\widehat{Y}_{H,t}$, $\widehat{Y}_{F,t}$, $\widehat{Y}^d_{H,t}$, $\widehat{Y}^d_{F,t}$ we get

$$\begin{aligned}w_t &= U_C\overline{C}\{\widehat{C}_t\cdot[n\Phi^H+(1-n)\Phi^F]+\frac{1}{2}(1-\rho)\widehat{C}^2_t \\ &\quad +\eta[\widehat{C}_t\overline{Y}^W_t+n(1-n)\widehat{T}_t\overline{Y}^R_t]-\frac{1}{2}[\widehat{C}^2_t+n(1-n)\widehat{T}^2_t] \\ &\quad -\frac{1}{2}\eta\cdot[\widehat{C}^2_t+n(1-n)\widehat{T}^2_t+2\widehat{C}_t g^W_t-2n(1-n)\widehat{T}_t g^R_t] \\ &\quad -\frac{1}{2}(\sigma^{-1}+\eta)\cdot[n\mathrm{var}_h\widehat{y}_t(h)+(1-n)\mathrm{var}_f\widehat{y}_t(f)]\} \\ &\quad +\text{t.i.p.}+o(\|\xi\|^3), \qquad (50)\end{aligned}$$

which can be written as

$$\begin{aligned} w_t &= -U_C\overline{C}\{-\widehat{C}_t \cdot [n\Phi^H + (1-n)\Phi^F] \\ &+\frac{1}{2}(\rho+\eta)[\widehat{C}_t - \widetilde{C}_t]^2 + \frac{1}{2}(1+\eta)n(1-n)[\widehat{T}_t - \widetilde{T}_t]^2 \\ &+\frac{1}{2}(\sigma^{-1}+\eta)\cdot[n\text{var}_h\widehat{y}_t(h) + (1-n)\text{var}_f\widehat{y}_t(f)]\} \\ &+\text{t.i.p.} + o(\|\xi\|^3). \end{aligned} \quad (51)$$

Where the natural rate of world consumption and of the term of trade, which will arise when prices are flexible, are defined as

$$\begin{aligned} \widetilde{C}_t^W &\equiv \frac{\eta}{\rho+\eta}(\overline{Y}_t^W - g_t^W), \\ \widetilde{T}_t &\equiv \frac{\eta}{1+\eta}(g_t^R - \overline{Y}_t^R). \end{aligned}$$

By using equations (42) and after having defined $c_t^W \equiv \widehat{C}_t^W - \widetilde{C}_t$ we obtain

$$\begin{aligned} w_t &= -U_C\overline{C}\{\frac{1}{2}(\rho+\eta)[c_t^W - \overline{c}^W]^2 + \frac{1}{2}(1+\eta)n(1-n)[\widehat{T}_t - \widetilde{T}_t]^2 \\ &+\frac{1}{2}(\sigma^{-1}+\eta)\cdot[n\text{var}_h\widehat{y}_t(h) + (1-n)\text{var}_f\widehat{y}_t(f)]\} \\ &+\text{t.i.p.} + o(\|\xi\|^3), \end{aligned} \quad (52)$$

where $\overline{c}^W \equiv -\ln\overline{C}/C^*$.

Here we derive $\text{var}_h\widehat{y}_t(h)$ and $\text{var}_f\widehat{y}_t(f)$. We have that

$$\text{var}_h\{\log y_t(h)\} = \sigma^2\text{var}_h\{\log p_t(h)\}.$$

Defining $\overline{p}_t \equiv \text{E}_h\log p_t(h)$, we have

$$\begin{aligned} \text{var}_h\{\log p_t(h)\} &= \text{var}_h\{\log p_t(h) - \overline{p}_{t-1}\} = \text{E}_h\{[\log p_t(h) - \overline{p}_{t-1}]^2\} - (\Delta\overline{p}_t)^2 \\ &= \alpha^H\text{E}_h\{[\log p_{t-1}(h) - \overline{p}_{t-1}]^2\} + (1-\alpha^H)[\log\widetilde{p}_t(h) - \overline{p}_{t-1}]^2 + \\ &\quad -(\Delta\overline{p}_t)^2 \\ &= \alpha^H\text{var}_h\{\log p_{t-1}(h)\} + (1-\alpha^H)[\log\widetilde{p}_t(h) - \overline{p}_{t-1}]^2 - (\Delta\overline{p}_t)^2. \end{aligned}$$

We have also that

$$\overline{p}_t - \overline{p}_{t-1} = (1-\alpha^H)[\log\widetilde{p}_t(h) - \overline{p}_{t-1}], \quad (53)$$

from which we obtain that

$$\text{var}_h\{\log p_t(h)\} = \alpha^H\text{var}_h\{\log p_{t-1}(h)\} + \frac{\alpha^H}{1-\alpha^H}(\Delta\overline{p}_t)^2.$$

But

$$\overline{p}_t = \log P_{H,t} + o(\|\xi\|^2),$$

which implies

$$\mathrm{var}_h\{\log p_t(h)\} = \alpha^H \mathrm{var}_h\{\log p_{t-1}(h)\} + \frac{\alpha^H}{1-\alpha^H}(\pi_t^H)^2 + o(\|\xi\|^3),$$

after integration of the above equation we obtain

$$\mathrm{var}_h\{\log p_t(h)\} = (\alpha^H)^{t+1}\mathrm{var}_h\{\log p_{-1}(h)\} + \sum_{s=0}^{t}(\alpha^H)^{t-s}\frac{\alpha^H}{1-\alpha^H}(\pi_t^H)^2 + o(\|\xi\|^3)$$

where we note that the first term in the right hand side is independent of the policy chosen after period $t \geq 0$. After taking the discounted value, with the discount factor β, we obtain

$$\sum_{t=0}^{\infty}\beta^t \mathrm{var}_h\{\log p_t(h)\} = \frac{\alpha^H}{(1-\alpha^H)(1-\alpha^H\beta)}\sum_{t=0}^{\infty}\beta^t(\pi_t^H)^2 + \text{t.i.p.} + o(\|\xi\|^3)$$

The same derivations apply also for the Foreign country. We define

$$\begin{aligned} d^H &\equiv \frac{\alpha^H}{(1-\alpha^H)(1-\alpha^H\beta)}, \\ d^F &\equiv \frac{\alpha^F}{(1-\alpha^F)(1-\alpha^F\beta)}. \end{aligned}$$

We can simplify (31) to

$$W_t = -\Omega\sum_{j=0}^{\infty}\beta^j L_{t+j} \quad (54)$$

where

$$\begin{aligned} L_{t+j} = {} & \Lambda[c_{t+j}^W - \bar{c}^W]^2 + n(1-n)\Gamma[\widehat{T}_{t+j} - \widetilde{T}_{t+j}]^2 + \\ & \gamma(\pi_{t+j}^H)^2 + (1-\gamma)(\pi_{t+j}^F)^2 + \text{t.i.p.} + o(\|\xi\|^3), \end{aligned}$$

which corresponds to the equation at page 28 of Benigno and Benigno (2000), where $c^W = y^W$ and $\bar{y}^W \equiv \bar{c}^W = 0$. Furthermore

$$\begin{aligned} \Omega &\equiv \frac{1}{2}U_C\overline{C}(nd^H + (1-n)d^F)\sigma(1+\sigma\eta) \\ \Lambda &\equiv \frac{k_C^H k_C^F}{\sigma}\frac{1}{nk_C^F + (1-n)k_C^H}, \\ \Gamma &\equiv \frac{k_T^H k_T^F}{\sigma}\frac{1}{nk_T^F + (1-n)k_T^H}, \\ \gamma &\equiv \frac{nd^H}{nd^H + (1-n)d^F}. \end{aligned}$$

We note that when the degrees of rigidity are the same, i.e $d^H = d^F$, γ coincides with n.

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