Workshop on

Structure and randomness in hypergraphs Proposed Problems

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Asaf Shapira

I will mention open problems in my talk, which can also be found in `https://arxiv.org/abs/1805.07781`.

Allan Lo

Topic: Codegree threshold for a large K_t^r-tiling.

Let H be an r-graph on n vertices, and K_t^r be the complete r-graph on t vertices. A K_t^r-tiling in H is a collection of vertex-disjoint copies of K_t^r.

Question: Does there exist a constant $C = C(r)$ such that: For large t, every r-graph H with codegree $\delta_{r-1}(H) \geq \left(1 - C\frac{\log t}{t^{r-1}} + \epsilon\right)|V(H)|$ contains a K_t^r-tiling covering all but at most $\epsilon' n$ vertices?

The statement holds for

$$\delta_{r-1}(H) \geq \left(1 - \frac{1}{\binom{t-1}{r-1}} + \epsilon\right)|V(H)|\,.$$

Note that this is the natural bound for constructing a K_t^r greedily. On the other hand,

$$\delta_{r-1}(H) \geq \left(1 - \frac{C' \log t}{t^{r-1}} + \epsilon\right)|V(H)|$$

implies a copy of K_t^r. (Answering this question would also lead to a codegree threshold for the existence of perfect K_t^r-tiling.)

Oliver Cooley and Mihyun Kang

Let $H_k(n,p)$ denote the binomial random k-uniform hypergraph, where the vertex set is $[n] := \{1,\dots,n\}$ and each k-element subset of $[n]$ forms a hyperedge with probability p independently. For $1 \le j \le k-1$, a *j-tight path* of length ℓ is a sequence $e_0, \dots, e_{\ell-1}$ of ℓ distinct hyperedges such that $e_i = \{v_{i(k-j)+1}, \dots v_{i(k-j)+k}\}$ for some distinct vertices for $0 \le i \le \ell - 1$.

(Q1) Determine the asymptotic behaviour of the number of j-tight paths/cycles of length ℓ in $H_k(n,p)$.

(Q2) Determine the asymptotic order of the longest j-tight path/cycle in $H_k(n,p)$.

Maya Stein

Problem 1: Given a k-uniform complete hypergraph on n vertices whose edges are coloured with two colours, we would like to cover all or almost all of its vertices by two disjoint monochromatic l-paths of distinct colours (an l-path is one where consecutive edge intersect in exactly l vertices). We know that we can cover all but $\approx 4k$ vertices if $l \le k/2$. We also know that for $k = 3$ (and $l = 1$ or $l = 2$) we can cover all but $o(n)$. (And the latter result implies that for $l = 2k/3$ we can also cover all but $o(n)$.) It would be nice to find some more results for $l > k/2$, when $k > 3$. (Also, we can ask the same question for paths, we know that all but $k-2$ vertices can be covered by two disjoint loose paths, and something similar is true for l-paths with l le $k/2$. For $k = 3$, we know how to cover all vertices with tight or loose paths.)

Problem 2: Given i and j, what is the maximum $n = n(i,j)$ such that K_n is the union of j triangle-free graphs H_j, and each edge of K_n is covered by at least i of the graphs H_j? If $i = 1$ this is the j-colour Ramsey number of the triangle minus 1, but if $i > 2j/3$, then $n(i,j) = 2$, so at some point the problem seems to become easier. Other known numbers are $n(2,3) = 4$ and $n(2,4) = 8$. The smallest not known pair (i,j) is $(2,5)$.

Oleg Pikhurko

1 The minimum number of triangles in graphs with n vertices $m = \binom{n}{2} - O(n)$ edges

A special case of the *Erdős-Rademacher problem* asks for $g_3(n,m)$, the minimum number of triangles in a graph G with n vertices and m edges. Due to the results in [Raz08, PR17, LPS17], we know $g_3(n,m)$ for all pairs (n,m) except when $m = \binom{n}{2} - o(n^2)$.

Erdős [Erd62] observed if $m \geq \binom{n}{2} - \frac{n}{2}$ then, trivially, all extremal graphs are obtained from the n-clique by removing a matching. While solving the range $m = \binom{n}{2} - O(n)$ seems to be as difficult as the whole range (see [LPS17, Lemma 10.1]), it may be worth to try to push Erdős' observation as far as possible.

References

[Erd62] P. Erdős, *On a theorem of Rademacher-Turán*, Illinois J. Math. **6** (1962), 122–127.

[LPS17] H. Liu, O. Pikhurko, and K. Staden, *The exact minimum number of triangles in graphs of given order and size*, E-print arxiv:1712.00633, 2017.

[PR17] O. Pikhurko and A. Razborov, *Asymptotic structure of graphs with the minimum number of triangles*, Combin. Probab. Computing **26** (2017), 138–160.

[Raz08] A. Razborov, *On the minimal density of triangles in graphs*, Combin. Probab. Computing **17** (2008), 603–618.

Open problem

Christina Goldschmidt

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Consider a random hypergraph on n vertices such that each possible 2-edge is present independently with probability a/n and every possible 3-edge is present independently with probability b/n^2, where $a > 0$ and $b \geq 0$ are fixed constants. A vertex chosen uniformly at random is infected with a disease. The disease always spreads along 2-edges but may only spread along a 3-edge if two of its three vertices have already been infected. We are interested in the size of the infected set. (Let $I(v)$ be the set of vertices infected by the vertex v and, more generally, $I(S)$ the set of vertices infected by a set S of initially infected vertices. One might want to think of $I(v)$ as the "component" of v, but this isn't particularly natural here because of a lack of symmetry: we may have $w \in I(v)$ but $v \notin I(w)$.)

If $a < 1$, then only a vanishing proportion of the vertices gets infected. If $a > 1$, on the other hand, there is strictly positive probability that a positive proportion of the vertices gets infected. This is, of course, essentially the emergence of the giant component in the Erdős-Rényi random graph $G(n, a/n)$: for $a > 1$, with non-vanishing probability, the infected vertex lies in the giant component of the underlying 2-graph and infects all of it. (Note, however, that the infected set for $b > 0$ is strictly larger than the 2-edge giant – the 3-edges do make a difference.)

The behaviour when $a = 1$ is much more delicate, and depends crucially on the value of b. The largest 2-edge components are on the order of $n^{2/3}$ now, and collectively contain a vanishing proportion of the vertices. So for $b = 0$ our single uniform infected vertex will whp infect only an $O(1)$ number of other vertices and it's not hard to convince yourself that the same is true for any $b > 0$ also. But for $b = 0$ if we allow ourselves to add $O(n^{1/3})$ infected vertices one by one, we will find some of the largest 2-components, and generate a total infection of size $\Theta(n^{2/3})$. That turns out to be true also for $0 < b < 1$. However, genuinely different behaviours are possible. If $b = 1$, by adding $O(n^{2/5})$ infected vertices one by one, we will generate a total infection of size $\Theta(n^{4/5})$. On the other hand, if $b > 1$, then adding $\Theta(n^{2/5})$ infected vertices one by one is sufficient to generate a total infection of size $\Theta(n)$. Indeed, the proportion of vertices infected converges in probability to $\theta > 0$ (where θ the solution to some equation). In this regime, the phase transition at $a = 1$ is *discontinuous*, although the formulation in which we keep adding single infected vertices until the process of infection "gets going" feels a little unsatisfactory. (Although it did help to get me a PhD...!)

Open problem. Suppose $a = 1$ and $b > 1$. What is the size of the smallest set $S_\star$ of vertices such that $|I(S_\star)|/n \xrightarrow{p} \theta$? Might it even be the case that there exists a vertex $v_\star$ such that $|I(v_\star)|/n \xrightarrow{p} \theta$?

(This model and its phase transition were first studied by Darling and Norris [2] and Darling, Levin and Norris [3]. The $a = 1$ size-scaling results were proved in [4] using an adapted version of Aldous' exploration-process approach to the critical Erdős-Rényi random graph [1]. There are analogous results, with a whole series of different critical size-scalings, in the setting where we allow possible k-edges to be present with probability a_k/n^{k-1} independently, for any $k \geq 2$.)

References

[1] D. Aldous, Brownian excursions, critical random graphs and the multiplicative coalescent, *Annals of Probability* **25** (1997), pp.812–854.

[2] R.W.R. Darling and J.R. Norris, Structure of large random hypergraphs, *Annals of Applied Probability* **15** (2005), pp.125–152.

[3] R.W.R. Darling, D.A. Levin and J.R. Norris, Continuous and discontinuous phase transitions in hypergraph processes, *Random Structures and Algorithms* **24** (2004), pp.347–419.

[4] C. Goldschmidt, Critical random hypergraphs: the emergence of a giant set of identifiable vertices, *Annals of Probability* **33** (2005), pp.1573–1600.

Let $\mathcal{F}$ be a collection of (hyper)graphs. We say that a graph is $\mathcal{F}$-free if it does not contain, as a subgraph, any member of $\mathcal{F}$. Further, we say that elements of $\mathcal{F}$ are forbidden subgraphs for the class of $\mathcal{F}$-free graphs.
Many families of graphs can be characterized in terms of forbidden substructures, for example:

- Forests, $\mathcal{F} = \{\text{cycles}\}$;
- Bipartite graphs, $\mathcal{F} := \{\text{odd cycles}\}$;

The examples above have in common that they have a specific structural characterization, but one can give an alternative definition in terms of forbidden subgraphs.
We are interested in generalize the characterization of bipartite graphs to higher uniformities, which, as far as we know, has not been studied yet.
Let $\mathcal{H}_3$ be the family of 3-partite 3-uniform graphs.

Question 0.1. *Is there an explicit family $\mathcal{F}$ such that $\mathcal{H}_3$ is the class of $\mathcal{F}$-free graphs?*

One could think that $\mathcal{H}_3$ is the class of $\mathcal{C}_{\not\equiv 3}$-free 3-graphs, where $\mathcal{C}_{\not\equiv 3}$ denotes the collection of 3-uniform tight cycles whose length is not divisible by 3. It is clear that every member of $\mathcal{H}_3$ is $\mathcal{C}_{\not\equiv 3}$-free. However, there are $\mathcal{C}_{\not\equiv 3}$-free graphs which are not 3-partite. For instance, let C be the 3-graph obtained by taking a tight cycle of length 11 and identifying each vertex at a position divisible by 3(see Figure 1 below).

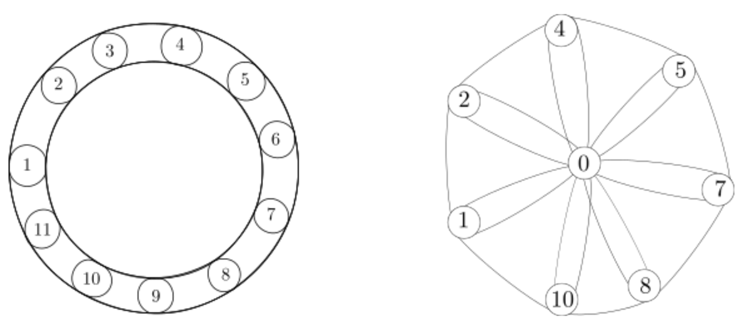

Figure 1: The vertices $3, 6, 9$ are mapped to 0.

Note that C is $\mathcal{C}_{\not\equiv 3}$-free but is not 3-partite. Therefore, $\mathcal{H}_3$ is strictly contained in the family of $\mathcal{C}_{\not\equiv 3}$-free graphs. On the other hand, observe that C

is actually an surjective homomorphism of $C_{11}^{(3)}$, the 3-uniform tight cycle of length 11. So, if $\overline{\mathcal{C}_{\not\equiv 3}}$ denotes the class of graphs which are obtained by a surjective homomorphism of members of $\mathcal{C}_{\not\equiv 3}$, so in particular $\mathcal{C}_{\not\equiv 3} \subset \overline{\mathcal{C}_{\not\equiv 3}}$, one could conjecture that $\mathcal{H}_3$ is characterized by forbidding all graphs from $\overline{\mathcal{C}_{\not\equiv 3}}$[1].

Let X and Y be disjoint sets and let H be 3-graph with vertex set $V(H) = X \cup Y$ and edge set consisting only of triples of the form $xx'y$ with $x, x' \in X$ and $y \in Y$. Note that H can only have cycles of length divisible by 3. Further, if some vertex $y \in Y$ has a non-bipartite link graph, then H is not 3-partite.

This leads to the following natural questions:

Question 0.2. *Is there a structural characterization for the family of $\overline{\mathcal{C}_{\not\equiv 3}}$-free graphs ?*

Question 0.3. *Generalize Question 0.1 and Question 0.2 to higher uniformities.*

[1]For bipartite graphs is equivalent to forbid odd cycles and homomorphisms of odd cycles.