THE CHROMATIC THRESHOLDS OF GRAPHS

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Abstract. The chromatic threshold $\delta_\chi(H)$ of a graph H is the infimum of $d > 0$ such that there exists $C = C(H,d)$ for which every H-free graph G with minimum degree at least $d|G|$ satisfies $\chi(G) \leqslant C$. We prove that

$$\delta_\chi(H) \in \left\{ \frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1} \right\}$$

for every graph H with $\chi(H) = r \geqslant 3$. We moreover characterise the graphs H with a given chromatic threshold, and thus determine $\delta_\chi(H)$ for every graph H. This answers a question of Erdős and Simonovits [Discrete Math. 5 (1973), 323–334], and confirms a conjecture of Łuczak and Thomassé [preprint (2010), 18pp].

1. Introduction

Two central problems in Graph Theory involve understanding the structure of graphs which avoid certain subgraphs, and bounding the chromatic number of graphs in a given family. For more than sixty years, since Zykov [42] and Tutte [5] first constructed triangle-free graphs with arbitrarily large chromatic number, the interplay between these two problems has been an important area of study. The generalisation of Zykov's result, by Erdős [12], to H-free graphs (for any non-acyclic H), was one of the first applications of the probabilistic method in combinatorics.

In 1973, Erdős and Simonovits [14] asked whether such constructions are still possible if one insists that the graph should have large minimum degree. As a way of investigating their problem, they implicitly defined what is now known as the *chromatic threshold* of a graph H as follows (see [14, Section 4]):

$$\delta_\chi(H) := \inf \Big\{ d : \exists C = C(H,d) \text{ such that if } G \text{ is a graph on } n \text{ vertices,} \\ \text{with } \delta(G) \geqslant dn \text{ and } H \not\subseteq G, \text{ then } \chi(G) \leqslant C \Big\}.$$

Date: August 16, 2011.

Key words and phrases. Chromatic threshold, minimum degree, graph colouring.

2010 Mathematics Subject Classification. Primary 05C35; Secondary 05C15.

PA was partially supported by FAPESP (Proc. 2010/09555-7); JB by FAPESP (Proc. 2009/17831-7); SG by CNPq (Proc. 500016/2010-2); YK by CNPq (Proc. 308509/2007-2); RM by a CNPq bolsa de Produtividade em Pesquisa. This research was supported by CNPq (Proc. 484154/2010-9). The authors are grateful to NUMEC/USP, Núcleo de Modelagem Estocástica e Complexidade of the University of São Paulo, and Project MaCLinC/USP, for supporting this research.

That is, for $d < \delta_\chi(H)$ the chromatic number of H-free graphs with minimum degree dn may be arbitrarily large, while for $d > \delta_\chi(H)$ it is necessarily bounded. In this paper we shall determine $\delta_\chi(H)$ for every graph H, and thus completely solve the problem of Erdős and Simonovits.

The chromatic threshold has been most extensively investigated for the triangle $H = K_3$, where in fact much more is known. Erdős and Simonovits [14] conjectured that $\delta_\chi(K_3) = \frac{1}{3}$, which was proven in 2002 by Thomassen [39], and that moreover if G is triangle-free and $\delta(G) > n/3$ then $\chi(G) \leqslant 3$. This stronger conjecture was disproved by Häggkvist [18], who found a $(10n/29)$-regular graph with chromatic number four. However, Brandt and Thomassé [8] recently showed that the conjecture holds with $\chi(G) \leqslant 3$ replaced by $\chi(G) \leqslant 4$. Hence the situation is now well-understood for triangle-free graphs G (see [3, 7, 8, 9, 18, 20, 30]), and can be summarised as follows:

$\delta(G) >$	$2n/5$	$10n/29$	$n/3$	$(1/3 - \varepsilon)n$
$\chi(G) \leqslant$	2	3	4	∞

For bipartite graphs H, it follows trivially from the Kövári-Sós-Turán Theorem [26] that $\delta_\chi(H) = 0$, and for larger cliques, Goddard and Lyle [16] determined the chromatic threshold, proving that $\delta_\chi(K_r) = \frac{2r-5}{2r-3}$ for every $r \geqslant 3$. Erdős and Simonovits also conjectured that $\delta_\chi(C_5) = 0$, which was proven (and generalised to all odd cycles) by Thomassen [40].

For graphs other than cliques and odd cycles, very little was known until the recent work of Lyle [31] and the breakthrough of Łuczak and Thomassé [30] who introduced a new technique which allows the study of $\delta_\chi(H)$ for more general classes of graphs. In order to motivate their results, let us summarise what was known previously for 3-chromatic graphs H. We have seen that there are such graphs with chromatic threshold 0 (the odd cycle C_5), and chromatic threshold $\frac{1}{3}$ (the triangle). A folklore result is that there are also 3-chromatic graphs with chromatic threshold $\frac{1}{2}$, such as the *octahedron* $K_{2,2,2}$. Indeed, given a graph H with $\chi(H) = r \geqslant 3$, the *decomposition family* $\mathcal{M}(H)$ of H is the set of bipartite graphs which are obtained from H by deleting $r - 2$ colour classes in some r-colouring of H. Observe that $K_{2,2,2}$ has the property that its decomposition family contains no forests (in fact, $\mathcal{M}(K_{2,2,2}) = \{C_4\}$). It is not difficult to show that whenever there are no forests in $\mathcal{M}(H)$, the graph H has chromatic threshold $\frac{1}{2}$ (see Proposition 5).

Thus it remains to consider those 3-chromatic graphs whose decomposition family does contain a forest; in other words, graphs which admit a partition into a forest and an independent set (such as all odd cycles). Lyle [31] proved that these graphs have chromatic threshold strictly smaller than $\frac{1}{2}$. In fact, as we shall show, they have chromatic threshold at most $\frac{1}{3}$ (see Theorem 2). Łuczak and Thomassé [30] described a large sub-family of these graphs with chromatic threshold strictly smaller than $\frac{1}{3}$. More precisely they considered triangle-free graphs which admit a partition into a (not necessarily perfect) matching and an independent set; they called a graph (such as C_5) *near-bipartite* if it is of this form, and proved that if H is near-bipartite then $\delta_\chi(H) = 0$.

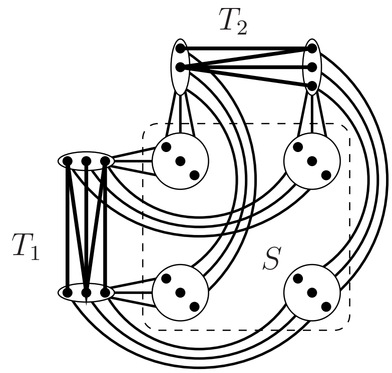

Figure 1. An illustration of near-acyclic graphs with partition into an independent set S and a forest F consisting of two trees T_1 and T_2.

However, Łuczak and Thomassé did not believe that the near-bipartite graphs are the only graphs with chromatic threshold zero. Generalising near-bipartite graphs, they defined H to be *near-acyclic* if $\chi(H) = 3$ and H admits a partition into a forest F and an independent set S such that every odd cycle of H meets S in at least two vertices. Equivalently, for each tree T in F with colour classes $V_1(T)$ and $V_2(T)$, there is no vertex of S with neighbours in both $V_1(T)$ and $V_2(T)$ (see also Figure 1). Observe that the near-bipartite graphs are precisely the near-acyclic graphs in which every tree is a single edge or vertex. The first graph in Figure 2 is near-acyclic (as illustrated by the highlighted forest), but has no matching in its decomposition family and thus is not near-bipartite.

Lyle [31] proved that $\delta_\chi(H) = 0$ for a sub-family of near-acyclic graphs H which are not necessarily near-bipartite, and Łuczak and Thomassé gave a construction (see Section 4) showing that every graph which is *not* near-acyclic has chromatic threshold at least $\frac{1}{3}$. They made the following conjecture.

Conjecture 1 (Łuczak and Thomassé [30])**.** *Let H be a graph with $\chi(H) = 3$. Then $\delta_\chi(H) = 0$ if and only if H is near-acyclic.*

We shall prove Conjecture 1, and moreover determine $\delta_\chi(H)$ for *every* graph H. In this theorem, we use the following generalisation of near-acyclic graphs. We call a graph H *r-near-acyclic* if $\chi(H) = r \geqslant 3$, and there exist $r-3$ independent sets in H whose removal yields a near-acyclic graph. Note in particular that if H is r-near-acyclic, then there is a forest in $\mathcal{M}(H)$. Our main theorem is as follows.

Theorem 2. *Let H be a graph with $\chi(H) = r \geqslant 3$. Then*

$$\delta_\chi(H) \in \left\{ \frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1} \right\} .$$

Moreover, $\delta_\chi(H) \neq \frac{r-2}{r-1}$ if and only if H has a forest in its decomposition family, and $\delta_\chi(H) = \frac{r-3}{r-2}$ if and only if H is r-near-acyclic.

For example, the *dodecahedron* is 3-chromatic and near-bipartite, hence it has chromatic threshold 0 (Figure 2 shows the dodecahedron together with a corresponding matching).

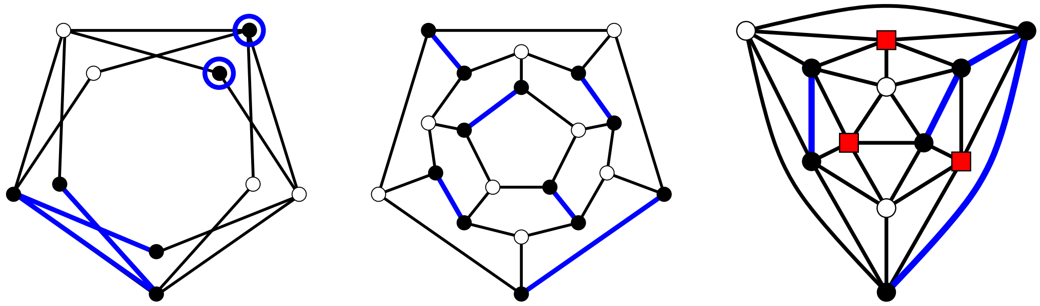

FIGURE 2. A near-acyclic graph that is not near-bipartite, the dodecahedron, and the icosahedron.

The *icosahedron* on the other hand has chromatic threshold $\frac{3}{5}$ because it is four-chromatic and has a forest in its decomposition family (a partition of the icosahedron into this forest and two independent sets is also displayed in Figure 2), but is not 4-near-acyclic.

For easier reference, given H with $\chi(H) = r$, we define

$$\theta(H) := \frac{r-3}{r-2}, \quad \lambda(H) := \frac{2r-5}{2r-3} \quad \text{and} \quad \pi(H) := \frac{r-2}{r-1}.$$

Observe that $\pi(H)$ is precisely the Turán density of H, and therefore the Erdős-Stone Theorem [15] yields $\delta_\chi(H) \leqslant \pi(H)$ for all H. Furthermore, the constructions giving the lower bounds in Theorem 2 are straightforward extensions of those given in [30, 31]. It follows that our main challenge is to prove that $\delta_\chi(H) \leqslant \lambda(H)$ when $\mathcal{M}(H)$ contains a forest, and that $\delta_\chi(H) \leqslant \theta(H)$ when H is r-near-acyclic.

The recent results of both Lyle [31] and Łuczak and Thomassé [30] contain important new techniques, which we re-use and extend here. Most significantly, Łuczak and Thomassé [30] introduced a concept which they termed *paired VC-dimension*, which is based on the classical Vapnik-Červonenkis dimension of a set-system [41]. Our proof of Conjecture 1 relies on an extension of this technique (see Section 6), together with a new embedding lemma (see Section 5) which allows us to find a copy of H in sufficiently many 'well-structured' copies of the 'Zykov graph', which is a universal near-bipartite graph.

Lyle [31] introduced a novel graph partitioning method based on the celebrated Szemerédi Regularity Lemma. We shall use his partition in Section 3, together with averaging arguments similar to those in [1], to prove that $\delta_\chi(H) \leqslant \lambda(H)$ for any graph H such that $\mathcal{M}(H)$ contains a forest. In Section 7 we shall combine and extend both techniques in order to generalise Conjecture 1 to arbitrary $r \geqslant 3$.

Organisation. In Section 2 we state the Regularity Lemma in the form in which we shall use it, together with some auxiliary tools, and provide some notes on notation. In Section 3 we prove that $\delta_\chi(H) \leqslant \lambda(H)$ for any graph H such that $\mathcal{M}(H)$ contains a forest. In addition we give a construction which shows that if $\mathcal{M}(H)$ does not contain a forest, then $\delta_\chi(H) \geqslant \pi(H)$. In Section 4 we provide a construction (using the Borsuk-Ulam Theorem) which shows that for graphs H which are not r-near-acyclic, we have $\delta_\chi(H) \geqslant \lambda(H)$. In Section 5 we introduce a generalisation of the class of Zykov graphs, a class of universal near-bipartite graphs which were used in [30]. We show that for every near-acyclic graph H, if G contains a suitably well-structured collection of Zykov graphs, then G contains H. Complementing this, in Section 6 we refine Łuczak and Thomassé's paired VC-dimension argument to show that every graph with linear minimum degree and sufficiently large chromatic number indeed contains such a well-structured collection of Zykov graphs. Completing the proof of Theorem 2, in Section 7 we give a construction which shows that for any r-chromatic graph H we have $\delta_\chi(H) \geqslant \theta(H)$, and combine the results of Sections 5 and 6 with the Regularity Lemma in order to show that for r-near-acyclic graphs H, we have $\delta_\chi(H) \leqslant \theta(H)$. Finally, in Section 8 we conclude with a collection of open problems.

2. Tools and the Regularity Lemma

In this section we shall state some of the tools used in the proof of Theorem 2. In particular, we shall recall the Szemerédi Regularity Lemma, which is one of the most powerful and important results in Graph Theory. Introduced in the 1970s by Szemerédi [38] in order to prove that sets of positive density in the integers contain arbitrarily long arithmetic progressions (a result known as Szemerédi's Theorem [37]), it says (roughly) that *any* graph can be approximated well by a bounded number of 'quasi-random' graphs. The lemma has turned out to have an enormous number of applications, and many important extensions and variations have been proved (see for example [17, 23, 29, 34] and the references therein). The reader who is unfamiliar with the Regularity Lemma is encouraged to see the excellent surveys [24, 25].

We begin by stating the Regularity Lemma in the form in which we shall use it. Let (A, B) be a pair of subsets of vertices of a graph G. We write $d(A, B) = \frac{e(A,B)}{|A||B|}$, and call $d(A, B)$ the *density* of the pair (A, B). (Here $e(A, B)$ denotes the number of edges with one endpoint in A and the other in B.) For each $\varepsilon > 0$, we say that (A, B) is *ε-regular* if $|d(A, B) - d(X, Y)| < \varepsilon$ for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$.

A partition $V_0 \cup V_1 \cup \ldots \cup V_k$ of $V(G)$ is said to be an *ε-regular partition* (or sometimes a Szemerédi partition of G for ε) if $|V_0| \leqslant \varepsilon n$, $|V_1| = \ldots = |V_k|$, and all but at most εk^2 of the pairs (V_i, V_j) are ε-regular. We will often refer to the partition classes $V_1, \ldots, V_k$ as the *clusters* of the regular partition. In its simplest form, the Regularity Lemma is as follows.

Szemerédi's Regularity Lemma. *For every $\varepsilon > 0$ and every $k_0 \in \mathbb{N}$, there exists a constant $k_1 = k_1(k_0, \varepsilon)$ such that the following holds. Every graph G on at least k_1 vertices has an ε-regular partition into k parts, for some $k_0 \leqslant k \leqslant k_1$.*

We shall in fact use a slight extension of the statement above, which follows easily from [24, Theorem 1.10] (a proof can be found in, e.g., [27, Proposition 9]). Given $0 < d < 1$ and a pair (A, B) of sets of vertices in a graph G, we say that (A, B) is *(ε, d)-regular* if it is ε-regular and has density at least d.

Given an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_k$ of $V(G)$ and $0 < d < 1$, we define a graph R, called the *reduced graph* of the partition, as follows: $V(R) = [k] = \{1, \ldots, k\}$ and

$ij \in E(R)$ if and only if (V_i, V_j) is an (ε, d)-regular pair. We shall occasionally omit the partition, and simply say that G has (ε, d)-*reduced graph* R.

Szemerédi's Regularity Lemma (minimum degree form). *Let $0 < \varepsilon < d < \delta < 1$, and let $k_0 \in \mathbb{N}$. There exists a constant $k_1 = k_1(k_0, \varepsilon, \delta, d)$ such that the following holds. Every graph G on $n > k_1$ vertices, with minimum degree $\delta(G) \geqslant \delta n$, has an (ε, d)-reduced graph R on k vertices, with $k_0 \leqslant k \leqslant k_1$ and $\delta(R) \geqslant \big(\delta - d - \varepsilon\big)k$.*

Thus the reduced graph R of G 'inherits' the high minimum degree of G. The main motivation for the definition of ε-regularity is the following so-called 'counting lemma' (see [24, Theorem 3.1], for example).

Counting Lemma. *Let G be a graph with (ε, d)-reduced graph R whose clusters contain m vertices, and suppose that there is a homomorphism $\phi\colon H \to R$. Then G contains at least*

$$\frac{1}{\big|\mathrm{Aut}(H)\big|}\big(d - \varepsilon|H|\big)^{e(H)} m^{|H|}$$

copies of H, each with the property that every vertex $x \in V(H)$ lies in the cluster corresponding to the vertex $\phi(x)$ of R.

Note that since we count unlabelled copies of H, it is necessary to correct for the possibility that two different maps from H to G may yield the same copy of H (precisely when they differ by some automorphism of H). In fact, such an automorphism of H must also preserve ϕ, but dividing by the number of elements of the full automorphism group $\mathrm{Aut}(H)$ provides a lower bound which is sufficient for our purposes. We state one more useful fact about subpairs of (ε, d)-regular pairs.

Fact 3. *Let (U, W) be an (ε, d)-regular pair and $U' \subseteq U$, $W' \subseteq W$ satisfy $|U'| \geqslant \alpha|U|$ and $|W'| \geqslant \alpha|W|$. Then (U', W') is $(\varepsilon/\alpha, d - \varepsilon)$-regular.*

We shall also use the following straightforward and well-known fact several times.

Fact 4. *Let F be a forest and G be a graph on $n \geqslant 1$ vertices. If $e(G) \geqslant |F|n$, then $F \subseteq G$.*

Proof. Since G has average degree at least $2|F|$, it contains a subgraph G' with minimum degree at least $|F|$. It is easy to show that G' contains F; for example, remove a leaf and apply induction. $\square$

2.1. **Notation.** We finish this section by describing some of the notation which we shall use throughout the paper. Most is standard (see [6], for example); we shall repeat non-standard definitions when they are first used.

For each $t \in \mathbb{N}$, let $[t] = \{1, \ldots, t\}$. We say that we *blow up a vertex $v \in V(G)$ to size t* if we replace v by an independent set of size t, and replace each edge containing v by a complete bipartite graph. Given disjoint sets X and Y, we shall write $K[X, Y]$ for the edge set of the complete bipartite graph on $X \cup Y$, that is, the set of all pairs with one end in X and the other in Y. We write $K_s(t)$ for the complete s-partite graph with t vertices in each part: that is, the graph obtained from K_s by blowing up each of its vertices to size t.

Given a graph G, we write $E(G)$ for the set of edges of G, and $e(G)$ for $|E(G)|$. We use both $|G|$ and $v(G)$ to denote the number of vertices of G. Given a set $X \subseteq V(G)$, we write $E(X)$ for the set of edges of G with both ends in X, and $N(X)$ for the set of *common* neighbours of the vertices in X. If $D \subseteq E(G)$, then $N(D)$ denotes the set of common neighbours of $V(D)$, the set of the endpoints of edges in D. (In particular, if $e = xy$ is an edge then $N(e) = N\big(\{x,y\}\big) = N(x) \cap N(y)$.) Further, we let $G[D]$ denote the subgraph of G with vertex set $V(D)$ and edge set D, and write $\overline{d}(D)$ for the average degree of $G[D]$ and $\delta(D)$ for the minimum degree of $G[D]$.

A graph G is said to be C-*degenerate* if there exists an ordering $(v_1, \dots, v_n)$ of $V(G)$ such that v_{k+1} has at most C neighbours in $\{v_1, \dots, v_k\}$ for every $1 \leqslant k \leqslant n-1$. Finally, if $e_1, \dots, e_\ell \in E(G)$, then we shall write $\mathbf{e}^\ell$ for the tuple $(e_1, \dots, e_\ell)$.

3. Graphs with large chromatic threshold

In this section we shall categorise the graphs H with chromatic threshold greater than $\lambda(H)$. First observe that a trivial upper bound on $\delta_\chi(H)$ is given by the *Turán density* of H,

$$\pi(H) \;=\; \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \;=\; \frac{\chi(H)-2}{\chi(H)-1},$$

since if $\delta(G) = \delta n$ with $\delta > \pi(H)n$ then $H \subseteq G$, by the Erdős-Stone Theorem [15]. Moreover, it is not hard to prove the following sufficient condition for equality, which can be found, for example, in [31]. Recall that $\mathcal{M}(H)$ denotes the decomposition family of H.

Proposition 5. *Let H be a graph with $\chi(H) = r \geqslant 3$. If $\mathcal{M}(H)$ does not contain a forest, then $\delta_\chi(H) = \pi(H) = \frac{r-2}{r-1}$.*

For each $k, \ell \in \mathbb{N}$, we shall call a graph G a (k,ℓ)-*Erdős graph* if it has chromatic number at least k, and girth (length of the shortest cycle) at least ℓ. In one of the first applications of the probabilistic method, Erdős [12] proved that such graphs exist for every k and ℓ.

Proof of Proposition 5. Let H be a graph with $\chi(H) = r \geqslant 3$ such that $\mathcal{M}(H)$ contains no forest, let $C \in \mathbb{N}$, and let G' be a $(C, |H|+1)$-Erdős graph; that is, $\chi(G') \geqslant C$ and $\mathrm{girth}(G') \geqslant |H|+1$. Let G be the graph obtained from the complete, balanced $(r-1)$-partite graph on $(r-1)|G'|$ vertices by replacing one of its partition classes with G'. Then $\delta(G) = \frac{r-2}{r-1}v(G)$, $H \not\subseteq G$, and $\chi(G) \geqslant C$. □

We remark that the same construction, with the complete balanced $(r-1)$-partite graph replaced by a complete balanced $(r-2)$-partite graph, shows that, whatever the structure of H, its chromatic threshold is at least $\frac{r-3}{r-2}$ (see Proposition 35).

Lyle [31] showed that the condition of Proposition 5 is necessary.

Theorem 6 (Lyle [31])**.** *If $\chi(H) = r \geqslant 3$, then $\delta_\chi(H) < \pi(H) = \frac{r-2}{r-1}$ if and only if the decomposition family of H contains a forest.*

In this section we shall strengthen this result by proving that if $\delta_\chi(H) < \pi(H)$, then it is at most $\lambda(H)$.

Theorem 7. *Let H be a graph with $\chi(H) = r \geqslant 3$. If $\mathcal{M}(H)$ contains a forest, then*

$$\delta_\chi(H) \leqslant \frac{2r-5}{2r-3}.$$

The proof of Theorem 7 is roughly as follows. Let $\gamma > 0$, and let G be a sufficiently large graph with

$$\delta(G) \geqslant \left(\frac{2r-5}{2r-3} + \gamma\right) v(G).$$

For some suitably small ε and d, let $V_0 \cup \ldots \cup V_k$ be the partition of $V(G)$ given by the minimum degree form of the Szemerédi Regularity Lemma, and R be the (ε, d)-reduced graph of this partition. Define, for each $I \subseteq [k]$,

$$X_I := \Big\{ v \in V(G) : \, i \in I \Leftrightarrow |N(v) \cap V_i| \geqslant d|V_i| \Big\}.$$

We remark that this partition was used by Lyle [31]. We show that $\chi\big(G[X_I]\big)$ is bounded if $H \not\subseteq G$. We distinguish two cases. If $|I| \geqslant (2r-4)/(2r-3)$, then it is straightforward to show that $R[I]$ contains a copy of K_{r-1}, and hence, by the Counting Lemma, that $N(x)$ contains 'many' (i.e., a positive density of) copies of $K_{r-1}(t)$ for every $x \in X_I$. We then use the pigeonhole principle (see Lemma 9, below), to show that either $|X_I|$ is bounded, or $H \subseteq G$.

If $|I| \leqslant (2r-4)/(2r-3)$, then set $V_I = \bigcup_{j \in I} V_j$, and observe that every pair $x, y \in X_I$ has 'many' common neighbours in V_I. We use a greedy algorithm (in the form of Lemma 9(a)) to conclude that every edge is contained in a positive density of copies of K_r. Finally, we shall use a counting version of a lemma of Erdős (Lemma 8) together with the pigeonhole principle to show that when $H \not\subseteq G$, $\chi(G[X_I])$ is bounded (see Lemma 10).

We begin with some preliminary lemmas. The following lemma from [1] will be an important tool in the proof; it is a counting version of a result of Erdős [13].

Lemma 8 (Lemma 7 of [1])**.** *For every $\alpha > 0$ and $s, t \in \mathbb{N}$ there is an $\alpha' = \alpha'(\alpha, s, t) > 0$ such that the following holds. Let G be a graph on n vertices with at least αn^s copies of K_s. Then G contains at least $\alpha' n^{st}$ copies of $K_s(t)$.*

We shall also use the following easy lemma, which is just an application of a greedy algorithm and the pigeonhole principle. Let $G + H$ denote the graph obtained by taking disjoint copies of G and H, and adding a complete bipartite graph between the two.

Lemma 9. *Let $\alpha, \delta > 0$ and $s, t \in \mathbb{N}$, let F be a forest, and suppose that $H \subseteq F + K_s(t)$. Let G be a graph on n vertices, and $X \subseteq V(G)$.*

(a) *If $\delta(G) \geqslant \delta n$ and*

$$|X| \geqslant \big(\alpha^{1/s} s + (1-\delta)(s-1)\big) n\,,$$

then $G[X]$ contains at least αn^s copies of K_s.

(b) *If $G\big[N(x)\big]$ contains at least $\alpha n^{(s+1)|H|}$ copies of $K_{s+1}(|H|)$ for every $x \in X$, then either $H \subseteq G$ or $|X| \leqslant |H|/\alpha$.*

Proof. For part (a), we construct copies of K_s in $G[X]$ using the following greedy algorithm: First choose an arbitrary vertex $x_1 \in X$, then a vertex $x_2 \in X$ in the neighbourhood of x_1, then $x_3 \in N(x_1) \cap N(x_2) \cap X$, and so on, until we find $x_s \in X$ in the common neighbourhood of $x_1, \dots, x_{s-1}$. Clearly, $G\big[\{x_1, \dots, x_s\}\big]$ is a copy of K_s.

Now we simply count: for choosing x_i we have at least

$$|X| - (i-1)(1-\delta)n \geqslant \alpha^{1/s} s \cdot n \geqslant (s!\alpha)^{1/s} \cdot n$$

possibilities, so in total we have at least $s!\alpha n^s$ choices. Since the algorithm can construct a particular copy of K_s at most in $s!$ different ways, we have found at least αn^s distinct K_s-copies in $G[X]$.

For part (b), simply observe that, by the pigeonhole principle, there is a copy T of $K_{s+1}(|H|)$ in G such that $T \subseteq G\big[N(x)\big]$ for at least $\alpha|X|$ vertices of X. Since $H \subseteq K_{s+2}(|H|)$ this implies that either $H \subseteq G$ or $\alpha|X| < |H|$. □

The following result follows easily from Lemma 8. For a forest F and $H \subseteq F + K_s(t)$, it will enable us to draw conclusions about the chromatic number of an H-free graph which contains many K_{s+2}-copies arranged in a suitable way.

Lemma 10. *For every $\alpha > 0$ and $s, t \in \mathbb{N}$, there exists $\alpha' = \alpha'(\alpha, s, t)$ such that for every forest F and every graph $H \subseteq F + K_s(t)$, the following holds. Let G be an H-free graph on n vertices, and let $X \subseteq V(G)$ be such that every edge $xy \in E\big(G[X]\big)$ is contained in at least αn^s copies of K_{s+2} in G.*

Then $G[X]$ is $(2|F|/\alpha')$-degenerate, and hence $\chi\big(G[X]\big) \leqslant (2|F|/\alpha') + 1$.

Given a subgraph K of G, we say that an edge $e = xy$ of G *extends to* $e + K$ if $V(K) \subseteq N(x, y)$. We say e *extends to a copy of* $e + K_s(t)$ if e extends to $e + K$ for some copy K of $K_s(t)$ in G.

Proof of Lemma 10. Let $\alpha' = \alpha'(\alpha, s, t)$ be the constant provided by Lemma 8, let $xy \in E(G[X])$ and let $G' = G[N(x) \cap N(y)]$. Then, by our assumption, G' contains at least αn^s copies of K_s. By Lemma 8, it follows that G' contains $\alpha' n^{st}$ copies of $K_s(t)$, so xy extends to at least $\alpha' n^{st}$ copies of $e + K_s(t)$ in G.

Now, let $x_1, \dots, x_{|X|}$ be an ordering of the vertices of X with the property that x_i has minimum degree in $G_i := G[X \setminus \{x_1, \dots, x_{i-1}\}]$ for each $1 \leqslant i \leqslant |X|$. In order to show that $G[X]$ is $(2|F|/\alpha')$-degenerate, it suffices to prove that

$$e(G_i) \leqslant \frac{|F|}{\alpha'} \cdot |G_i|,$$

since then $\delta(G_i) \leqslant 2|F|/\alpha'$, as desired.

Since each edge $e \in E(G_i)$ extends to at least $\alpha' n^{st}$ copies of $e + K_s(t)$, it follows, by the pigeonhole principle, that there is a copy K' of $K_s(t)$ and a set $E_i \subseteq E(G_i)$ with $|E_i| \geqslant \alpha' e(G_i)$, such that e extends to $e + K'$ for every $e \in E_i$. Let G_i^* be the graph with vertex set $V(G_i)$ and edge set E_i. Since $H \not\subseteq G$, it follows that $F \not\subseteq G_i^*$. Thus, by Fact 4, we have

$$|F| \cdot |G_i| > e(G_i^*) \geqslant \alpha' e(G_i),$$

as required. □

We are ready to prove Theorem 7. We shall apply the minimum degree form of the Szemerédi Regularity Lemma, together with the Counting Lemma and Lemmas 9 and 10.

Proof of Theorem 7. Let F be a forest, let $r \geqslant 3$, and let H be a graph with $\chi(H) = r$ and $F \in \mathcal{M}(H)$. Observe that we have $H \subseteq F + K_{r-2}(|H|)$. Let $\gamma > 0$, and let G be an H-free graph with

$$\delta(G) \,\geqslant\, \left(\frac{2r-5}{2r-3} + 2\gamma\right) n,$$

where $n = |G|$. We shall show that $\chi(G)$ is bounded above by some constant $C = C(H,\gamma)$.

The first step is to apply the minimum degree form of Szemerédi's Regularity Lemma to G, with

$$d := \frac{\gamma}{2}, \qquad k_0 := r^2 \qquad \text{and} \qquad \varepsilon := \min\left\{\frac{\gamma}{2}, \frac{d^2}{2d+2|H|}\right\}. \tag{1}$$

We obtain a partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_k$, where $k_0 \leqslant k \leqslant k_1 = k_1(\varepsilon, d, k_0)$, with an (ε, d)-reduced graph R such that

$$\delta(R) \,\overset{(1)}{\geqslant}\, \left(\frac{2r-5}{2r-3} + \gamma\right) k.$$

We now partition the vertices of $V(G)$ depending upon the collection of the sets V_i to which they send 'many' edges. More precisely, define $V(G) = \bigcup_{I \subseteq [k]} X_I$ by setting

$$X_I := \Big\{ v \in V(G) :\, i \in I \Leftrightarrow |N(v) \cap V_i| \geqslant d|V_i| \Big\},$$

for each $I \subseteq [k]$. We claim that $\chi\big(G[X_I]\big) \leqslant \max\{C_1, C_2 + 1\}$ for all $I \subseteq [k]$, where $C_1 = C_1(H,\gamma)$ and $C_2 = C_2(H,\gamma)$ are constants defined below. Since the X_I form a partition, this implies that $\chi(G) \leqslant 2^k \max\{C_1, C_2+1\} \leqslant 2^{k_1} \max\{C_1, C_2+1\} = C(H,\gamma)$ as desired.

In order to establish this claim we distinguish two cases.

Case 1: $|I| \,\geqslant\, \left(\dfrac{2r-4}{2r-3}\right) k.$

In this case we shall show that $|X_I| \leqslant C_1$ (where C_1 is a constant defined below, and independent of n), and thus trivially $\chi\big(G[X_I]\big) \leqslant C_1$. We first claim that $R[I]$ contains a copy of K_{r-1}. Indeed, by our minimum degree condition on R, we have

$$\delta\big(R[I]\big) \,\geqslant\, \delta(R) - \big(k - |I|\big) \,\geqslant\, |I| - \left(\frac{2}{2r-3} - \gamma\right) k \,\geqslant\, \left(\frac{r-3}{r-2} + \gamma\right) |I|.$$

Thus, by Turán's Theorem (or simply by proceeding greedily), $R[I]$ contains a copy of K_{r-1}, as claimed. Let $\{W_1, \ldots, W_{r-1}\} \subseteq \{V_1, \ldots, V_k\}$ be the set of parts corresponding to this copy of K_{r-1}.

Now let $x \in X_I$, set $W_i' = N(x) \cap W_i$ for each $i \in [r-1]$, and note that $|W_i'| \geqslant d|W_i|$, by the definition of X_I. By Fact 3, each pair (W_i', W_j'), $i \neq j$, is $(\varepsilon/d, d-\varepsilon)$-regular. By the Counting Lemma and (1), it follows that $G[N(x)]$ contains at least

$$\alpha_1 n^{(r-1)|H|}$$

copies of $K_{r-1}(|H|)$, for some $\alpha_1 = \alpha_1(H,\gamma) > 0$.

Thus, by Lemma 9(b) (applied with $\alpha = \alpha_1$, $s = r-2$ and $X = X_I$), we have either $H \subseteq G$, a contradiction, or $|X_I| \leqslant |H|/\alpha_1 = C_1(H,\gamma)$, as claimed.

Case 2: $|I| \leqslant \left(\dfrac{2r-4}{2r-3}\right) k$.

In this case we shall show that $G[X_I]$ is C_2-degenerate (where C_2 is defined below and independent of n), using Lemma 10. It follows that $\chi\big(G[X_I]\big) \leqslant C_2 + 1$. First, we shall show that every edge of $G[X_I]$ is contained in at least $\gamma^{r-2} n^{r-2}$ copies of K_r.

Let $V_I := \bigcup_{i \in I} V_i$ denote the set of vertices in clusters corresponding to I, and let $xy \in E(G[X_I])$. By the definition of X_I, x and y each have at most $(d+\varepsilon)n$ neighbours outside V_I, and thus, since $d + \varepsilon < \gamma$, they each have at least $\big(\frac{2r-5}{2r-3} + \gamma\big) n$ neighbours in V_I. It follows that they have at least $2\big(\frac{2r-5}{2r-3} + \gamma\big) - |V_I|$ common neighbours in V_I. Finally, since $|I| \leqslant \big(\frac{2r-4}{2r-3}\big)k$, we have $|V_I| \leqslant \big(\frac{2r-4}{2r-3}\big)n$, and thus x and y have at least

$$\left(\frac{2r-6}{2r-3} + \gamma\right) n$$

common neighbours in V_I.

Now, apply Lemma 9(a) with $\alpha = \gamma^{r-2}$, $\delta = \frac{2r-5}{2r-3} + \gamma$, $s = r-2$, and $X = N(x) \cap N(y)$. We have

$$\frac{2r-6}{2r-3} + \gamma \; = \; (r-2)\gamma + (r-3)\left(1 - \frac{2r-5}{2r-3} - \gamma\right),$$

and so $G[N(x) \cap N(y)]$ contains at least $\gamma^{r-2} n^{r-2}$ copies of K_{r-2}, i.e., every edge of $G[X_I]$ is contained in $\gamma^{r-2} n^{r-2}$ copies of K_r. Hence, by Lemma 10 (applied with $\alpha = \gamma^{r-2}$, $s = r-2$ and $t = |H|$), there exists $\alpha' = \alpha'(\gamma^{r-2}, r-2, |H|) > 0$ such that $G[X_I]$ is $2|H|/\alpha' = C_2$-degenerate, and so $\chi\big(G[X_I]\big) \leqslant C_2 + 1$, as required. $\square$

4. Borsuk-Hajnal graphs

In this section we shall describe the constructions (based on those in [30]) which provide the lower bounds on $\delta_\chi(H)$ in Theorem 2. One of the main building blocks in these constructions is a class of graphs which also mark the first (and most famous) application of algebraic topology in combinatorics: the *Kneser graphs* $\mathrm{Kn}(n,k)$, which are defined as follows. Given $n, k \in \mathbb{N}$, let $\mathrm{Kn}(n,k)$ have vertex set $\binom{[n]}{k}$, the family of k-vertex subsets of $[n]$, and let $\{S, T\}$ be an edge if and only if S and T are disjoint. (For example, $\mathrm{Kn}(5,2)$ is the well-known Petersen graph.) These graphs were first studied by Kneser [21], who conjectured that $\chi\big(\mathrm{Kn}(n,k)\big) = n - 2k + 2$ for every n and k. This problem stood open for 23 years, until it was solved by Lovász [28], whose proof led eventually to a new area, known as Topological Combinatorics (see [32], for example).

Hajnal (see [14]) used the Kneser graphs in order to give the first examples of dense triangle-free graphs with high chromatic number. Given $k, \ell, m \in \mathbb{N}$ such that $2m + k$ divides ℓ, let the *Hajnal graph*, denoted $H(k, \ell, m)$, be the graph obtained as follows: first take a copy of $\mathrm{Kn}(2m + k, m)$, and a complete bipartite graph $K_{2\ell,\ell}$, with vertex set $A \cup B$, where $|A| = 2\ell$, and $|B| = \ell$; next partition A into $2m + k$ equally sized pieces $A_1, \ldots, A_{2m+k}$; finally, add an edge between $S \in V\big(\mathrm{Kn}(2m + k, m)\big)$ and $y \in A_j$ whenever $j \in S$.

The following theorem, which appeared in [14], implies that $\delta_\chi(K_3) \geqslant 1/3$.

Theorem 11 (Hajnal, 1973)**.** *For all $\nu > 0$ and $k \in \mathbb{N}$ there exist integers m and ℓ_0 such that, for every $\ell \geqslant \ell_0$, the Hajnal graph $G = H(k, \ell, m)$ satisfies $v(G) = 3\ell + \binom{2m+k}{m}$, $\chi(G) \geqslant k + 2$ and $\delta(G) \geqslant (\frac{1}{3} - \nu)v(G)$, and is triangle-free.*

In order to generalise Theorem 11 from triangles to arbitrary 3-chromatic graphs which are not near-acyclic, Łuczak and Thomassé [30] defined the so-called Borsuk-Hajnal graphs. We shall next describe their construction.

The *Borsuk graph* $\mathrm{Bor}(k, \varepsilon)$ has vertex set S^k, the k-dimensional unit sphere, and edge set $\{xy \colon \measuredangle(x, y) \geqslant \pi - \varepsilon\}$, where $\measuredangle(x, y)$ denotes the angle between the vectors x and y. It follows from the Borsuk-Ulam Theorem (see [32], for example) that $\chi\big(\mathrm{Bor}(k, \varepsilon)\big) \geqslant k + 2$ for any $\varepsilon > 0$.

In order to construct Borsuk-Hajnal graphs from Borsuk graphs, we also need the following theorem, which follows easily from a result of of Nešetřil and Zhu [33].

Theorem 12. *Given $\ell \in \mathbb{N}$ and a graph G, there exists a graph G' with girth at least ℓ, such that $\chi(G') = \chi(G)$, and such that there exists a homomorphism ϕ from G' to G.*

Now, given $k \in \mathbb{N}$, a set $W \subseteq S^k$ with $|W|$ even, and $\varepsilon, \delta > 0$, we define the Borsuk-Hajnal graph, $\mathrm{BH} = \mathrm{BH}(W; k, \varepsilon, \delta)$, as follows.

First, let $B = \mathrm{Bor}(k, \varepsilon)$ be the Borsuk graph, and let $U \subseteq S^k = V(B)$ be a finite set, with U chosen such that $\chi(B[U]) = k + 2$. (This is possible by the de Bruijn-Erdős Theorem [11], which states that every infinite graph with chromatic number k' has a finite subgraph with chromatic number k'.) Let B' denote the graph given by Theorem 12, applied with $G = B[U]$ and $\ell = k$, let ϕ be the corresponding homomorphism from B' to $B[U]$, and let U' be the vertex set of B'.

Let X be a set of size $|W|/2$, and recall that $K[W, X]$ denotes the edge set of the complete bipartite graph with parts W and X.

Definition 13 (The Borsuk-Hajnal graph)**.** Define $\mathrm{BH} = \mathrm{BH}(W; k, \varepsilon, \delta)$ to be the graph on vertex set $U' \cup W \cup X$, where U', W and X are pairwise disjoint and as described above, with the following edges:

$$E(B') \cup K[W, X] \cup \Big\{ \{u, w\} : u \in U',\ w \in W \text{ and } \measuredangle\big(\phi(u), w\big) < \frac{\pi}{2} - \delta \Big\}.$$

Observe that $\chi(\mathrm{BH}) \geqslant \chi\big(B[U]\big) > k$.

Theorem 14 (Łuczak and Thomassé [30])**.** *For every $k \in \mathbb{N}$ and $\nu > 0$, there exist $\varepsilon, \delta > 0$ and $W \subseteq S^k$, such that, setting $\mathrm{BH} = \mathrm{BH}(W; k, \varepsilon, \delta)$, we have*

$$\chi(\mathrm{BH}) \,\geqslant\, k \qquad \textit{and} \qquad \delta(\mathrm{BH}) \,\geqslant\, \left(\frac{1}{3} - \nu\right) v(\mathrm{BH}).$$

Moreover every subgraph $H \subseteq \mathrm{BH}$ with $v(H) < k$ and $\chi(H) = 3$ is near-acyclic.

Hence, for any H with $\chi(H) = 3$ which is not near-acyclic, we have $\delta_\chi(H) \geqslant 1/3$.

We shall generalise the Łuczak-Thomassé construction further, as follows, to give our claimed lower bound on $\delta_\chi(H)$ for r-chromatic H which are not r-near-acyclic.

Definition 15 (The r-Borsuk-Hajnal graph)**.** Define $\mathrm{BH}_r(W; k, \varepsilon, \delta)$ to be the graph obtained from the Borsuk-Hajnal graph $\mathrm{BH} = \mathrm{BH}(W; k, \varepsilon, \delta)$ by adding $r - 3$ independent sets $Y_1, \ldots, Y_{r-3}$ of size $|Y_1| = \ldots = |Y_{r-3}| = |W|$, and the following edges:

$$\bigcup_{1 \leqslant i < j \leqslant r-3} K[Y_i, Y_j] \,\cup\, \bigcup_{i=1}^{r-3} K[Y_i, V(\mathrm{BH})].$$

That is, we add the complete $(r-2)$-partite graph on $V(\mathrm{BH}) \cup Y_1 \cup \ldots \cup Y_{r-3}$.

The following result extends Theorem 14 to arbitrary $r \geqslant 3$.

Theorem 16. *For every $r \geqslant 3$, $k \in \mathbb{N}$ and $\nu > 0$, there exist $\varepsilon, \delta > 0$ and $W \subseteq S^k$, such that, setting $\mathrm{BH}_r = \mathrm{BH}_r(W; k, \varepsilon, \delta)$, we have*

$$\chi(\mathrm{BH}_r) \,\geqslant\, k \qquad \textit{and} \qquad \delta(\mathrm{BH}_r) \,\geqslant\, \left(\frac{2r-5}{2r-3} - \nu\right) v(\mathrm{BH}_r).$$

Moreover every subgraph $H \subseteq \mathrm{BH}_r$ with $v(H) < k$ and $\chi(H) = r$ is r-near-acyclic.

Hence, for any H with $\chi(H) = r$ which is not r-near-acyclic, we have $\delta_\chi(H) \geqslant \frac{2r-5}{2r-3}$.

Theorem 16 follows easily from Łuczak and Thomassé's argument for Theorem 14; for completeness, we shall provide a proof here.

Proof of Theorem 16. As noted above, we have $\chi\big(\mathrm{BH}_r(W; k, \varepsilon, \delta)\big) > k$ for every choice of W, ε and δ. For the other properties, we shall choose W randomly, and $\varepsilon, \delta > 0$ as follows.

Let $r \geqslant 3$, $k \in \mathbb{N}$, and $\nu > 0$ be arbitrary, and choose $\delta > 0$ such that the spherical cap of S^k (centred around the pole) with polar angle $\frac{\pi}{2} - \delta$ covers a $(\frac{1}{2} - \frac{\nu}{2})$-fraction of S^k. Set $\varepsilon = \delta/(2k)$,

$$u_0 \,:=\, v\big(\mathrm{BH}_r(\emptyset; k, \varepsilon, \delta)\big) \,=\, |U'|,$$

and choose w_0 sufficiently large such that

$$\text{(2)} \quad \exp\left(-\frac{\nu^2 w_0}{4}\right) < \frac{1}{u_0} \quad \text{and} \quad \left(\frac{2r-5}{2} - \nu\right) w_0 \geqslant \left(\frac{2r-5}{2r-3} - \nu\right)\left(\frac{2r-3}{2} w_0 + u_0\right),$$

which is possible because $(2r-3)/2 > 1$. Draw w_0 points uniformly at random from S^k, call the resulting set W and consider the graph $\mathrm{BH}_r = \mathrm{BH}_r(W; k, \varepsilon, \delta)$.

We show first that, with positive probability, BH_r has the desired minimum degree. Let $Y = Y_1 \cup \ldots \cup Y_{r-3}$, and recall that BH_r has vertex set $U' \cup W \cup X \cup Y$ and that $|U'| = u_0$.

Claim 17. With positive probability the following holds. For every $v \in V(\mathrm{BH}_r)$,

$$\deg_{\mathrm{BH}_r}(v) \geqslant \left(\frac{2r-5}{2} - \nu\right)|W|.$$

Proof of Claim 17. For $v \in W \cup X \cup Y$, it is easy to check that $\deg_{\mathrm{BH}_r}(v) \geqslant \left(\frac{2r-5}{2}\right)|W|$. Moreover, if $v \in U'$ then $Y \subseteq N(v)$ and $|Y| = (r-3)|W|$. Thus it will suffice to show that the following event σ holds with positive probability: For every $v \in U'$ we have $\deg_{\mathrm{BH}_r}(v, W) \geqslant (\frac{1}{2} - \nu)|W|$.

To this end observe that, for a given $v \in U'$, the value of $\deg_{\mathrm{BH}_r}(v, W)$ is a random variable B with distribution $\mathrm{Bin}\big(|W|, \frac{1}{2} - \frac{\nu}{2}\big)$. This follows because W was chosen uniformly at random from S^k, by our choice of δ, and since by Definition 13, v is adjacent to $w \in W$ if and only if $\measuredangle(\phi(v), w) \leqslant \frac{\pi}{2} - \delta$. Thus, by Chernoff's inequality (see, e.g., [19, Chapter 2]),

$$\mathbb{P}\Big(\deg_{\mathrm{BH}_r}(v, W) < \big(\tfrac{1}{2} - \nu\big)|W|\Big) \leqslant \exp\left(-\frac{\nu^2|W|}{4}\right) \stackrel{(2)}{<} \frac{1}{|U'|}.$$

By the union bound, the event σ holds with positive probability, as required. □

Using (2), we have

$$\left(\frac{2r-5}{2} - \nu\right)|W| \geqslant \left(\frac{2r-5}{2r-3} - \nu\right)v(\mathrm{BH}_r),$$

and so the desired lower bound on $\delta(\mathrm{BH}_r)$ follows immediately from the claim.

Finally, let us show that every subgraph $H \subseteq \mathrm{BH}_r$ with $v(H) < k$ and $\chi(H) = r$ is r-near-acyclic. We begin by showing that $H' := H[U' \cup W \cup X]$ is near-acyclic.

Observe first that $H'[W]$ is independent, and recall (from Definition 13) that $\mathrm{BH}_r[U']$ has girth at least $k > v(H)$. Thus $H'[U' \cup X]$ is a forest, since all of its edges are contained in U'. It therefore suffices to prove the following claim.

Claim 18. Every odd cycle in H' contains at least two vertices of W.

Proof of Claim 18. Let C be an odd cycle in H'. (Hence $v(C) < k$.) If $V(C) \cap X \neq \emptyset$ then $|V(C) \cap W| \geqslant 2$ since $e(U', X) = 0$ and X is independent. Thus we may assume that $V(C) \cap X = \emptyset$. Similarly, since $H'[U']$ is a forest, we must have $|V(C) \cap W| \geqslant 1$.

Let $P = v_1 \dots v_p$ be a path in U' with $p < k$ and p even. Recall that $\phi(v_1), \dots, \phi(v_p)$ are vectors from S^k such that $\measuredangle\big(\phi(v_i), \phi(v_{i+1})\big) \geqslant \pi - \varepsilon$ for all $i \in [p-1]$. We shall show that $N_{\mathrm{BH}_r}(v_1) \cap N_{\mathrm{BH}_r}(v_p) \cap W = \emptyset$, i.e., that $\phi(v_1)$ and $\phi(v_p)$ do not lie in a common spherical cap with angle $\frac{\pi}{2} - \delta$. Indeed, we have $\measuredangle\big(\phi(v_1), \phi(v_3)\big) \leqslant 2\varepsilon$, and, in general, $\measuredangle\big(\phi(v_1), \phi(v_{2j})\big) \geqslant \pi - 2j\varepsilon$ for every $j \in [p/2]$. Hence $\measuredangle\big(\phi(v_1), \phi(v_p)\big) \geqslant \pi - k\varepsilon > 2(\frac{\pi}{2} - \delta)$, and so, by Definition 13, v_1 and v_p do not have a common neighbour in W, as required.

This implies that $V(C) \cap U'$ cannot be a path on $v(C) - 1$ vertices and thus we conclude $|V(C) \cap W| \geqslant 2$. □

Finally, note that as $H[Y_i]$ is an independent set for each $i \in [r-3]$, and H' is obtained by removing these sets, H is indeed r-near-acyclic, as required. □

5. Zykov graphs

In this section we shall prove a key result on Zykov graphs (see Definition 19 and Proposition 22, below), which will be an important tool in our proof of Theorem 2. Let G' be a graph with connected components $C_1, \ldots, C_m$, and let G be the graph obtained from G' by adding, for each m-tuple $\mathbf{u} = (u_1, \ldots, u_m) \in C_1 \times \cdots \times C_m$, a vertex $v_{\mathbf{u}}$ adjacent to each u_j. This construction was introduced by Zykov [42] in order to obtain triangle-free graphs with high chromatic number.

We shall use the following slight modification of Zykov's construction. Recall that to blow up a vertex $v \in V(G)$ to size t means to replace v by an independent set of size t, and replace each edge containing v by a complete bipartite graph, and that $K(v, X)$ denotes the set of pairs $\{vx : x \in X\}$.

Definition 19 (Modified Zykov graphs)**.** Let $T_1, \ldots, T_\ell$ be (disjoint) trees, and let T_j have bipartition $A_j \cup B_j$. We define $Z_\ell(T_1, \ldots, T_\ell)$ to be the graph on vertex set

$$V\big(Z_\ell(T_1, \ldots, T_\ell)\big) := \left(\bigcup_{j \in [\ell]} A_j \cup B_j\right) \cup \big\{u_I \colon I \subseteq [\ell]\big\}$$

and with edge set

$$E\big(Z_\ell(T_1, \ldots, T_\ell)\big) := \bigcup_{j=1}^{\ell} \left(E(T_j) \cup \bigcup_{j \in I \subseteq [\ell]} K\big(u_I, A_j\big) \cup \bigcup_{j \notin I \subseteq [\ell]} K\big(u_I, B_j\big) \right).$$

For each $r \geqslant 3$ and $t \in \mathbb{N}$, the *modified Zykov graph* $Z_\ell^{r,t}(T_1, \ldots, T_\ell)$ is the graph obtained from $Z_\ell(T_1, \ldots, T_\ell)$ by performing the following two operations:

(a) Add vertices $W = \{w_1, \ldots, w_{r-3}\}$, and all edges with an endpoint in W.

(b) Blow up each vertex u_I with $I \subseteq [\ell]$ and each vertex w_j in W to a set S_I or S'_j, respectively, of size t.

Finally, we shall write $Z_\ell^{r,t}$ for the modified Zykov graph obtained when each tree T_i, $i \in [\ell]$, is a single edge; that is, $Z_\ell^{r,t} = Z_\ell^{r,t}(e_1, \ldots, e_\ell)$.

Note that $Z_\ell^{r,t}$ has $(2^\ell + r - 3)t + 2\ell$ vertices, and that, in the special case $r = 3$ and $t = 1$, the graph $Z_\ell^{r,t}$ coincides with that obtained by Zykov's construction (described above) applied to a matching of size ℓ.

The following observation motivates (and follows immediately from) Definition 19.

Observation 20. *Let $\chi(H) = r$. Then H is r-near-acyclic if and only if there exist trees $T_1, \ldots, T_\ell$ and $t \in \mathbb{N}$ such that H is a subgraph of $Z_\ell^{r,t}(T_1, \ldots, T_\ell)$.*

Proof. Recall that H is r-near-acyclic if and only if there exist $r - 2$ independent sets $U_1, \ldots, U_{r-3}, W$ such that $H \setminus \big(W \cup \bigcup_j U_j\big)$ is a forest F whose components are trees $T_1, \ldots, T_\ell$ with the following property. For each $i \in [\ell]$, there is no vertex of W adjacent to vertices in both partition classes of T_i. If $H \subseteq Z_\ell^{r,t}(T_1, \ldots, T_\ell)$ then we can take $W = \bigcup_{I \subseteq [\ell]} S_I$ and $U_1, \ldots, U_{r-3}$ to be the sets $S'_1, \ldots, S'_{r-3}$, and so H is r-near-acyclic, as claimed.

Conversely if H is r-near-acyclic, then $H \subseteq Z_\ell^{r,t}(T_1, \dots, T_\ell)$, where $t = |H|$ and $T_1, \dots, T_\ell$ are the components of the forest F. $\square$

It will be convenient for us to provide a compact piece of notation for the adjacencies in $Z_\ell^{r,t}$. For this purpose, given a graph G and a set $Y \subseteq V(G)$, and integers $\ell, t \in \mathbb{N}$ and $r \geqslant 3$, define $\mathcal{G}_\ell^{r,t}(Y)$ to be the collection of functions

$$S \,:\, 2^{[\ell]} \cup [r-3] \,\to\, \binom{Y}{t}.$$

It is natural to think of S as a family $\{S_I : I \subseteq [\ell]\} \cup \{S'_j : j \in [r-3]\}$ of subsets of Y of size t. We say that $S \in \mathcal{G}_\ell^{r,t}(Y)$ is *proper* if these sets are pairwise disjoint and $E(G)$ contains all edges xy with $x \in S_I \cup S'_j$ and $y \in S'_{j'}$, whenever $j \neq j'$. We shall write $\mathcal{F}_\ell^{r,t}(Y)$ for the collection of proper functions in $\mathcal{G}_\ell^{r,t}(Y)$. The idea behind this definition is that we will later want to consider a vertex set $Y \subseteq V(G)$ and a family of disjoint subsets $\{S_I \colon I \subseteq [\ell]\} \cup \{S'_j \colon j \in [r-3]\}$ of size t in Y that we want to extend to a copy of $Z_\ell^{r,t}$.

For an ordered pair (x, y) of vertices of G, a function $S \in \mathcal{F}_\ell^{r,t}(Y)$, and $i \in [\ell]$, we write $(x, y) \to_i S$, if $S'_j \subseteq N(x, y)$ for every $j \in [r-3]$ and

$$\bigcup_{I \,:\, i \in I} S_I \subseteq N(x) \qquad \text{and} \qquad \bigcup_{I \,:\, i \notin I} S_I \subseteq N(y)\,.$$

For an edge $e = xy \in E(G)$, we write $e \to_i S$ if either $(x, y) \to_i S$ or $(y, x) \to_i S$. Recall that $\mathbf{e}^\ell$ denotes the ℓ-tuple $(e_1, \dots, e_\ell)$, with $\mathbf{e}^0$ the empty tuple. Define

$$\mathbf{e}^\ell \to S \qquad \Leftrightarrow \qquad e_i \to_i S \quad \text{for each } i \in [\ell]\,.$$

Observe that the graph $Z_\ell^{r,t}$ consists of a set of pairwise disjoint edges $e_1, \dots, e_\ell$ and an $S \in \mathcal{F}_\ell^{r,t}(Y)$ such that $\mathbf{e}^\ell \to S$. An advantage of this notation is that we can write $\mathbf{e}^\ell \to S$ even if the edges in $\mathbf{e}^\ell$ are not pairwise disjoint. This will greatly clarify our proofs.

In Section 6, we shall show how to find a well-structured set of many copies of $Z_\ell^{r,t}$ inside a graph with high minimum degree and high chromatic number. The following definition (in which we shall make use of the compact notation just defined) makes the concept of 'well-structured' precise. Recall that, given $X \subseteq V(G)$, we write $E(X)$ for the edge set of $G[X]$, and that if $D \subseteq E(G)$, then $\delta(D)$ denotes the minimum degree of the graph $G[D]$.

Definition 21 ((C, α)-rich in copies of $Z_\ell^{r,t}$)**.** Let X and Y be disjoint vertex sets in a graph G, let $C \in \mathbb{N}$ and $\alpha > 0$, and let $s := (2^\ell + r - 3)t$. We say that (X, Y) is (C, α)-*rich* in copies of $Z_\ell^{r,t}$ if

$$\exists\, D = D(\mathbf{e}^0) \subseteq E(X)\, \forall\, e_1 \in D\, \exists\, D(\mathbf{e}^1) \subseteq E(X)\, \forall\, e_2 \in D(\mathbf{e}^1) \quad \dots$$
$$\dots \quad \forall\, e_{\ell-1} \in D(\mathbf{e}^{\ell-2})\, \exists\, D(\mathbf{e}^{\ell-1}) \subseteq E(X)\, \forall\, e_\ell \in D(\mathbf{e}^{\ell-1})$$

the following properties hold:

(a) $\delta(D), \delta\big(D(\mathbf{e}^1)\big), \dots, \delta\big(D(\mathbf{e}^{\ell-1})\big) > C$, and
(b) $\big|\big\{S \in \mathcal{F}_\ell^{r,t}(Y) \,:\, \mathbf{e}^\ell \to S\big\}\big| \geqslant \alpha |Y|^s$.

If (X, Y) is (C, α)-rich in copies of $Z_\ell^{r,t}$, then, for each $q \in [\ell]$, define

$$\mathcal{D}_q(X, Y) := \Big\{ \mathbf{e}^q \in E(X)^q \,:\, e_j \in D(\mathbf{e}^{j-1}) \text{ for each } j \in [q] \Big\}, \tag{3}$$

where $D(\mathbf{e}^0) := D$.

The aim of this section is to prove the following proposition, which says that if some pair (X, Y) in G is (C, α)-rich in copies of $Z_\ell^{r,t}$ (where $\alpha > 0$ and C is sufficiently large), then for any 'small' $T_1, \ldots, T_\ell$ we have $Z_\ell^{r,t}(T_1, \ldots, T_\ell) \subseteq G$, and hence (by Observation 20) G is not H-free for any r-near-acyclic graph H.

Proposition 22. *Let G be a graph, and let X and Y be disjoint subsets of its vertices. Let $r, \ell, t \in \mathbb{N}$, with $r \geqslant 3$, and let $\alpha > 0$. Let $T_1, \ldots, T_\ell$ be trees, and set $C := 2^{\ell+3} \alpha^{-1} \sum_{i=1}^{\ell} |T_i|$. If (X, Y) is (C, α)-rich in copies of $Z_\ell^{r,t}$, then $Z_\ell^{r,t}(T_1, \ldots, T_\ell) \subseteq G$.*

The proof of Proposition 22 uses a double counting argument and proceeds by induction. We shall find a set of functions $\mathcal{S} \subseteq \mathcal{F}_\ell^{r,t}(Y)$ such that, for each $S \in \mathcal{S}$, we can construct (one by one) a collection of subgraphs $E_1, \ldots, E_\ell$ of $G[X]$ with the following properties: each subgraph has large average degree, and for *any* choice $e_1 \in E_1, \ldots, e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$. Recall that this simply says that $e_i \to_i S$ for every $e_i \in E_i$.

Let the graph G, disjoint subsets $X, Y \subseteq V(G)$, constants $\alpha > 0$ and $r, \ell, t \in \mathbb{N}$ with $r \geqslant 3$, and trees $T_1, \ldots, T_\ell$ be fixed for the rest of the section. Set $C := 2^{\ell+3} \alpha^{-1} \sum_{i=1}^{\ell} |T_i|$ and $s := (2^\ell + r - 3)t$, and let $0 \leqslant q \leqslant \ell$. For our induction hypothesis we use the following definition.

Definition 23 (Good function, (C, α)-dense)**.** A function $S \in \mathcal{F}_\ell^{r,t}(Y)$ is (r, ℓ, t, C, α)-*good* for a tuple $\mathbf{e}^q$ and (X, Y) if there exist sets

$$E_{q+1}, \ldots, E_\ell \subseteq E(X), \quad \text{with} \quad \overline{d}(E_j) \geqslant 2^{-\ell} \alpha C \quad \text{for each } q+1 \leqslant j \leqslant \ell,$$

such that for every $e_{q+1} \in E_{q+1}, \ldots, e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$.

When the constants (r, ℓ, t, C, α) and the sets (X, Y) are clear from the context, we shall omit them. We shall abbreviate '(r, ℓ, t, C, α)-good for $\mathbf{e}^0$ and (X, Y)' to '(r, ℓ, t, C, α)-good for (X, Y)'.

The pair (X, Y) is (C, α)-*dense* in copies of $Z_\ell^{r,t}$ if there exist at least $2^{-\ell} \alpha |Y|^s$ families $S \in \mathcal{F}(Y)$ which are (r, ℓ, t, C, α)-good for (X, Y).

The next lemma constitutes the inductive argument in the proof of Proposition 22. The final assertion we shall also need in Section 7.

Lemma 24. *For any $0 \leqslant q \leqslant \ell$, if (X, Y) is (C, α)-rich in copies of $Z_\ell^{r,t}$, then*

$$\Big| \Big\{ S \in \mathcal{F}_\ell^{r,t}(Y) \,:\, S \text{ is good for } \mathbf{e}^q \Big\} \Big| \geqslant 2^{q-\ell} \alpha |Y|^s$$

for every $\mathbf{e}^q \in \mathcal{D}_q(X, Y)$.

In particular, if (X, Y) is (C, α)-rich in copies of $Z_\ell^{r,t}$, then (X, Y) is (C, α)-dense in copies of $Z_\ell^{r,t}$.

Proof. The proof is by induction on $\ell - q$. The base case, $q = \ell$, follows immediately from the definition of (C,α)-rich. Indeed, a function S is good for $\mathbf{e}^\ell$ if and only if $\mathbf{e}^\ell \to S$, and by Property (b) in Definition 21, we have $\big|\big\{S \in \mathcal{F}_\ell^{r,t}(Y) : \mathbf{e}^\ell \to S\big\}\big| \geqslant \alpha |Y|^s$.

So let $0 \leqslant q < \ell$, assume that the lemma holds for $q+1$, and let $\mathbf{e}^q \in \mathcal{D}_q(X,Y)$. Set $\beta = \alpha 2^{q-\ell}$. By Definition 21, there exists a set $D(\mathbf{e}^q) \subseteq E(X)$ with $\delta\big(D(\mathbf{e}^q)\big) > C$, such that $\mathbf{e}^{q+1} \in \mathcal{D}_{q+1}(X,Y)$ for every $e_{q+1} \in D(\mathbf{e}^q)$. Thus, by the induction hypothesis,

$$\forall e_{q+1} \in D(\mathbf{e}^q) \text{ at least } 2\beta|Y|^s \text{ functions } S \in \mathcal{F}_\ell^{r,t}(Y) \text{ are good for } \mathbf{e}^{q+1}\,, \tag{4}$$

that is, for each such S there exist sets

$$E_{q+2}, \ldots, E_\ell \subseteq E(X), \quad \text{with} \quad \overline{d}(E_j) \geqslant 2^{-\ell}\alpha C \quad \text{ for each } q+2 \leqslant j \leqslant \ell,$$

such that for every $e_{q+2} \in E_{q+2}, \ldots, e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$. It is crucial to observe that given S, if the edge sets $E_{q+2}, \ldots, E_\ell$ have this last property for *some* $e_{q+1} \in D(\mathbf{e}^q)$ with $\mathbf{e}^{q+1} \to S$, then $E_{q+2}, \ldots, E_\ell$ have this property for *all* $e_{q+1} \in D(\mathbf{e}^q)$ with $\mathbf{e}^{q+1} \to S$.

The $S \in \mathcal{F}_\ell^{r,t}(Y)$ that will be good for $\mathbf{e}^q$ are those which are good for many $\mathbf{e}^{q+1}$. More precisely, for each $S \in \mathcal{F}_\ell^{r,t}(Y)$, let

$$W_S := \big\{e_{q+1} \in D(\mathbf{e}^q) : S \text{ is good for } \mathbf{e}^{q+1}\big\},$$

and let $Z = \big\{S \in \mathcal{F}_\ell^{r,t}(Y) : |W_S| \geqslant \beta|D(\mathbf{e}^q)|\big\}$. By (4) the number of pairs (e_{q+1}, S) with e_{q+1} in W_S is at least $|D(\mathbf{e}^q)| \cdot 2\beta|Y|^s$. On the other hand, for every $S \in Z$ there are at most $|D(\mathbf{e}^q)|$ pairs (e_{q+1}, S) with e_{q+1} in W_S, and for every $S \in \mathcal{F}_\ell^{r,t}(Y) \setminus Z$, there are (by definition of Z) at most $\beta|D(\mathbf{e}^q)|$ such pairs. Putting these together, we obtain

$$|D(\mathbf{e}^q)| \cdot 2\beta|Y|^s \leqslant |D(\mathbf{e}^q)||Z| + \beta|D(\mathbf{e}^q)||Y|^s$$

and hence $|Z| \geqslant \beta|Y|^s$.

We claim that every $S \in Z$ is good for $\mathbf{e}^q$. Indeed, fix $S \in Z$. Set $E_{q+1} = W_S$, and let $E_{q+2}, \ldots, E_\ell$ be the sets defined above (for any, and thus all, $e_{q+1} \in E_{q+1}$), i.e., those obtained by the induction hypothesis. Since $S \in Z$ we have $|W_S| \geqslant \beta|D(\mathbf{e}^q)|$, so it follows from $\delta\big(D(\mathbf{e}^q)\big) > C$ that $\overline{d}(E_{q+1}) \geqslant \beta C \geqslant 2^{-\ell}\alpha C$. Since S is good for $\mathbf{e}^{q+1}$ for every $e_{q+1} \in E_{q+1}$, we have $e_i \to_i S$ for every $1 \leqslant i \leqslant q+1$, and by the induction hypothesis, we have $e_i \to_i S$ for every $e_i \in E_i$ and every $q+2 \leqslant i \leqslant \ell$. Thus $\mathbf{e}^\ell \to S$ for every such $\mathbf{e}^\ell$, as required. Since $|Z| \geqslant \beta|Y|^s$, this completes the induction step, and hence the proof of the lemma. □

Lemma 24 shows that richness in copies of $Z_\ell^{r,t}$ implies denseness in copies of $Z_\ell^{r,t}$. Observe that if (X,Y) is dense in copies of $Z_\ell^{r,t}$, then in particular there is a function $S \in \mathcal{F}_\ell^{r,t}(Y)$ which is good for (X,Y). The next lemma now shows that in this case we have $Z_\ell^{r,t}(T_1, \ldots, T_\ell) \subseteq G$.

Lemma 25. *Let X and Y be disjoint vertex sets in G. Given $r, \ell, t \in \mathbb{N}$, $\alpha > 0$, and trees $T_1, \ldots, T_\ell$, if $C \geqslant 2^{\ell+3}\alpha^{-1}\sum_{i=1}^{\ell}|T_i|$ and $S \in \mathcal{F}_\ell^{r,t}(Y)$ is (r,ℓ,t,C,α)-good for (X,Y), then $Z_\ell^{r,t}(T_1, \ldots, T_\ell) \subseteq G$.*

Proof. Let S be (r,ℓ,t,C,α)-good for (X,Y). Then there exist sets

$$E_1,\dots,E_\ell \subseteq E(X), \quad \text{with} \quad \overline{d}(E_j) \geqslant 2^{-\ell}\alpha C \quad \text{for each } 1 \leqslant j \leqslant \ell,$$

such that for every $e_1 \in E_1,\dots,e_\ell \in E_\ell$, we have $\mathbf{e}^\ell \to S$.

For each $j \in [\ell]$ and each edge $e \in E_j$, let $e = xy$ be such that $(x,y) \to_j S$, and orient the edge e from x to y. Recall that $C \geqslant 2^{\ell+3}\alpha^{-1}\sum_{i=1}^{\ell}|T_i|$, and so $\overline{d}(E_j) \geqslant 8\sum_{i=1}^{\ell}|T_i|$ for each $j \in [\ell]$. For each $j \in [\ell]$, by choosing a maximal bipartite subgraph of E_j, and then removing at most half the edges, we can find a set $E_j' \subseteq E_j$ such that,

(*a*) E_j' is bipartite, with bipartition (A_j,B_j),
(*b*) every edge $e \in E_j'$ is oriented from A_j to B_j, and
(*c*) $\overline{d}(E_j') \geqslant 2\sum_{i=1}^{\ell}|T_i|$.

Thus, by Fact 4, there exists, for each $j \in [\ell]$, a copy T_j' of T_j in

$$E_j' - \big(V(T_1') \cup \ldots \cup V(T_{j-1}')\big),$$

since removing a vertex can only decrease the average degree by at most two. These trees, together with S, form a copy of $Z_\ell^{r,t}(T_1,\dots,T_\ell)$ in G, so we are done. □

It is now easy to deduce Proposition 22 from Lemma 24 and Lemma 25.

Proof of Proposition 22. Let (X,Y) be (C,α)-rich in copies of $Z_\ell^{r,t}$, and apply Lemma 24 to (X,Y) with $q=0$. Note that $\mathcal{D}_0(X,Y) = \{\mathbf{e}^0\}$ consists of the tuple of length zero, and let $\mathcal{S} = \big\{S \in \mathcal{F}_\ell^{r,t}(Y) : S \text{ is good for } \mathbf{e}^0\big\}$. Then $|\mathcal{S}| \geqslant \alpha 2^{-\ell}|Y|^s$, and so in particular $\mathcal{S}$ is non-empty. Let $S \in \mathcal{S}$, and apply Lemma 25 to obtain a copy of $Z_\ell^{r,t}(T_1,\dots,T_\ell)$ in G. □

6. The paired VC-dimension argument

In this section we shall modify and extend a technique which was introduced by Łuczak and Thomassé [30], and used by them to prove Conjecture 1 in the case where H is near-bipartite. This technique is based on the concept of *paired VC-dimension*, which generalises the well-known Vapnik-Červonenkis dimension of a set-system (see [35, 41]). We shall not state our proof in the abstract setting of paired VC-dimension, which is more general than that which we shall require, but we refer the interested reader to [30] for the definition and further details.

We shall use the paired VC-dimension (or 'booster tree') argument of Łuczak and Thomassé in order to prove the following result, which may be thought of as a 'counting version' of Theorem 5 in [30]. The case $r=3$ of Theorem 2 will follow as an easy consequence of Propositions 26 and 22 (see Section 7).

Proposition 26. *For every $\ell,t \in \mathbb{N}$ and $d>0$, there exists $\alpha>0$ such that, for every $C \in \mathbb{N}$, there exists $C' \in \mathbb{N}$ such that the following holds. Let G be a graph and let X and Y be disjoint subsets of $V(G)$, such that $|N(x) \cap Y| \geqslant d|Y|$ for every $x \in X$.*

Then either $\chi\big(G[X]\big) \leqslant C'$, or (X,Y) is (C,α)-rich in copies of $Z_\ell^{3,t}$.

In order to prove Proposition 26, we shall break X up into a bounded number of suitable pieces, $X_1, \ldots, X_m$, and show that either $G[X_j]$ has bounded chromatic number, or (X_j, Y) is (C, α)-rich in copies of $Z_\ell^{3,t}$.

Let $d(x, Y) := d\big(\{x\}, Y\big) = \big|N(x) \cap Y\big|/|Y| = e\big(\{x\}, Y\big)/|Y|$ be the density of the neighbours of x in Y. The key definition, which will allow us to choose the sets X_j, is as follows.

Definition 27 (Boosters)**.** Let G be a graph, let X and Y be disjoint subsets of $V(G)$, and let $\varepsilon > 0$. We say that $x \in X$ is *ε-boosted* by $Y' \subseteq Y$ if $d(x, Y') \geqslant (1 + \varepsilon) d(x, Y)$.

Now let $C, p \in \mathbb{N}$, and let $\beta > 0$. Let $Y_1 \cup \ldots \cup Y_p$ of Y be a partition of Y, and $X_0 \cup \ldots \cup X_p$ be a partition of X. We say that $(X_0, \emptyset), (X_1, Y_1), \ldots, (X_p, Y_p)$ is a *$(p, C, \varepsilon, \beta)$-booster* of (X, Y) if

- (a) $G[X_0]$ is C-degenerate.
- (b) Every $x \in X_j$ is ε-boosted by Y_j, for each $j \in [p]$.
- (c) $|Y_j| \geqslant \beta |Y|$ for every $j \in [p]$.

We say that a partition $\{Y_1, \ldots, Y_p\}$ of Y induces a $(p, C, \varepsilon, \beta)$-booster of (X, Y) if there exists a partition $X_0 \cup \ldots \cup X_p$ of X such that $(X_0, \emptyset), (X_1, Y_1), \ldots, (X_p, Y_p)$ is a $(p, C, \varepsilon, \beta)$-*booster* of (X, Y).

We remark that this is slightly different from the definition of a p-booster in Section 5 of [30], where Condition (a) was replaced by '$G[X_0]$ is independent', and Condition (c) was missing.

Using Definition 27, we can now state the second key definition.

Definition 28 (Booster trees)**.** Let $C, p_0 \in \mathbb{N}$ and $\beta, \varepsilon > 0$. A *$(p_0, C, \varepsilon, \beta)$-booster tree* for (X, Y) is an oriented rooted tree $\mathcal{T}$, whose vertices are pairs (X', Y') such that $X' \subseteq X$ and $Y' \subseteq Y$, and all of whose edges are oriented away from the root, such that the following conditions hold:

- (a) The root of $\mathcal{T}$ is (X, Y).
- (b) No vertex of $\mathcal{T}$ has more than p_0 out-neighbours.
- (c) The out-neighbourhood of each non-leaf (X', Y') of $\mathcal{T}$ forms a $(p, C, \varepsilon, \beta)$-booster of (X', Y') for some $p \leqslant p_0$.
- (d) If (X', Y') is a leaf of $\mathcal{T}$, then either $G[X']$ is C-degenerate, or there does not exist a $(p, C, \varepsilon, \beta)$-booster for (X', Y') for any $p \leqslant p_0$.

A vertex (X', Y') of $\mathcal{T}$ is called *degenerate* if it is a leaf of $\mathcal{T}$ and $G[X']$ is C-degenerate.

The following lemma is immediate from the definitions.

Lemma 29. *Let $C, p_0 \in \mathbb{N}$, and $\beta, \varepsilon > 0$, let G be a graph, and let X and Y be disjoint subsets of $V(G)$. If $d(x, Y) \geqslant d$ for every $x \in X$, then there exists a $(p_0, C, \varepsilon, \beta)$-booster tree $\mathcal{T}$ for (X, Y) such that $|\mathcal{T}|$ is bounded as a function of ε, d and p_0.*

Moreover, if (X', Y') is a non-degenerate vertex of $\mathcal{T}$, then $d(x, Y') \geqslant d$ for all $x \in X'$, and $|Y'| \geqslant \beta^{|\mathcal{T}|} |Y|$.

Proof. We construct $\mathcal{T}$, with root (X,Y), as follows: We simply repeatedly choose a (p,C,ε,β)-booster $(X_0,\emptyset),(X_1,Y_1),\dots,(X_p,Y_p)$ for each leaf (X',Y') of $\mathcal{T}$ such that $G[X']$ is not C-degenerate, until this is no longer possible for any $p \leqslant p_0$. We add to $\mathcal{T}$ the vertices $(X_0,\emptyset),(X_1,Y_1),\dots,(X_p,Y_p)$ as out-neighbours of (X',Y').

By the definition of 'ε-boosted' and the construction of $\mathcal{T}$, if (X',Y') is a non-degenerate vertex of $\mathcal{T}$ at distance t from the root, we have $d(x,Y') \geqslant (1+\varepsilon)^t d$ for every $x \in X'$, and we have $|Y'| \geqslant \beta^t|Y| \geqslant \beta^{|\mathcal{T}|}|Y|$. This both establishes that the height $h(\mathcal{T})$ of $\mathcal{T}$ is bounded in terms of ε and d, and that we have $d(x,Y') \geqslant d$ for every $x \in X'$. Since $\mathcal{T}$ has no vertex of out-degree greater than p_0 and $h(\mathcal{T})$ is bounded by a function of ε and d, it follows that $|\mathcal{T}|$ is bounded as a function of ε, d and p_0. □

The following lemma is the key step in the proof of Proposition 26.

Lemma 30. *Let $\ell,t \in \mathbb{N}$ and $d > 0$. Let $\beta = \left(\frac{d}{4}\right)^\ell$ and $\varepsilon = \beta/2$. There exists $\alpha > 0$ such that the following holds for every $C \in \mathbb{N}$. Let G be a graph, let X and Y be disjoint subsets of $V(G)$, and suppose that $|N(x) \cap Y| \geqslant d|Y|$ for every $x \in X$.*

If there does not exist a (p,C,ε,β)-booster of (X,Y) for any $p \leqslant 2^\ell$, then (X,Y) is (C,α)-rich in copies of $Z_\ell^{3,t}$.

Lemma 30 is proved by repeating a fairly straightforward algorithm ℓ times, at each step $q \in [\ell]$ finding a set $D(\mathbf{e}^q)$ as in the definition of (C,α)-richness (see Definition 21). In order to make the proof more transparent, we shall state a slightly more technical lemma, which is proved by induction on q, and from which Lemma 30 follows immediately.

The following definition will simplify the statement. It is a slight strengthening of the concept of (C,α)-richness in the case $r = 3$. Recall that $\mathbf{e}^\ell$ is just a shorthand for $(e_1,\dots,e_\ell)$ and $\mathbf{e}^0$ is the empty tuple. Further, recall the definitions of $\mathcal{F}_\ell^{r,t}$ and $\mathbf{e}^\ell \to S$ from Section 5.

Definition 31 ((C,α,ℓ)-Zykov)**.** Let X and Y be disjoint vertex sets in a graph G, let $C,\ell \in \mathbb{N}$ and $\alpha > 0$. We say that (X,Y) is (C,α,ℓ)-*Zykov* if

$$\exists\, D = D(\mathbf{e}^0) \subseteq E(X)\ \forall\, e_1 \in D\ \exists\, D(\mathbf{e}^1) \subseteq E(X)\ \forall\, e_2 \in D(\mathbf{e}^1) \quad \dots$$
$$\dots \quad \forall\, e_{\ell-1} \in D(\mathbf{e}^{\ell-2})\ \exists\, D(\mathbf{e}^{\ell-1}) \subseteq E(X)\ \forall\, e_\ell \in D(\mathbf{e}^{\ell-1})$$

the following properties hold:

(*a*) $\delta\big(D\big), \delta\big(D(\mathbf{e}^1)\big), \dots, \delta\big(D(\mathbf{e}^{\ell-1})\big) > C$, and

(*b*) $\exists\, S \in \mathcal{F}_\ell^{3,\alpha|Y|}(Y)$ such that $\mathbf{e}^\ell \to S$.

We remark that the requirement (*b*) of Definition 21 that each tuple $\mathbf{e}^\ell$ should extend to *many* copies of $Z_\ell^{3,t}$ is replaced in this definition by the requirement that $\mathbf{e}^\ell$ should extend to *one* much bigger copy of $Z_\ell^{3,\alpha|Y|}$. In particular, if (X,Y) is (C,α,ℓ)-Zykov, then, for any $t \in \mathbb{N}$, it is (C,α')-rich in copies of $Z_\ell^{3,t}$, where $\alpha' = \left(\frac{\alpha}{t}\right)^{2^\ell t}$ (this is shown in the proof of Lemma 30).

Lemma 32. *Let $\ell \in \mathbb{N}$ and $d > 0$. For $\beta = \left(\frac{d}{4}\right)^\ell$ and $\varepsilon = \beta/2$, the following holds for every $C \in \mathbb{N}$. Let G be a graph, let X and Y be disjoint subsets of $V(G)$, and suppose that $|N(x) \cap Y| \geqslant d|Y|$ for every $x \in X$.*

If there does not exist a $(p, C, \varepsilon, \beta)$-booster of (X, Y) for any $p \leqslant 2^\ell$, then (X, Y) is (C, α_q, q)-Zykov for every $q \in [\ell]$, where $\alpha_q = \left(\frac{d}{4}\right)^q$.

Proof. Let $C, \ell \in \mathbb{N}$ and $d > 0$, and let $\beta = \left(\frac{d}{4}\right)^\ell$ and $\varepsilon = \beta/2$. Let G and X, Y be as described in the statement, and suppose that there does not exist a $(p, C, \varepsilon, \beta)$-booster of (X, Y) for any $p \leqslant 2^\ell$. We proceed by induction.

We begin with the base case, $q = 1$. We are required to find a set $D = D(\mathbf{e}^0) \subseteq E(X)$, with $\delta(D) > C$, such that, for every $e_1 = xy \in D$, there exists $S(\mathbf{e}^1) \in \mathcal{F}_1^{3,\alpha_1|Y|}$ such that $\mathbf{e}^1 \to S(\mathbf{e}^1)$; that is, there exist disjoint sets $S_\emptyset(\mathbf{e}^1)$ and $S_{\{1\}}(\mathbf{e}^1)$ in Y, both of size $\alpha_1|Y| = \frac{d}{4}|Y|$, such that $S_{\{1\}}(\mathbf{e}^1) \subseteq N(x)$ and $S_\emptyset(\mathbf{e}^1) \subseteq N(y)$. Since there is no $(1, C, \varepsilon, \beta)$-booster of (X, Y), it follows that $G[X]$ is not C-degenerate, and so there exists a subgraph $G_0 \subseteq G[X]$ with $\delta(G_0) > C$. We choose $D := E(G_0)$. Now for each $e_1 = xy \in D$, let $A_\emptyset(\mathbf{e}^1) := N(y) \cap Y$ and $A_{\{1\}}(\mathbf{e}^1) := N(x) \cap Y$. Since $|A_\emptyset(\mathbf{e}^1)|, |A_{\{1\}}(\mathbf{e}^1)| \geqslant d|Y|$ by the assumption of the lemma, there exist disjoint sets $S_\emptyset(\mathbf{e}^1) \subseteq A_\emptyset(\mathbf{e}^1)$ and $S_{\{1\}}(\mathbf{e}^1) \subseteq A_{\{1\}}(\mathbf{e}^1)$ with $|S_\emptyset(\mathbf{e}^1)|, |S_{\{1\}}(\mathbf{e}^1)| = \frac{d}{4}|Y|$, as required.

For the induction step, let $1 < q \leqslant \ell$ and assume that the result holds for $q-1$. By this induction hypothesis

$$\exists D(\mathbf{e}^0) \subseteq E(X) \forall e_1 \in D(\mathbf{e}^0)\, \exists D(\mathbf{e}^1) \subseteq E(X)\, \forall e_2 \in D(\mathbf{e}^1) \quad \dots$$
$$\dots \quad \forall e_{q-2} \in D(\mathbf{e}^{q-3})\, \exists D(\mathbf{e}^{q-2}) \subseteq E(X)\, \forall e_{q-1} \in D(\mathbf{e}^{q-2})$$

we have

(a^*) $\delta\big(D(\mathbf{e}^0)\big), \delta\big(D(\mathbf{e}^1)\big), \dots, \delta\big(D(\mathbf{e}^{q-2})\big) > C$, and

(b^*) $\exists\, S(\mathbf{e}^{q-1}) \in \mathcal{F}_{q-1}^{3,\alpha_{q-1}|Y|}(Y)$ such that $\mathbf{e}^{q-1} \to S(\mathbf{e}^{q-1})$.

As in Definition 21, set

$$\mathcal{D}_q(X, Y) := \Big\{ \mathbf{e}^q \in E(X)^q \,:\, e_j \in D(\mathbf{e}^{j-1}) \text{ for each } j \in [q] \Big\}.$$

We shall show that for every $\mathbf{e}^{q-1} \in \mathcal{D}_{q-1}(X, Y)$, there exists a set of edges $D(\mathbf{e}^{q-1}) \subseteq E(X)$, with $\delta\big(D(\mathbf{e}^{q-1})\big) > C$, such that for every $e_q \in D(\mathbf{e}^{q-1})$ there exists an $S(\mathbf{e}^q) \in \mathcal{F}_q^{3,\alpha_q|Y|}(Y)$ such that $\mathbf{e}^q \to S(\mathbf{e}^q)$.

Indeed, given $\mathbf{e}^{q-1} \in \mathcal{D}_{q-1}(X, Y)$, by (b^*) there exists

$$\big\{ S_I(\mathbf{e}^{q-1}) \subseteq Y : I \subseteq [q-1] \big\} \;=\; S(\mathbf{e}^{q-1}) \;\in\; \mathcal{F}_{q-1}^{3,\alpha_{q-1}|Y|}(Y)$$

with $\mathbf{e}^{q-1} \to S(\mathbf{e}^{q-1})$. In particular, note that by definition of $\mathcal{F}_{q-1}^{3,\alpha_{q-1}|Y|}(Y)$, the sets $S_I(\mathbf{e}^{q-1})$ are disjoint, and that $|S_I(\mathbf{e}^{q-1})| = \alpha_{q-1}|Y|$ for every $I \subseteq [q-1]$. Let $R = Y \setminus \bigcup_I S_I(\mathbf{e}^{q-1})$, and recall that $q \leqslant \ell$, and that there is no $(p, C, \varepsilon, \beta)$-booster of (X, Y) for any $p \leqslant 2^\ell$. Thus the partition $S(\mathbf{e}^{q-1}) \cup \{R\}$ of Y does not induce a $(p, C, \varepsilon, \beta)$-booster of (X, Y).

By our choice of β, we have $\beta \leqslant \alpha_{q-1}$, and thus $|S_I(\mathbf{e}^{q-1})| \geqslant \beta|Y|$ for all $I \subseteq [q-1]$. Similarly, since $2^{q-1}\alpha_{q-1} = 2^{q-1}(d/4)^{q-1} \leqslant 1/2 < 1 - \beta$, we have $|R| > \beta|Y|$. Let $X' \subseteq X$ be the set of vertices which are not ε-boosted by any of the sets $S(\mathbf{e}^{q-1}) \cup \{R\}$. Since $S(\mathbf{e}^{q-1}) \cup \{R\}$ does not induce a $(2^{q-1} + 1, C, \varepsilon, \beta)$-booster of (X, Y), the graph $G[X']$

is not C-degenerate, and hence there exists a set of edges $D(\mathbf{e}^{q-1}) \subseteq E(X')$ such that $\delta\big(D(\mathbf{e}^{q-1})\big) > C$. We claim that this is the set we are looking for.

In order to verify this, let $e_q = xy \in D(\mathbf{e}^{q-1})$ be arbitrary. Our task is to show that there exists $S(\mathbf{e}^q) \in \mathcal{F}_q^{3,\alpha_q|Y|}$ such that $\mathbf{e}^q \to S(\mathbf{e}^q)$. Recall that x and y are not ε-boosted by $S(\mathbf{e}^{q-1}) \cup \{R\}$. Hence $d(x,U) < (1+\varepsilon)d(x,Y)$ for each $U \in S(\mathbf{e}^{q-1}) \cup \{R\}$, and so, for every $I \subseteq [q-1]$,

$$\begin{aligned} e\big(x, S_I(\mathbf{e}^{q-1})\big) \;&=\; e(x,Y) - e\big(x, Y \setminus S_I(\mathbf{e}^{q-1})\big) \\ &\geqslant\; e(x,Y) - \big(1+\varepsilon\big)\left(1 - \left(\tfrac{d}{4}\right)^{q-1}\right) e(x,Y) \;\geqslant\; \frac{1}{2}\left(\frac{d}{4}\right)^{q-1} e(x,Y)\,, \end{aligned} \tag{5}$$

where we used $|S_I(\mathbf{e}^{q-1})| = \alpha_{q-1}|Y| = \left(\frac{d}{4}\right)^{q-1}|Y|$ for the first inequality, and $\varepsilon = \frac{1}{2}\left(\frac{d}{4}\right)^{\ell}$ for the second. By the same argument, y has at least $\frac{1}{2}\left(\frac{d}{4}\right)^{q-1} e(y,Y)$ neighbours in $S_I(\mathbf{e}^{q-1})$ for each $I \subseteq [q-1]$.

Define, for each $I \subseteq [q]$, the set $A_I(\mathbf{e}^q) \subseteq Y$ as follows:

$$\begin{aligned} A_I(\mathbf{e}^q) &:= N(x) \cap S_{I\setminus\{q\}}(\mathbf{e}^{q-1}) \qquad && \text{if } q \in I \\ A_I(\mathbf{e}^q) &:= N(y) \cap S_I(\mathbf{e}^{q-1}) && \text{if } q \notin I. \end{aligned} \tag{6}$$

Since $e(x,Y), e(y,Y) \geqslant d|Y|$, we conclude from (5), that we have $|A_I(\mathbf{e}^q)| \geqslant 2\left(\frac{d}{4}\right)^q |Y|$ for every $I \subseteq [q]$. Moreover, the sets $A_I(\mathbf{e}^q)$ and $A_J(\mathbf{e}^q)$ are disjoint unless $I \setminus \{q\} = J \setminus \{q\}$. Hence we may choose disjoint sets $S_I(\mathbf{e}^q) \subseteq A_I(\mathbf{e}^q)$ with $|S_I(\mathbf{e}^q)| = \left(\frac{d}{4}\right)^q |Y|$ for each $I \subseteq [q]$.

Let $S(\mathbf{e}^q) = \big\{S_I(\mathbf{e}^q) : I \subseteq [q]\big\}$. We claim that this is the desired family; that is, that $S(\mathbf{e}^q) \in \mathcal{F}_q^{3,\alpha_q|Y|}(Y)$ and $\mathbf{e}^q \to S(\mathbf{e}^q)$. Indeed, the sets $S_I(\mathbf{e}^q)$ are disjoint, and

$$|S_I(\mathbf{e}^q)| = \left(\frac{d}{4}\right)^q |Y| = \alpha_q|Y|$$

for each $I \subseteq [q]$, by construction. Finally, we prove that $\mathbf{e}^q \to S(\mathbf{e}^q)$, i.e., that $e_i \to_i S(\mathbf{e}^q)$ for each $i \in [q]$. For $i \leqslant q-1$, this follows because $\mathbf{e}^{q-1} \to S(\mathbf{e}^{q-1})$, and

$$S_I(\mathbf{e}^q) \cup S_{I\cup\{q\}}(\mathbf{e}^q) \subseteq S_I(\mathbf{e}^{q-1})$$

for every $I \subseteq [q-1]$ by (6). For $i = q$, it follows since $S_I(\mathbf{e}^q) \subseteq N(x)$ if $q \in I \subseteq [q]$ and $S_I(\mathbf{e}^q) \subseteq N(y)$ if $q \notin I \subseteq [q]$ by (6). Hence $\mathbf{e}^q \to S(\mathbf{e}^q)$, as required. This completes the induction step, and hence the proof of the lemma. $\square$

We can now easily deduce Lemma 30.

Proof of Lemma 30. By Lemma 32 (applied with $q = \ell$), it suffices to show that if (X,Y) is (C,α_ℓ,ℓ)-Zykov, then it is (C,α)-rich in copies of $Z_\ell^{3,t}$, where $\alpha_\ell = \left(\frac{d}{4}\right)^{\ell}$ and $\alpha = \left(\frac{\alpha_\ell}{t}\right)^{2^\ell t}$. In other words, we want to prove that if there exists $S \in \mathcal{F}_\ell^{3,\alpha_\ell|Y|}(Y)$ with $\mathbf{e}^\ell \to S$, then

$$\big|\big\{S' \in \mathcal{F}_\ell^{3,t}(Y) \,:\, \mathbf{e}^\ell \to S'\big\}\big| \;\geqslant\; \alpha|Y|^s\,,$$

where $s = 2^\ell t$. Indeed, this is true because $|S_I| = \alpha_\ell |Y|$ for $I \subseteq [\ell]$, and the number of ways of choosing, for each $I \subseteq [\ell]$, a t-subset of S_I is

$$\prod_{I \subseteq [\ell]} \binom{|S_I|}{t} = \binom{\alpha_\ell |Y|}{t}^{2^\ell} \geqslant \left(\frac{\alpha_\ell |Y|}{t}\right)^{2^\ell t} = \alpha |Y|^s \,,$$

as claimed. □

It is now straightforward to prove Proposition 26.

Proof of Proposition 26. Let $C, \ell, t \in \mathbb{N}$ and $d > 0$, and set $\beta = \left(\frac{d}{4}\right)^\ell$ and $\varepsilon = \beta/2$. Let G and (X, Y) be as described in the statement, so $|N(x) \cap Y| \geqslant d|Y|$ for every $x \in X$. By Lemma 29 there exists a $(2^\ell, C, \varepsilon, \beta)$-booster tree for (X, Y), and moreover $|\mathcal{T}|$ is bounded as a function of d, ε and ℓ.

Recall that the leaves of $\mathcal{T}$ correspond to a partition of X (and a partition of Y). If every leaf (X', Y') of $\mathcal{T}$ is degenerate then $\chi(G[X]) \leqslant |\mathcal{T}|(C+1) =: C'$, where C' depends only upon C, ℓ and d. So we may assume that some leaf $(X', Y') \in V(\mathcal{T})$ is not degenerate.

By the definition of a $(2^\ell, C, \varepsilon, \beta)$-booster tree, it follows that there is no $(p, C, \varepsilon, \beta)$-booster of (X', Y') for any $p \leqslant 2^\ell$. Further, $|N(x) \cap Y'| = d(x, Y')|Y'| \geqslant d|Y'|$ for every $x \in X'$ by Lemma 29. Then, by Lemma 30 (applied with ℓ, t and d), (X', Y') is (C, α')-rich in copies of $Z_\ell^{3,t}$, for some $\alpha' = \alpha'(d, \ell, t) > 0$. Since (again by Lemma 29) $|Y'| \geqslant \beta^{|\mathcal{T}|}|Y|$, it follows that (X, Y) is (C, α)-rich in copies of $Z_\ell^{3,t}$, where $\alpha = \alpha' \beta^{|T|s}$ is a constant depending only on d, ℓ and t, as required. □

7. The proof of Theorem 2

In this section we shall complete the proof of Theorem 2. As a warm-up, we begin with the case $r = 3$, which is an almost immediate consequence of the results of the last four sections.

The following theorem proves Conjecture 1. The proof does not use the Regularity Lemma; it follows from Propositions 22 and 26.

Theorem 33. *If H is a near-acyclic graph, then $\delta_\chi(H) = 0$.*

Proof. Let H be a near-acyclic graph (so in particular $\chi(H) = 3$), let $\gamma > 0$ be arbitrary, and let G be an H-free graph on n vertices, with $\delta(G) \geqslant 2\gamma n$. We shall prove that the chromatic number of G is at most C', for some $C' = C'(H, \gamma)$.

First, using Observation 20, choose $t \in \mathbb{N}$ and a collection $T_1, \ldots, T_\ell$ of trees such that $H \subseteq Z_\ell^{3,t}(T_1, \ldots, T_\ell)$. Choose a maximal bipartition (X, Y) of G, assume without loss of generality that $\chi(G[X]) \geqslant \chi(G[Y])$, and note that $|N(x) \cap Y| \geqslant \gamma |Y|$ for every $x \in X$.

Let $\alpha > 0$ be given by Proposition 26 (applied with ℓ, t and γ), let $C := 2^{\ell+3}\alpha^{-1}\sum_{i=1}^{\ell} |T_i|$, and apply Proposition 26. We obtain a $C' = C'(H, \gamma) > 0$ such that either $\chi(G) \leqslant 2\chi\big(G[X]\big) \leqslant 2C'$, or (X, Y) is (C, α)-rich in copies of $Z_\ell^{3,t}$.

In the former case we are done, and so let us assume the latter. By Proposition 22 and our choice of C, it follows that $Z_\ell^{3,t}(T_1, \ldots, T_\ell) \subseteq G$. But then $H \subseteq G$, which is a contradiction. Thus $\chi(G)$ is bounded, as claimed. □

The case $r = 3$ of Theorem 2 now follows from Proposition 5, and Theorems 7, 14 and 33.

Proof of the case $r = 3$ of Theorem 2. Let H be a graph with $\chi(H) = 3$, and recall that $\mathcal{M}(H)$ denotes the decomposition family of H. By Proposition 5, if $\mathcal{M}(H)$ does not contain a forest then $\delta_\chi(H) = \frac{1}{2}$, and by Theorem 7, if $\mathcal{M}(H)$ does contain a forest then $\delta_\chi(H) \leqslant \frac{1}{3}$.

Now, by Theorem 14, if H is not near-acyclic then $\delta_\chi(H) \geqslant \frac{1}{3}$, and by Theorem 33, if H is near-acyclic then $\delta_\chi(H) = 0$. Thus

$$\delta_\chi(H) \,\in\, \big\{0,\, 1/3,\, 1/2\big\},$$

where $\delta_\chi(H) \neq \frac{1}{2}$ if and only if H has a forest in its decomposition family, and $\delta_\chi(H) = 0$ if and only if H is near-acyclic, as required. $\square$

The rest of this section is devoted to the proof of the following theorem, which generalises Theorem 33 to arbitrary $r \geqslant 3$.

Theorem 34. *Let H be a graph with $\chi(H) = r \geqslant 3$. If H is r-near-acyclic, then*

$$\delta_\chi(H) \,=\, \frac{r-3}{r-2}\,.$$

We begin with the lower bound, which follows by essentially the same construction as in Proposition 5.

Proposition 35. *For any graph H with $\chi(H) = r \geqslant 3$, we have $\delta_\chi(H) \geqslant \frac{r-3}{r-2}$.*

Proof. We claim that, for any such H, n_0 and C, there exist H-free graphs on $n \geqslant n_0$ vertices, with minimum degree $\frac{r-3}{r-2}n$, and chromatic number at least C. Recall that we call a graph a (k,ℓ)-Erdős graph if it has chromatic number at least k and girth at least ℓ, and that such graphs exist for every $k, \ell \in \mathbb{N}$.

Let G' be a $(C, |H|+1)$-Erdős graph on at least n_0 vertices, and let G be the graph obtained from the complete, balanced $(r-2)$-partite graph on $(r-2)|G'|$ vertices by replacing one of its partition classes with G'. Then G is H-free, since every $|H|$-vertex subgraph of G has chromatic number at most $r-1$. Moreover, $\delta(G) = \frac{r-3}{r-2}n$ and $\chi(G) \geqslant C$, as required. $\square$

We now sketch the proof of the upper bound of Theorem 34. Let G be an n-vertex, H-free graph with minimum degree $\big(\frac{2r-5}{2r-3} + 3\gamma\big)n$. Let $T_1, \ldots, T_\ell$ be such that $H \subseteq Z_\ell^{r,t}(T_1, \ldots, T_\ell)$. First, we take an (ε, d)-regular partition, using the degree form of the Regularity Lemma (where ε and d will be chosen sufficiently small given γ). We then construct a second partition $\mathcal{P}$ of $V(G)$, similar to that used in the proof of Theorem 7. Our aim is to show that $\chi(G[X]) \leqslant C'$ for each $X \in \mathcal{P}$.

In the next step, we observe that the minimum degree condition guarantees that for each $X \in \mathcal{P}$, there are clusters Y and $Z_1, \ldots, Z_{r-3}$ of the (ε, d)-regular partition with the following properties. First, for each $v \in X$ we have $d_Y(v) \geqslant \gamma|Y|$, and for each $i \in [r-3]$ we have $d_{Z_i}(v) \geqslant \big(\frac{1}{2} + \gamma\big)|Z_i|$. Second, $Y, Z_1, \ldots, Z_{r-3}$ forms a clique in the reduced graph of the (ε, d)-regular partition.

Now recall that $Z_\ell^{r,t}(T_1,\dots,T_\ell)$ contains independent sets S_I for each $I \subseteq [\ell]$, and independent sets S_i for each $i \in [r-3]$. The idea now is to show that if $\chi(G[X]) \leqslant C'$ does not hold, then we find a copy of $Z_\ell^{r,t}(T_1,\dots,T_\ell)$ in which the trees $T_1,\dots,T_\ell$ lie in X, the independent sets S_I lie in Y, and S_i lies in Z_i for each $i \in [r-3]$, which contradicts the assumption that G is H-free.

In order to achieve this, we work as follows. We apply the paired VC-dimension argument (Proposition 26) to (X,Y), with constants ℓ^* and t^* which are much larger than ℓ and t, and a very large C^*. This yields our C' and an $\alpha > 0$ such that either $\chi(G[X]) \leqslant C'$ (in which case we are done), or (X,Y) is (C^*,α)-rich in copies of $Z_{\ell^*}^{3,t^*}$.

In the latter case, we apply Lemma 24 to conclude that (X,Y) is (C^*,α)-dense in copies of $Z_{\ell^*}^{3,t^*}$. The main work of this section then is to show (in Proposition 36) that this implies that there is an $S \in \mathcal{F}_\ell^{r,t}(Y \cup Z_1 \cup \dots \cup Z_{r-3})$ such that S is (r,ℓ,t,C,α)-good for $(X, Y \cup Z_1 \cup \dots \cup Z_{r-3})$. Finally, applying Lemma 25 we find that there is a copy of $Z_\ell^{r,t}(T_1,\dots,T_\ell)$ in G.

As just explained, the following proposition is the main missing tool for the proof of Theorem 34.

Proposition 36. *For every $r > 3$, $\ell, t \in \mathbb{N}$ and $d, \gamma > 0$ there exist $\ell^*, t^* \in \mathbb{N}$ such that for every $\alpha > 0$ and $C \in \mathbb{N}$, there exist $\varepsilon_1 > 0$ and $C^* \in \mathbb{N}$, such that for every $0 < \varepsilon < \varepsilon_1$ the following holds.*

Let G be a graph, and let X, Y and $Z_1,\dots,Z_{r-3}$ be disjoint subsets of $V(G)$, with $|Y| = |Z_j|$ for each $j \in [r-3]$. Let $Z := Z_1 \cup \dots \cup Z_{r-3}$. Suppose that (Y, Z_j) and (Z_i, Z_j) are (ε, d)-regular for each $i \neq j$, and that

$$|N(x) \cap Z_j| \geqslant \left(\frac{1}{2} + \gamma\right) |Z_j|$$

for every $x \in X$ and $j \in [r-3]$.

If (X,Y) is (C^*,α)-dense in copies of $Z_{\ell^*}^{3,t^*}$, then there is some $S \in \mathcal{F}_\ell^{r,t}(Y \cup Z)$ such that S is (r,ℓ,t,C,α)-good for $(X, Y \cup Z)$.

For the proof of this proposition, we combine an application of the Counting Lemma and two uses of the pigeonhole principle. As a preparation for these steps we need to show that there exists a family $S^* \in \mathcal{F}_{\ell^*}^{3,t^*}$ which is $(3,\ell^*,t^*,C^*,\alpha)$-good for (X,Y) and 'well-behaved' in the following sense. For each of the sets $S_I^* \subseteq Y$ given by S_I^* only a small positive fraction of the $(r-3)t$-element sets in Z has a common neighbourhood in S_I^* of less than t vertices. To this end we shall use the following lemma.

Recall that for a set T of vertices in a graph G, we write

$$N(T) \colon = \bigcap_{x \in T} N(x)\,.$$

Lemma 37. *For all $r, t \in \mathbb{N}$ and $\mu, d > 0$, there exist $t^* = t^*(r,t,\mu,d) \in \mathbb{N}$ and $\varepsilon_0 = \varepsilon_0(r,t,\mu,d) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.*

Let G be a graph, and suppose that Y and $Z_1, \dots, Z_{r-3}$ are disjoint subsets of $V(G)$ such that (Y, Z_j) is (ε, d)-regular for each $j \in [r-3]$. Let $Z := Z_1 \cup \ldots \cup Z_{r-3}$, and define

$$\mathcal{B}(S) := \Big\{ T \in \binom{Z}{(r-3)t} \,:\, |N(T) \cap S| < t \Big\}$$

for each $S \subseteq Y$. Then we have

$$\mathcal{S} := \Big\{ S \in \binom{Y}{t^*} \,:\, |\mathcal{B}(S)| \geqslant \mu |Z|^{(r-3)t} \Big\} \;\leqslant\; \sqrt{\varepsilon} |Y|^{t^*} .$$

Proof. Choose t^* sufficiently large such that

$$\mathbb{P}\Big(\mathrm{Bin}\big(t^*, (d/2)^{(r-3)t}\big) < t \Big) \leqslant \mu \,, \tag{7}$$

where $\mathrm{Bin}(n, p)$ denotes a random variable with binomial distribution, and set

$$\varepsilon_0 := \min \Big\{ \Big(\frac{d}{2}\Big)^{t^*}, \big(t^* \cdot 2^{t^*} (r-3) \big)^{-2} \Big\} . \tag{8}$$

In the first part of the proof we shall construct a family $\mathcal{S}'$ of at least $\binom{|Y|}{t^*} - \sqrt{\varepsilon}|Y|^{t^*}$ sets $S \in \binom{Y}{t^*}$. In the second part we will then show that $\mathcal{S}' \subseteq \binom{Y}{t^*} \setminus \mathcal{S}$, which proves the lemma. For constructing the sets $S \in \mathcal{S}'$ we proceed inductively and shall choose the vertices $v_1, \dots, v_{t^*}$ of S one by one, in each step $k \in [t^*]$ avoiding a set $Y_k \subseteq Y$ of size at most $\varepsilon 2^k (r-3)|Y|$. Clearly, by (8), this gives at least $\binom{|Y|}{t^*} - \sqrt{\varepsilon}|Y|^{t^*}$ choices for S as desired.

Indeed, suppose we have already chosen the vertices $v_1, \dots, v_{k-1}$. In addition we have chosen for each $j \in [r-3]$ a partition P_j^{k-1} of Z_j with the following property (we shall make use of these partitions in part two of the proof): for each $I \subseteq \{v_1, \dots, v_{k-1}\}$ we have chosen a part $P_j^{k-1}(I)$ of size $(d-\varepsilon)^{|I|}(1-d+\varepsilon)^{k-1-|I|}|Z_j|$ such that $P_j^{k-1}(I) \subseteq N(I)$. Now we will explain how v_k can be chosen together with partitions P_j^k satisfying the above conditions. For this purpose consider the set $Y_k \subseteq Y$ of vertices y such that for some $j \in [r-3]$ and some $I \subseteq \{v_1, \dots, v_{k-1}\}$ we have

$$|N(y) \cap P_j^{k-1}(I)| < (d-\varepsilon)^k |Z_j| \,,$$

where $P_j^0 := \{Z_j\}$ is the trivial partition of Z_j. The possible choices for v_k now are the vertices in $Y \setminus Y_k$. The partitions P_j^k with $j \in [r-3]$ are defined as follows. For each $I' \subseteq \{v_1, \dots, v_{k-1}\}$ we choose an arbitrary subset P of $N(v_k) \cap P_j^{k-1}(I)$ with $|P| = (d-\varepsilon)^k |Z_j|$, which is possible by the choice of v_k, and set

$$P_j^k(I') := P_j^{k-1}(I') \setminus P \quad \text{and} \quad P_j^k\big(I' \cup \{z_k\}\big) := P \,.$$

Clearly, the partitions defined in this way satisfy that each part $P_j^k(I)$ is of size $(d-\varepsilon)^{|I|}(1-d+\varepsilon)^{k-|I|}|Z_j|$ and that $P_j^k(I) \subseteq N(I)$ as desired.

It remains to show that $|Y_k| \leqslant \varepsilon 2^k (r-3)|Y|$ as claimed above. If this is not true, then for some $j \in [r-3]$ and $I \subseteq [k-1]$, there exist $\varepsilon|Y|$ vertices in Y which have at most $(d-\varepsilon)|P_j^k(I)|$ neighbours in $P_j^{k-1}(I)$. Since $|P_j^{k-1}(I)| \geqslant (d-\varepsilon)^{k-1}|Z_j| \geqslant \varepsilon |Z_j|$ by (8), this contradicts (ε, d)-regularity of (Y, Z_j).

We now turn to the second part of the proof: We claim that for every $S \in \mathcal{S}'$ we have $|\mathcal{B}(S)| < \mu |Z|^{(r-3)t}$. To see this, simply choose a random multiset $T \subseteq Z$ of size $(r-3)t$, and observe that $N(T) \cap S$ is given by the intersection of $(r-3)t$ sets $S_1, \dots, S_{(r-3)t} \subseteq S$ chosen (independently) according to the distribution

$$\mathbb{P}\big(S_i = I\big) = \frac{\big|\{z \in Z \,:\, I = N(z) \cap S\}\big|}{|Z|} \qquad \text{for } I \subseteq S\,.$$

By construction we have $|P_j^{t^*}(I)| = (d-\varepsilon)^{|I|}(1-d+\varepsilon)^{t^*-|I|}|Z_j|$ for every $j \in [r-3]$ and $I \subseteq S$. Hence

$$\mathbb{P}(I \subseteq S_i) \geqslant \frac{\big|\bigcup_{j=1}^{r-3}\bigcup_{I \subseteq I' \subseteq S} P_j^{t^*}(I')\big|}{|Z|} = \sum_{I \subseteq I' \subseteq S} (d-\varepsilon)^{|I'|}(1-d+\varepsilon)^{t^*-|I'|} = (d-\varepsilon)^{|I|}\,.$$

This implies that for every $I \subseteq S$, we have $\mathbb{P}(I \subseteq S_i) \geqslant \mathbb{P}(I \subseteq S_i')$ for the random variable S_i' with the following distribution: for every $u \in S$ we take $u \in S_i'$ independently with probability $d - \varepsilon$. We conclude that

$$\begin{aligned} \mathbb{P}\big(|S_1 \cap \ldots \cap S_{(r-3)t}| \geqslant t\big) &= \mathbb{P}\big(I \subseteq S_1 \cap \cdots \cap S_{(r-3)t} \quad \text{for some } I \subseteq S \text{ with } |I| \geqslant t\big) \\ &\geqslant \mathbb{P}\big(I \subseteq S_1' \cap \cdots \cap S_{(r-3)t}' \quad \text{for some } I \subseteq S \text{ with } |I| \geqslant t\big) \\ &= \mathbb{P}\big(|S_1' \cap \ldots \cap S_{(r-3)t}'| \geqslant t\big) = \mathbb{P}\Big(\mathrm{Bin}\big(t^*, (d-\varepsilon)^{(r-3)t}\big) \geqslant t\Big) \\ &\geqslant 1 - \mu\,, \end{aligned}$$

where the last inequality follows from (7). This proves $|\mathcal{B}(S)| < \mu|Z|^{(r-3)t}$ and hence finishes the proof of the lemma. $\square$

We shall now prove Proposition 36.

Proof of Proposition 36. We start by defining the constants. Given $r > 3$, $\ell, t \in \mathbb{N}$ and $\gamma, d > 0$, we set

$$\mu := \frac{\gamma^{(r-3)t}}{8\big((r-3)t\big)!(r-3)^{(r-3)t}}\left(\frac{d}{2}\right)^{\binom{r-3}{2}t^2} \quad \text{and} \quad \ell^* := \frac{\ell}{2\mu}\,. \tag{9}$$

Let t^* and ε_0 be given by Lemma 37 with input $r, t, \mu' := 2^{-\ell^*}\mu, d$. Given $\alpha > 0$ and C, we choose

$$\varepsilon_1 := \min\left(\frac{\alpha^2}{2^{4\ell^*+1}}, \frac{d\gamma}{4(\gamma+1)(r-3)t}, \varepsilon_0\right) \quad \text{and} \quad C^* := \frac{2^{\ell^*}C}{\alpha\mu}\,. \tag{10}$$

Now let $0 < \varepsilon < \varepsilon_1$, let G be a graph, and let X, Y and $Z_1, \dots, Z_{r-3}$ be disjoint subsets of $V(G)$ as described in the statement, so in particular, (X, Y) is (C^*, α)-dense in copies of $Z_{\ell^*}^{3,t^*}$. The goal is to show that there exists $S \in \mathcal{F}_\ell^{r,t}(Y \cup Z)$ such that S is (r, ℓ, t, C, α)-good for $(X, Y \cup Z)$.

Our first step is to show that there is a 'well-behaved' function $S^* \in \mathcal{F}_{\ell^*}^{3,t^*}(Y)$.

Claim 38. There is a function $S^* \in \mathcal{F}_{\ell^*}^{3,t^*}(Y)$ which is $(3,\ell^*,t^*,C^*,\alpha)$-good for (X,Y) and has the property that for every $I \subseteq [\ell^*]$, the set

$$\mathcal{B}(S_I^*) = \left\{ T \in \binom{Z}{(r-3)t} : \big|N(T) \cap S_I^*\big| \leqslant t \right\}$$

in $\binom{Z}{(r-3)t}$ has size at most $2^{-\ell^*}\mu|Z|^{(r-3)t}$.

Proof of Claim 38. By Lemma 37 (with input $r, t, \mu' = 2^{-\ell^*}\mu, d$), the total number of 'bad' t^*-subsets S' of Y, i.e., those for which $\mathcal{B}(S') \geqslant 2^{-\ell^*}\mu|Z|^{(r-3)t}$, is at most $\sqrt{\varepsilon}|Y|^{t^*}$. Let $\mathcal{S}$ be the set of functions S^* in $\mathcal{F}_{\ell^*}^{3,t^*}(Y)$ which do *not* have the property that for every $I \subseteq [\ell^*]$ we have $\mathcal{B}(S_I^*) < 2^{-\ell^*}\mu|Z|^{(r-3)t}$. We can obtain any function S^* in $\mathcal{S}$ by taking a set $I \subseteq [\ell^*]$ and one of the at most $\sqrt{\varepsilon}|Y|^{t^*}$ 'bad' t^*-sets to be S_I^*, and choosing the $2^{\ell^*}-1$ remaining sets of S^* in any way from $\binom{Y}{t^*}$. It follows that

$$|\mathcal{S}| \leqslant 2^{\ell^*}\sqrt{\varepsilon}|Y|^{t^*}|Y|^{(2^{\ell^*}-1)t^*} = 2^{\ell^*}\sqrt{\varepsilon}|Y|^{2^{\ell^*}t^*}.$$

Since (X,Y) is (C^*,α)-dense in copies of $Z_{\ell^*}^{3,t^*}$, there are at least $2^{-\ell^*}\alpha|Y|^{2^{\ell^*}t^*}$ functions in $\mathcal{F}_{\ell^*}^{3,t^*}(Y)$ which are $(3,\ell^*,t^*,C^*,\alpha)$-good for (X,Y). Since by (10) we have $2^{-\ell^*}\alpha > 2^{\ell^*}\sqrt{\varepsilon}$, at least one of these functions is not in $\mathcal{S}$, as required. □

For the remainder of the proof, S^* will be a fixed function satisfying the conclusion of Claim 38. Since S^* is $(3,\ell^*,t^*,C^*,\alpha)$-good for (X,Y), there exist sets

$$E_1^*,\ldots,E_{\ell^*}^* \subseteq E(X), \quad \text{with} \quad \overline{d}(E_j^*) \geqslant 2^{-\ell^*}\alpha C^* \quad \text{for each} \quad 1 \leqslant j \leqslant \ell^*,$$

such that for every $e_1 \in E_1^*,\ldots,e_{\ell^*} \in E_{\ell^*}^*$, we have $\mathbf{e}^{\ell^*} \to S^*$.

Our next claim comprises two applications of the pigeonhole principle to find a copy of $K_{r-3}(t)$ in Z.

Claim 39. There exists a copy T of $K_{r-3}(t)$ with t vertices in Z_j for each $j \in [r-3]$, and a set $L \subseteq [\ell^*]$ of size $|L| = \ell$ such that:

- (i) $|N(T) \cap S_I^*| \geqslant t$ for every $I \subseteq [\ell^*]$,
- (ii) $N(T)$ contains at least $\mu|E_j^*|$ edges of E_j^*, for each $j \in L$.

Proof of Claim 39. By assumption, for every $x \in X$ and $j \in [r-3]$ we have

$$|N(x) \cap Z_j| \geqslant \left(\frac{1}{2} + \gamma\right)|Z_j|,$$

and so each edge $e \in E_1^* \cup \ldots \cup E_{\ell^*}^*$ has at least $\gamma|Z_j|$ common neighbours in Z_j. By Fact 3, the common neighbours of e in Z_i and Z_j form an $(\varepsilon/\gamma, d-\varepsilon)$-regular pair for each $1 \leqslant i < j \leqslant r-3$. By (10) we have $d-\varepsilon-(r-3)t\varepsilon/\gamma > d/2$. Hence, applying the Counting Lemma with d replaced by $d-\varepsilon$ and ε replaced by ε/γ to the graph $H = K_{r-3}(t)$, it follows that there are at least

$$\frac{1}{\mathrm{Aut}(H)}\Big(d-\varepsilon-\frac{\varepsilon}{\gamma}|H|\Big)^{e(H)}\Big(\frac{\gamma|Z|}{r-3}\Big)^{|H|} \geqslant \frac{1}{\big((r-3)t\big)!}\Big(\frac{d}{2}\Big)^{\binom{r-3}{2}t^2}\Big(\frac{\gamma|Z|}{r-3}\Big)^{(r-3)t} \overset{(9)}{\geqslant} 8\mu|Z|^{(r-3)t}$$

copies of $K_{r-3}(t)$ in $N(e)\cap Z$, each with t vertices in each Z_j.

There are therefore, for each $j\in[\ell^*]$, at least $8\mu|Z|^{(r-3)t}|E_j^*|$ pairs (e,T), where $e\in E_j^*$ and T is a copy of $K_{r-3}(t)$ as described, such that $T\subseteq N(e)$, or equivalently $e\subseteq N(T)$. Since we have

$$8\mu|Z|^{(r-3)t}|E_j^*| = 4\mu|Z|^{(r-3)t}|E_j^*| + 4\mu|E_j^*||Z|^{(r-3)t}\,,$$

by the pigeonhole principle, it follows that there are at least $4\mu|Z|^{(r-3)t}$ copies of $K_{r-3}(t)$ in Z each of which has at least $4\mu|E_j^*|$ edges of E_j^* in its common neighbourhood. Let us denote by $\mathcal{T}_j$ the collection of such copies of $K_{r-3}(t)$. For a copy T of $K_{r-3}(t)$, let $L(T)=\big\{j : T\in\mathcal{T}_j\big\}$.

We claim that there is a set $\mathcal{T}$ containing at least $2\mu|Z|^{(r-3)t}$ copies T of $K_{r-3}(t)$ in Z, each with $|L(T)|\geqslant\ell$. Indeed, this follows once again by the pigeonhole principle, since there are at least

$$\ell^*\cdot 4\mu|Z|^{(r-3)t} \overset{(9)}{=} \ell|Z|^{(r-3)t} + \ell^*\cdot 2\mu|Z|^{(r-3)t}$$

pairs (T,j) with $T\in\mathcal{T}_j$.

Now, recall that S^* satisfies the conclusion of Claim 38, i.e., for each $I\subseteq[\ell^*]$, there are at most $2^{-\ell^*}\mu|Z|^{(r-3)t}$ sets $T\in\binom{Z}{(r-3)t}$ such that $|N(T)\cap S_I^*|\leqslant t$. Since $|\mathcal{T}|\geqslant 2\mu|Z|^{(r-3)t}$, there is a copy T of $K_{r-3}(t)\in\mathcal{T}$ such that for each $I\subseteq[\ell^*]$, we have $|N(T)\cap S_I^*|\geqslant t$. If we let L be any subset of $L(T)$ of size ℓ, then T and L satisfy the conclusions of the claim. $\square$

Let T and L be as given by Claim 39 and for each $j\in L$ let $E_j\subseteq X$ be a set of $\mu|E_j^*|$ edges of E_j^* contained in $N(T)$ as promised by Claim 39(ii). We construct a function $S\in\mathcal{F}_\ell^{r,t}(Y)$ by choosing, for each $I\subseteq L$, a subset $S_I\subseteq S_I^*$ of size t in $N(T)\cap Y$ (which is possible by Claim 39(i)), and letting the sets S_i, $i\in[r-3]$, be the parts of T.

Claim 40. S is (r,ℓ,t,C,α)-good for $(X,Y\cup Z)$.

Proof of Claim 40. Recall that $|L|=\ell$, and assume without loss of generality that $L=\{1,\dots,\ell\}$. By the choice of T and the definition of the sets S_I with $I\subseteq L$ and the sets S_i with $i\in[r-3]$, we have that S_i is completely adjacent to each $S_{i'}$ with $i\neq i'$, to each S_I, and to each edge $e\in\bigcup_{j\in L}E_j$. Since $\mathbf{e}^{\ell^*}\to S^*$ for each $\mathbf{e}^{\ell^*}\in E_1^*\times\ldots\times E_{\ell^*}^*$, it follows that $\mathbf{e}^{\ell}\to S$ for each $\mathbf{e}^{\ell}\in E_1\times\ldots\times E_\ell$. Finally, for each $j\in L$, since $|E_j|\geqslant\mu|E_j^*|$, we have

$$\overline{d}(E_j)\geqslant\mu\overline{d}(E_j^*)\geqslant\mu 2^{-\ell^*}\alpha C^* \overset{(10)}{=} C\,,$$

as required. $\square$

Thus there exists a function $S \in \mathcal{F}_\ell^{r,t}(Y)$ which is (r,ℓ,t,C,α)-good for $(X, Y \cup Z)$, as required. $\square$

Remark 41. It is possible to strengthen the conclusion of Proposition 36: under the same conditions, $(X, Y \cup Z_1 \cup \cdots \cup Z_{r-3})$ is (C,α')-dense in copies of $Z_\ell^{r,t}$, for some $\alpha' = \alpha'(r,\ell,t,d,\gamma,\alpha) > 0$. To see this, observe that the proofs of Claims 38 and 39 both in fact yield a positive density of functions S^* in $\mathcal{F}_{\ell^*}^{3,t^*}(Y)$ and of copies T of $K_{r-3}(t)$, respectively. From any such S^* and T can be obtained a function S which is (r,ℓ,t,C,α)-good for $(X, Y \cup Z)$.

We can now deduce Theorem 34.

Proof of Theorem 34. The lower bound is given by Proposition 35, so we are only required to prove the upper bound. Let H be an r-near-acyclic graph, with $r \geqslant 4$, and let $\gamma > 0$. Because H is r-near-acyclic, by Observation 20 there exist trees $T_1, \ldots, T_\ell$ and a number $t \in \mathbb{N}$ such that $H \subseteq Z_\ell^{r,t}(T_1,\ldots,T_\ell)$. We now set constants as follows. First, we choose $d = \gamma$. Given r, ℓ, t, d and γ, Proposition 36 returns integers ℓ^* and t^*. Now Proposition 26, with input ℓ^*, t^* and d, returns $\alpha > 0$. Next, consistent with Lemma 25 we set $C := 2^{\ell+3}\alpha^{-1}\sum_{i=1}^{\ell}|T_i|$. Feeding α and C into Proposition 36 yields $\varepsilon_1 > 0$ and C^*. Putting C^* into Proposition 26 yields a constant C'. We choose

$$k_0 := 2r/\gamma \quad \text{and} \quad \varepsilon := \min(\varepsilon_1, \gamma)\,. \tag{11}$$

Finally, from the minimum degree form of the Szemerédi Regularity Lemma, with input ε, d, $\delta = (\frac{r-3}{r-2} + 3\gamma)$ and k_0, we obtain a constant k_1.

Let G be an H-free graph on $n > k_1$ vertices, with $\delta(G) \geqslant \left(\frac{r-3}{r-2} + 3\gamma\right)n$. We shall prove that $\chi(G) \leqslant 2 \cdot 2^{2k_1}C'$. First, applying the minimum degree form of the Szemerédi Regularity Lemma, we obtain a partition $V_0 \cup \ldots \cup V_k$ of $V(G)$, with reduced graph R, where $\delta(R) \geqslant \left(\frac{r-3}{r-2} + \gamma\right)k$. We form a second partition by setting

$$\begin{aligned} X(I_1,I_2) := \Big\{ v \in V(G) :\ & i \in I_1 \quad \Leftrightarrow \quad |N(v) \cap V_i| \geqslant \gamma |V_i| \\ & \text{and} \quad i \in I_2 \quad \Leftrightarrow \quad |N(v) \cap V_i| \geqslant \left(\frac{1}{2} + \gamma\right)|V_i| \Big\} \end{aligned}$$

for each pair of sets $I_2 \subseteq I_1 \subseteq [k]$. It obviously suffices to establish that for each I_1 and I_2 we have $\chi\big(G[X(I_1,I_2)]\big) \leqslant 2C'$.

Hence let $I_2 \subseteq I_1 \subseteq [k]$ be fixed. Since $\chi\big(G[X(I_1,I_2)]\big) \leqslant 2C'$ is obvious when $X(I_1,I_2)$ is empty, assume it is non-empty. Then the minimum degree condition on G allows us to establish the following claim.

Claim 42. There exist distinct clusters $Y, Y' \in I_1$ and $Z_1, Z_1', \ldots, Z_{r-3}, Z_{r-3}' \in I_2$ such that $(Y, Z_i), (Y', Z_i'), (Z_i, Z_j)$ and (Z_i', Z_j') are (ε, d)-regular for every pair $\{i,j\} \subseteq [r-3]$.

Proof of Claim 42. Let x be any vertex in $X(I_1,I_2)$, and let $m = |V_1| = \cdots = |V_k|$. By the definition of $X(I_1,I_2)$, we have $|N(x) \cap V_i| \geqslant \gamma m$ iff $i \in I_1$, and thus

$$\left(\tfrac{r-3}{r-2} + 3\gamma\right)n \leqslant \delta(G) \leqslant d(x) \leqslant \varepsilon n + \big(k - |I_1|\big)\gamma m + |I_1| m \leqslant (\varepsilon + \gamma)n + |I_1|\tfrac{n}{k}\,.$$

Since by (11) we have $\varepsilon < \gamma$, we deduce $|I_1| \geqslant \big(\frac{r-3}{r-2}+\gamma\big)k$. Similarly, we have $|N(x)\cap V_i| \geqslant (\frac{1}{2}+\gamma)m$ iff $i \in I_2$ and therefore

$$\big(\tfrac{r-3}{r-2}+3\gamma\big)n \leqslant d(x) \leqslant \varepsilon n + \big(k-|I_2|\big)\big(\tfrac{1}{2}+\gamma\big)m + |I_2|m \leqslant (\varepsilon+\tfrac{1}{2}+\gamma)n + |I_2|\tfrac{n}{2k}\,,$$

from which we obtain $|I_2| \geqslant \big(\frac{r-4}{r-2}+\gamma\big)k$.

Since $\delta(R) \geqslant \big(\frac{r-3}{r-2}+\gamma\big)k$, each cluster in R has at most $\big(\frac{1}{r-2}-\gamma\big)k$ non-neighbours. It follows that

$$\delta\big(R[I_2]\big) \geqslant |I_2| - \tfrac{k}{r-2} + \gamma k \geqslant \big(\tfrac{r-5}{r-4}+\gamma\big)|I_2|\,,$$

so by Turán's theorem, $R[I_2]$ contains a copy of K_{r-3}. We let its clusters be $Z_1,\dots,Z_{r-3}$. Since each Z_i is non-adjacent to at most $\big(\frac{1}{r-2}-\gamma\big)k$ cluster in I_1, there is a cluster Y in I_1 adjacent in R to each Z_i with $i\in[r-3]$. Since $k\geqslant k_0$, by (11) we have $\gamma k-(r-2)\geqslant \gamma k/2$ and therefore

$$\delta\big(R[I_2\setminus\{Y,Z_1,\dots,Z_{r-3}\}]\big) \geqslant |I_2| - \tfrac{k}{r-2} + \tfrac{1}{2}\gamma k \geqslant \big(\tfrac{r-5}{r-4}+\tfrac{1}{2}\gamma\big)\big|I_2\setminus\{Y,Z_1,\dots,Z_{r-3}\}\big|\,.$$

Thus we can again apply Turán's theorem to $R[I_2\setminus\{Y,Z_1,\dots,Z_{r-3}\}]$ to obtain a clique $Z'_1,\dots,Z'_{r-3}$ in I_2, which has a common neighbour $Y'\in I_1\setminus\{Y,Z_1,\dots,Z_{r-3}\}$, as required. □

Let $Y,Y'\in I_1$ and $Z_1,Z'_1,\dots,Z_{r-3},Z'_{r-3}\subseteq I_2$ be the clusters given by Claim 42. Let $X = X(I_1,I_2)\cap(Y'\cup Z'_1\cup\dots\cup Z'_{r-3})$, and $X' = X(I_1,I_2)\setminus X$. Observe that $X,Y,Z_1,\dots,Z_{r-3}$ are pairwise disjoint (as are $X',Y',Z'_1,\dots,Z'_{r-3}$). Our goal now is to show that $\chi\big(G[X]\big)\leqslant C'$. Since an analogous argument gives $\chi\big(G[X']\big)\leqslant C'$ and we have $X(I_1,I_2)=X\cup X'$, this will imply $\chi\big(G[X(I_1,I_2)]\big)\leqslant 2C'$, and thus complete the proof.

We apply Proposition 26, with input ℓ^*, t^*, d and C^*, to (X,Y). Observe that, since $Y\in I_1$ and $X\subseteq X(I_1,I_2)$, we have $|N(x)\cap Y|\geqslant d|Y|$ for each $x\in X$. Recall that α and C' were defined such that the conclusion of Proposition 26 is the following. Either $\chi\big(G[X]\big)\leqslant C'$, or (X,Y) is (C^*,α)-rich in copies of $Z^{3,t^*}_{\ell^*}$. In the first case we are done, so we assume the latter. We will show that this contradicts our assumption that G is H-free.

By Lemma 24 the pair (X,Y) is (C^*,α)-dense in copies of $Z^{3,t^*}_{\ell^*}$. We now apply Proposition 36, with input r, ℓ, t, d, γ, α, C, and ε to $X,Y,Z_1,\dots,Z_{r-3}$. Observe that since $Z_1,\dots,Z_{r-3}\in I_2$, we have $|N(x)\cap Z_i|\geqslant(\frac{1}{2}+\gamma)|Z_i|$ for each $x\in X$ and $i\in[r-3]$. Moreover, by Claim 42, any pair of $Y,Z_1,\dots,Z_{r-3}$ is (ε,d)-regular. Recall that ℓ^*, t^*, ε_1 and C^* were defined such that the conclusion of Proposition 36 is that there exists a function $S\in\mathcal{F}^{r,t}_{\ell}(Y\cup Z_1\cup\dots\cup Z_{r-3})$ which is (r,ℓ,t,C,α)-good for $(X,Y\cup Z_1\cup\dots\cup Z_{r-3})$. Finally, we apply Lemma 25, with input r,ℓ,t,α and $T_1,\dots,T_\ell$, to X and $Y\cup Z_1\cup\dots\cup Z_{r-3}$. By the definition of C, this lemma gives that $H\subseteq Z^{r,t}_{\ell}(T_1,\dots,T_\ell)$ is contained in G, a contradiction. □

Finally, we put the pieces together and complete the proof of Theorem 2.

Proof of Theorem 2. Let H be a graph with $\chi(H)=r\geqslant 3$, and recall that $\mathcal{M}(H)$ denotes the decomposition family of H. By Proposition 5, if $\mathcal{M}(H)$ does not contain a forest then $\delta_\chi(H)=\frac{r-2}{r-1}$, and by Theorem 7, if $\mathcal{M}(H)$ does contain a forest then $\delta_\chi(H)\leqslant\frac{2r-5}{2r-3}$.

Now, by Theorem 16, if H is not r-near-acyclic then $\delta_\chi(H) \geqslant \frac{2r-5}{2r-3}$, and by Theorem 34, if H is r-near-acyclic then $\delta_\chi(H) = \frac{r-3}{r-2}$. Thus

$$\delta_\chi(H) \in \left\{ \frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1} \right\},$$

where $\delta_\chi(H) \neq \frac{r-2}{r-1}$ if and only if H has a forest in its decomposition family, and $\delta_\chi(H) = \frac{r-3}{r-2}$ if and only if H is r-near-acyclic, as required. $\square$

8. Open questions

Although we have determined $\delta_\chi(H)$ for every graph H, there are still many important questions left unresolved. In this section we shall discuss some of these. We begin by conjecturing that the assumption on the minimum degree can be weakened to force the boundedness of the chromatic number, as Brandt and Thomassé [8] proved in the case of the triangle.

Conjecture 43. *For every graph H with $\delta_\chi(H) = \lambda(H)$, there exists a constant $C(H)$ such that the following holds. If G is an H-free graph on n vertices and $\delta(G) > \lambda(H)n$, then $\chi(G) \leqslant C(H)$.*

We mention that an analogous statement is not true for H with $\delta_\chi(H) \in \{\theta(H), \pi(H)\}$ as a simple modification of our constructions for Propositions 5 and 35 shows: we merely need to make the partite graphs used in these constructions slightly unbalanced and to guarantee that the Erdős graphs cover the whole partition class they are pasted into and have a sufficient minimum degree.

For graphs H with $\delta_\chi(H) = 0$ one could still ask whether the minimum degree condition can be weakened to some function $f(n) = o(n)$. The following well-known fact shows that this is not the case.

Proposition 44. *Let H be a graph with $\chi(H) \geqslant 3$, and let $f(n) = o(n)$. For every C and n_1, there exist H-free graphs G on at least n_1 vertices with $\delta(G) \geqslant f\big(v(G)\big)$ and $\chi(G) \geqslant C$.*

Proof. Given H, f, C and n_1, let G_0 be a $(C, v(H)+1)$-Erdős graph. Without loss of generality, we may assume $\delta(G_0) \geqslant 1$. Let n_0 be such that $f(n) \leqslant n/v(G_0)$ for each $n \geqslant n_0$. Let G be obtained from G_0 by blowing up each vertex to a set of size $\max(n_0, n_1)$. Then G has at least n_1 vertices, and we have $\delta(G) \geqslant v(G)/v(G_0) \geqslant f\big(v(G)\big)$. Since G_0 contains no cycle on $v(H)$ or fewer vertices, G contains no odd cycle with $v(H)$ or fewer vertices. In particular, every $v(H)$-vertex subgraph of G is bipartite, and hence G is H-free. $\square$

Proposition 44 also implies that for graphs H with $\delta_\chi(H) = 0$ the upper bound on $\chi(G)$ for H-free graphs G with $\delta(G) \geqslant \varepsilon n$ increases as ε goes to zero. This suggests the following problem. Set

$$\delta_\chi(H, k) \ := \ \inf \Big\{ d \,:\, \delta(G) \geqslant d|G| \ \text{ and } \ H \not\subseteq G \ \ \Rightarrow \ \ \chi(G) \leqslant k \Big\},$$

or, equivalently,

$$\chi_\delta(H,d) \;:=\; \max\Big\{\chi(G) \,:\, \delta(G) \geqslant d|G| \;\text{ and }\; H \not\subseteq G\Big\},$$

and call this the *chromatic profile* of H.

Problem 45. *Determine the chromatic profile for every graph H.*

As noted in the Introduction, we have, by the results of Andrásfai, Erdős and Sós [3], Brandt and Thomassé [8], Häggkvist [18] and Jin [20], that

$$\delta_\chi(K_3,2) = \frac{2}{5}, \quad \delta_\chi(K_3,3) = \frac{10}{29} \quad \text{ and } \quad \delta_\chi(K_3,k) = \frac{1}{3} \quad \text{for every } k \geqslant 4.$$

We remark that this problem was also asked by Erdős and Simonovits [14], who remarked that it seemed (in full generality) 'too complicated' to study; despite the progress made in recent years, we still expect it to be extremely difficult. Note that although our results give explicit upper bounds on $\chi_\delta(H,d)$ for every graph H, even in the case $\delta_\chi(H) = 0$, where we do not use the Szemerédi Regularity Lemma, these bounds are very weak.

Łuczak and Thomassé [30] suggested the following more general problem. Given a (without loss of generality monotone) family $\mathcal{F}$ of graphs, we define

$$\delta_\chi(\mathcal{F}) \;:=\; \inf\Big\{\delta \,:\, \exists\, C = C(\mathcal{F},\delta) \text{ such that if } G \in \mathcal{F} \text{ is a graph on } n \text{ vertices with } \delta(G) \geqslant \delta n, \text{ then } \chi(G) \leqslant C\Big\}.$$

Problem 46. *What values can $\delta_\chi(\mathcal{F})$ take?*

Our results settle this question completely when $\mathcal{F}$ is defined by finitely many minimal forbidden subgraphs (in which case $\delta_\chi(\mathcal{F})$ is precisely the minimum of $\delta_\chi(H)$ over all minimal forbidden subgraphs H). For families $\mathcal{F}$ defined by infinitely many forbidden subgraphs, however, this minimum provides only an upper bound on $\delta_\chi(\mathcal{F})$.

Łuczak and Thomassé [30] suggested in particular to determine $\delta_\chi(\mathcal{B})$, where $\mathcal{B}$ is the family of graphs G such that for every vertex $v \in G$, the graph $G\big[N(v)\big]$ is bipartite (as a natural generalisation of the family of triangle-free graphs, in which every neighbourhood is an independent set). This family is indeed defined by infinitely many forbidden subgraphs: to be precise, by the odd wheels. Łuczak and Thomassé gave a construction showing that $\delta_\chi(\mathcal{B}) \geqslant \frac{1}{2}$, and conjectured that $\delta_\chi(\mathcal{B}) = \frac{1}{2}$. Since the wheel W_5 (i.e., the graph obtained from C_5 by adding a vertex adjacent to all its vertices) is a forbidden graph for $\mathcal{B}$, and $\delta_\chi(W_5) = \frac{1}{2}$ by Theorem 2, our results confirm that their conjecture is true.

One can generalise the concept of chromatic threshold to uniform hypergraphs. Recently, Balogh, Butterfield, Hu, Lenz and Mubayi [4] extended the Łuczak-Thomassé method to uniform hypergraphs $\mathcal{H}$, and thereby proved that $\delta_\chi(\mathcal{H}) = 0$ for a large family of such $\mathcal{H}$. To quote from their paper, 'Many open problems remain'.

Finally, we would like to introduce a new class of problems relating to the chromatic threshold. There has been a recent trend in Combinatorics towards proving 'random analogues' of extremal results in Graph Theory and Additive Number Theory (see, for example, the recent breakthroughs of Conlon and Gowers [10] and Schacht [36]). We propose the following variation on this theme: for each graph H and every function $p = p(n) \in [0,1]$, define

$$\delta_\chi\big(H,p\big) \;:=\; \inf\Big\{d \,:\, \exists\, C(H,d) \text{ such that for } G = G_{n,p}, \text{ asymptotically almost surely,} \\ \text{if } G' \subseteq G,\; \delta(G') \geqslant dpn \;\text{ and }\; H \not\subseteq G',\; \text{then } \chi(G') \leqslant C(H,d)\Big\},$$

where $G_{n,p}$ is the Erdős-Rényi random graph. Note that when $p(n) = 1$, we recover the definition of $\delta_\chi(H)$.

Problem 47. *Determine $\delta_\chi(H,p)$ for every graph H, and every $p = p(n)$.*

In a forthcoming paper [2] we intend to show that for every constant $p > 0$ and every graph H, we have $\delta_\chi(H) = \delta_\chi(H,p)$. This is of course trivial in the case $\delta_\chi(H) = 0$, when it follows from the results of this paper together with the well-known fact that for constant p, the minimum degree of $G_{n,p}$ is asymptotically almost surely at least $pn/2$. In the case $\delta_\chi(H) > 0$, the result is not trivial: but much of the machinery developed in this paper can be used unchanged. The following construction shows that the result is best possible, in the sense that it fails to hold for $p = o(1)$.

Theorem 48. *Let $r \geqslant 3$ and $C \in \mathbb{N}$, and let H be a graph with $\chi(H) = r$ and $\delta_\chi(H) \geqslant \lambda(H) = \frac{2r-5}{2r-3}$. If $\varepsilon > 0$ is sufficiently small, the following holds. If $n^{-\varepsilon} < p < \varepsilon^2$, then asymptotically almost surely the graph $G = G_{n,p}$ contains an H-free subgraph G' with $\chi(G') \geqslant C$ and $\delta(G') \geqslant (1-\varepsilon)\frac{r-2}{r-1}pn$.*

Proof (sketch). Given r, H and C, we let F be a fixed $(C, v(H)+1)$-Erdős graph. We choose a sufficiently small $\varepsilon > 0$.

We now construct an H-free subgraph G' of $G = G_{n,p}$ as follows. Let $V_1, \ldots, V_{r-1}$ be an arbitrary balanced partition of $[n]$. We fix a copy of F within $G[V_1]$ (which exists asymptotically almost surely). Then we delete all edges within each part V_i with $i \in [r-1]$, except those in the copy of F. Moreover, for each pair of vertices $u, v \in V(F)$, we delete the edges from u and v to the common neighbours of u and v in each of $V_2, \ldots, V_{r-1}$.

It follows that $\chi(G') \geqslant C$ and that G' asymptotically almost surely has minimum degree $(1-\varepsilon)\frac{r-2}{r-1}pn$. In addition it can easily be checked from our characterisation of graphs H with $\delta_\chi(H) \geqslant \lambda(H)$ that G' is also H-free. $\square$

Theorem 48 can be significantly strengthened, and we intend to do so in [2]. However, results of Kohayakawa, Rödl and Schacht [22] show that we cannot increase the value $\frac{r-2}{r-1}$ in the minimum degree, i.e, $\delta_\chi(H,p) \leqslant \pi(H)$. Thus, by Theorem 48, and in contrast to the $p = \Theta(1)$ case, if $\delta_\chi(H) \geqslant \lambda(H)$ and $n^{-o(1)} < p = o(1)$, then $\delta_\chi(H,p) = \frac{r-2}{r-1} = \pi(H)$.

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