Minimum degree conditions for large subgraphs

Peter Allen [1]

DIMAP
University of Warwick
Coventry, United Kingdom

Julia Böttcher and Jan Hladký [2,3]

Zentrum Mathematik
Technische Universität München
Garching bei München, Germany

Oliver Cooley [4]

School of Mathematics
University of Birmingham
Birmingham, United Kingdom

Abstract

Much of extremal graph theory has concentrated either on finding very small subgraphs of a large graph (such as Turán's theorem [10]) or on finding spanning subgraphs (such as Dirac's theorem [3] or more recently work of Komlós, Sárközy and Szemerédi [7,8] towards a proof of the Pósa-Seymour conjecture). Only a few results give conditions to obtain some intermediate-sized subgraph. We contend that this neglect is unjustified. To support our contention we focus on the illustrative case of minimum degree conditions which guarantee squared-cycles of various lengths, but also offer results, conjectures and comments on other powers of paths and cycles, generalisations thereof, and hypergraph variants.

Keywords: extremal graph theory, minimum degree, large subgraphs

1 Introduction

We recall that the classic theorems of Turán [10] and Erdős and Stone [4] provide best possible (in the latter case, up to an $o(n)$ error term) minimum degree conditions (and indeed density conditions) for a large graph G to contain respectively a clique and a fixed small subgraph H. In the first theorem the extremal graphs are Turán graphs; in the second they are Turán graphs with at most $o(n^2)$ edges changed.

We define the *bottle graph* $B_k(n,\delta)$ to be a complete k-partite graph on n vertices whose smallest part is as small as possible subject to the minimum degree of the graph being δ. We note that when $\delta = \lfloor \frac{k-1}{k} n \rfloor$ the smallest part is at most one vertex smaller than the largest, and we obtain the Turán graph $T_k(n)$.

Dirac's theorem [3] provides a best possible minimum degree condition for a graph to contain a Hamilton cycle; the extremal graphs are bottle graphs.

We define the k-th power of a graph F, F^k, to be that graph with vertex set $V(F^k) = V(F)$ and edge set $E(F^k) = \{uv\colon \mathrm{dist}_F(u,v) \le k\}$. Generalising Dirac's theorem, Pósa (for $k = 2$) and Seymour (for general k) gave conjectured best possible minimum degree conditions for a graph G to contain the k-th power of a Hamilton cycle; these conjectures have recently been proved for large graphs by Komlós, Sárközy and Szemerédi [7,8]. Again the extremal graphs are bottle graphs.

We say a graph G contains an *H-packing* on v vertices when there exist in G vertex-disjoint copies of the graph H covering in total v vertices of G. Hajnal and Szemerédi [5] gave for each k best possible minimum degree conditions for a graph G to contain a perfect K_k-packing (i.e. a packing covering all but at most $k-1$ vertices of G). The extremal graphs are bottle graphs.

Finally, Kühn and Osthus [9] gave for each fixed H best possible (up to an $o(n)$ error term) minimum degree conditions for an n-vertex graph G to contain a perfect H-packing. Yet again, their extremal graphs are obtained

[1] Email: p.d.allen@warwick.ac.uk . Supported by DIMAP—Centre for Discrete Mathematics and its Applications at the University of Warwick.
[2] Email: boettche@ma.tum.de . Partially supported by DFG grant TA 309/2-1.
[3] Email: hladk@seznam.cz . Supported in part by the grants GAUK 202-10/258009 and DAAD.
[4] Email: cooleyo@maths.bham.ac.uk .

by changing at most $o(n^2)$ edges of suitable bottle graphs.

2 Large subgraphs

If one desires to find in an n-vertex graph G not a perfect H-packing but only a partial H-packing, then for $H = K_k$ the Hajnal-Szemerédi theorem already gives a best possible minimum degree condition:

Theorem 2.1 *(Hajnal, Szemerédi) If G is any n-vertex graph with minimum degree $\delta(G) \in (\frac{k-2}{k-1}n, \frac{k-1}{k}n)$ we are guaranteed a K_k-packing covering at least the number of vertices covered by a maximal K_k-packing in $B_k(n, \delta)$.*

In 2000 Komlós [6] generalised this (albeit with an $o(n)$ error term) to general partial H-packings; as with the result of Kühn and Osthus the statement is not short, but amounts to a linear relationship (up to $o(n)$ error) between the number of vertices guaranteed in an H-packing and the minimum degree; the only real 'surprise' with H-packings is that for some H there is a 'plateau'; that is, an $\Theta(n)$-sized interval of values of δ over which we are guaranteed that if $\delta(G) = \delta$ then G has an almost-perfect packing but we are not guaranteed a perfect packing. The extremal graphs for H-packings are not necessarily bottle graphs, but they differ only in at most $o(n^2)$ edges.

We wish to give a similar generalisation of Pósa's conjecture. We note that it is straighforward to extend Komlós, Sárközy and Szemerédi's proof to:

If G is a sufficiently large n-vertex graph which has minimum degree $\delta(G) \geq 2n/3$ then G contains $C_{3\ell}^2$ for each $3\ell \in [3, n]$, and if the inequality is strict also C_ℓ for each $\ell \in \{3, 4\} \cup [6, n]$. It is this stronger statement that we will generalise (for large graphs).

We wish to find a (best possible) function $\mathrm{sc}(n, \delta)$ such that any n-vertex graph G with minimum degree $\delta(G) \geq \delta$ contains $C_{3\ell}^2$ for each $3\ell \leq sc(n, \delta)$. There are two reasonable conjectures one might make—both false.

The first is that the extremal graphs should (yet again) be bottle graphs and thus we should find $\mathrm{sc}(n, \delta) \approx 6\delta - 3n$. This is far too optimistic.

The second is that the method used for previous spanning subgraph and partial packing results should work: start a construction at any vertex and continue cleverly until stuck, when the best possible result will be achieved. This would suggest that $\mathrm{sc}(n, \delta) \approx 3\delta - 3n/2$ (provided $\delta = n/2 + \Omega(n)$)—which is almost always far too pessimistic.

In fact, the extremal graphs—which we call $\mathrm{SC}(n, \delta)$—are far from being bottle graphs, and the function $\mathrm{sc}(n, \delta)$ is far from linear: it has (as n grows) an unbounded number of 'jumps'; that is, places where when δ increases by one,

$\mathrm{sc}(n,\delta)$ increases by $\Theta(n)$. An interesting feature of the extremal graphs is that—even though we cannot guarantee $C_\ell^2 \subset G$ for any $\ell \geq 4$ not divisible by 3, since G could be tripartite—they do in fact contain C_ℓ^2 for every $\ell \leq \mathrm{sc}(n,\delta)$. One might think that excluding all, or at least one, of these 'extra' squared-cycles with chromatic number four would not change the result by very much—but this is again false. We will describe the graphs $\mathrm{SC}(n,\delta)$ and show that, when $\delta = n/2+\Omega(n)$, $\mathrm{sc}(n,\delta)$ is equal to the length of the longest squared-cycle of $\mathrm{SC}(n,\delta)$, proving the following theorem.

Theorem 2.2 *(Allen, Böttcher, Hladký) Given any $\gamma > 0$ there is n_0 such that the following hold for any graph G on $n \geq n_0$ vertices with $\delta(G) = \delta \in (n/2 + \gamma n, 2n/3)$.*

First, $C_{3\ell}^2 \subset G$ for each $3\ell \leq \mathrm{sc}(n,\delta)$. Second, either $C_\ell^2 \subset G$ for each $\ell \in \{3,4\} \cup [6, \mathrm{sc}(n,\delta)]$, or $C_{3\ell}^2 \subset G$ for each $3\ell \leq 6\delta - 3n - \gamma n$.

We conjecture that the γn error term can be removed from the second condition, but it is necessary that $\delta(G)$ is not too close to $n/2$. Our results give also best-possible conditions for squared-paths.

Surprisingly, even for simple cycles no such best possible theorem has previously been given. We give the correct result.

Theorem 2.3 *(Allen) Given an integer $k \geq 2$ there is $n_0 = O(k^{20})$ such that the following hold for any graph G on $n \geq n_0$ vertices with $\delta(G) = \delta \geq n/k$.*

First, $C_\ell \subset G$ for every even $4 \leq \ell \leq \lceil \frac{n}{k-1} \rceil$. Second, either $C_\ell \subset G$ for each $\lfloor \frac{2n}{\delta} \rfloor - 1 \leq \ell \leq \lceil \frac{n}{k-1} \rceil$, or $C_\ell \subset G$ for every even $4 \leq \ell \leq 2\delta$.

3 Further work

At the time of writing the correct generalisation of Theorem 2.2 to higher powers of paths or cycles remains a conjecture. However there are further questions in this area—Bollobás and Komlós suggested a further generalisation of the Seymour conjecture to objects more complicated than powers of cycles (which was recently proved by Böttcher, Schacht and Taraz [1]); we believe that the related large subgraph questions exhibit interesting phenomena which are not found with powers of cycles.

Finally, there has recently been progress in understanding extremal phenomena in r-uniform hypergraphs; unsurprisingly the situation here is much more complex than with simple graphs (there are for example multiple reasonable generalisations of the notion of minimum degree). Nevertheless, we offer some observations on matchings and paths in this setting.

References

[1] Böttcher, J., M. Schacht and A. Taraz, *Proof of the bandwidth conjecture of Bollobás and Komlós*, Math. Ann **343(1)** (2009), 175–205.

[2] Corrádi, K. and A. Hajnal, *On the maximal number of independent circuits of a graph*, Acta Math. Acad. Sci. Hungar. **14** (1963), 423–443.

[3] Dirac, G.A., *Some theorems on abstract graphs*, Proc. London Math. Soc. **s3-2** (1952), 69–81.

[4] Erdős, P. and A. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091.

[5] Hajnal, A. and E. Szemerédi, *Proof of a conjecture of Erdős*, Combinatorial theory and its applications, Vol. 2, North-Holland (1970), 601–623.

[6] Komlós, J., *Tiling Turán theorems*, Combinatorica **20** (2000), 203–218.

[7] Komlós, J., G. N. Sárközy and E. Szemerédi, *On the square of a Hamiltonian cycle in dense graphs*,, Random Struct. Algorithms **9** (1996), 193-211.

[8] Komlós, J., G. N. Sárközy and E. Szemerédi, *Proof of the Seymour Conjecture for large graphs*, Ann. Comb. **2** (1998), 43–60.

[9] Kühn, D. and D. Osthus, *The minimum degree threshold for perfect graph packings*, Combinatorica, to appear.

[10] Turán, P., *On an extremal problem in graph theory* (in Hungarian), Matematiko Fizicki Lapok **48** (1941), 436–452.