# A density Corrádi-Hajnal theorem

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**Abstract**

For $n$ sufficiently large, we determine the density threshold for an $n$-vertex graph to contain $k$ vertex-disjoint triangles, where $0 \leq k \leq \frac{n}{3}$. This extends results by Erdős and by Moon, and can be viewed as a density version of the Corrádi-Hajnal theorem.

*Keywords:* extremal graph theory, tiling problems, Corrádi-Hajnal theorem, Mantel's theorem

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## 1 Introduction

A classic result of Mantel [9] asserts that each $n$-vertex graph $G$ with more than $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges contains a triangle[5]. What happens when the threshold $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ is exceeded? Can we quantify the presence of triangles in $G$?

One natural approach to this broad question is to determine how many triangles is $G$ guaranteed to have, as a parameter of the edge density of $G$. Solving a long-standing open problem, Razborov [11] determined a tight bound $f(\alpha)$ such that each $n$-vertex graph with $\alpha n^2$ edges contains at least $(f(\alpha) + o_{n\to\infty}(1))n^3$ triangles. Note that $f(\alpha) = 0$ for $\alpha \in [0, \frac{1}{4}]$, while by Mantel's theorem and the Supersaturation Theorem [4], $f(\alpha) > 0$ for $\alpha \in (\frac{1}{4}, \frac{1}{2})$. It is striking that the function $f(\alpha)$ exhibits very complicated behaviour.

In this abstract, we deal with a different measure of the presence of triangles. We ask what edge density in an $n$-vertex graph guarantees $k$ *vertex-disjoint* triangles. Such a collection of triangles is often called a *tiling*. Prior to our work this question was considered by Erdős [2] and by Moon [10]; the former proved the exact result when $n \geq 400k^2$, and the latter when $n \geq 9k/2+4$. Interestingly, although Moon states that his result 'almost certainly remains valid for somewhat smaller values of $n$ also', in fact he almost reaches a natural barrier: the graph which Moon proved to be extremal (the first in Figure 1 below) is only extremal when $n \geq 9k/2 + 3$. We give a precise answer to the question for all values of $k$ when $n$ is greater than an absolute constant $n_0$ in Theorem 2.1 below.

Tiling questions, which can of course be formulated for other graphs than triangles, have received a great deal of attention for a long time already. They typically fall into the following class of problems.

**Problem 1.1** *Suppose that a density condition $\mathcal{C}$ is satisfied for a graph $G$. How many vertex-disjoint copies of a graph $H$ are then guaranteed in $G$?*

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[5] This was later extended by Turán [13]. Turán's result is considered to be the starting point of the field of Extremal Graph Theory. However we restrict ourselves to investigating extremal problems concerning only triangles, and in that sense Mantel's theorem is more relevant.

The density condition $\mathcal{C}$ is usually parametrised by the average degree of $G$ (as is the case in our Theorem 2.1) or by the minimum degree of $G$.

Erdős and Gallai [3] gave a tight bound on the size of a maximum matching (i.e., a tiling with edges) as a function of the average degree. This was recently extended by Grosu and Hladký [5] who determined the asymptotic size of a tiling with a fixed bipartite graph $H$ guaranteed in a graph of given density.

Theorems of Corrádi and Hajnal [1], Hajnal and Szemerédi [6], Komlós [7] and Kühn and Osthus [8] answer Problem 1.1 when the condition $\mathcal{C}$ concerns the minimum degree. For example, the Corrádi-Hajnal theorem in its original form asserts that an $n$-vertex graph with minimum degree at least $\frac{2n}{3}$ contains a triangle tiling which covers all but at most two vertices. It is straightforward to deduce the following generalisation. Every $n$-vertex graph $G$ with $\frac{n}{2} < \delta(G) < \frac{2n}{3}$ contains a triangle tiling with at least $2\delta(G) - n$ triangles. This bound is tight, as is shown by unbalanced complete tripartite graphs. Our main result, Theorem 2.1 below, is therefore a density version of the Corrádi-Hajnal theorem.

## 2 Result

Given an integer $\ell$ and a graph $H$, we write $\ell \times H$ for the disjoint union of $\ell$ copies of $H$. A graph is $\ell \times H$*-free* if it does not contain $\ell$ vertex disjoint copies of $H$. In Theorem 2.1 we determine the maximal number of edges in a $(k+1) \times K_3$-free graph on $n$ vertices for every $1 \le k \le \frac{n}{3}$. To this end we identify the extremal structures for this problem, i.e., the graphs which attain this maximal number of edges. These are as follows (see also Figure 1).

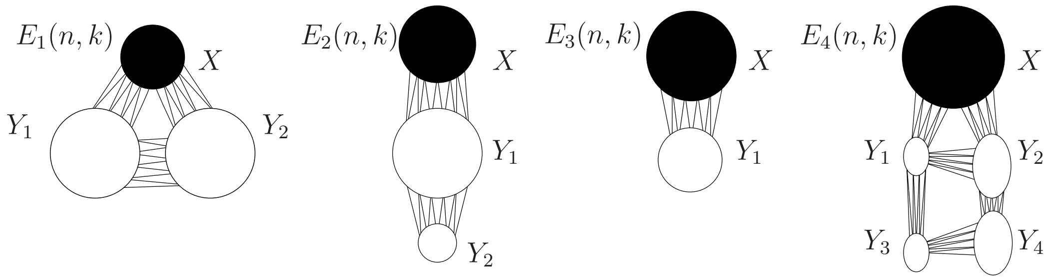


Fig. 1. The extremal graphs.

$E_1(n,k)$: Let $X \dot\cup Y_1 \dot\cup Y_2$ with $|X| = k$, $|Y_1| = \lceil \frac{n-k}{2} \rceil$, and $|Y_2| = \lfloor \frac{n-k}{2} \rfloor$ be the vertices of $E_1(n,k)$. Insert all edges intersecting $X$, and between $Y_1$ and $Y_2$.

$E_2(n,k)$: Let $X \dot\cup Y_1 \dot\cup Y_2$ with $|X| = 2k+1$, $|Y_1| = \lfloor \frac{n}{2} \rfloor$, and $|Y_2| = \lceil \frac{n}{2} \rceil -$

$2k-1$ (or $|Y_1| = \lceil \frac{n}{2} \rceil$, and $|Y_2| = \lfloor \frac{n}{2} \rfloor - 2k - 1$) be the vertices of $E_2(n,k)$. Insert all edges within $X$, and between $Y_1$ and $X \cup Y_2$. If $n$ is odd, this construction captures two graphs, if $n$ is even just one.

$E_3(n,k)$: Let $X \dot\cup Y_1$ with $|X| = 2k+1$ and $|Y_1| = n-2k-1$ be the vertices of $E_3(n,k)$. Insert all edges intersecting $X$.

$E_4(n,k)$: The fourth class of extremal graphs is defined only for $k \geq \frac{n}{6} - 2$. The vertex set is formed by five disjoint sets $X$, $Y_1$, $Y_2$, $Y_3$, and $Y_4$, with $|Y_1| = |Y_3|$, $|Y_2| = |Y_4|$, $|Y_1| + |Y_2| = n - 3k - 2$, and $|X| = 6k - n + 4$. Insert exactly all edges in $X$, between $X$ and $Y_1 \cup Y_2$, and between $Y_1 \cup Y_4$ and $Y_2 \cup Y_3$. Thus the choice of $|Y_1|$ determines a particular graph in the class $E_4(n,k)$. All graphs in $E_4(n,k)$ have the same number of edges.

It is straightforward to check that these graphs are edge-maximal subject to not containing $(k+1) \times K_3$. Our theorem now states that for each value of $k$ one of these constructions is extremal.

**Theorem 2.1** *There exists $n_0$ such that the following holds for all $n \geq n_0$ and $k \leq \frac{n}{3}$. Let $G$ be a $(k+1) \times K_3$-free graph on $n$ vertices. Then*

$$e(G) \leq \max_{j \in [4]} e\big(E_j(n,k)\big) . \tag{1}$$

Comparing the numbers of edges of the extremal graphs reveals that, as $k$ grows from 1 to $\frac{n}{3}$, the extremal graphs dominate in the following order (for $n$ sufficiently large). In the beginning $E_1(n,k)$ has the most edges, but at $k \approx \frac{2n}{9}$ it is surpassed by $E_2(n,k)$. At $k \approx \frac{n}{4}$ the structure $E_2(n,k)$ ceases to exist and is replaced by $E_3(n,k)$, and finally at $k \approx (5+\sqrt{3})n/22$ the structure $E_4(n,k)$ takes over. The exact transition values are given in the following table.

| $E_1(n,k) \to E_2(n,k)$ | $E_2(n,k) \to E_3(n,k)$ | $E_3(n,k) \to E_4(n,k)$ |
|---|---|---|
| $\frac{2n-6}{9}$ | $\frac{n-1}{4}$ | $\frac{5n-12+\sqrt{3n^2-10n+12}}{22}$ |

## 3 Sketch of the proof of Theorem 2.1

We call a pair $(\mathcal{T}', \mathcal{M}')$ a *tiling pair* if $\mathcal{T}'$ is a collection of vertex-disjoint triangles in $G$, if $\mathcal{M}'$ is a matching in $G$, and if $\mathcal{T}'$ and $\mathcal{M}'$ do not share vertices. Among all the tiling pairs choose a pair $(\mathcal{T}, \mathcal{M})$ which $(i)$ maximises $|\mathcal{T}|$, and among all such pairs, $(ii)$ one which maximises $|\mathcal{M}|$. Clearly, the set of vertices $\mathcal{I}$ not covered by $\mathcal{T} \cup \mathcal{M}$ is independent. By assumption $|\mathcal{T}| \leq k$.

The proof idea is simple. To obtain a bound on $e(G)$, we work with the quantities $e(G[\mathcal{T}])$, $e(G[\mathcal{M}])$, and further $e(G[\mathcal{T}, \mathcal{M}])$, $e(G[\mathcal{T}, \mathcal{I}])$, and

$e(G[\mathcal{M}, \mathcal{I}])$. To get the bound (1) we aim to establish inequalities involving combinations of the edge counts above.

In fact, we will need to split the set $\mathcal{T}$ further as follows. We say that an edge $e$ of $\mathcal{M}$ *sees* a triangle $T$ of $\mathcal{T}$ when a vertex of $T$ forms a triangle with $e$, and similarly that a vertex $v$ of $\mathcal{I}$ *sees* $T$ when $v$ together with two vertices of $T$ forms a triangle. Then we set $\mathcal{T}_1$ to be the triangles of $\mathcal{T}$ which are seen by at least two $\mathcal{M}$-edges and $\mathcal{T}_2$ the triangles of $\mathcal{T} \setminus \mathcal{T}_1$ which are seen by either at least two $\mathcal{I}$-vertices, or one $\mathcal{I}$-vertex and one $\mathcal{M}$-edge.

Let us illustrate our methods for the proof of Theorem 2.1 by establishing the bound $e(G[\mathcal{T}_1]) \leq 7\binom{|\mathcal{T}_1|}{2} + 3|\mathcal{T}_1|$ (where $|\mathcal{T}_1|$ counts the triangles in $\mathcal{T}_1$), which is one of the easier bounds we use. For this bound it suffices to show that between any pair of triangles of $\mathcal{T}_1$ there are at most seven edges. Suppose, then, that there are two triangles $uvw$ and $u'v'w'$ of $\mathcal{T}_1$ with at least eight edges between them. By definition of $\mathcal{T}_1$, there are distinct (and hence disjoint) edges $xy$ and $x'y'$ of $\mathcal{M}$ which see respectively $uvw$ and $u'v'w'$; let us assume that they form triangles with $u$ and $u'$ respectively. Now $v, w, v'$ and $w'$ induce a subgraph of $G$ with at least five edges and thus containing a triangle, say $vwv'$. Finally $vwv'$, $xyu$, $x'y'u'$ and $\mathcal{T} \setminus \{uvw, u'v'w'\}$ form a triangle tiling with more triangles than $\mathcal{T}$, contradicting the definition of $(\mathcal{T}, \mathcal{M})$.

We are able to obtain all bounds involving $\mathcal{T}_1$ and $\mathcal{T}_2$ by similarly short arguments. However, just as the extremal structure $E_4(n,k)$ is the most complicated of our four structures, so we need significantly more complex arguments to handle the triangles $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ to which it corresponds. We partition these remaining triangles of $\mathcal{T}$ into a 'sparse part' $\mathcal{T}_3$ and a 'dense part' $\mathcal{T}_4$ by applying the following algorithm. We start with $D$ equal to the set of all triangles in $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$, and $S = \emptyset$. If there is a triangle in $D$ which sends at most $8(|D|-1)$ edges to the other triangles in $D$, we move it to $S$. We repeat until $D$ contains no more such triangles. We then set $\mathcal{T}_3 = S$ and $\mathcal{T}_4 = D$.

The motivation behind this last partition is the following. The construction of $\mathcal{T}_3$ already guarantees that $e(G[\mathcal{T}_3]) + e(G[\mathcal{T}_3, \mathcal{T}_4])$ is small enough for our purposes. On the other hand, within the set $\mathcal{T}_4$ we have not only a high density of edges but even a 'minimum degree' condition between triangles: every triangle in $\mathcal{T}_4$ sends more than $8(|\mathcal{T}_4|-1)$ edges to the other triangles in $\mathcal{T}_4$. It is much easier to work with this latter condition than simply an edge density condition. This enables us to find complex structures in $\mathcal{T}_4$ (using up to 27 triangles) whose existence together with the maximality of $\mathcal{T}$ and a substantial amount of additional technical work yields the required upper bound on $e(G[\mathcal{T}_4 \cup \mathcal{M} \cup \mathcal{I}])$.

Last, let us remark that we were able to prove (1) only for large graphs. This is caused by the usage of the Stability Method of Simonovits [12] in our proof. It is plausible that the bound holds for *all* graphs and the assumption is only an artefact of our proof.

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