BANDWIDTH, EXPANSION, TREEWIDTH, SEPARATORS, AND UNIVERSALITY FOR BOUNDED DEGREE GRAPHS

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Abstract. We establish relations between the bandwidth and the treewidth of bounded degree graphs G, and relate these parameters to the size of a separator of G as well as the size of an expanding subgraph of G. Our results imply that if one of these parameters is sublinear in the number of vertices of G then so are all the others. This implies for example that graphs of fixed genus have sublinear bandwidth or, more generally, a corresponding result for graphs with any fixed forbidden minor. As a consequence we establish a simple criterion for universality for such classes of graphs and show for example that for each $\gamma > 0$ every n-vertex graph with minimum degree $(\frac{3}{4}+\gamma)n$ contains a copy of every bounded-degree planar graph on n vertices if n is sufficiently large.

1. Introduction

There are a number of different parameters in graph theory which measure how well a graph can be organised in a particular way, where the type of desired arrangement is often motivated by geometrical properties, algorithmic considerations, or specific applications. Well-known examples of such parameters are the genus, the bandwidth, or the treewidth of a graph. The topic of this paper is to discuss the relations between such parameters. We would like to determine how they influence each other and what causes them to be large. To this end we will mostly be interested in distinguishing between the cases when these parameters are linear or sublinear in the number of vertices in the graph.

We start with a few simple observations. Let $G = (V, E)$ be a graph on n vertices. The *bandwidth* of G is denoted by $\mathrm{bw}(G)$ and defined to be the minimum positive integer b, such that there exists a labelling of the vertices in V by numbers $1, \dots, n$ so that the labels of every pair of adjacent vertices differ by at most b. Clearly one reason for a graph to have high bandwidth are vertices of high degree as $\mathrm{bw}(G) \geq \lceil \Delta(G)/2 \rceil$, where $\Delta(G)$ is the maximum degree of G. It is also clear that not all graphs of bounded maximum degree have sublinear bandwidth: Consider for example a random bipartite graph G on n vertices with bounded maximum degree. Indeed, with high probability, G does not have small bandwidth since in any linear ordering of its vertices there will be an edge between the first $n/3$ and the last $n/3$ vertices in this ordering. The reason behind this obstacle is that G has good expansion properties (definitions and exact statements are provided in Section 2). This implies that graphs with sublinear bandwidth cannot exhibit good

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expansion properties. One may ask whether the converse is also true, i.e., whether the absence of big expanding subgraphs in bounded-degree graphs must lead to small bandwidth. We will prove that this is indeed the case via the existence of certain separators.

In fact, we will show a more general theorem in Section 2 (Theorem 8) which proves that the concepts of sublinear bandwidth, sublinear treewidth, bad expansion properties, and sublinear separators are equivalent for graphs of bounded maximum degree. In order to prove this theorem, we will establish quantitative relations between the parameters involved (see Theorem 5, Theorem 6, and Proposition 7).

As a byproduct of these relations we obtain sublinear bandwidth bounds for several graph classes (see Section 4). Since planar graphs are known to have small separators [19] for example, we get the following result: The bandwidth of a planar graph on n vertices with maximum degree at most Δ is bounded from above by $\mathrm{bw}(G) \le \frac{15n}{\log_\Delta(n)}$. This extends a result of Chung [8] who proved that any n-vertex tree T with maximum degree Δ has bandwidth at most $5n/\log_\Delta(n)$. Similar upper bounds can be formulated for graphs of any fixed genus and, more generally, for any graph class defined by a set of forbidden minors (see Section 4.1).

In Section 4.2 we conclude by considering applications of these results to the domain of universal graphs and derive implications such as the following. If n is sufficiently large then any n-vertex graph with minimum degree slightly above $\frac{3}{4}n$ contains every planar n-vertex graphs with bounded maximum degree as a subgraph (cf. Corollary 19).

2. Definitions and Results

In this section we formulate our main results which provide relations between the bandwidth, the treewidth, the expansion properties, and separators of bounded degree graphs. We need some further definitions. For a graph $G = (V, E)$ and disjoint vertex sets $A, B \subseteq V$ we denote by $E(A, B)$ the set of edges with one vertex in A and one vertex in B and by $e(A, B)$ the number of such edges.

Next, we will introduce the notions of *tree decomposition* and *treewidth*. Roughly speaking, a tree decomposition tries to arrange the vertices of a graph in a tree-like manner and the treewidth measures how well this can be done.

Definition 1 (treewidth)**.** *A tree decomposition of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\},\ T = (I, F))$ where $\{X_i : i \in I\}$ is a family of subsets $X_i \subseteq V$ and $T = (I, F)$ is a tree such that the following holds:*

(a) $\bigcup_{i\in I} X_i = V$,
(b) *for every edge $\{v, w\} \in E$ there exists $i \in I$ with $\{v, w\} \subseteq X_i$,*
(c) *for every $i, j, k \in I$ the following holds: if j lies on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.*

The width of $(\{X_i : i \in I\}, T = (I, F))$ is defined as $\max_{i\in I} |X_i| - 1$. The treewidth $\mathrm{tw}(G)$ of G is the minimum width of a tree decomposition of G.

It follows directly from the definition that $\mathrm{tw}(G) \le \mathrm{bw}(G)$ for any graph G: if the vertices of G are labelled by numbers $1, \dots, n$ such that the labels of adjacent vertices differ by at most b, then $I := [n - b]$, $X_i := \{i, \dots, i + b\}$ for $i \in I$ and $T := (I, F)$ with $F := \{\{i - 1, i\} : 2 \le i \le n - b\}$ define a tree decomposition of G with width b.

A *separator* in a graph is a small cut-set that splits the graph into components of limited size.

Definition 2 (separator, separation number)**.** *Let $\frac{1}{2} \leq \alpha < 1$ be a real number, $s \in \mathbb{N}$ and $G = (V, E)$ a graph. A subset $S \subseteq V$ is said to be an (s, α)-separator of G, if there exist subsets $A, B \subseteq V$ such that*

(a) $V = A \dot\cup B \dot\cup S$,
(b) $|S| \leq s$, $|A|, |B| \leq \alpha|V|$, *and*
(c) $E(A, B) = \emptyset$.

We also say that S separates G into A and B. The separation number $\mathrm{s}(G)$ *of G is the smallest s such that all subgraphs G' of G have an $(s, 2/3)$-separator.*

A vertex set is said to be expanding, if it has many external neighbours. We call a graph *bounded*, if every sufficiently large subgraph contains a subset which is not expanding.

Definition 3 (expander, bounded)**.** *Let $\varepsilon > 0$ be a real number, $b \in \mathbb{N}$ and consider graphs $G = (V, E)$ and $G' = (V', E')$. We say that G' is an ε-expander if all subsets $U \subseteq V'$ with $|U| \leq |V'|/2$ fulfil $|N(U)| \geq \varepsilon|U|$. (Here $N(U)$ is the set of neighbours of vertices in U that lie outside of U.) The graph G is called (b, ε)-bounded, if no subgraph $G' \subseteq G$ with $|V'| \geq b$ vertices is an ε-expander. Finally, we define the ε-boundedness $\mathrm{b}_\varepsilon(G)$ of G to be the minimum b for which G is $(b + 1, \varepsilon)$-bounded.*

There is a wealth of literature on expander graphs (see, e.g., [15]). In particular, it is known that for example (bipartite) random graphs with bounded maximum degree form a family of ε-expanders. We also loosely say that such graphs have *good expansion properties*.

As indicated earlier, our aim is to provide relations between the parameters we defined above. A well known example of a result of this type is the following theorem due to Robertson and Seymour which relates the treewidth and the separation number of a graph.[1]

Theorem 4 (treewidth→separator, [20])**.** *All graphs G have separation number*

$$\mathrm{s}(G) \leq \mathrm{tw}(G) + 1.$$

This theorem states that graphs with small treewidth have small separators. By repeatedly extracting separators, one can show that (a qualitatively different version of) the converse also holds: $\mathrm{tw}(G) \leq \mathcal{O}(\mathrm{s}(G) \log n)$ for a graph G on n vertices (see, e.g., [4, Theorem 20]). In this paper, we use a similar but more involved argument to show that one can establish the following relation linking the separation number with the bandwidth of graphs with bounded maximum degree.

Theorem 5 (separator→bandwidth)**.** *For each $\Delta \geq 2$ every graph G on n vertices with maximum degree $\Delta(G) \leq \Delta$ has bandwidth*

$$\mathrm{bw}(G) \leq \frac{6n}{\log_\Delta(n/\mathrm{s}(G))}.$$

[1]In fact, their result states that any graph G has a $(\mathrm{tw}(G) + 1, 1/2)$-separator, and does not talk about subgraphs of G. But since every subgraph of G has treewidth at most $\mathrm{tw}(G)$, it thus also has a $(\mathrm{tw}(G) + 1, 1/2)$-separator and the result, as stated here, follows.

The proof of this theorem is provided in Section 3.2. Observe that Theorems 4 and 5 together with the obvious inequality $\operatorname{tw}(G) \leq \operatorname{bw}(G)$ tie the concepts of treewidth, bandwidth, and separation number well together. Apart from the somewhat negative statement of *not having* a small separator, what can we say about a graph with large tree- or bandwidth? The next theorem states that such a graph must contain a big expander.

Theorem 6 (bounded→treewidth)**.** *Let $\varepsilon > 0$ be constant. All graphs G on n vertices have treewidth $\operatorname{tw}(G) \leq 2\,\mathrm{b}_\varepsilon(G) + 2\varepsilon n$.*

A result with similar implications was recently proved by Grohe and Marx in [13]. It shows that $b_\varepsilon(G) < \varepsilon n$ implies $\operatorname{tw}(G) \leq 2\varepsilon n$. For the sake of being self-contained we present our (short) proof of Theorem 6 in Section 3.3. In addition, it is not difficult to see that conversely the boundedness of a graph can be estimated via its bandwidth – which we prove in Section 3.3, too.

Proposition 7 (bandwidth→bounded)**.** *Let $\varepsilon > 0$ be constant. All graphs G on n vertices have $\mathrm{b}_\varepsilon(G) \leq 2\operatorname{bw}(G)/\varepsilon$.*

A qualitative consequence summarising the four results above is given in the following theorem. It states that if one of the parameters bandwidth, treewidth, separation number, or boundedness is sublinear for a family of bounded degree graphs, then so are the others.

Theorem 8 (sublinear equivalence theorem)**.** *Let Δ be an arbitrary but fixed positive integer and consider a hereditary class of graphs $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have maximum degree at most Δ. Denote by $\mathcal{C}_n$ the set of those graphs in $\mathcal{C}$ with n vertices. Then the following four properties are equivalent:*

(1) *For all $\beta_1 > 0$ there is n_1 s.t. $\operatorname{tw}(G) \leq \beta_1 n$ for all $G \in \mathcal{C}_n$ with $n \geq n_1$.*
(2) *For all $\beta_2 > 0$ there is n_2 s.t. $\operatorname{bw}(G) \leq \beta_2 n$ for all $G \in \mathcal{C}_n$ with $n \geq n_2$.*
(3) *For all $\beta_3, \varepsilon > 0$ there is n_3 s.t. $\mathrm{b}_\varepsilon(G) \leq \beta_3 n$ for all $G \in \mathcal{C}_n$ with $n \geq n_3$.*
(4) *For all $\beta_4 > 0$ there is n_4 s.t. $\mathrm{s}(G) \leq \beta_4 n$ for all $G \in \mathcal{C}_n$ with $n \geq n_4$.*

The paper is organized as follows. Section 3 contains the proofs of all the results mentioned so far: First we derive Theorem 8 from Theorems 4, 5, 6 and Proposition 7. Then Section 3.2 is devoted to the proof of Theorem 5, whereas Section 3.3 contains the proofs of Theorem 6 and Proposition 7. Finally, in Section 4, we apply our results to deduce that certain classes of graphs have sublinear bandwidth and can therefore be embedded as spanning subgraphs into graphs of high minimum degree.

3. Proofs

3.1. Proof of Theorem 8.

Proof. (1)⇒(4): Given $\beta_4 > 0$ set $\beta_1 := \beta_4/2$, let n_1 be the constant from (1) for this β_1, and set $n_4 := \max\{n_1, 2/\beta_4\}$. Now consider $G \in \mathcal{C}_n$ with $n \geq n_4$. By assumption we have $\operatorname{tw}(G) \leq \beta_1 n$ and thus we can apply Theorem 4 to conclude that $\mathrm{s}(G) \leq \operatorname{tw}(G) + 1 \leq \beta_1 n + 1 \leq (\beta_4/2 + 1/n)n \leq \beta_4 n$.

(4)⇒(2): Given $\beta_2 > 0$ let $d := \max\{2, \Delta\}$, set $\beta_4 := d^{-6/\beta_2}$, get n_4 from (4) for this β_4, and set $n_2 := n_4$. Let $G \in \mathcal{C}_n$ with $n \geq n_2$. We conclude from (4) and

Theorem 5 that

$$\mathrm{bw}(G) \le \frac{6n}{\log_d n - \log_d \mathrm{s}(G)} \le \frac{6n}{\log_d n - \log_d (d^{-6/\beta_2} n)} = \beta_2 n.$$

(2)⇒(3): Given $\beta_3, \varepsilon > 0$ set $\beta_2 := \varepsilon\beta_3/2$, get n_2 from (2) for this β_2 and set $n_3 := n_2$. By (2) and Proposition 7 we get for $G \in \mathcal{C}_n$ with $n \ge n_3$ that $\mathrm{b}_\varepsilon(G) \le 2\,\mathrm{bw}(G)/\varepsilon \le 2\beta_2 n/\varepsilon \le \beta_3 n$.

(3)⇒(1): Given $\beta_1 > 0$, set $\beta_3 := \beta_1/4$, $\varepsilon := \beta_1/4$ and get n_3 from (3) for this β_3 and ε, and set $n_1 := n_3$. Let $G \in \mathcal{C}_n$ with $n \ge n_1$. Then (3) and Theorem 6 imply $\mathrm{tw}(G) \le 2\,\mathrm{b}_\varepsilon(G) + 2\varepsilon n \le 2\beta_3 n + 2(\beta_1/4)n = \beta_1 n$. □

3.2. **Separation and bandwidth.** For the proof of Theorem 5 we will use the following decomposition result which roughly states the following. If the removal of a small separator S decomposes the vertex set of a graph G into relatively small components $R_i \dot\cup P_i$ such that the vertices in P_i form a "buffer" between the vertices in the separator S and the set of remaining vertices R_i in the sense that $\mathrm{dist}_G(S, R_i)$ is sufficiently big, then the bandwidth of G is small.

Lemma 9 (decomposition lemma)**.** *Let $G = (V, E)$ be a graph and S, P, and R be vertex sets such that $V = S \dot\cup P \dot\cup R$. For $b, r \in \mathbb{N}$ with $b \ge 3$ assume further that there are decompositions $P = P_1 \dot\cup \ldots \dot\cup P_b$ and $R = R_1 \dot\cup \ldots \dot\cup R_b$ of P and R, respectively, such that the following properties are satisfied:*

(i) $|R_i| \le r$,
(ii) $e(R_i \dot\cup P_i, R_j \dot\cup P_j) = 0$ *for all* $1 \le i < j \le b$,
(iii) $\mathrm{dist}_G(u, v) \ge \lfloor b/2 \rfloor$ *for all* $u \in S$ *and* $v \in R_i$ *with* $i \in [b]$.

Then $\mathrm{bw}(G) < 2(|S| + |P| + r)$.

Proof. Assume we have $G = (V, E)$, $V = S \dot\cup P \dot\cup R$ and $b, r \in \mathbb{N}$ with the properties stated above. Our first goal is to partition V into pairwise disjoint sets $B_1, \ldots, B_b$, which we call *buckets*, and that satisfy the following property:

$$\text{If } \{u, v\} \in E \text{ for } u \in B_i \text{ and } v \in B_j \text{ then } |i - j| \le 1. \tag{1}$$

To this end all vertices of R_i are placed into bucket B_i for each $i \in [b]$ and the vertices of S are placed into bucket $B_{\lceil b/2 \rceil}$. The remaining vertices from the sets P_i are distributed over the buckets according to their distance from S: vertex $v \in P_i$ is assigned to bucket $B_{j(v)}$ where $j(v) \in [b]$ is defined by

$$j(v) := \begin{cases} i & \text{if } \mathrm{dist}(S, v) \ge |\lceil b/2 \rceil - i|, \\ \lceil b/2 \rceil - \mathrm{dist}(S, v) & \text{if } \mathrm{dist}(S, v) < \lceil b/2 \rceil - i, \\ \lceil b/2 \rceil + \mathrm{dist}(S, v) & \text{if } \mathrm{dist}(S, v) < i - \lceil b/2 \rceil. \end{cases} \tag{2}$$

This placement obviously satisfies

$$|B_i| \le |S| + |P| + |R_i| \le |S| + |P| + r \tag{3}$$

by construction and condition (i). Moreover, we claim that it guarantees condition (1). Indeed, let $\{u, v\} \in E$ be an edge. If u and v are both in S then clearly (1) is satisfied. Thus it remains to consider the case where, without loss of generality, $u \in R_i \dot\cup P_i$ for some $i \in [b]$. By condition (ii) this implies $v \in S \dot\cup R_i \dot\cup P_i$. First assume that $v \in S$. Thus $\mathrm{dist}(u, S) = 1$ and from condition (iii) we infer that $u \in P_i$. Accordingly u is placed into bucket $B_{j(u)} \in \{B_{\lceil b/2 \rceil - 1}, B_{\lceil b/2 \rceil}, B_{\lceil b/2 \rceil + 1}\}$ by (2) and v is placed into bucket $B_{\lceil b/2 \rceil}$ and so we also get (1) in this case. If both $u, v \in R_i \dot\cup P_i$, on the other hand, we are clearly done if $u, v \in R_i$. So assume without loss of generality, that $u \in P_i$. If $v \in P_i$ then we conclude from $|\operatorname{dist}(S,u) - \operatorname{dist}(S,v)| \leq 1$ and (2) that u is placed into bucket $B_{j(u)}$ and v into $B_{j(v)}$ with $|j(u) - j(v)| \leq 1$. If $v \in R_i$, finally, observe that $|\operatorname{dist}(S,u) - \operatorname{dist}(S,v)| \leq 1$ together with condition (iii) implies that $\operatorname{dist}(S,u) \geq \lfloor b/2 \rfloor - 1$ and so u is placed into bucket $B_{j(u)}$ with $|j(u) - i| \leq 1$, where i is the index such that $v \in B_i$, by (2). Thus we also get (1) in this last case.

Now we are ready to construct an ordering of V respecting the desired bandwidth bound. We start with the vertices in bucket B_1, order them arbitrarily, proceed to the vertices in bucket B_2, order them arbitrarily, and so on, up to bucket B_b. By condition (1) this gives an ordering with bandwidth at most twice as large as the largest bucket and thus we conclude from (3) that $\operatorname{bw}(G) < 2(|S| + |P| + r)$. □

A decomposition of the vertices of G into buckets as in the proof of Lemma 9 is also called a path partition of G and appears, e.g., in [11].

Before we get to the proof of Theorem 5, we will establish the following technical observation about labelled trees.

Proposition 10. *Let b be a positive real, $T = (V, E)$ be a tree with $|V| \geq 3$, and $\ell : V \to [0, 1]$ be a real valued labelling of its vertices such that $\sum_{v \in V} \ell(v) \leq 1$. Denote further for all $v \in V$ by $L(v)$ the set of leaves that are adjacent to v and suppose that $\ell(v) + \sum_{u \in L(v)} \ell(u) \geq |L(v)|/b$. Then T has at most b leaves in total.*

Proof. Let $L \subseteq V$ be the set of leaves of T and $I := V \setminus L$ be the set of internal vertices. Clearly

$$1 \geq \sum_{v \in V} \ell(v) = \sum_{v \in I} \left(\ell(v) + \sum_{u \in L(v)} \ell(u) \right) \geq \sum_{v \in I} \frac{|L(v)|}{b} = \frac{|L|}{b}$$

which implies the assertion. □

The idea of the proof of Theorem 5 is to repeatedly extract separators from G and the pieces that result from the removal of such separators. We denote the union of these separators by S, put all remaining vertices with small distance from S into sets P_i, and all other vertices into sets R_i. Then we can apply the decomposition lemma (Lemma 9) to these sets S, P_i, and R_i. This, together with some technical calculations, will give the desired bandwidth bound for G.

of Theorem 5. Let $G = (V, E)$ be a graph on n vertices with maximum degree $\Delta \geq 2$. Observe that the desired bandwidth bound is trivial if $\Delta = 2$ or if $\log_\Delta n - \log_\Delta \operatorname{s}(G) \leq 6$, so assume in the following that $\Delta \geq 3$ and $\log_\Delta n - \log_\Delta \operatorname{s}(G) > 6$. Define

$$\beta := \log_\Delta n - \log_\Delta \operatorname{s}(G) \qquad \text{and} \qquad b := \lfloor \beta \rfloor \geq 6 \tag{4}$$

and observe that with this choice of β our aim is to show that $\operatorname{bw}(G) \leq 6n/\beta$.

The goal is to construct a partition $V = S \dot\cup P \dot\cup R$ with the properties required by Lemma 9. For this purpose we will recursively use the fact that G and its subgraphs have separators of size at most $\operatorname{s}(G)$. In the i-th round we will identify separators $S_{i,k}$ in G whose removal splits G into parts $V_{i,1}, \dots, V_{i,b_i}$. The details are as follows.

In the first round let $S_{1,1}$ be an arbitrary $(\operatorname{s}(G), 2/3)$-separator in G that separates G into $V_{1,1}$ and $V_{1,2}$ and set $b_1 := 2$. In the i-th round, $i > 1$, consider each of the sets $V_{i-1,j}$ with $j \in [b_{i-1}]$. If $|V_{i-1,j}| \le 2n/b$ then let $V_{i,j'} := V_{i-1,j}$, otherwise choose an $(\mathrm{s}(G), 2/3)$-separator $S_{i,k}$ that separates $G[V_{i-1,j}]$ into sets $V_{i,j'}$ and $V_{i,j'+1}$ (where k and j' are appropriate indices, for simplicity we do not specify them further). Let S_i denote the union of all separators constructed in this way (and in this round). This finishes the i-th round. We stop this procedure as soon as all sets $V_{i,j'}$ have size at most $2n/b$ and denote the corresponding i by i^*. Then b_{i^*} is the number of sets $V_{i^*,j'}$ we end up with in the last iteration. Let further x_S be the number of separators $S_{i,k}$ extracted from G during this process in total.

Claim 11. *We have $b_{i^*} \le b$ and $x_S \le b-1$.*

We will postpone the proof of this fact and first show how it implies the theorem. Set $S := \bigcup_{i\in[i^*]} S_i$, for $j \in [b_{i^*}]$ define

$$P_j := \{v \in V_{i^*,j} : \mathrm{dist}(v,S) < \lfloor b/2 \rfloor\} \qquad \text{and} \qquad R_j = V_{i^*,j} \setminus P_j,$$

set $P_j = R_j = \emptyset$ for $b_{i^*} < j \le b$ and finally define $P := \bigcup_{j\in[b]} P_j$ and $R := \bigcup_{j\in[b]} R_j$.

We claim that $V = S \,\dot\cup\, P \,\dot\cup\, R$ is a partition that satisfies the requirements of the decomposition lemma (Lemma 9) with parameter b and $r = 2n/b$. To check this, observe first that for all $i \in [i^*]$ and $j, j' \in [b_i]$ we have $e(V_{i,j}, V_{i,j'}) = 0$ since $V_{i,j}$ and $V_{i,j'}$ were separated by some $S_{i',k}$. It follows that $e(R_j \,\dot\cup\, P_j, R_{j'} \,\dot\cup\, P_{j'}) = e(V_{i^*,j}, V_{i^*,j'}) = 0$ for all $j, j' \in [b_{i^*}]$. Trivially $e(R_j \,\dot\cup\, P_j, R_{j'} \,\dot\cup\, P_{j'}) = 0$ for all $j \in [b]$ and $b_{i^*} < j' \le b$ and therefore we get condition (ii) of Lemma 9. Moreover, condition (iii) is satisfied by the definition of the sets P_j and R_j above. To verify condition (i) note that $|R_j| \le |V_{i^*,j}| \le 2n/b = r$ for all $j \in [b_{i^*}]$ by the choice of i^* and $|R_j| = 0$ for all $b_{i^*} < j \le b$. Accordingly we can apply Lemma 9 and infer that

$$\mathrm{bw}(G) \le 2\left(|S| + |P| + \frac{2n}{b}\right). \tag{5}$$

In order to establish the desired bound on the bandwidth, we thus need to show that $|S|+|P| \le n/\beta$. We first estimate the size of S. By Claim 11 at most $x_S \le b-1$ separators have been extracted in total, which implies

$$|S| \le x_S \cdot \mathrm{s}(G) \le (b-1)\,\mathrm{s}(G). \tag{6}$$

Furthermore all vertices $v \in P$ satisfy $\mathrm{dist}_G(v,S) \le \lfloor b/2 \rfloor - 1$ by definition. As G has maximum degree Δ there are at most $|S|(\Delta^{\lfloor b/2 \rfloor} - 1)/(\Delta - 1)$ vertices $v \in V \setminus S$ with this property and hence

$$\begin{aligned} |S| + |P| &\le |S|\left(1 + \frac{\Delta^{\lfloor b/2 \rfloor} - 1}{\Delta - 1}\right) \le |S| \frac{\Delta^{\beta/2}}{\Delta - 3/2} \\ &\le \frac{(b-1)\,\mathrm{s}(G)}{(\Delta - 3/2)} \sqrt{\frac{n}{\mathrm{s}(G)}} = \frac{(b-1)n}{(\Delta - 3/2)} \sqrt{\frac{\mathrm{s}(G)}{n}} \end{aligned}$$

where the second inequality holds for any $\Delta \ge 3$ and $b \ge 6$ and the third inequality follows from (4) and (6). It is easy to verify that for any $\Delta \ge 3$ and $x \ge \Delta^6$ we have $(\Delta - 3/2)\sqrt{x} \ge \frac{9}{8} \log_\Delta^2 x$. This together with (4) gives $(\Delta - 3/2)\sqrt{n/\mathrm{s}(G)} \ge \frac{9}{8}\beta^2$ and hence we get

$$|S| + |P| \le \frac{8(b-1)n}{9\beta^2}. \tag{7}$$

As $6 \le b = \lfloor \beta \rfloor$ it is not difficult to check that

$$\frac{8(b-1)}{9\beta^2} + \frac{2}{b} \le \frac{3}{\beta}\,.$$

Together with (5) and (7) this gives our bound.

It remains to prove Claim 11. Notice that the process of repeatedly separating G and its subgraphs can be seen as a binary tree T on vertex set W whose internal nodes represent the extraction of a separator $S_{i,k}$ for some i (and thus the separation of a subgraph of G into two sets $V_{i,j}$ and $V_{i,j'}$) and whose leaves represent the sets $V_{i,j}$ that are of size at most $2n/b$. Clearly the number of leaves of T is b_{i^*} and the number of internal nodes x_S. As T is a binary tree we conclude $x_S = b_{i^*} - 1$ and thus it suffices to show that T has at most b leaves in order to establish the claim. To this end we would like to apply Proposition 10. Label an internal node of T that represents a separator $S_{i,k}$ with $|S_{i,k}|/n$, a leaf representing $V_{i,j}$ with $|V_{i,j}|/n$ and denote the resulting labelling by ℓ. Clearly we have $\sum_{w\in W} \ell(w) = 1$. Moreover we claim that

$$\ell(w) + \sum_{u\in L(w)} \ell(w) \ge |L(w)|/b \qquad \text{for all } w \in W \tag{8}$$

where $L(w)$ denotes the set of leaves that are children of w. Indeed, let $w \in W$, notice that $|L(w)| \le 2$ as T is a binary tree, and let u and u' be the two children of w. If $|L(w)| = 0$ we are done. If $|L(w)| > 0$ then w represents a $(2/3, s(G))$-separator $S(w) := S_{i-1,k}$ that separated a graph $G[V(w)]$ with $V(w) := V_{i-1,j} \ge 2n/b$ into two sets $U(w) := V_{i,j'}$ and $U'(w) := V_{i,j'+1}$ such that $|U(w)| + |U'(w)| + |S(w)| = |V(w)|$. In the case that $|L(w)| = 2$ this implies

$$\ell(w) + \ell(u) + \ell(u') = \frac{|S(w)| + |U(w)| + |U'(w)|}{n} = \frac{|V(w)|}{n} \ge 2/b$$

and thus we get (8). If $|L(w)| = 1$ on the other hand then, without loss of generality, u is a leaf of T and $|U'(w)| > 2n/b$. Since $S(w)$ is a $(2/3, s(G))$-separator however we know that $|V(w)| \ge \frac{3}{2}|U'(w)|$ and hence

$$\begin{aligned}\ell(w) + \ell(u) &= \frac{|S(w)| + |U(w)|}{n} = \frac{|S(w)| + |V(w)| - |U'(w)| - |S(w)|}{n} \\ &\ge \frac{\frac{3}{2}|U'(w)| - |U'(w)|}{n} \ge \frac{\frac{1}{2}(2n/b)}{n}\end{aligned}$$

which also gives (8) in this case. Therefore we can apply Proposition 10 and infer that T has at most b leaves as claimed. □

3.3. **Boundedness.** In this section we study the relation between boundedness, bandwidth and treewidth. We first give a proof of Proposition 7.

of Proposition 7. We have to show that for every graph G and every $\varepsilon > 0$ the inequality $b_\varepsilon(G) \le 2\,\mathrm{bw}(G)/\varepsilon$ holds. Suppose that G has n vertices and let $\sigma : V \to [n]$ be an arbitrary labelling of G. Furthermore assume that $V' \subseteq V$ with $|V'| = b_\varepsilon(G)$ induces an ε-expander in G. Define $V^* \subseteq V'$ to be the first $b_\varepsilon(G)/2 = |V'|/2$ vertices of V' with respect to the ordering σ. Since V' induces an ε-expander in G there must be at least $\varepsilon b_\varepsilon(G)/2$ vertices in $N^* := N(V^*) \cap V'$. Let u be the vertex in N^* with maximal $\sigma(u)$ and $v \in V^* \cap N(u)$. As $u \notin V^*$ and $\sigma(u') > \sigma(v')$ for all $u' \in N^*$ and $v' \in V^*$ by the choice of V^* we have $|\sigma(u) - \sigma(v)| \ge |N^*| \ge \varepsilon b_\varepsilon(G)/2$. Since this is true for every labelling σ we can deduce that $b_\varepsilon(G) \le 2\,\mathrm{bw}(G)/\varepsilon$. □

The remainder of this section is devoted to the proof of Theorem 6. We will use the following lemma which establishes a relation between boundedness and certain separators.

Lemma 12 (bounded→separator)**.** *Let G be a graph on n vertices and let $\varepsilon > 0$. If G is $(n/2, \varepsilon)$-bounded then G has a $(2\varepsilon n/3, 2/3)$-separator.*

Proof. Let $G = (V, E)$ with $|V| = n$ be $(n/2, \varepsilon)$-bounded for $\varepsilon > 0$. It follows that every subset $V' \subseteq V$ with $|V'| \geq n/2$ induces a subgraph $G' \subseteq G$ with the following property: there is $W \subseteq V'$ such that $|W| \leq |V'|/2$ and $|N_{G'}(W)| \leq \varepsilon|W|$. We use this fact to construct a $(2\varepsilon n/3, 2/3)$-separator in the following way:

(1) Define $V_1 := V$ and $i := 1$.
(2) Let $G_i := G[V_i]$.
(3) Find a subset $W_i \subseteq V_i$ with $|W_i| \leq |V_i|/2$ and $|N_{G_i}(W_i)| \leq \varepsilon|W_i|$.
(4) Set $S_i := N_{G_i}(W_i)$, $V_{i+1} := V_i \setminus (W_i \cup S_i)$.
(5) If $|V_{i+1}| \geq \frac{2}{3}n$ then set $i := i + 1$ and go to step (2).
(6) Set $i^* := i$ and return

$$A := \bigcup_{i=1}^{i^*} W_i, \quad B := V_{i^*+1}, \quad S := \bigcup_{i=1}^{i^*} S_i.$$

This construction obviously returns a partition $V = A \dot\cup B \dot\cup S$ with $|B| < \frac{2}{3}n$. Moreover, $|V_{i^*}| \geq \frac{2}{3}n$ and $|W_{i^*}| \leq |V_{i^*}|/2$ and hence

$$|A| = n - |B| - |S| = n - |V_{i^*+1}| - |S| =$$
$$n - (|V_{i^*}| - |W_{i^*}| - |S_{i^*}|) - |S| \leq n - \frac{|V_{i^*}|}{2} \leq \frac{2}{3}n.$$

The upper bound on $|S|$ follows easily since

$$|S| = \sum_{i=1}^{i^*} |N_{G_i}(W_i)| \leq \sum_{i=1}^{i^*} \varepsilon|W_i| = \varepsilon|A| \leq \frac{2}{3}\varepsilon n.$$

It remains to show that S separates G. This is indeed the case as $N_G(A) \subseteq S$ by construction and thus $E(A, B) = \emptyset$. □

Now we can prove Theorem 6. As remarked earlier, Grohe and Marx [13] independently gave a proof of an equivalent result which employs similar ideas but does not use separators explicitly.

of Theorem 6. Let $G = (V, E)$ be a graph on n vertices, $\varepsilon > 0$, and let $b \geq b_\varepsilon(G)$. It follows immediately from the definition of boundedness that every subgraph $G' \subseteq G$ with $G' = (V', E')$ and $|V'| \geq 2b$ also has $b_\varepsilon(G') \leq b$.

We now prove Theorem 6 by induction on the size of G. The relation $\mathrm{tw}(G) \leq 2\varepsilon n + 2b$ trivially holds if $n \leq 2b$. So let G have $n > 2b$ vertices and assume that the theorem holds for all graphs with less than n vertices. Then G is (b, ε)-bounded and thus has a $(2\varepsilon n/3, 2/3)$-separator S by Lemma 12. Assume that S separates G into the two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $(\mathcal{X}_1, T_1)$ and $(\mathcal{X}_2, T_2)$ be tree decompositions of G_1 and G_2, respectively, such that $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$. We use them to construct a tree decomposition $(\mathcal{X}, T)$ of G as follows. Let $\mathcal{X} = \{X_i \cup S : X_i \in \mathcal{X}_1\} \cup \{X_i \cup S : X_i \in \mathcal{X}_2\}$ and $T = (I_1 \cup I_2, F = F_1 \cup F_2 \cup \{e\})$ where e is an arbitrary edge between the two trees. This is indeed a tree decomposition of G:

Every vertex $v \in V$ belongs to at least one $X_i \in \mathcal{X}$ and for every edge $\{v, w\} \in E$ there exists $i \in I$ (where I is the index set of $\mathcal{X}$) with $\{v, w\} \subseteq X_i$. This is trivial for $\{v, w\} \subseteq V_i$ and follows from the definition of $\mathcal{X}$ for $v \in S$ and $w \in V_i$. Since S separates G there are no edges $\{v, w\}$ with $v \in V_1$ and $w \in V_2$. For the same reason the third property of a tree decomposition holds: if j lies on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$ as the intersection is S if X_i, X_k are subsets of V_1 and V_2 respectively.

We have seen that $(\mathcal{X}, T)$ is a tree decomposition of G and can estimate its width as follows: $\mathrm{tw}(G) \leq \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\} + |S|$. With the induction hypothesis we get

$$\begin{aligned}\mathrm{tw}(G) &\leq \max\{2\varepsilon \cdot |V_1| + 2b,\ 2\varepsilon \cdot |V_2| + 2b\} + |S| \\ &\leq 2\varepsilon n + 2b.\end{aligned}$$

where the second inequality follows from $|V_i| \leq (2/3)n$ and $|S| \leq (2\varepsilon n)/3$. □

4. Applications

For many interesting bounded degree graph classes (non-trivial) upper bounds on the bandwidth are not at hand. A wealth of results however has been obtained about the existence of sublinear separators. This illustrates the importance of Theorem 8. In this section we will give examples of such separator theorems and provide applications of them in conjunction with Theorem 8.

4.1. Separator theorems.

A classical result in the theory of planar graphs concerns the existence of separators of size $2\sqrt{2n}$ in any planar graph on n vertices proved by Lipton and Tarjan [19] in 1977. Clearly, together with Theorem 5 this result implies the following theorem.

Corollary 13. *Let G be a planar graph on n vertices with maximum degree at most $\Delta \geq 2$. Then the bandwidth of G is bounded from above by*

$$\mathrm{bw}(G) \leq \frac{15n}{\log_\Delta(n)}.$$

It is easy to see that the bound in Corollary 13 is sharp up to the multiplicative constant – since the bandwidth of any graph G is bounded from below by $(n-1)/\mathrm{diam}(G)$, it suffices to consider for example the complete binary tree on n vertices. Corollary 13 is used in [7] to infer a result about the geometric realisability of planar graphs $G = (V, E)$ with $|V| = n$ and $\Delta(G) \leq \Delta$.

This motivates why we want to consider some generalisations of the planar separator theorem in the following. The first such result is due to Gilbert, Hutchinson, and Tarjan [12] and deals with graphs of arbitrary genus. [2]

Theorem 14 ([12])**.** *An n-vertex graph G with genus $g \geq 0$ has separation number $\mathrm{s}(G) \leq 6\sqrt{gn} + 2\sqrt{2n}$.*

For fixed g the class of all graphs with genus at most g is closed under taking minors. Here H is a minor of G if it can be obtained from a subgraph of G by a sequence of edge deletions and contractions. A graph G is called H-minor free if H

[2] Again, the separator theorems we refer to bound the size of a separator in G. Since the class of graphs with genus less than g (or, respectively, of H-minor free graphs) is closed under taking subgraphs however, this theorem can also be applied to such subgraphs and thus the bound on $\mathrm{s}(G)$ follows.

is no minor of G. The famous graph minor theorem by Robertson and Seymour [21] states that any minor closed class of graphs can be characterised by a finite set of forbidden minors (such as $K_{3,3}$ and K_5 in the case of planar graphs). The next separator theorem by Alon, Seymour, and Thomas [2] shows that already forbidding one minor enforces a small separator.

Theorem 15 ([2]). *Let H be an arbitrary graph. Then any n-vertex graph G that is H-minor free has separation number $\mathrm{s}(G) \leq |H|^{3/2}\sqrt{n}$.*

We can apply these theorems to draw the following conclusion concerning the bandwidth of bounded-degree graphs with fixed genus or some fixed forbidden minor from Theorem 5.

Corollary 16. *Let g be a positive integer, $\Delta \geq 2$ and H be an h-vertex graph and G an n-vertex graph with maximum degree $\Delta(G) \leq \Delta$.*

(a) *If G has genus g then $\mathrm{bw}(G) \leq 15n/\log_{\Delta}(n/g)$.*
(b) *If G is H-minor free then $\mathrm{bw}(G) \leq 12n/\log_{\Delta}(n/h^3)$.*

4.2. **Embedding problems and universality.** A graph H that contains copies of all graphs $G \in \mathcal{G}$ for some class of graphs $\mathcal{G}$ is called *universal for $\mathcal{G}$*. The construction of sparse universal graphs for certain families $\mathcal{G}$ has applications in VLSI circuit design and was extensively studied (see, e.g., [1] and the references therein). In contrast to these results our focus is not on minimising the number of edges of H, but instead we are interested in giving a relatively *simple* criterion for universality for $\mathcal{G}$ that is satisfied by *many* graphs H of the same order as the largest graph in $\mathcal{G}$.

The setting with which we are concerned here are embedding results that guarantee that a bounded-degree graph G can be embedded into a graph H with sufficiently high minimum degree, even when G and H have the same number of vertices. Dirac's theorem [10] concerning the existence of Hamiltonian cycles in graphs of minimum degree $n/2$ is a classical example for theorems of this type. It was followed by results of Corrádi and Hajnal [9], Hajnal and Szemerédi [14] about embedding K_r-factors, and more recently by a series of theorems due to Komlós, Sarközy, and Szemerédi and others which deal with powers of Hamiltonian cycles, trees, and H-factors (see, e.g., the survey [17]). Along the lines of these results the following unifying conjecture was made by Bollobás and Komlós [16] and recently proved by Böttcher, Schacht, and Taraz [5].

Theorem 17 ([5]). *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If G is an r-chromatic graph on n vertices with $\Delta(G) \leq \Delta$ and bandwidth at most βn and if H is a graph on n vertices with minimum degree $\delta(H) \geq (\frac{r-1}{r} + \gamma)n$, then G can be embedded into H.*

The proof of Theorem 17 heavily uses the bandwidth constraint insofar as it constructs the required embedding sequentially, following the ordering given by the vertex labels of G. Here it is of course beneficial that the neighbourhood of every vertex v in G is confined to the βn vertices which immediately precede or follow v.

Also, it is not difficult to see that the statement in Theorem 17 becomes false without the constraint on the bandwidth: Consider $r = 2$, let G be a random bipartite graph with bounded maximum degree and let H be the graph formed by two cliques of size $(1/2 + \gamma)n$ each, which share exactly $2\gamma n$ vertices. Then H

cannot contain a copy of G, since in G every vertex set of size $(1/2-\gamma)n$ has more than $2\gamma n$ neighbours. The reason for this obstruction is again that G has good expansion properties.

On the other hand, Theorem 8 states that in bounded degree graphs, the existence of a big expanding subgraph is in fact the only obstacle which can prevent sublinear bandwidth and thus the only possible obstruction for a universality result as in Theorem 17. More precisely we immediately get the following corollary from Theorem 8.

Corollary 18. *If the class $\mathcal{C}$ meets one (and thus all) of the conditions in Theorem 8, then the following is also true. For every $\gamma > 0$ and $r \in \mathbb{N}$ there exists n_0 such that for all $n \geq n_0$ and for every graph $G \in \mathcal{C}_n$ with chromatic number r and for every graph H on n vertices with minimum degree at least $(\frac{r-1}{r}+\gamma)n$, the graph H contains a copy of G.*

By Corollary 13 we infer as a special case that all sufficiently large graphs with minimum degree $(\frac{3}{4}+\gamma)n$ are universal for the class of bounded-degree planar graphs. Universal graphs for bounded degree planar graphs have also been studied in [3, 6].

Corollary 19. *For all $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds:*

(a) *Every 3-chromatic planar graph on n vertices with maximum degree at most Δ can be embedded into every graph on n vertices with minimum degree at least $(\frac{2}{3}+\gamma)n$.*
(b) *Every planar graph on n vertices with maximum degree at most Δ can be embedded into every graph on n vertices with minimum degree at least $(\frac{3}{4}+\gamma)n$.*

This extends a result by Kühn, Osthus, and Taraz [18], who proved that for every graph H with minimum degree at least $(\frac{2}{3}+\gamma)n$ *there exists a particular* spanning triangulation G that can be embedded into H. Using Corollary 16 it is moreover possible to formulate corresponding generalisations for graphs of fixed genus and for H-minor free graphs for any fixed H.

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