Bandwidth, treewidth, separators, expansion, and universality

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Abstract

We prove that planar graphs with bounded maximum degree have sublinear bandwidth. As a consequence for each $\gamma > 0$ every n-vertex graph with minimum degree $(\frac{3}{4} + \gamma)n$ contains a copy of every bounded-degree planar graph on n vertices. The proof relies on the fact that planar graphs have small separators. Indeed, we show more generally that for any class of bounded-degree graphs the concepts of sublinear bandwidth, sublinear treewidth, the absence of big expanders as subgraphs, and the existence of small separators are equivalent.

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1 Bandwidth and universality for planar graphs

Let $G = (V, E)$ be a graph on $n = |V|$ vertices. The *bandwidth* of G is denoted by $\mathrm{bw}(G)$ and defined to be the minimum $b \in \mathbb{N}$, such that there exists a labelling of the vertices in V by numbers $1, \ldots, n$ so that the labels of adjacent vertices differ by at most b. In [2] Chung proved that any tree T with n vertices and maximum degree Δ has bandwidth at most $5n/\log_\Delta(n)$, and it is easy to see that this bound is sharp up to the multiplicative constant. Our first theorem extends Chung's result to planar graphs.

Theorem 1.1 *Let G be a planar graph on n vertices with maximum degree Δ. Then the bandwidth of G is bounded from above by*

$$\mathrm{bw}(G) \leq \frac{15n}{\log_\Delta(n)}.$$

Our main motivation for studying the bandwidth is that it turns out to be helpful when embedding a bounded-degree graphs G into a graph H with sufficiently high minimum degree, even when G and H have the same number of vertices. Dirac's theorem [4] concerning the existence of Hamiltonian cycles in graphs of minimum degree $n/2$ is a classical example for theorems of this type. It was followed by results of Corrádi and Hajnal [3], Hajnal and Szemerédi [5] about embedding K_r-factors, and more recently by a series of theorems due to Komlós, Sarközy, and Szemerédi and others (see e.g. [7,8,9]) which deal with powers of Hamiltonian cycles, trees, and H-factors. Along the lines of these results the following unifying theorem was conjectured by Bollobás and Komlós [6] and recently proven by Böttcher, Schacht, and Taraz [1].

Theorem 1.2 *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If G is an r-chromatic graph on n vertices with $\Delta(G) \leq \Delta$ and bandwidth at most βn and if H is a graph on n vertices with minimum degree $\delta(H) \geq (\frac{r-1}{r} + \gamma)n$, then G can be embedded into H.*

Combining Theorems 1.1 and 1.2 immediately yields the following result which states that all sufficiently large graphs with minimum degree $(\frac{3}{4} + \gamma)n$ are universal for the class of bounded-degree planar graphs.

Corollary 1.3 *For all $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds:*

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(*a*) *Every* 3*-chromatic planar graph on* n *vertices with maximum degree at most* Δ *can be embedded into every graph on* n *vertices with minimum degree at least* $(\frac{2}{3}+\gamma)n$*.*

(*b*) *Every planar graph on* n *vertices with maximum degree at most* Δ *can be embedded into every graph on* n *vertices with minimum degree at least* $(\frac{3}{4}+\gamma)n$*.*

This extends a result by Kühn, Osthus, and Taraz [10], who proved that for every graph H with minimum degree at least $(\frac{2}{3}+\gamma)n$ *there exists* a particular spanning triangulation G that can be embedded into H.

2 Expansion, separators, and treewidth

In the first section we observed the connection between results about the bandwidth of a class of graphs and embedding problems due to Theorem 1.2. This raises the question which rôle the bandwidth plays in this theorem. It is not difficult to see that the condition on the bandwidth of G in Theorem 1.2 is necessary already for the case $r = 2$: Let G be a random bipartite graph with bounded maximum degree and let H be the graph formed by two cliques of size $(1/2+\gamma)n$ each, which share exactly $2\gamma n$ vertices. Then H cannot contain a copy of G, since in G every vertex set of size $(1/2-\gamma)n$ has more than $2\gamma n$ neighbours. The reason for this obstacle is that H has good expansion properties. In light of Theorem 1.2 the same example shows that graphs with sublinear bandwidth (as in Theorem 1.1) cannot exhibit good expansion properties (definitions and exact statements are provided below). One may ask whether the converse is also true, i.e. whether bad expansion properties in bounded-degree graphs lead to small bandwidth. We will show that this is indeed the case via the existence of certain separators. A similar approach can be used to prove Theorem 1.1. In fact, we will show a more general theorem (Theorem 2.4) which proves that the concepts of sublinear bandwidth, sublinear treewidth, bad expansion properties, and sublinear separators are equivalent for graphs of bounded maximum degree. Since planar graphs have sublinear separators [11] Theorem 1.1 is a direct consequence of this theorem. For the precise statement of Theorem 2.4, we need the following definitions.

We start with the notions of *tree decompositions* and *treewidth*. Roughly speaking, a tree decomposition tries to arrange the vertices of a graph in a tree-like manner and the treewidth measures how well this can be done.

Definition 2.1 (treewidth) A tree decomposition of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\}, T = (I, F))$ where $\{X_i : i \in I\}$ is a family of subsets $X_i \subseteq V$

with $\bigcup_{i \in I} X_i = V$, and where $T = (I, F)$ is a tree such that for every edge $\{v, w\} \in E$ there exists $i \in I$ with $\{v, w\} \subseteq X_i$ and for every $i, j, k \in I$ such that j lies on the path from i to k in T we have $X_i \cap X_k \subseteq X_j$. The width of $(\{X_i : i \in I\}, T = (I, F))$ is defined as $\max_{i \in I} |X_i| - 1$. The treewidth $\mathrm{tw}(G)$ of G is the minimum width of a tree decomposition of G.

A vertex set is said to be expanding, if it has many external neighbours. We call a graph *bounded*, if every sufficiently large subgraph contains a subset which is not expanding.

Definition 2.2 (bounded) Let $0 < \varepsilon < 1$ be a real number, $b \in \mathbb{N}$ and $G = (V, E)$ be a graph. G is called (b, ε)-bounded, if for every subgraph $G' \subseteq G$ with $|V(G')| \geq b$ vertices there exists a subset $U \subseteq V(G')$ such that $|U| \leq |V(G')|/2$ and $|N(U)| \leq \varepsilon|U|$. (Here $N(U)$ is the set of neighbours of vertices in U that lie outside of U.)

Finally, a *separator* in a graph is a small cut-set that splits the graph into components of limited size.

Definition 2.3 (separator) Let $0 < \alpha < 1$ be a real number, $b \in \mathbb{N}$ and $G = (V, E)$ a graph. A subset $S \subseteq V$ is said to be a (b, α)-separator of G, if there exist subsets $A, B \subseteq V$ such that $V = A \dot{\cup} B \dot{\cup} S$, $|S| \leq b$, $|A|, |B| \leq \alpha|V(G)|$, and $E(A, B) = \emptyset$, where $E(A, B)$ denotes the set of edges with one end in A and one in B.

With these definitions at hand, we are ready to state our main theorem.

Theorem 2.4 *Let Δ be an arbitrary but fixed positive integer and consider a class of graphs $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have maximum degree at most Δ. Denote by $\mathcal{C}_n$ the set of those graphs in $\mathcal{C}$ with n vertices. Then the following four properties are equivalent:*

(1) *For every $\beta_1 > 0$ there exists n_0 such that for all $n \geq n_0$ every graph in $\mathcal{C}_n$ has treewidth at most $\beta_1 n$.*

(2) *For every $\beta_2 > 0$ there exists n_0 such that for all $n \geq n_0$ every graph in $\mathcal{C}_n$ has bandwidth at most $\beta_2 n$.*

(3) *For every $\beta_3 > 0$ and every $\varepsilon > 0$ there exists n_0 such that for all $n \geq n_0$ every graph in $\mathcal{C}_n$ is $(\beta_3 n, \varepsilon)$-bounded.*

(4) *For every $\beta_4 > 0$ there exists n_0 such that for all $n \geq n_0$ every graph in $\mathcal{C}_n$ has a $(\beta_4 n, 2/3)$-separator.*

If the class $\mathcal{C}$ meets one (and thus all) of the above conditions, then the following is also true.

(*5*) *For every $\gamma > 0$ and $r \in \mathbb{N}$ there exists n_0 such that for all $n \geq n_0$ and for every graph $G \in \mathcal{C}_n$ with chromatic number r and for every graph H on n vertices with minimum degree at least $(\frac{r-1}{r} + \gamma)n$, the graph H contains a copy of G.*

3 Proofs

We conclude by briefly sketching the main ideas for the proof of Theorem 2.4.

(*1*)⇒(*4*) This follows immediately from a result by Robertson and Seymour [12], which says that a graph with treewidth less than k has a $(k+1, 1/2)$-separator.

(*4*)⇒(*2*) The rough idea is to repeatedly extract separators from G until the remaining components $R_1, \ldots, R_t$ are of size ρn at most where ρ is such that $\rho + c \cdot \log(2/\rho)\beta_4 = \beta_2/2$ and the constant c will be determined later. Denote the union of the vertices in these separators by S and observe that vertices in R_i are not adjacent to vertices in R_j for $j \neq i$. It is not difficult to see that $|S| \leq \log(2/\rho)\beta_4 n$. We now partition the graph into classes $V_1, \ldots, V_{t+1}$ as follows. First add all vertices from S to V_1. For $v \in R_i$ let $j = \min\{i, \mathrm{dist}(v, S)\}$ and add v to V_{j+1}. Observe that there are at most $c|S|$ such vertices with $j \neq i$ where c is a constant depending on Δ only. It follows that we obtain a partition $V_1, \ldots, V_t$ with classes of size $|V_i| \leq \rho n + c \cdot \log(2/\rho)\beta_4 n = \beta_2 n/2$ and edges that only run within the V_i and between V_i and V_{i+1}. Thus the labelling constructed by first labelling the vertices in V_1, then in V_2, and so on shows that G has bandwidth at most $\beta_2 n$.

(*2*)⇒(*1*) This is obvious, because the treewidth of a graph is bounded from above by its bandwidth.

(*2*)⇒(*3*) This follows rather trivially directly from the definitions: for β_3 sufficiently small a labelling of G that respects the bandwidth bound requires that the neighbours of any set $U \subseteq V(G)$ of size at most $\beta_3 n/2$ are not too far away from U, and hence there cannot be too many of them.

(*3*)⇒(*4*) For G with property (3) we can explicitly construct the required separator as follows. We apply (3) in order to determine a set $S \subseteq V(G)$ with $N(S) \leq \varepsilon|V(G)|$. Then we consider $V(G) \setminus S$, apply (3) to find $S' \subseteq V(G) \setminus S$ with $N(S') \leq \varepsilon|V(G) \setminus S|$, and add the vertices in S' to S. We repeat this process until the remaining graph has less than $2n/3$ vertices. It is not difficult to see that $N(S)$ is a $(\beta_4 n, 2/3)$-separator for appropriate β_4 depending on β_3 only.

(2)⇒*(5)* This is exactly the assertion of Theorem 1.2.

A closer look at the proof of the implication (4) ⇒ (2) together with the fact that planar graphs have $(O(\sqrt{n}), 2/3)$-separators [11], yields Theorem 1.1.

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