# The tripartite Ramsey number for trees

Julia Böttcher [1,4], Jan Hladký [2,4], Diana Piguet [3,4],

*Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D-85747 Garching bei München, Germany*

**Abstract**

We prove that for every $\varepsilon > 0$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. For any two-colouring of the edges of $K_{n,n,n}$ one colour contains copies of all trees $T$ of order $k \leq (3-\varepsilon)n/2$ and with maximum degree $\Delta(T) \leq n^{\alpha}$. This answers a conjecture of Schelp.

*Keywords:* Ramsey theory, trees, regularity lemma, connected matchings

## 1 Introduction and results

The famous theorem of Ramsey [8] states that for any finite family of graphs $\mathcal{F}$ there is an integer $m$ such that in any edge-colouring of $K_m$ with red and blue there are copies of all members of $\mathcal{F}$ in red or copies of all of them in blue. In this case we also write $K_m \to \mathcal{F}$ and say that $K_m$ is *Ramsey* for $\mathcal{F}$. We write $R(\mathcal{F})$ for the smallest integer $m$ such that $K_m \to \mathcal{F}$. Let $\mathcal{T}_k$ denote the class of trees of order $k$, $\mathcal{T}_k^{\Delta}$ is its restriction to trees of maximum degree at

[1] Email:boettche@ma.tum.de
[2] Email:hladk@seznam.cz
[3] Email:diana@kam.mff.cuni.cz
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most $\Delta$. Ajtai, Komlós, Simonovits, and Szemerédi [1] announced a proof of a long-standing conjecture by Erdős and Sós, which implies that $K_{2k-2} \to \mathcal{T}_k$ for large even $k$ and $K_{2k-3} \to \mathcal{T}_k$ for large odd $k$. This bound is best possible.

The graph $K_{R(\mathcal{F})}$ is obviously a Ramsey graph for $\mathcal{F}$ with as few vertices as possible. However, one may still ask whether there exist graphs with fewer edges that are Ramsey for $\mathcal{F}$. The minimal number of edges of such a graph is also called *size Ramsey number* for $\mathcal{F}$ and denoted by $R_s(\mathcal{F})$. Trivially $R_s(\mathcal{F}) \leq \binom{R(\mathcal{F})}{2}$, but it turns out that this inequality is often far from tight. The investigation of size Ramsey numbers recently experienced much attention. Trees are considered in [5,2]. Recently progress on determining the size Ramsey number for classes of bounded degree graphs was made in [6].

A question of similar flavour is what happens when we do not confine ourselves to finding Ramsey graphs for $\mathcal{F}$ with few edges, but require in addition that they are proper subgraphs of $K_m$ with $m$ very close to $R(\mathcal{F})$. This question has two aspects: a quantitative one (i.e., how many edges can be deleted from $K_m$ so that the remaining graph is still Ramsey) and a structural one (i.e., what is the structure of edges which may be deleted). Questions of similar nature were explored in [4] for odd cycles and in [3] for paths. Our focus in this paper is on the case when $\mathcal{F}$ is a class of trees.

Schelp [9] posed the following Ramsey-type conjecture about trees in tripartite graphs: For $n$ sufficiently large the tripartite graph $K_{n,n,n}$ is Ramsey for the class $\mathcal{T}_k^{\Delta}$ of trees on $k \leq (3-\varepsilon)n/2$ vertices with maximum degree at most $\Delta$ for constant $\Delta$. This conjecture asserts that we can delete three cliques of size $m/3$ from a graph $K_m$ with $m$ being roughly $R(\mathcal{T}_k^{\Delta})$, while maintaining the Ramsey property. In addition Schelp asked whether the same remains true when the constant maximum degree bound in the conjecture above is replaced by $\Delta \leq \frac{2}{3}k$ (which is easily seen to be best possible). Our main result is situated in-between these two cases, solving the problem for trees of maximum degree $n^{\alpha}$ for some small $\alpha$ and hence, in particular, answering the conjecture above.

**Theorem 1.1** *For every $\varepsilon > 0$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that*

$$K_{n,n,n} \to \mathcal{T}_k^{\Delta},$$

*for all $n \geq n_0$, $\Delta \leq n^{\alpha}$, and $k \leq (3-\varepsilon)n/2$.*

As we shall explain more in detail in the next section, our proof of Theorem 1.1 uses Szemerédi's regularity lemma [10]. Due to the nature of the methods related to this lemma, it follows that Theorem 1.1 remains true when $K_{n,n,n}$ is replaced by a much sparser graph: For any fixed $p \in (0,1]$ a random subgraph of $K_{n,n,n}$ with edge probability $p$ allows for the same conclusion (as long as $n$ is sufficiently large).

On the way to our proof of Theorem 1.1 we obtain an embedding lemma (Lemma 2.1) that provides rather general conditions for the embedding of trees with growing maximum degree.

## 2 The main lemmas

As indicated earlier we obtain Theorem 1.1 via Szemerédi's regularity lemma which provides us with a regular partition of the edge coloured $K_{n,n,n}$. In this section we shall introduce the two other main ingredients for the proof: An embedding result for trees in regular partitions with certain properties (Lemma 2.1) and a purely structural result for coloured tripartite graphs (Lemma 2.2).

One of the main strengths of the regularity method for embedding problems is that we can often reduce the original problem to a simpler one: Instead of looking for a particular substructure in a host graph $G$ we search for a member of a much more general class of substructures in the cluster graph corresponding to $G$. One prominent example is the concept of a *connected matching* introduced in [7], i.e., a matching contained in one component. The existence of a connected matching in the cluster graph $\mathbb{G}$ of $G$ allows for the embedding of certain graphs in $G$. In particular, a connected matching is a suitable structure for embedding balanced trees. Here, we generalise this concept in the following way. An odd connected matching $M$ in a graph $G = (V, E)$ is a collection of vertex disjoint edges such that between each two vertices in $V(M)$ there is an odd as well as an even length walk in $G$. A *connected fork system* $F$ in $G$ is a set of pairwise vertex disjoint graphs $K_{1,r(i)}$, $i \in [k]$ such that for each pair $i, i' \in [k]$ there is a walk of even length between a leaf of $K_{1,r(i)}$ and a leaf of $K_{1,r(i')}$. We call the sum of all the $r(i)$'s the *size* of $F$ and say that $F$ has *ratio* at most $r$ if $r(i) \leq r$ for all $i \in [k]$.

The following lemma states that the appearance of either an odd connected matching or a connected fork system in a cluster graph allows us to embed a variety of large trees. In contrast to connected matchings we can deal with unbalanced trees now.

**Lemma 2.1 (embedding lemma)** *For all $d, \mu > 0$ there is $\varepsilon = \varepsilon(d, \mu) > 0$ such that for every $N \in \mathbb{N}$ there are $\alpha = \alpha(N) > 0$ and $n_0 = n_0(\mu, \varepsilon, N) \in \mathbb{N}$ such that for all $n \geq n_0$ and $r \in \mathbb{N}$ the following holds. Let $G$ be a graph of order $n$ with an $(\varepsilon, d)$-regular partition with cluster graph $\mathbb{G} = ([N], E(\mathbb{G}))$.*

(a) *If $\mathbb{G}$ contains an odd connected matching $M$ of size at least $m$ then $G$ contains all trees $T$ with $\Delta(T) \leq n^\alpha$ and $|V(T)| \leq (1-\mu)2m\frac{n}{N}$.*

(b) *If $\mathbb{G}$ contains a connected fork system $F$ with ratio $r$ and size at least $f$, then $G$ contains all trees $T$ with $\Delta(T) \leq n^\alpha$ and colour class sizes $t_1 \leq t_2$ with $t_1 \leq t$ and $t_2 \leq t/r$, where $t = (1-\mu)f\frac{n}{N}$.*

When applying the regularity lemma to an edge 2-coloured complete tripartite graph we obtain an edge 2-coloured almost complete tripartite cluster graph $\mathbb{K}$. An edge $e$ in $\mathbb{K}$ is coloured by the majority colour in the regular pair corresponding to $e$. In order to apply Lemma 2.1 to $\mathbb{K}$ we need a result that provides us with an odd connected matching or a connected fork system in one of the colours of $\mathbb{K}$. This is taken care of by the following lemma which states that such structures exist in almost complete tripartite coloured graphs.

**Lemma 2.2** *For every $\eta' > 0$ there are $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Let $K$ be a spanning subgraph of $K_{n,n,n}$ that is given with a (red,blue)-colouring of its edges and such that each vertex has at least $2n - \eta n$ neighbours. Then one of the graphs $G$ induced by the red or by the blue edges in $K$ satisfies at least one of the following.*

(a) *$G$ contains an odd connected matching with at least $(1-\eta')\frac{3}{4}n$ edges.*

(b) *$G$ contains a connected fork system of ratio 1 and size at least $(1-\eta')n$ and a connected fork system of ratio 3 and size at least $(1-\eta')\frac{3}{2}n$.*

It turns out that odd matchings or fork systems as provided by Lemma 2.2 are exactly the right structures we need to plug into Lemma 2.1 in order to embed the trees we are considering in Theorem 1.1. In the last section we sketch the proofs of these two main lemmas.

## 3 Outline of the proof

**Proof of Lemma 2.1.** To prove this lemma we remove a few vertices from the tree $T$ such that the resulting forest $\mathcal{T}$ contains only small components. Then we assign each tree in $\mathcal{T}$ to an edge of $M$ or $F$, respectively, in such a way that no cluster gets overfilled with vertices assigned to it. Since the edges of $M$ or $F$ correspond to dense regular pairs, we could now easily embed all these small trees with the help of a simple greedy algorithm. However, we need to "connect" them. For this we use the walks connecting $M$ or $F$ and reassign some vertices to these walks. The fact that these walks may be long implies our requirement on the maximum degree of $T$.

**Proof of Lemma 2.2.** With the help of a case analysis we show that the graph $G$, unless it is in some very special configuration, contains an odd connected matching in one of the colours. But this matching may not have the required size yet. In this case we show that there is a larger odd connected matching in the other colour. Iterating this argument will give the desired big odd connected matching. It remains to analyse the special configuration which turns out to contain suitable connected fork systems.

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