Forcing spanning subgraphs via Ore type conditions

Julia Böttcher [1,3], Sybille Müller [2]

Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D-85747 Garching bei München, Germany

Abstract

We determine an Ore type condition that allows the embedding of 3-colourable bounded degree graphs of sublinear bandwidth: For all $\Delta, \gamma > 0$ there are $\beta, n_0 > 0$ such that for all $n \geq n_0$ the following holds. Let $G = (V, E)$ and H be n-vertex graphs such that H is 3-colourable, has maximum degree $\Delta(H) \leq \Delta$ and bandwidth $\mathrm{bw}(H) \leq \beta n$, and G satisfies $\deg(u) + \deg(v) \geq (\frac{4}{3} + \gamma)n$ for all $uv \notin E$. Then G contains a copy of H.

This improves on the Bollobás-Komlós conjecture for 3-chromatic graphs proven by Böttcher, Schacht, and Taraz [J. Combin. Theory, Ser. B, 98(4), 752–777, 2008] and applies a result of Kierstaed and Kostochka [J. Comb. Theory, Ser. B, 98(1), 226–234, 2008] about the existence of spanning triangle factors under Ore type conditions.

Keywords: extremal graph theory, regularity lemma, spanning subgraphs, Ore condition

[1] Email:boettche@ma.tum.de

[2] Email:sybille.mueller@mytum.de

[3] The first author was partially supported by DFG grant TA 309/2-1.

1 Introduction and Results

One central topic in extremal combinatorics studies how global structural properties are enforced by certain local conditions. In graph theory for example a number of results concern the existence of particular spanning subgraphs (global structures), under various vertex degree conditions (local conditions). One of the classical theorems in this area is due to Dirac and states that any n-vertex graph with minimum degree at least $n/2$ contains a Hamilton cycle, i.e., a cycle on n vertices. Ore [7] realised that this conclusion remains true even under a weaker condition. He showed that it is not necessary to control the degree of every vertex independently but that it suffices to guarantee a high degree sum for pairs of non-adjacent vertices.

Theorem 1.1 (Ore [7]) *Every n-vertex graph $G = (V, E)$ with $\deg(u) + \deg(v) \geq n$ for all $xy \notin E$ contains a Hamilton cycle.*

In the following we will call degree conditions of this type *Ore conditions* and define the *Ore degree* $\delta_{\varnothing}(G)$ to denote the maximum number q such that all pairs of non-adjacent vertices uv of G satisfy $\deg(u)+\deg(v) \geq q$. Recently, Kierstead and Kostochka [4] considered the replacement of the minimum degree condition by an Ore condition in yet another famous theorem that can be stated as a result concerning spanning subgraphs: The theorem of Hajnal and Szemerédi [3] states that any n-vertex graph G with minimum degree $\delta(G) \geq \frac{r-1}{r}n$ contains a so-called *K_r-factor*, that is, a family of $\lfloor n/r \rfloor$ vertex disjoint r-cliques.

Theorem 1.2 (Kierstead, Kostochka [4]) *For all r, every n-vertex graph G with $\delta_{\varnothing}(G) \geq 2\frac{r-1}{r}n - 1$ contains a K_r-factor.*

In this paper we are interested in deriving an Ore type result that allows us to embed much larger families of graphs than cycles or K_r-factors. More precisely we focus on the 3-chromatic version of the Bollobás-Komlós conjecture recently resolved by Böttcher, Schacht, and Taraz [2]: A graph is said to have bandwidth at most b if there exists a labelling of the vertices by numbers $1, \ldots, n$, such that for every edge $\{i, j\}$ of the graph we have $|i - j| \leq b$. Now let H be a 3-colourable n vertex graph with bounded maximum degree and bandwidth $o(n)$ and let G be an n-vertex graph with minimum degree $\delta(G) \geq (\frac{2}{3}+\gamma)n$. Then the main theorem from [2] asserts that H is a subgraph of G as long as n is sufficiently large. Our main result replaces the minimum degree condition in this conjecture by an Ore condition. We establish the following theorem.

Theorem 1.3 *For all $\Delta, \gamma > 0$ there are $\beta, n_0 > 0$ such that for all $n \geq n_0$ the following holds. Let G and H be n-vertex graphs such that H is 3-colourable, has maximum degree $\Delta(H) \leq \Delta$ and bandwidth $\mathrm{bw}(H) \leq \beta n$, and G satisfies $\delta_\varnothing(G) \geq (\frac{4}{3} + \gamma)n$. Then G contains a copy of H.*

As is shown in [1] all bounded degree graph classes characterised by forbidden induced minors have sublinear bandwidth. Accordingly Theorem 1.3 applies for example to all 3-colourable bounded degree planar graphs H. In contrast to the Ore type results mentioned above we use Szemerédi's celebrated regularity lemma [8] for proving this theorem. The proof method is similar to the one in [2] but we need to cope with the weaker Ore condition now and hence new ideas are necessary.

2 Outline of the proof

As indicated in the introduction one central ingredient to the proof of Theorem 1.3 is Szemerédi's regularity lemma. When applying the regularity lemma in graph embedding problems one typically makes use of the fact that the reduced graph R "inherits" certain properties from the host graph G. For example, it is well known that minimum degree conditions translate from G to R in this way. One of the important observations for our proof is that this is also true for Ore conditions.

On the way to the proof of Theorem 1.3 we first show the following lemma which may be regarded as a special case of Theorem 1.3. As sketched below this special case helps us then to deduce the general case. The *square* of a graph H is obtained by adding edges between all those vertices with distance 2 to H. A *square-path* on ℓ vertices is the square of a path on ℓ vertices.

Lemma 2.1 *For all $\gamma, \mu > 0$ there is an n_0 such that for all $n \geq n_0$ every n-vertex graph $G = (V, E)$ satisfying $\delta_\varnothing(G) \geq (\frac{4}{3} + \gamma)n$ contains a square-path on at least $(1 - \mu)n$ vertices.*

The question which minimum degree condition enforces a *spanning* square of a cycle (and hence also the square of a path) is subject of a well-known conjecture of Pósa that was resolved for large n by Komlós, Sárközy, and Szemerédi [5] with the help of the regularity lemma. In order to prove Lemma 2.1 we use a strategy similar to the one in [5]. Again, several new ideas are necessary in order to deal with the Ore condition. We outline them now.

The Proof of Lemma 2.1. The existence of a square-path in Lemma 2.1 is verified via a joint application of the regularity lemma and the blow-up lemma [6] and can roughly be described as follows. An application of the regularity lemma yields a regular partition of G with reduced graph R. With the help Theorem 1.2 it is then possible to infer that R can be covered with triangles and not difficult to modify the regular partition such that all pairs in these triangles are super-regular. In a third step, we show that each pair of these triangles can be connected by a short sequence of triangles within R. We will make this step more precise in Lemma 2.2 below. It implies that for each pair of super-regular triangles in R there is a constant length square-path in G connecting these two triangles. Finally, for each super-regular triangle in R the blow-up lemma asserts that one can find a square-path in G covering the clusters of this triangle such that these square-paths together with the connecting square-paths give a square-path covering almost all vertices of G.

For making step three of this procedure more precise, as promised, it is convenient to introduce some notation. A *triangle walk* in G of length $p-1$ is a sequence of edges $e_1, \ldots, e_p$ in G such that e_i and e_{i+1} share a triangle in G for all $i \in [p-1]$. The next lemma states that under a suitable Ore condition vertex disjoint triangles can be connected by short triangle walks.

Lemma 2.2 *An n-vertex graph R with $\delta_\varnothing(R) > \frac{4}{3}n$ has the property that between any edges of two vertex disjoint triangles in R there is a triangle walk of length at most 7.*

The proof of this lemma is centered around structural information implied by the following proposition. Its first part states that an Ore condition induces a (much weaker) minimum degree condition.

Proposition 2.3 *An n-vertex graph G has minimum degree $\delta(G) \geq \delta_\varnothing(G) - n$. Moreover, the set of vertices in G with degree less than $\delta_\varnothing(G)/2$ forms a clique.*

The Proof of Theorem 1.3. Lemma 2.1 asserts that inside a graph with sufficiently high Ore degree we can find a very rigid structure: an almost spanning square-path. In the proof of Theorem 1.3, which again combines the regularity lemma and the blow-up lemma, we now want to use this result on the cluster graph R of our host graph G. This is possible because, as we explained earlier, R "inherits" the Ore condition of G. A large square-path P (as provided by Lemma 2.1) in a cluster graph is a structure suitable for the embedding of 3-colourable bounded degree graphs of sublinear bandwidth as was illustrated in [2]. However, what makes this task quite demanding is that (in contrast to the proof of Lemma 2.1) we want to embed a spanning graph H in a host graph G now and hence need to use all vertices of G. The main difficulties here are to incorporate the vertices from the exceptional set into the regular-partition (that needs to be super-regular on certain spots as we want to apply the blow-up lemma later) and to produce a partition P_H of H that is "compatible" with the regular-partition P_G of G in the following sense: P_H allows edges only between partition classes that correspond to regular pairs in P_G, and P_G needs to be changed only slightly to achieve that the partition class sizes match those in P_H exactly. Finally, we need to make sure that these slight changes can be performed.

For showing that H and G have "compatible" partitions we prove, somewhat more generally, that the following holds for any (cluster) graph R. Assume R contains a triangle factor T such that each pair of triangles in T is connected by a triangle walk (which is true in particular if R contains a spanning square-path), and R contains a copy of K_4. Then any 3-colourable graph with small bandwidth admits a homomorphism h to R such that most edges of H end up on edges of T and we can control how many vertices of H are assigned by h to each vertex of R. It turns out that this captures the notion of "compatible" partitions that we need for applying the blow-up lemma.

References

[1] Böttcher, J., K. P. Pruessmann, A. Taraz and A. Würfl, *Bandwidth, expansion, treewidth, separators, and universality for bounded degree graphs*, submitted.

[2] Böttcher, J., M. Schacht and A. Taraz, *Spanning 3-colourable subgraphs of small bandwidth in dense graphs.*, J. Comb. Theory, Ser. B **98** (2008), pp. 752–777.

[3] Hajnal, A. and E. Szemerédi, *Proof of a conjecture of P. Erdős*, in: *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, North-Holland, Amsterdam, 1970 pp. 601–623.

[4] Kierstead, H. and A. Kostochka, *An Ore-type theorem on equitable coloring.*, J. Comb. Theory, Ser. B **98** (2008), pp. 226–234.

[5] Komlós, J., G. N. Sárközy and E. Szemerédi, *On the square of a Hamiltonian cycle in dense graphs*, Random Structures Algorithms **9** (1996), pp. 193–211.

[6] Komlós, J., G. N. Sárközy and E. Szemerédi, *Blow-up lemma*, Combinatorica **17** (1997), pp. 109–123.

[7] Ore, Ø., *Note on Hamilton circuits*, Am. Math. Mon. **67** (1960), p. 55.

[8] Szemerédi, E., *Regular partitions of graphs.*, Problèmes combinatoires et théorie des graphes, Orsay 1976, Colloq. int. CNRS No.260, 399-401 (1978). (1978).