SPANNING 3-COLOURABLE SUBGRAPHS OF SMALL BANDWIDTH IN DENSE GRAPHS

JULIA BÖTTCHER, MATHIAS SCHACHT, AND ANUSCH TARAZ

ABSTRACT. A conjecture by Bollobás and Komlós states the following: *For every $\gamma > 0$ and integers $r \geq 2$ and Δ, there exists $\beta > 0$ with the following property. If G is a sufficiently large graph with n vertices and minimum degree at least $(\frac{r-1}{r} + \gamma)n$ and H is an r-chromatic graph with n vertices, bandwidth at most βn and maximum degree at most Δ, then G contains a copy of H.*

This conjecture generalises several results concerning sufficient degree conditions for the containment of spanning subgraphs. We prove the conjecture for the case $r = 3$.

1. Introduction and results

The study of sufficient degree conditions which imply that a given graph G satisfies a certain property is one of the central themes in extremal graph theory. In this paper we are concerned with conditions on the minimum degree of G which guarantee that G contains a copy of a particular spanning subgraph H.

A well known example of such a result is Dirac's theorem [13]. It asserts that any graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains a spanning, so called Hamiltonian, cycle. Another classical result of that type by Corrádi and Hajnal [9] states that every graph G with n vertices and $\delta(G) \geq 2n/3$ contains $\lfloor n/3 \rfloor$ vertex disjoint triangles. This was generalised by Hajnal and Szemerédi [19], who proved that every graph G with $\delta(G) \geq (r-1)n/r$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r.

Pósa (see, e.g., [14]) and Seymour [36] indicated how these theorems could actually fit into a common framework. They conjectured that, at the same threshold $\delta(G) \geq (r-1)n/r$, one can in fact ask for 'well-connected' cliques, more precisely that such a graph G contains a copy of the $(r-1)$-st power of a Hamiltonian cycle (where the $(r-1)$-st power of an arbitrary graph is obtained by inserting an edge between every two vertices of distance at most $r-1$ in the original graph). The following approximate version of this conjecture for the case $r = 3$ was proved by Fan and Kierstead [17], and independently, by Komlós, Sárközy, and Szemerédi [26].

Theorem 1 ([17, 26]). *For every constant $\gamma > 0$ there is a constant n_0 such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (2/3 + \gamma)n$ contains the square of a Hamiltonian cycle.*

Fan and Kierstead [18] also gave a proof for the exact statement (i.e., with $\gamma = 0$ and $n_0 = 1$) for the square of a Hamiltonian path.[1] Moreover, Komlós, Sárközy,

Date: August 23, 2006.

The second author was supported by DFG grant SCHA 1263/1-1.

[1] In fact, Fan and Kierstead [18] showed that for the existence of a square of a Hamiltonian path $\delta(G) \geq (2n-1)/3$ is a sufficient and

and Szemerédi [26] proved the approximate version concerning the $(r-1)$-st power of a Hamiltonian cycle. Finally, the same authors [23, 27] gave a proof of the sharp Pósa–Seymour conjecture for sufficiently large graphs G and general r.

Recently, several other results of a similar flavour have been obtained which deal with a variety of spanning subgraphs H, such as, e.g., trees, F-factors, and planar graphs [3, 5, 6, 7, 10, 11, 22, 28, 29, 31, 32, 33, 37].

Facing this wealth of results, there seems to be a need for a unifying generalisation. Which parameter(s) of H determine the minimum degree threshold for G to guarantee a spanning copy of H as a subgraph? The results above indicate that the chromatic number of H plays a crucial rôle. Obviously, by the classical results of Turán [39] and of Erdős, Stone and Simonovits [15, 16], any graph H of *constant size* with $\chi(H) = r$, is forced to appear as a subgraph in any sufficiently large graph G if $\delta(G) \geq (\frac{r-2}{r-1} + \gamma)n$. However, if H has *as many vertices as* G and if in every r-colouring of H the colour classes are of the same size, then it is clear that we do indeed need $\delta(G) \geq \frac{r-1}{r}n$. For example, let G be the complete r-partite graph with partition classes almost, but not exactly, of the same size and let H be the union of vertex disjoint r-cliques. (See, e.g., [22, 32, 37] for a more detailed discussion how a less balanced r-colouring of H can lead to a smaller minimum degree threshold between $\frac{r-2}{r-1}n$ and $\frac{r-1}{r}n$.)

Thus, in an attempt to move away from results that concern only graphs H with a special, rigid structure, a naïve conjecture could be that $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$ suffices to guarantee that G contains a spanning copy of any r-chromatic graph H of bounded maximum degree. While the results mentioned above are in accordance with this idea, it is known that it fails in general as the following simple example shows. Let H be a random bipartite graph with bounded maximum degree and partition classes of size $n/2$ each, and let G be the graph formed by two cliques of size $(1/2 + \gamma)n$ each, which share exactly $2\gamma n$ vertices. It is then easy to see that G cannot contain a copy of H, since in H every set of vertices of size $(1/2 - \gamma)n$ has more than $2\gamma n$ neighbours.

One way to rule out such expansion properties for H, is to restrict the *bandwidth* of H. A graph is said to have bandwidth at most b, if there exists a labelling of the vertices by numbers $1, \ldots, n$, such that for every edge $\{i, j\}$ of the graph we have $|i - j| \leq b$. Bollobás and Komlós [21, Conjecture 16] conjectured that every r-chromatic graph on n vertices of bounded degree and bandwidth limited by $o(n)$, can be embedded into any graph G on n vertices with $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$. In this paper we give a proof of this conjecture for the case $r = 3$.

Theorem 2. *For all $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds.*

If H is a 3-chromatic graph on n vertices with $\Delta(H) \leq \Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (2/3 + \gamma)n$, then G contains a copy of H.

We note that our proof can be turned into an algorithm. More precisely, an embedding of H can be found in $O(n^{3.376})$ if H is given along with a valid 3-colouring and a labelling of the vertices respecting the bandwidth bound βn (see the last paragraph of Section 4 for more details).

---

sharp minimum degree condition.

Theorem 2 embraces a fairly large class of 3-chromatic graphs H. In fact, most graphs H considered so far were of constant bandwidth, whereas Theorem 2 includes for example (higher-dimensional) grid graphs as possible graphs H.

The analogue of Theorem 2 for bipartite H was announced by Abbasi [1] in 1998, and can now easily be obtained by our methods (see [20]), too. In [2] it is shown that in this case no sharp version of Theorem 2 (with $\gamma = 0$) is possible. More precisely, it is shown that if $\gamma \to 0$ and $\Delta \to \infty$ then β must tend to 0 in Theorem 2. However, the bound on β coming from our proof is rather poor, having a tower-type dependence on $1/\gamma$.

The proof of Theorem 2 is based on the *regularity method* and uses, in particular, the regularity lemma [38] and the blow-up lemma [24] together with Theorem 1. There is a well established strategy for proofs of this kind, which, as described by Komlós in his survey [21], proceeds in several steps: First, prepare the graph H by dividing it into a constant number of smaller pieces, which is usually possible and not too difficult by calling upon the structural properties guaranteed for H. Secondly, prepare the graph G by applying the regularity lemma and thus obtaining a sufficiently regular vertex partition. Thirdly, find an assignment that maps vertices of H to the partition classes of G. Fourthly, ensure that the edges between the different parts of H are mapped to edges in G. Finally, complete the embedding by applying the blow-up lemma to the individual pieces of H and their counterparts in G.

Steps 2, 3, and 5 have been standardised by the use of the powerful tools mentioned above, but the proofs are still technically rather involved: although H and G have been 'prepared' roughly for each other, there is still a great deal of details that have to be carefully adjusted and fitted, especially in step 4. Since, in our case, we have very little control about the structure of H, this difficulty becomes particularly pressing. In order to avoid the looming threat of many cases, we have pushed the agenda described above a bit further.

We will prove two main lemmas. While they will deal exclusively with the graph G and the graph H respectively, they are linked to each other in the following way: the *lemma for* G (Lemma 11) will suggest a partition of G and communicate the structure of this partition (but not the graph G) to the *lemma for* H. The lemma for H (Lemma 12) will then try to find a partition of H with a very similar structure, and return the sizes of its partition classes to the lemma for G. The latter will then adjust its partition classes by shifting a few vertices of G, until they fit exactly the class sizes of H. The embedding of H into G can then be found using (a slight variant of) the embedding lemma (Lemma 10) first used by Chvátal et al. for step 4 and the blow-up lemma (Theorem 9) for step 5.

This approach provides a very modular proof strategy that can easily be checked and may be of further use for other similar problems. For example, our current work-in-progress indicates that a proof of the Bollobás-Komlós conjecture for general r-chromatic graphs H is now within reach.

This paper is organised as follows. In the next section, Section 2, we introduce the regularity lemma together with the two embedding lemmas mentioned above. In Section 3, we state and explain our two main lemmas, the lemma for G and the lemma for H. Here we also *outline* how Theorem 2 can be deduced from these lemmas, while the the full details of the proof are given in Section 4. Finally, we prove the lemma for G and the lemma for H in Sections 5 and 6, respectively.

2. The regularity method

In this section we recall the notation needed for Szemerédi's regularity lemma and the blow-up lemma. We also prove a few simple facts concerning ε-regular pairs, which will be useful in the proofs of Theorem 2 and the lemma for G. We would advise a reader familiar with Szemerédi's regularity lemma to skip this section at the first reading and go directly to the outline of the proof of Theorem 2 in Section 3.

We start with some basic definitions. Our general aim is to find a *copy* of some graph H in some other graph G, by which we mean that G contains a subgraph which is isomorphic to H. In other words, we are looking for an *embedding* of H into G, i.e., an injective function $f\colon V(H) \to V(G)$ such that for every edge $\{u, v\} \in E(H)$ we have $\{f(u), f(v)\} \in E(G)$.

2.1. **Szemerédi's regularity lemma.** One of the main tools in our proof is the regularity lemma [38] of Szemerédi, which pivots around the concept of an ε-regular pair. Let $G = (V, E)$ be a graph. For a vertex $v \in V$ we write $d_G(v) := |N_G(V)|$ for the degree of v in G. Let A, $B \subseteq V$ be disjoint vertex sets. We denote the number of edges with one end in A and the other end in B by $e(A, B)$. The ratio $d(A, B) := e(A, B)/(|A||B|)$ is called the *density* of (A, B). The pair (A, B) is *ε-regular*, if for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ it is true that $|d(A, B) - d(A', B')| < \varepsilon$. An ε-regular pair (A, B) is called (ε, d)-regular, if it has density at least d. The following is the so-called degree form of Szemerédi's regularity lemma (see, e.g., [30, Theorem 1.10]).

Theorem 3 (Regularity lemma)**.** *For every $\varepsilon > 0$ and every integer k_0 there is an $K_0 = K_0(\varepsilon, k_0)$ such that for every $d \in [0, 1]$ and for every graph G on at least K_0 vertices there exists a partition of $V(G)$ into $V_0, V_1, \ldots, V_k$ and a spanning subgraph G' of G such that the following holds:*

(i) $k_0 \leq k \leq K_0$,
(ii) $d_{G'}(x) > d_G(x) - (d + \varepsilon)|V(G)|$ *for all vertices* $x \in V(G)$,
(iii) *for all* $i \geq 1$ *the induced subgraph* $G'[V_i]$ *is empty,*
(iv) $|V_0| \leq \varepsilon|V(G)|$,
(v) $|V_1| = |V_2| = \cdots = |V_k|$,
(vi) *all pairs* (V_i, V_j) *with* $1 \leq i < j \leq k$ *are either* (ε, d)*-regular or* $G'[V_i, V_j]$ *is empty.*

The sets V_i in Theorem 3 are called *clusters* and the set V_0 is the *exceptional set*. Given a partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ as in Theorem 3, the *reduced graph* R_k is the graph on vertices $[k]$ and with edges $\{i, j\}$ for $1 \leq i, j \leq k$ for exactly those pairs (V_i, V_j) that are (ε, d)-regular in G'. Thus, $\{i, j\}$ is an edge of R_k if and only if G' has an edge between V_i and V_j. On the other hand, for a graph $G = (V, E)$ and a graph R_k on the vertex set $[k]$ we say that $V_1 \dot\cup \cdots \dot\cup V_k$ is *(ε, d)-regular on R_k* if (V_i, V_j) is (ε, d)-regular for every $\{i, j\} \in E(R_k)$. We will also use the following simple corollary of Theorem 3 (see, e.g., [33, Proposition 9]).

Corollary 4. *For every $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ and every integer k_0 there exist K_0 so that the following holds.*

For every $c \geq 0$, an application of Theorem 3 to a graph G of minimum degree at least $(c + \gamma)|V(G)|$ yields a partition $V_0, V_1, \ldots, V_k$ of $V(G)$ and a subgraph G' of G so that additionally to properties (i)–(vi) the following holds:

(vii) *the reduced graph* R_k *has minimum degree at least* $(c + \gamma/2)k$.

2.2. **Super-regular pairs.** For the blow-up lemma we need the concept of a *super-regular pair*. Roughly speaking a regular pair is super-regular if every vertex has a sufficiently large degree.

Definition 5 (super-regular pair)**.** *Let ε, $d > 0$ and let (A, B) be an (ε, d)-regular pair in a graph G. We say (A, B) is* (ε, d)-super-regular *if, in addition, every $v \in A$ has at least $d|B|$ neighbours in B and every $v \in B$ has at least $d|A|$ neighbours in A.*

Moreover, for a graph $G = (V, E)$ and a graph R_k on vertex set $[k]$ we say $V_1 \dot\cup \cdots \dot\cup V_k \subseteq V$ is (ε, d)-super-regular on R_k *if (V_i, V_j) is (ε, d)-super-regular for every $\{i, j\} \in E(R_k)$.*

Proposition 6 implies that every (ε, d)-regular pair (A, B) contains a "large" $(2\varepsilon, d - 2\varepsilon)$-super-regular sub-pair (A', B').

Proposition 6. *Let (A, B) be an (ε, d)-regular pair and B' be a subset of B of size at least $\varepsilon|B|$. Then there are at most $\varepsilon|A|$ vertices v in A with $|N(v) \cap B'| < (d - \varepsilon)|B'|$.*

Proof. Let $A' = \{v \in A : |N(v)| \cap B' < (d - \varepsilon)|B'|\}$ and assume to the contrary that $|A'| > \varepsilon|A|$. But then $d(X, Y) < ((d - \varepsilon)|A'||B'|)/(|A'||B'|) = d - \varepsilon$ which is a contradiction since (A, B) is (ε, d)-regular. $\square$

Repeating the last observation a fixed number of times, we obtain the following proposition, which we will later combine with Corollary 4.

Proposition 7. *With the notation of Corollary 4, let R' be a subgraph of the reduced graph R with $\Delta(R') \leq \Delta$. Then for each vertex i of R', the corresponding set V_i contains a subset V_i' of size $(1-\varepsilon\Delta)|V_i|$ such that for every edge $\{i, j\} \in E(R')$ the pair (V_i', V_j') is $(\varepsilon/(1 - \varepsilon\Delta), d - \Delta\varepsilon)$-super-regular. Moreover, for every edge $\{i, j\}$ of the original reduced graph R, the pair (V_i', V_j') is still $(\varepsilon/(1 - \varepsilon\Delta), d - \Delta\varepsilon)$-regular.*

For the simple proof of Proposition 7 we refer to [33, Proposition 8]. We close this section with the following useful observation. It states that the notion of regularity is "robust" in view of small alterations of the respective vertex sets.

Proposition 8. *Let (A, B) be an (ε, d)-regular pair and let $(\hat{A}, \hat{B})$ be a pair such that $|\hat{A} \triangle A| \leq \hat{\alpha}|\hat{A}|$ and $|\hat{B} \triangle B| \leq \hat{\beta}|\hat{B}|$ for some $0 \leq \hat{\alpha}, \hat{\beta} \leq 1$. Then, $(\hat{A}, \hat{B})$ is an $(\hat{\varepsilon}, \hat{d})$-regular pair with*

$$\hat{\varepsilon} := \varepsilon + 3\big(\sqrt{\hat{\alpha}} + \sqrt{\hat{\beta}}\big) \quad \text{and} \quad \hat{d} := d - 2(\hat{\alpha} + \hat{\beta})$$

If, moreover, (A, B) is (ε, d)-super-regular and each vertex v in $\hat{A}$ has at least $d|\hat{B}|$ neighbours in $\hat{B}$ and each vertex v in $\hat{B}$ has at least $d|\hat{A}|$ neighbours in $\hat{A}$, then $(\hat{A}, \hat{B})$ is $(\hat{\varepsilon}, \hat{d})$-super-regular with $\hat{\varepsilon}$ and $\hat{d}$ as above.

Proof. Let A, B, $\hat{A}$ and $\hat{B}$ be as above. First we estimate the density of $(\hat{A}, \hat{B})$. Let $d' := d(A, B) \geq d$ be the density of (A, B). If $(\hat{A}, \hat{B})$ had the same density as (A, B), we would have $e(\hat{A}, \hat{B}) = d'|\hat{A}||\hat{B}|$. The actual value of $e(\hat{A}, \hat{B})$ can deviate by at most

$$\begin{aligned} |\hat{A} \triangle A| \cdot |\hat{B} \cup B| + |\hat{B} \triangle B| \cdot |\hat{A} \cup A| &\leq \hat{\alpha}|\hat{A}| \cdot (1 + \hat{\beta})|\hat{B}| + \hat{\beta}|\hat{B}| \cdot (1 + \hat{\alpha})|\hat{A}| \\ &\leq 2(\hat{\alpha} + \hat{\beta})|\hat{A}||\hat{B}| \end{aligned}$$

from this value. So, clearly

$$\hat{d} = d - 2(\hat{\alpha} + \hat{\beta}) \leq d' - 2(\hat{\alpha} + \hat{\beta}) \leq d(\hat{A}, \hat{B}) \leq d' + 2(\hat{\alpha} + \hat{\beta})\,.$$

Now let $\hat{A}' \subseteq \hat{A}$ and $\hat{B}' \subseteq \hat{B}$ be sets of sizes $|\hat{A}'| \geq \hat{\varepsilon}|\hat{A}|$ and $|\hat{B}'| \geq \hat{\varepsilon}|\hat{B}|$. Denote $\hat{A}' \cap A$ by A' and $\hat{B}' \cap B$ by B' and observe that

$$|A'| \geq |\hat{A}'| - \hat{\alpha}|\hat{A}| \geq (\hat{\varepsilon} - \hat{\alpha})|\hat{A}| \geq (\varepsilon + \sqrt{\hat{\alpha}})|\hat{A}| \geq \varepsilon(1 + \hat{\alpha})|\hat{A}| \geq \varepsilon|A|.$$

Similarly, $|B'| \geq \varepsilon|B|$. It follows that $d' - \varepsilon \leq d(A', B') \leq d' + \varepsilon$. Moreover, $|A'| \leq |\hat{A}'|$ and

$$|A'| \geq |\hat{A}'| - \hat{\alpha}|\hat{A}| \geq |\hat{A}'| - \hat{\alpha}\frac{|\hat{A}'|}{\hat{\varepsilon}} \geq (1 - \sqrt{\hat{\alpha}})|\hat{A}'|\,,$$

where the last inequality follows from the definition of $\hat{\varepsilon}$. The same calculations yield

$$(1 - \sqrt{\hat{\beta}})|\hat{B}'| \leq |B'| \leq |\hat{B}'|\,.$$

For the number of edges between A' and B' we therefore get

$$\begin{aligned} e(\hat{A}', \hat{B}') &\geq e(A', B') \geq (d' - \varepsilon)|A'||B'| \geq (d' - \varepsilon)(1 - \sqrt{\hat{\alpha}})(1 - \sqrt{\hat{\beta}})|\hat{A}'||\hat{B}'| \\ &\geq (d' - \varepsilon - \sqrt{\hat{\alpha}} - \sqrt{\hat{\beta}})|\hat{A}'||\hat{B}'| \end{aligned}$$

since $\hat{\alpha}, \hat{\beta} \leq 1$. Similarly,

$$\begin{aligned} e(\hat{A}', \hat{B}') &\leq e(A', B') + (|\hat{A}'| - |A'|)|\hat{B}'| + (|\hat{B}'| - |B'|)|\hat{A}'| \\ &\leq (d' + \varepsilon)|A'||B'| + \sqrt{\hat{\alpha}}|\hat{A}'||\hat{B}'| + \sqrt{\hat{\beta}}|\hat{A}'||\hat{B}'| \\ &\leq (d' + \varepsilon + \sqrt{\hat{\alpha}} + \sqrt{\hat{\beta}})|\hat{A}'||\hat{B}'|\,. \end{aligned}$$

With this we can now compare the densities of $(\hat{A}', \hat{B}')$ and $(\hat{A}, \hat{B})$:

$$\begin{aligned} d(\hat{A}, \hat{B}) - d(\hat{A}', \hat{B}') &\leq (d' + 2(\hat{\alpha} + \hat{\beta})) - (d' - \varepsilon - \sqrt{\hat{\alpha}} - \sqrt{\hat{\beta}}) \leq \hat{\varepsilon}, \\ d(\hat{A}', \hat{B}') - d(\hat{A}, \hat{B}) &\geq (d' + \varepsilon + \sqrt{\hat{\alpha}} + \sqrt{\hat{\beta}}) - (d' - 2(\hat{\alpha} - \hat{\beta})) \leq \hat{\varepsilon}, \end{aligned}$$

This implies that $(\hat{A}, \hat{B})$ is $(\hat{\varepsilon}, \hat{d})$-regular. The second part of the proposition follows immediately from Definition 5, since $\hat{d}|\hat{A}| \leq d|\hat{A}|$ and $\hat{d}|\hat{B}| \leq d|\hat{B}|$. □

2.3. **Embedding results for regular pairs.** The important feature of super-regular pairs is that a powerful theorem, the so-called *blow-up lemma* proven by Komlós, Sárközy and Szemerédi [24] (see also [34] for an alternative proof), guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently super-regular pairs. In fact, the statement is more general and allows the embedding of r-chromatic graphs into the union of r vertex classes that form $\binom{r}{2}$ super-regular pairs, but we will only use this lemma in the following restricted form for 3-chromatic graphs.

Theorem 9 (Blow-up lemma [24])**.** *For every d, Δ, $c > 0$ there exist constants $\varepsilon_{\mathrm{BL}} = \varepsilon_{\mathrm{BL}}(d, \Delta, c)$ and $\alpha_{\mathrm{BL}} = \alpha_{\mathrm{BL}}(d, \Delta, c)$ such that the following holds.*

Let n_1, n_2, and n_3 be arbitrary positive integers, $0 < \varepsilon < \varepsilon_{\mathrm{BL}}$, and $G = (V_1 \dot\cup V_2 \dot\cup V_3, E)$ be a 3-partite graph with $|V_i| = n_i$ for $i \in [3]$ and with all pairs (V_i, V_j) being (ε, d)-super-regular for $1 \leq i < j \leq 3$, i.e., $V_1 \dot\cup V_2 \dot\cup V_3$ is (ε, d)-super-regular on K_3.

Suppose H is a 3-partite graph on vertex classes $W_1\dot\cup W_2\dot\cup W_3$ of sizes n_1, n_2, and n_3 with $\Delta(H)\le\Delta$. Moreover, suppose that in each class W_i there is a set of at most $\alpha_{\mathrm{BL}}n_i$ special *vertices y, each of them equipped with a set $C_y\subseteq V_i$ with $|C_y|\ge cn_i$.*

Then there is an embedding of H into G such that every special vertex y is mapped to a vertex in C_y.

We say that the special vertices y in Theorem 9 are *image restricted to C_y*.

For some technical reasons (see Step 4 in the overview of the proof of Theorem 2 discussed in Section 1) we also need the following weaker embedding lemma (concerning only linear sized, but not spanning embeddings) in the less restrictive environment of (ε,d)-regular pairs. Such a lemma, in a slightly different context, was first obtained by Chvátal, Rödl, Szemerédi, and Trotter [8] (see also [12, Lemma 7.5.2]). The only difference between Lemma 10 and their embedding lemma is that we only embed *some* of the vertices of a given graph B into G and reserve sufficiently many places in G for a future embedding of the remaining vertices of B.

Lemma 10 (Partial embedding lemma)**.** *For every integer $\Delta\ge 2$ and every $d\in(0,1]$ there exist constants $c=c(\Delta,d)$ and $\varepsilon_{\mathrm{PEL}}=\varepsilon_{\mathrm{PEL}}(\Delta,d)$ such that for all $\varepsilon\le\varepsilon_{\mathrm{PEL}}$ the following is true.*

Let R_k be a graph with $V(R_k)=[k]$ and G be a graph with $V(G)=V_1\dot\cup\cdots\dot\cup V_k$, such that $|V_i|\ge(1-\varepsilon_{\mathrm{PEL}})n/k$ for all $i\in[k]$ and $V_1\dot\cup\cdots\dot\cup V_k$ is (ε,d)-regular on R_k. Let, furthermore, B be a graph with $V(B)=X\dot\cup Y$ and $f\colon V(B)\to V(R_k)=[k]$ be a mapping with $\{f(b),f(b')\}\in E(R_k)$ for all $\{b,b'\}\in E(B)$.

If $|V(B)|\le\varepsilon_{\mathrm{PEL}}n/k$ and $\Delta(B)\le\Delta$, then there exists an injective mapping $g\colon X\to V(G)$ with $g(x)\in V_{f(x)}$ for all $x\in X$ such that for all $y\in Y$ there exist sets $C_y\subseteq V_{f(y)}\setminus g(X)$ such that

(*i*) *if $x,x'\in X$ and $\{x,x'\}\in E(B)$ then $\{g(x),g(x')\}\in E(G)$,*
(*ii*) *for all $y\in Y$ we have $C_y\subseteq N_G(g(x))$ for all $x\in N_B(y)\cap X$, and*
(*iii*) *$|C_y|\ge c|V_{f(y)}|$ for every $y\in Y$.*

In other words, Lemma 10 provides a mapping g for those vertices $x\in X$ of B into the cluster $V_{f(x)}$ required by f, respecting the edges between such vertices. Moreover, for the other vertices $y\in Y$ of B, it prepares sufficiently large sets $C_y\subseteq V_{f(y)}\setminus g(X)$ such that, no matter where y will later be embedded in C_y, it will be adjacent to any of its already embedded neighbours $x\in N_B(y)\cap X$.

The proof of Lemma 10 follows very much along the lines of the embedding lemma from [8]. We also proceed iteratively, embedding the vertices in X into G one by one.

Proof. Given Δ and d, choose $c:=(d/2)^{\Delta}/2$ and $\varepsilon_{\mathrm{PEL}}:=c/\Delta$. Note, that this implies $\varepsilon_{\mathrm{PEL}}\le(d/2)^{\Delta}/4\le d/2$. Let $0<\varepsilon\le\varepsilon_{\mathrm{PEL}}$, and G, R_k and B with $V(B)=X\dot\cup Y$ be graphs as required. For the size of X we have for all $i\in[k]$

$$|X|\le|V(B)|\le\varepsilon_{\mathrm{PEL}}n/k\le|V_i|\varepsilon_{\mathrm{PEL}}/(1-\varepsilon_{\mathrm{PEL}})\le 2\varepsilon_{\mathrm{PEL}}|V_i|\,.$$

We now construct the embedding $g:X\to V(G)$. For this, we will create sets C_b not only for the vertices in Y, but for all vertices $b\in V(B)$. First, set $C_b:=V_{f(b)}$ for all $b\in V(B)$. Then, repeat the following steps for each $x\in X$:

(*a*) For all $b\in N_B(x)$, delete all vertices $v\in C_x$ with $|N_G(v)\cap C_b|<(d-\varepsilon)|C_b|$.

(*b*) Then, choose one of the vertices remaining in C_x as $g(x)$.
(*c*) For all $b \in N_B(x)$, delete all vertices $v \in C_b$ with $v \notin N_G(g(x))$.
(*d*) For all $b \in V(B)$, delete $g(x)$ from C_b.

We claim, that at the end of this procedure, g and the C_y with $y \in Y$ are as desired. Indeed, conditions (*i*) and (*ii*) are satisfied by construction. It remains, to prove that condition (*iii*) is satisfied and that $g(x)$ can be chosen in step (*a*) throughout the entire procedure.

We start, by showing, that we always have $|C_b| \geq c|V_{f(b)}|$ for all $b \in V(B)$. This implies condition (*iii*). In total, step (*d*) removes at most $|X|$ vertices from each C_b. By the choice of $g(x)$ in step (*a*) and (*b*), an application of step (*c*) to a vertex $b \in N_B(x)$, reduces the size of C_b at most by a factor of $d-\varepsilon$. Since each vertex in $b \in B$ has at most Δ neighbours, we always have

$$|C_b| \geq (d-\varepsilon)^{\Delta}|V_{f(b)}| - |X| \geq ((d/2)^{\Delta} - 2\varepsilon_{\mathrm{PEL}})|V_{f(b)}| \geq \tfrac{1}{2}(d/2)^{\Delta}|V_{f(b)}| = c|V_{f(b)}|.$$

Finally we consider step (*a*). The last inequality shows that we always have $|C_b| \geq c|V_{f(b)}| \geq \varepsilon|V_{f(b)}|$ for every vertex $b \in V(B)$. Consequently, by Proposition 6, at most $\Delta\varepsilon|V_{f(x)}|$ vertices are deleted from C_x in step (*a*). Since $\Delta\varepsilon|V_{f(x)}| \leq (c/2)|V_{f(x)}| < |C_x|$, the set C_x doesn't become empty and thus $g(x)$ can be chosen in step (*b*). □

3. Main lemmas and outline of the proof

In this section we introduce the central lemmas that are needed for the proof of our main theorem. Our emphasis in this section is to explain how they work together to give the proof of Theorem 2, which itself is then presented in full detail in the subsequent section, Section 4.

Our first lemma incorporates the regularity lemma, but before we can state it we will need a few more definitions. For all $n, k \in \mathbb{N}$ with k divisible by 3, we call an integer partition $n_1 + \cdots + n_k = n$ (with $n_i \in \mathbb{N}$ for all $i \in [k]$) *equitriangular*, if $|n_{3(j-1)+l} - n_{3(j-1)+l'}| \leq 1$ for all $j \in [k/3]$ and $l, l' \in [3]$. We denote by $R_k^* = ([k], E(R_k^*))$ the square of the Hamiltonian cycle with edges $\{\{i, i+1\} : i = 1, \ldots, k-1\} \cup \{\{1, k\}\}$. Moreover, we write R_k^{**} for the subgraph of R_k^* consisting of the family of $k/3$ vertex disjoint triangles in R_k^* with vertex sets $3(j-1)+1$, $3(j-1)+2$, and $3(j-1)+3$ for $j \in [k/3]$.

We can now state (and then explain) our first main lemma which 'prepares' the graph G for the embedding of H into G.

Lemma 11 (Lemma for G)**.** *For all $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there exist K_0 and $\xi_0 > 0$ such that for all $n \geq K_0$ and for every graph G on vertex set $[n]$ with $\delta(G) \geq (2/3 + \gamma)n$ there exist $k \in \mathbb{N} \setminus \{0\}$ and a graph R_k on vertex set $[k]$ with*

(*R1*) *$k \leq K_0$ and $3|k$,*
(*R2*) *$\delta(R_k) \geq (2/3 + \gamma/2)k$,*
(*R3*) *$R_k^{**} \subseteq R_k^* \subseteq R_k$, and*
(*R4*) *there is an equitriangular integer partition $m_1 + \cdots + m_k$ of n with $m_i \geq (1-\varepsilon)n/k$ such that the following holds.*

For every partition $n = n_1 + \cdots + n_k$ with $m_i - \xi_0 n \leq n_i \leq m_i + \xi_0 n$ there exists a partition $V_1 \dot\cup \cdots \dot\cup V_k$ of V with

(*V1*) $|V_i| = n_i$,

(V2) $V_1 \dot\cup \cdots \dot\cup V_k$ *is* (ε, d)*-regular on* R_k*, and*
(V3) $V_1 \dot\cup \cdots \dot\cup V_k$ *is* (ε, d)*-super-regular on* R_k^{**}*.*

In order to understand what this lemma says, let us first ignore property (*R4*), the two lines thereafter, and property (*V1*), and instead propose that the sizes $|V_i|$ form an equitriangular partition of n. In this case, Lemma 11 could be considered a standard corollary of the regularity lemma, Theorem 3, and Theorem 1 for graphs G with $\delta(G) \geq (2/3 + \gamma)n$ (cf. Corollary 4 and Proposition 7). Here it would guarantee a partition of the vertex set of G in such a way that the partition classes form many (super-)regular pairs, and that these pairs are organised in a sort of backbone, namely in the form of a square of a Hamiltonian cycle R_k^* for the regular pairs, and, contained therein, a spanning family R_k^{**} of disjoint triangles for the super-regular pairs.

However, the lemma says more. When we come to the point (*R4*), the lemma 'has in mind' the partition we just described, but doesn't exhibit it. Instead, it only discloses the sizes m_i and allows us to wish for small amendments: for every $i \in [k]$, we can now look at the value m_i and ask for the size of the i-th partition class to be adjusted to a new value n_i, differing from m_i by at most $\xi_0 n$.

When proving Lemma 11, one needs to alter the partition by shifting a few vertices. Note that while (ε, d)-regularity is very robust towards such small alterations, (ε, d)-*super*-regularity is not, so this is where the main difficulty lies (cf. Proposition 8). We give the proof of Lemma 11, which borrows ideas from [29], in Section 5.

Now we come to the second main lemma. It prepares the graph H so that it can be embedded into G. This is exactly the place where, given the values m_i, the new values n_i in the setting described above are specified.

Lemma 12 (Lemma for H)**.** *Let $k \geq 1$ be an integer and let β, $\xi > 0$ satisfy $\beta \leq \xi^2/10^4$. Let H be a 3-chromatic graph on n vertices with bandwidth at most βn and let R_k be a graph with $V(R_k) = [k]$ such that $\delta(R_k) > 2k/3$ and $R_k^{**} \subseteq R_k^* \subseteq R_k$. Furthermore, suppose $m_1 + \cdots + m_k$ is an equitriangular integer partition of n with $m_i \geq \beta n$ for every $i \in [k]$.*

Then there exists a mapping $f\colon V(H) \to [k]$ and a set of special vertices $X \subseteq V(H)$ with the following properties

(a) $|X| \leq k\xi n$,
(b) $m_i - \xi n \leq |W_i| := |f^{-1}(i)| \leq m_i + \xi n$ *for every* $i \in [k]$,
(c) *for every edge* $\{u, v\} \in E(H)$ *we have* $\{f(u), f(v)\} \in E(R_k)$*, and*
(d) *if* $\{u, v\} \in E(H)$ *and, moreover,* u *and* v *are both in* $V(H) \setminus X$*, then* $\{f(u), f(v)\} \in E(R_k^{**})$*.*

In other words, Lemma 12 receives a graph H as input and, from Lemma 11, a reduced graph R_k (with $R_k^{**} \subseteq R_k^* \subseteq R_k$), an equitriangular partition $n = m_1 + \cdots + m_k$, and a parameter ξ.

Again we emphasise that this is all what Lemma 12 needs to know about G. It then provides us with a function f which maps the vertices of H onto the vertex set $[k]$ of R_k in such a way that $i \in [k]$ receives $n_i := |W_i|$ vertices from H, with $|n_i - m_i| \leq \xi n$. Although the vertex partition of G is not known exactly at this point, we already have its reduced graph R_k. Lemma 12 guarantees that the endpoints of an edge $\{u, v\}$ of H get mapped into vertices $f(u)$ and $f(v)$ of R_k, representing future partition classes $V_{f(u)}$ and $V_{f(v)}$ in G which will form a *super-regular* pair (see (d)) – except for those few edges with one or both endpoints in some small special set X. But even these edges will be mapped into pairs of classes in G that will form at least *regular* pairs (see (c)). Lemma 12 will then return the values n_i to Lemma 11, which will finally produce a corresponding partition of the vertices of G.

If we consider the triangles $3(j-1)+1$, $3(j-1)+2$, and $3(j-1)+3$ for every $j \in [k/3]$ that form the edge set of R_k^{**}, then the blow-up lemma, Theorem 9, would immediately give us an embedding of

$$H[W_{3(j-1)+1}, W_{3(j-1)+2}, W_{3(j-1)+3}] \quad \text{into} \quad G[V_{3(j-1)+1}, V_{3(j-1)+2}, V_{3(j-1)+3}]$$

that takes care of all edges of $H[V(H) \setminus X]$.

Edges of H with one or both vertices in the special set X will need some special treatment. However, due to part (a) of Lemma 12 the size of X is quite small. In particular we will be able to ensure that $|X| \ll n/k$. Our strategy will be first to find an embedding g of the vertices of X into $V(G)$ such that for every $y \in N_H(X) := \{y \in V(H) \setminus X \colon \exists\ xy \in E(H) \text{ with } x \in X\}$ the set $C_y := V_{f(y)} \cap \bigcap_{x \in N_H(y) \cap X} N_G(g(x))$ is sufficiently large. The partial embedding lemma, Lemma 10, guarantees the existence of such an embedding g of X. Once we have applied it, we can complete the partial embedding g with the blow-up lemma, which will 'respect' the image restriction to C_y for every $y \in N_H(X)$. In the next section we give the precise details how Theorem 2 can be deduced from Lemma 11 and Lemma 12 following the outline discussed above.

4. Proof of Theorem 2

In this section we give the proof of Theorem 2 based on Theorem 9, Lemma 10, Lemma 11, and Lemma 12 from Section 2.3 and Section 3. In particular, we will use Lemma 11 for partitioning G, and Lemma 12 for assigning the vertices of H to the parts of G. For this, it will be necessary to split the application of Lemma 11 into two phases. The first phase is used to set up the parameters for Lemma 12. With this input, Lemma 12 then defines the sizes of the parts of G that are constructed during the execution of the second phase of Lemma 11.

Finally, H is embedded into G by using the blow-up lemma, Theorem 9, on the partition of G and by treating the special vertices $X \subseteq V(H)$ from Lemma 12 with the help of the partial embedding lemma, Lemma 10.

Here is how the constants that appear in the proof are related:

$$\frac{1}{\Delta}, \gamma \gg d \gg \varepsilon \gg \frac{1}{K_0} \gg \xi \gg \beta, \qquad \text{as well as} \qquad c \gg \varepsilon \gg \alpha\,.$$

Proof of Theorem 2. Given Δ and γ, let ε_0 and d be as asserted by Lemma 11 for input γ. Let $c = c(\Delta, d)$ and $\varepsilon_{\mathrm{PEL}} = \varepsilon_{\mathrm{PEL}}(\Delta, d)$ be as given by Lemma 10, and $\varepsilon_{\mathrm{BL}} = \varepsilon_{\mathrm{BL}}(d, \Delta, c)$ and $\alpha_{\mathrm{BL}} = \alpha_{\mathrm{BL}}(d, \Delta, c)$ as given by Theorem 9. Set

$$\varepsilon := \min\{\varepsilon_0, \varepsilon_{\mathrm{PEL}}/2, \varepsilon_{\mathrm{BL}}/2, d/4\}\,. \tag{1}$$

Then, the lemma for G (Lemma 11) provides constants K_0 and ξ_0 for this ε. We define

$$\xi := \min\{\xi_0, 1/(4K_0), \varepsilon/(K_0^2(\Delta+1)), \alpha_{\mathrm{BL}}/(2K_0^2(\Delta+1))\} \tag{2}$$

as well as $n_0 := K_0$, $\beta := \min\{\xi^2/2940, (1-\varepsilon)/K_0\}$ and consider arbitrary graphs H and G on $n \geq n_0$ vertices that meet the conditions of Theorem 2.

Applying Lemma 11 to G we get an integer k with $0 < k \le K_0$, graphs $R_k^{**} \subseteq R_k^* \subseteq R_k$ on vertex set $[k]$, and an equitriangular partition $m_1 + \cdots + m_k$ of n such that *(R1)*–*(R4)* are satisfied.

Before continuing with Lemma 11, we apply the lemma for H (Lemma 12). Note that due to *(R4)* and the choice of β above, we have $m_i \ge (1-\varepsilon)n/k \ge \beta n$ for every $i \in [k]$. Consequently, for constants k, β, and ξ, graphs H and $R_k^{**} \subseteq R_k^* \subseteq R_k$, and the equitriangular integer partition $m_1 + \cdots + m_k = n$ we can apply Lemma 12. This yields a mapping $f\colon V(H) \to [k]$ and a set of special vertices $X \subseteq V(H)$. These will be needed later. For the moment we are only interested in the sizes $n_i := |W_i| = |f^{-1}(i)|$ for $i \in [k]$. Condition *(b)* of Lemma 12 and the choice of $\xi \le \xi_0$ in (2) imply that the partition $n = n_1 + \cdots + n_k$ satisfies $m_i - \xi_0 n \le m_i - \xi n \le n_i \le m_i + \xi n \le m_i + \xi_0 n$ for every $i \in [k]$. Accordingly, we can continue with Lemma 11 to obtain a partition $V = V_1 \dot\cup \cdots \dot\cup V_k$ with $|V_i| = n_i$ that satisfies conditions *(V1)*–*(V3)* of Lemma 11. Note that

$$\begin{aligned} |V_i| = n_i \ge m_i - \xi n &\overset{(R4)}{\ge} (1-\varepsilon)\frac{n}{k} - \xi n = (1-(\varepsilon + \xi k))\frac{n}{k} \\ &\overset{(1),(2)}{\ge} (1-\varepsilon_{\mathrm{PEL}})\frac{n}{k} \ge \frac{1}{2}\frac{n}{k}. \end{aligned} \tag{3}$$

Now, we have partitions $W_1 \dot\cup \cdots \dot\cup W_k$ of H and $V_1 \dot\cup \cdots \dot\cup V_k$ of G with $|W_i| = |V_i| = n_i$ for all $i \in [k]$. We will build the embedding of H into G such that each vertex $v \in W_i \subseteq V(H)$ will be embedded into the corresponding set $V_i \subseteq V(G)$ for $i \in [k]$.

For embedding the special vertices X of H in G, we use the partial embedding lemma (Lemma 10). We provide Lemma 10 with constants Δ, d, and k, the graph R_k, the graph G with vertex partition $V_1 \dot\cup \cdots \dot\cup V_k = V(G)$, the graph $B := H[X \dot\cup Y]$ where $Y := N_H(X)$ consists of the neighbours of vertices of X outside X, and the mapping f restricted to $X \dot\cup Y$. By *(V2)* of Lemma 11 and *(c)* of Lemma 12, G and f fulfil the requirements of Lemma 10. Moreover, since $\Delta(B) \le \Delta(H) \le \Delta$

$$|X| + |Y| = |V(B)| \le (\Delta+1)|X| \le (\Delta+1)k\xi n \overset{(2)}{\le} \varepsilon\frac{n}{k} \tag{4}$$

by *(a)* of Lemma 12. Accordingly, since $\varepsilon \le \varepsilon_{\mathrm{PEL}}$ we can apply Lemma 10 for obtaining an embedding g of the vertices in X, and for every $y \in Y$ sets C_y such that

$$C_y \subseteq V_{f(y)} \setminus g(X) \qquad \text{and} \qquad |C_y| \ge c|V_{f(y)}| \ge c|V_{f(y)} \setminus g(X)|\,.$$

The sets C_y will be used in the blow-up lemma for the image restriction of the vertices in $Y = N_H(X)$. We first check that there are not too many of these restrictions. Let $W_i' := W_i \setminus X$, $V_i' := V_i \setminus g(X)$ and $n_i' := |W_i'| = |V_i'|$ for each $i \in [k]$. Observe that

$$|X| + |Y| \overset{(4)}{\le} (\Delta+1)k\xi n \overset{(2)}{\le} \frac{\alpha_{\mathrm{BL}}}{2k} n \overset{(3)}{\le} \alpha_{\mathrm{BL}} n_i,$$

and hence

$$|N_H(X)| = |Y| \le \alpha_{\mathrm{BL}} n_i - |X| \le \alpha_{\mathrm{BL}}(n_i - |X|) \le \alpha_{\mathrm{BL}} n_i'.$$

For any $j \in [k/3]$ we apply the blow-up lemma, Lemma 9, and find an embedding of $H[W'_{3(j-1)+1}, W'_{3(j-1)+2}, W'_{3(j-1)+3}]$ into $G[V'_{3(j-1)+1}, V'_{3(j-1)+2}, V'_{3(j-1)+3}]$ in such a way that every $y \in N_H(X)$ will be embedded into C_y. It is easy to check the the respective conditions are satisfied. Indeed, recall that by (*V3*) the pair $(V_{3(j-1)+l}, V_{3(j-1)+l'})$ is (ε, d)-super-regular and that $V'_i = V_i \setminus g(X)$ for every $i \in [k]$. Hence it follows directly from the definition of a super-regular pair and (3), (4), and $\varepsilon \leq d/4$, that $(V'_{3(j-1)+l}, V'_{3(j-1)+l'})$ is $(2\varepsilon, d/2)$-super-regular with $\varepsilon \leq \varepsilon_{\mathrm{BL}}/2$ (see (1)).

Having applied the blow-up lemma for every $j \in [k/3]$, we have obtained a bijection

$$h\colon\ W'_1\dot\cup\cdots\dot\cup W'_k \to V'_1\dot\cup\cdots\dot\cup V'_k \quad \text{with} \quad h(W'_i) = V'_i \text{ for every } i \in [k]$$

such that

$$\begin{aligned} h(y) \in C_y \quad &\text{for every } y \in N_H(X) \\ \text{and} \quad H[W'_1\dot\cup\cdots\dot\cup W'_k] &\subseteq G[h(W'_1)\dot\cup\cdots\dot\cup h(W'_k)]. \end{aligned} \tag{5}$$

Now we finish the proof by checking that the united embedding $\bar{h}\colon\ V(H) \to V(G)$ defined by

$$v \mapsto \bar{h}(v) := \begin{cases} h(v) & \text{if } v \in V(H) \setminus X \\ g(v) & \text{if } v \in X \end{cases}$$

is indeed an embedding of H into G. Let $e = \{u, v\}$ be an edge of H. We distinguish three cases.

If $u, v \in X$, then $\{\bar{h}(u), \bar{h}(v)\} = \{g(u), g(v)\}$, which is an edge in G since g is an embedding of $H[X]$ into G by the partial embedding lemma.

If $u \in X$ and $v \in V(H) \setminus X$, then $v \in N_H(u) \subseteq N_H(X)$, so we have $h(v) \in C_v \subseteq N_G(g(u))$ by (5) and part (*ii*) of Lemma 10, thus $\{\bar{h}(u), \bar{h}(v)\} = \{g(u), h(v)\} \in E(G)$.

If, finally, $u, v \in V(H)\setminus X$, then by part (*d*) of Lemma 12, $\{f(u), f(v)\} \in E(R^{**}_k)$. In other words, there exists a $j \in [k/3]$, such that

$$\{u, v\} \quad \text{is contained in} \quad H[W'_{3(j-1)+1}, W'_{3(j-1)+2}, W'_{3(j-1)+3}]$$

and hence $\{\bar{h}(u), \bar{h}(v)\} = \{h(u), h(v)\} \in E(G)$ by (5). □

Algorithmic embeddings. We note that the proof of Theorem 2 presented above yields an algorithm, which finds an embedding of H into G, if H is given along with a valid 3-colouring and a labelling of the vertices respecting the bandwidth bound βn. This follows from the observation that the proof above is constructive, and all the lemmas used in the proof (Theorem 9, Lemma 10, Lemma 11, and Lemma 12) have algorithmic proofs. Algorithmic versions of the blow-up lemma, Theorem 9, were obtained in [25, 35]. In [25] a running time of order $O(\max\{n_1, n_2, n_3\}^{3.376})$ was proved. The key ingredient of Lemma 11 is Szemerédi's regularity lemma for which a $O(n^{2.376})$ algorithm exists due to [4]. All other arguments in the proof of Lemma 11 can be done algorithmically in $O(n^2)$ (see Section 5). Similarly, the proof of Lemma 12 is constructive if a 3-colouring of H and a bandwidth ordering is given (see Section 6). Finally, we note that the proof of Lemma 10 (following along the lines of [8]) gives rise to a $O(n^3)$ algorithm. Thus there is a $O(k \times ((1/k+\xi_0)n)^{3.376} + n^{2.376} + n^3) = O(n^{3.376})$ embedding algorithm, where the implicit constant depends on γ and Δ only.

5. Lemma for G

The main ingredients for the proof of Lemma 11 are Szemerédi's regularity lemma which provides a reduced graph R_k for G, Theorem 1 which guarantees the square of a Hamiltonian cycle in R_k, and a strategy for moving vertices between the clusters of R_k in order to adjust the sizes of these clusters. We first prove Lemma 11 for the special case that $n_i = m_i$ for all $i \in [k]$.

Proposition 13. *For all $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there exists K_0 such that for all $n \geq K_0$ and for every graph G on vertex set $[n]$ with $\delta(G) \geq (2/3 + \gamma)n$ there exists $k \in \mathbb{N} \setminus \{0\}$, and a graph R_k on vertex set $[k]$ with*

- (*R1*) $k \leq K_0$ *and* $3|k$,
- (*R2*) $\delta(R_k) \geq (2/3 + \gamma/2)k$,
- (*R3*) $R_k^{**} \subseteq R_k^* \subseteq R_k$, *and*
- (*R4*) *there is an equitriangular integer partition $m_1 + \cdots + m_k$ of n with $m_i \geq (1 - \varepsilon)n/k$ such that the following holds.*

There is a partition $U_1 \dot\cup \cdots \dot\cup U_k = V$ with

- (*U1*) $|U_i| = m_i$,
- (*U2*) $U_1 \dot\cup \cdots \dot\cup U_k$ *is (ε, d)-regular on R_k*,
- (*U3*) $U_1 \dot\cup \cdots \dot\cup U_k$ *is (ε, d)-super-regular on R_k^{**}.*

Notice that once we have Proposition 13, the only thing that is left to be done when proving Lemma 11 is to show that the sizes of the classes U_i can be slightly changed from m_i to n_i without "destroying" properties (*U2*) and (*U3*).

In the proof of Proposition 13 we proceed in three steps. From the regularity lemma we first obtain a partition $U_0' \dot\cup U_1' \dot\cup \cdots \dot\cup U_k'$ of $V(G)$ with reduced graph R_k such that $R_k^{**} \subseteq R_k^* \subseteq R_k$. We then use Proposition 7 to get a new partition $U_0'' \dot\cup U_1'' \dot\cup \cdots \dot\cup U_k''$ that is super-regular on R_k^{**} (and still regular on R_k). In a last step we distribute the vertices in U_0'' to the sets U_i'' with $i \in [k]$, while maintaining the super-regularity. The partition obtained in this way will be the desired equitriangular partition $U_1 \dot\cup \cdots \dot\cup U_k$.

Proof of Proposition 13. We first fix all constants necessary for the proof. Let $\gamma > 0$ be given. The regularity lemma in form of Corollary 4 applied with $\gamma' = \gamma/2$ yields positive constants d' and ε_0'. We fix the promised constants d and ε_0 for Proposition 13 by setting

$$d := \min\left\{\frac{d'}{3}, \gamma\right\} \quad \text{and} \quad \varepsilon_0 := \varepsilon_0' \,. \tag{6}$$

Now let some positive $\varepsilon \leq \varepsilon_0$ be given, for which Proposition 13 asks us to define K_0. For that let k_0 be sufficiently large so that we can apply Theorem 1 to graphs R_k on $k \geq k_0$ vertices with minimum degree $\delta(R_k) \geq (2/3 + \gamma/2)k$. We then define some auxiliary constants ε' and k_0' by

$$\varepsilon' := \min\left\{\frac{\varepsilon^4}{12^4}, \frac{(d')^2}{12^2}, \frac{\gamma^2}{24^2}, \frac{1}{8}\right\} \quad \text{and} \quad k_0' := \max\left\{k_0, \frac{8}{\gamma}, \frac{2}{\varepsilon'}\right\} + 2\,. \tag{7}$$

Let K_0' be given by Corollary 4 applied with γ', ε', and k_0'. We finally set $K_0 := K_0'$ for Proposition 13. After we have defined K_0, let $G = (V, E)$ be a graph satisfying the assumptions of Proposition 13.

Since $\varepsilon' \le \varepsilon \le \varepsilon_0 = \varepsilon'_0$, by the choice of ε'_0 and d', Corollary 4 applied with input γ', ε', k'_0 and $c' := 2/3 + \gamma/2$ yields a partition $U'_0 \dot\cup U'_1 \dot\cup \cdots \dot\cup U'_{k'} = V$ and a subgraph G' so that properties (i)–(vi) of Theorem 3 and (vii) from Corollary 4 hold. In particular, $k'_0 \le k' \le K'_0$, the set U'_0 is the exceptional set and there is a reduced graph $\tilde{R}_{k'}$ such that $U'_1 \dot\cup \cdots \dot\cup U'_{k'}$ is (ε', d')-regular on $\tilde{R}_{k'}$ and such that $\delta(\tilde{R}_{k'}) \ge (2/3 + \gamma/2 + \gamma/4)k'$.

Let $L' := |U'_1| = \cdots = |U'_{k'}|$ and note that $|U'_0| \le \varepsilon' n$ implies that

$$(1 - \varepsilon')n/k' \le |L'| \le n/k'. \tag{8}$$

Let $k := 3 \cdot \lfloor k'/3 \rfloor$ and R_k be the graph induced by the vertices $[k]$ in $\tilde{R}_{k'}$. Observe, that $k \le k' \le K'_0 = K_0$ and that 3 divides k. Therefore R_k satisfies property $(R1)$ of Proposition 13. Moreover, R_k is a reduced graph for $G[U'_1 \dot\cup \cdots \dot\cup U'_k]$ with

$$|V(R_k)| = k \ge k' - 2 \ge k'_0 - 2 \overset{(7)}{\ge} k_0 \tag{9}$$

and

$$\delta(R_k) \ge \delta(\tilde{R}_{k'}) - 2 \ge (2/3 + \gamma/2 + \gamma/4)k' - 2 \overset{(7)}{\ge} (2/3 + \gamma/2)k.$$

Thus, we also have property $(R2)$. By (9) and the choice of k_0, Theorem 1 implies that $R^*_k \subseteq R_k$. Moreover, $R^{**}_k \subseteq R^*_k$ since $3|k$ and thus we get $(R3)$.

Proposition 7 applied with $R' := R^{**}_k$ and accordingly $\Delta(R') = 2$ asserts that for every $i \in [k]$ there are subsets U''_i of U'_i of size

$$L'' := |U''_1| = \cdots = |U''_k| = (1 - 2\varepsilon')L'\,,$$

such that $U''_1 \dot\cup \cdots \dot\cup U''_k$ is $(\varepsilon'/(1-2\varepsilon'), d' - 2\varepsilon')$-regular on R_k, and $(\varepsilon'/(1-2\varepsilon'), d' - 2\varepsilon')$-super-regular on R^{**}_k. By (7) we have $\varepsilon'/(1-2\varepsilon') \le 2\varepsilon'$ and $d' - 2\varepsilon' \ge d'/2$. This implies that $U''_1 \dot\cup \cdots \dot\cup U''_k$ is $(2\varepsilon', d'/2)$-regular on R_k, and $(2\varepsilon', d'/2)$-super-regular on R^{**}_k. Moreover,

$$\begin{aligned} \frac{n}{k} \ge L'' = (1 - 2\varepsilon')L' &\overset{(8)}{\ge} (1 - 2\varepsilon')(1 - \varepsilon')\frac{n}{k'} \ge (1 - 3\varepsilon')\frac{n}{k+2} \\ &\overset{(7)}{\ge} (1 - 3\varepsilon')\frac{n}{k + \varepsilon' k} = \left(\frac{1 - 3\varepsilon'}{1 + \varepsilon'}\right)\frac{n}{k} \ge (1 - 4\varepsilon')\frac{n}{k}. \end{aligned} \tag{10}$$

Now we collect all vertices from V not contained in $U''_1 \dot\cup \cdots \dot\cup U''_k$ in a set U''_0, i.e., let

$$U''_0 := V \setminus \bigcup_{i \in [k]} U''_i.$$

It follows that

$$|U''_0| = n - \sum_{i \in [k]} |U''_i| \overset{(10)}{\le} n - k(1 - 4\varepsilon')n/k = 4\varepsilon' n. \tag{11}$$

In order to obtain the required partition of V with clusters U_i for $i \in [k]$ we will distribute the vertices in U''_0 to the clusters U''_i so that the resulting partition is equitriangular and still (ε, d)-regular on R_k and (ε, d)-super-regular on R^{**}_k.

For this purpose, let u be a vertex in U''_0. A triangle $i, i+1, i+2$ of R^{**}_k is called *u-friendly*, if u has at least dn/k neighbours in each of the clusters U''_i, U''_{i+1}, and U''_{i+2}. We claim that each $u \in U''_0$ has at least $\gamma k/3$ u-friendly triangles. Indeed, assume for a contradiction that there were only $x < \gamma k/3$ u-friendly triangles for some u. Then, since u has less than $2L''+dn/k$ neighbours in clusters of triangles that are not u-friendly, we can argue that

$$|N_G(u)| < x\cdot 3L'' + \left(\frac{k}{3}-x\right)\left(2L''+\frac{dn}{k}\right) + |U_0''| \le xL'' + \frac{2k}{3}L'' + \frac{d}{3}n + 4\varepsilon' n$$
$$\overset{(10)}{<} \frac{\gamma k}{3}\frac{n}{k} + \frac{2k}{3}\frac{n}{k} + \frac{d}{3}n + 4\varepsilon' n \overset{(6),(7)}{\le} \left(\frac{2}{3}+\gamma\right)n,$$

which is a contradiction.

In a first step we now assign the vertices $u \in U_0''$ as evenly as possible to u-friendly triangles in R_k^{**}. Since each vertex $u \in U_0''$ has at least $\gamma k/3$ u-friendly triangles, each triangle of R_k^{**} gets assigned at most $3|U_0''|/(\gamma k)$ vertices.

Then in the second step, in each triangle we distribute the vertices that have been assigned to this triangle as evenly as possible among the three clusters of this triangle. It follows immediately that the resulting partition is equitriangular. Moreover, every cluster U_i'' with $i \in [k]$ gains at most

$$\frac{3|U_0''|}{\gamma k} \overset{(11)}{\le} \frac{12\varepsilon' n}{\gamma k} \overset{(10)}{\le} \frac{12\varepsilon'}{\gamma(1-4\varepsilon')}L'' \overset{(7)}{\le} 24\frac{\varepsilon'}{\gamma}|U_i''| \overset{(6),(7)}{\le} \sqrt{\varepsilon'}|U_i''| \tag{12}$$

vertices from U_0'' during this process. We claim that the resulting partition $U_1\dot\cup\cdots\dot\cup U_k$ of V satisfies properties (*U1*)–(*U3*). For that we first define

$$m_i := |U_i| \ge |U_i''| = L'' \overset{(10)}{\ge} (1-4\varepsilon')n/k \ge (1-\varepsilon)n/k,$$

and note that for this choice (*R4*) and (*U1*) of Proposition 13 hold. Moreover, recall that $U_1''\dot\cup\cdots\dot\cup U_k''$ is $(2\varepsilon', d'/2)$-regular on R_k and $(2\varepsilon', d'/2)$-super-regular on R_k^{**}. By (12), Proposition 8 with $\hat\alpha = \hat\beta = \sqrt{\varepsilon'}$ assures that $U_1\dot\cup\cdots\dot\cup U_k$ is $(\hat\varepsilon, \hat d)$-regular on R_k and $(\hat\varepsilon, \hat d)$-super-regular on R_k^{**}, where

$$\hat\varepsilon := 2\varepsilon' + 6\sqrt[4]{\varepsilon'} \quad \text{and} \quad \hat d := \frac{d'}{2} - 2\sqrt{\varepsilon'}.$$

Since $2\varepsilon' + 6\sqrt[4]{\varepsilon'} \le \varepsilon$ and $d'/2 - 2\sqrt{\varepsilon'} \ge d'/3 \ge d$ by (6) and (7), this implies (*U2*) and (*U3*) and concludes the proof of Proposition 13. □

Next we deduce the lemma for G (Lemma 11) from Proposition 13.

Proof of Lemma 11. Again we first fix the constants involved in the proof. Let $\gamma > 0$ be given by Lemma 11. For γ, Proposition 13 yields constants $d' > 0$ and $\varepsilon_0' > 0$. For Lemma 11 we set

$$\varepsilon_0 := \min\{\varepsilon_0', d'/8\} \quad \text{and} \quad d := d'/2\,. \tag{13}$$

For given $\varepsilon \le \varepsilon_0$, we fix

$$\varepsilon' := \min\left\{\frac{\varepsilon}{6\sqrt{2}}, \sqrt{\frac{d}{8}}\right\} \tag{14}$$

and note that $0 < \varepsilon' \le \varepsilon \le \varepsilon_0 \le \varepsilon_0'$. Therefore we can apply Proposition 13 with γ and ε' to obtain K_0'. Finally, we define the constants K_0 and ξ_0 promised by Lemma 11 and set

$$K_0 := K_0' \qquad \text{and} \qquad \xi_0 := \left(\frac{\varepsilon'}{2K_0}\right)^2. \tag{15}$$

Having fixed all the constants, let $G = (V, E)$ be a graph on $n \ge K_0$ vertices. We now apply Proposition 13 with γ and ε' to the input graph G and get a positive integer $k \le K_0'$, a graph R_k, and a partition $U_1\dot\cup\dots\dot\cup U_k = V$ so that (*R1*)–(*R4*) and (*U1*)–(*U3*) of Proposition 13 hold with ε replaced by ε' and d replaced by d'. Since $K_0 = K_0'$ and $\varepsilon \ge \varepsilon'$, this shows that k, R_k, and $m_i = |U_i|$ for all $i \in [k]$ also satisfy properties (*R1*)–(*R4*) of Lemma 11.

It remains to prove the 'second part' of Lemma 11. For that let $n_1 + \dots + n_k$ be an integer partition of $n = |V|$ satisfying $n_i = m_i \pm \xi_0 n$ for every $i \in [k]$. Our goal is to modify the partition $U_1\dot\cup\dots\dot\cup U_k = V$ to obtain a partition $V_1\dot\cup\dots\dot\cup V_k = V$ that satisfies (*V1*)–(*V3*) for ε and d.

The problem that occurs here is the following. Although a pair remains almost as *regular* as before when a few vertices leave or enter a cluster, the property of being *super-regular* is not that robust: *every* vertex that is moved to a new cluster which is part of a super-regular triangle must make sure that it has sufficiently many neighbours inside the neighbouring clusters within the triangle.

We first set $V_i := U_i$ for all $i \in [k]$. In the following, we will perform several steps to move vertices out of some clusters and into some other clusters. During this process we will call a cluster V_i *deficient*, if $|V_i| < n_i$, and *excessive*, if $|V_i| > n_i$. In the end we will neither have deficient clusters nor excessive clusters and thus obtain the desired partition.

In the following the cyclic structure of R_k^* will be important. To simplify the arguments, we will therefore allow the index i of a cluster V_i to become negative or bigger than k. Thus V_0 will denote cluster V_k, V_{-1} cluster V_{k-1}, and V_{k+1} cluster V_1, and so on.

Note that $\sigma\colon [k] \to [3]$ with

$$\sigma(3j + l) := l \text{ for } j \in \{0, \dots, (k/3) - 1\} \text{ and } l \in [3]$$

is a valid 3-colouring of R_k^*. We will also say that cluster V_i *has colour* $\sigma(i)$.

The following facts will allow us to balance deficient and excessive clusters. The first observation will be useful to address imbalances within clusters of colour 1 or 3.

Fact 14. *Suppose that $|V_i| \ge (1 - \varepsilon)n/k$ for all $i \in [k]$, and that $V_1\dot\cup\dots\dot\cup V_k$ is (ε, d)-regular on R_k. Then, for each $j \in \{0, \dots, (k/3) - 1\}$, there are at least $(1-3\varepsilon)n/k$ "good" vertices $v \in V_{3j+1}$ that have at least $dn/(2k)$ neighbours in each of $V_{3(j-1)+2}$ and $V_{3(j-1)+3}$. Similarly, there are at least $(1-3\varepsilon)n/k$ "good" vertices $v \in V_{3j+3}$ that have at least $dn/(2k)$ neighbours in each of $V_{3(j+1)+1}$ and $V_{3(j+1)+2}$.*

Proof of Fact 14. Note that the four pairs

$$\{3j+1, 3(j-1)+2\}, \{3j+1, 3(j-1)+3\}, \{3j+3, 3(j+1)+1\}, \{3j+3, 3(j+1)+2\}$$

are all edges of R_k^*. Since $V_1\dot\cup\dots\dot\cup V_k$ is (ε, d)-regular on R_k^* we can apply Proposition 6 once with input ε, d, $A = V_{3j+1}$, and $B = B' = V_{3(j-1)+2}$ and once with input ε, d, $A = V_{3j+1}$, and $B = B' = V_{3(j-1)+3}$. This asserts that at least $|V_{3j+1}| - 2\varepsilon|V_{3j+1}|$ vertices of V_{3j+1} have more than $(d-\varepsilon)|V_{3(j-1)+2}|$ neighbours in $V_{3(j-1)+2}$ and more than $(d-\varepsilon)|V_{3(j-1)+2}|$ neighbours in $V_{3(j-1)+3}$. This implies the first part of Fact 14, because

$$|V_{3j+1}| - 2\varepsilon|V_{3j+1}| \ge (1-2\varepsilon)(1-\varepsilon)\frac{n}{k} \ge (1-3\varepsilon)\frac{n}{k} \tag{16}$$

and

$$(d-\varepsilon)|V_{3(j-1)+2}| \ge (d-\varepsilon)(1-\varepsilon)\frac{n}{k} \ge (d-2\varepsilon)\frac{n}{k} \overset{(13)}{\ge} \frac{dn}{2k}.$$

The second part concerning vertices in V_{3j+3} follows analogously. □

Before we continue, let us briefly illustrate how Fact 14 is used later. Suppose that for some $j, j' \in \{0, \ldots, (k/3) - 1\}$ the set V_{3j+1} is an excessive cluster and $V_{3j'+1}$ is a deficient cluster, both of colour $\sigma(3j+1) = 1$. Then by Fact 14 there is some vertex v (in fact $(1-3\varepsilon)n/k$ vertices) in V_{3j+1} which has "many" neighbours in $V_{3(j-1)+2}$ and $V_{3(j-1)+3}$. Hence, we move v from V_{3j+1} to $V_{3(j-1)+1}$ without loosing the super-regularity of the resulting partition on R_k^{**}, nor the regularity on R_k^*. Recall that R_k^* was the square of a Hamiltonian *cycle*. Hence, repeating this process by moving a vertex from $V_{3(j-1)+1}$ to $V_{3(j-2)+1}$ and so on, we will eventually reach $V_{3j'+1}$. Observe that it is of course not necessarily the vertex $v \in V_{3j+1}$ we started with, which is really moved all the way to $V_{3j'+1}$ during this process, but rather a sequence of vertices each moving one cluster further. The crucial thing to note is that whenever we move a vertex from one cluster to another, it still has many neighbours in the new neighbouring clusters within R_k^{**}. Therefore, after such a sequence of applications of Fact 14, we end up with a new partition $V_1 \dot\cup \cdots \dot\cup V_k$ with the following properties. The cardinality of V_{3j+1} decreased by one and $|V_{3j'+1}|$ increased by one. For all $i \in [k]$ different from $3j+1$ and $3j'+1$ the size of V_i remains the same. We say then that we *moved a vertex along colour class* 1 *of* R_k^* *from* V_{3j+1} *to* $V_{3j'+1}$ and if, as assumed above, V_{3j+1} was excessive and $V_{3j'+1}$ was deficient, then such a move decreases the imbalances within clusters of colour 1. Similarly, we can apply the second part of Fact 14, for moving vertices along colour class 3 of R_k^*.

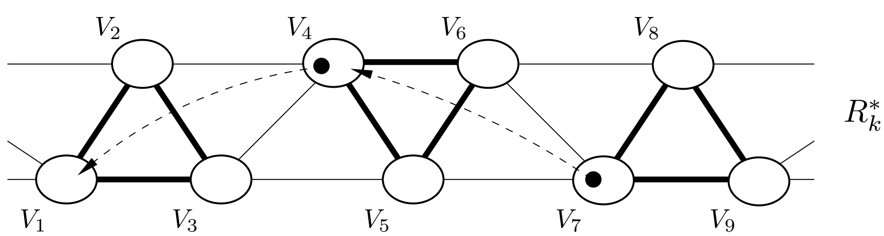

FIGURE 1. Moving a vertex from V_7 to V_1 along colour class 1 of R_k^* and thus decreasing the size of V_7 and increasing the size of V_1.

The clusters of colour 2 however need special treatment. Consider e.g. V_{3j+2}. Unfortunately we have no other vertex in R_k^* that is adjacent to $3j+1$ and $3j+3$. Hence vertices cannot be moved analogously along colour class 2 of R_k^*.

Therefore the following observation will be useful, which will allow us to deal with deficient clusters V_{3j+2} of colour 2.

Fact 15. *Suppose that* $|V_i| \geq (1-\varepsilon)n/k$ *for all* $i \in [k]$, *and that* $V_1 \dot\cup \cdots \dot\cup V_k$ *is* (ε, d)*-regular on* R_k. *For each* $j \in \{0, \ldots, (k/3)-1\}$ *there is an* $i \in [k]$ *with* $\sigma(i) \neq 2$, *such that* $3j+1, 3j+3 \in N_{R_k}(i)$ *and there are at least* $(1-3\varepsilon)n/k$ *"good" vertices* $v \in V_i$ *that have at least* $dn/(2k)$ *neighbours in each of* V_{3j+1} *and* V_{3j+3}.

Proof of Fact 15. Since $\delta(R_k) \geq (2/3 + \gamma/2)k$, the joint neighbourhood of $3j+1$ and $3j+3$ has size at least $(1/3+\gamma)k > k/3$. Hence, there must be a joint neighbour which is not of colour 2, and therefore i can be chosen. The existence of the vertices v follows as in the proof of Fact 14. □

This fact will be used for moving a vertex v from a cluster of V_i of colour 1 or 3 to a deficient cluster V_{3j+2} (of colour 2) for some $j \in \{0,\dots,(k/3)-1\}$.

The last simple fact allows to address imbalances across different colours. More precisely, it will be used for moving a vertex v from cluster V_i to any of the clusters V_{3j+1}, V_{3j+2}, or V_{3j+3}.

Fact 16. *Suppose that $|V_i| \geq (1-\varepsilon)n/k$ for all $i \in [k]$, and that $V_1\dot\cup\cdots\dot\cup V_k$ is (ε,d)-regular on R_k. For each $i \in [k]$ there is a $j \in \{0,\dots,(k/3)-1\}$, such that $3j+1, 3j+2, 3j+3 \in N_{R_k}(i)$ and there are at least $(1-4\varepsilon)n/k$ "good" vertices $v \in V_i$ that have at least $dn/(2k)$ neighbours in each of V_{3j+1}, V_{3j+2}, and V_{3j+3}.*

Proof of Fact 16. Since $\delta(R_k) \geq (2/3+\gamma/2)k > 2k/3$, there must be at least one triangle $3j+1, 3j+2, 3j+3$ in R_k^{**} such that all three vertices of this triangle are adjacent to i in R_k. The existence of the vertices v follows similar as in the proof of Fact 14. Indeed, by Proposition 6, there are at least

$$|V_i| - 3\varepsilon|V_i| \geq (1-3\varepsilon)(1-\varepsilon)n/k \geq (1-4\varepsilon)n/k$$

such vertices (cf. (16)). □

Now, we are ready to describe the process for eliminating deficient and excessive clusters. In a first phase, we deal with the deficient clusters of colour 2. One iteration of this phase is as follows. Let V_{3j+2} with $j \in \{0,\dots,(k/3)-1\}$ be such a cluster. By Fact 15, there is an $i \in [k]$ with $\sigma(i) \neq 2$ such that we can move a vertex from V_i to V_{3j+2}. We repeat this step, until no deficient cluster of colour 2 remains.

An iteration of the second phase performs the following steps. Choose an arbitrary excessive cluster V_i and a deficient cluster $V_{i'}$. Note that there are deficient clusters as long as there are excessive clusters by definition, and vice versa. Note further, that $\sigma(i') \neq 2$ by phase one. We distinguish two cases.

If $\sigma(i) = \sigma(i')$ and, hence, $\sigma(i') \neq 2$, we use Fact 14 for moving a vertex along colour class $\sigma(i)$ of R_k^* from cluster V_i to cluster $V_{i'}$.

Otherwise, we first apply Fact 16 to cluster V_i, which gives us a $j \in \{0,\dots,(k/3)-1\}$, so that we can move a vertex from cluster V_i to $V_{3j+\sigma(i')}$. Then, we can proceed as in the previous case and move a vertex along colour class $\sigma(i')$ of R_k^* from cluster $V_{3j+\sigma(i')}$ to $V_{i'}$ with Fact 14.

In total we have to move at most

$$\sum_{i=1}^{k} |n_i - m_i| \leq k\xi_0 n$$

vertices in order to guarantee that $|V_i| = n_i$, hence at most $k\xi_0 n$ iterations have to be performed in the first phase and at most $k\xi_0 n$ in the second phase. Moreover, in each iteration not more than one vertex gets moved out of each V_i with $i \in [k]$, and at most one vertex gets moved into each V_i. So, throughout the process we have

$$|U_i \triangle V_i| \leq 2\cdot 2k\xi_0 n \overset{(15)}{\leq} (\varepsilon')^2 n/k, \tag{17}$$

for all $i \in [k]$.

Note that since by (13) we have $(1-4\varepsilon)n/k \geq \varepsilon'^2 n/k$, in every step of phase one and two the "moving" vertex v can be chosen from the set of $(1-4\varepsilon)n/k$ "good" vertices guaranteed by Facts 14–16.

In addition it follows that

$$|V_i| \geq |U_i| - |U_i \triangle V_i| \overset{(R4),(17)}{\geq} \left(1 - \varepsilon' - (\varepsilon')^2\right) \frac{n}{k} \overset{(14)}{\geq} (1-\varepsilon)\frac{n}{k} \tag{18}$$

after phase one and two for all $i \in [k]$. Recall that $U_1 \dot\cup \cdots \dot\cup U_k$ is (ε', d')-regular on R_k^* and (ε', d')-super-regular on R_k^{**}. Therefore, we can apply Proposition 8 with input ε', d', $A = U_i$, $\hat{A} := V_i$, and $B := U_{i'}$, $\hat{B} := V_{i'}$ for any $\{i, i'\} \in E(R_k)$. For this, we set

$$\hat{\alpha} := \hat{\beta} := 2(\varepsilon')^2 \geq \frac{(\varepsilon')^2}{1-\varepsilon} \overset{(18),(17)}{\geq} \frac{|U_i \triangle V_i|}{|V_i|}. \tag{19}$$

Since

$$\hat{\varepsilon} = \varepsilon' + 3\left(\sqrt{\hat{\alpha}} + \sqrt{\hat{\beta}}\right) \overset{(19)}{=} \varepsilon' + 6\sqrt{2}\varepsilon' \overset{(14)}{\leq} \varepsilon$$

and

$$\hat{d} = d' - 2(\hat{\alpha} - \hat{\beta}) \overset{(19)}{=} d' - 8(\varepsilon')^2 \overset{(13),(14)}{\geq} d.$$

we deduce from Proposition 8 that $V_1 \dot\cup \cdots \dot\cup V_k$ remains (ε, d)-regular on R_k^* and, since we only moved "good" vertices, $V_1 \dot\cup \cdots \dot\cup V_k$ remains (ε, d)-super-regular on R_k^{**} throughout the entire process.

This also justifies that we could indeed apply Facts 14, 15, and 16 throughout the entire process. Therefore $V_1 \dot\cup \cdots \dot\cup V_k$ satisfies (*V1*)–(*V3*) and this concludes the proof of Lemma 11. □

6. Lemma for H

In order to prove the lemma for H (Lemma 12), we need to exhibit a mapping $f\colon V(H) \to [k]$ with properties (*a*)–(*d*). Basically, we would like to use the fact that H is 3-colourable, visit the vertices of H in bandwidth order and arrange that f maps the first vertices of colour 1 to 1, the first vertices of colour 2 to 2, and the first vertices of colour 3 to 3. It would be ideal if, at more or the less same moment, we would have dealt with m_1 vertices of colour 1, m_2 vertices of colour 2 and m_3 vertices of colour 3, since we could then move and let f assign vertices to 4, 5 and 6.

Now the problem is that the m_i are equitriangular, i.e., almost identical, but the colour classes of H may vary a lot in size. Therefore, our first step towards the proof of Lemma 12 will be to show that we can find a recolouring of H with more or less balanced colour classes. We cut H into pieces of length ξn and find a 3-colouring for each of these pieces, such that for all i the *largest* colour class of the union of pieces 1 to i has the same colour as the *smallest* colour class of the $(i+1)$-st piece, and vice versa. In order to glue these colourings together and obtain a proper colouring of the whole graph H, we need to assign the new colour 0 to some of the vertices. We start with three simple observations, which will be helpful later in the proof of Lemma 12.

Observation 17. *Let a, a', b, b', c, c' and x be positive integers with $a \leq b \leq c \leq a + x$ and $c' \leq b' \leq a' \leq c' + x$. If we set $A := a + a'$, $B := b + b'$, and $C := c + c'$, then* $\max\{A, B, C\} \leq \min\{A, B, C\} + x$.

Proof. Indeed, $a + a' \leq b + c' + x$ and therefore $A \leq b + b' + x = B + x$ and $A \leq c + c' + x = C + x$. Similarly, $B \leq a + x + a' = A + x$, $B \leq c + c' + x = C + x$, $C \leq b + x + b' = B + x$, and $C \leq a + x + a' = A + x$. □

We say that a graph H on vertex set $[n]$ with bandwidth at most b is given *in bandwidth order*, if the vertex labels $1,\dots,n$ satisfy that for every edge $\{i,j\}\in E(H)$ we have $|i-j|\le b$.

Observation 18. *Let H be a 3-colourable graph on vertex set $[n]$ with bandwidth at most βn and suppose that the vertices are in bandwidth order. Let $s\in[n]$ and suppose $\sigma\colon [n]\to\{0,\dots,3\}$ is a proper 4-colouring of $V(H)$ such that $\sigma(u)\neq 0$ for all vertices $u>s-2\beta n$. Then for any two colours $l,l'\in[3]$ the mapping $\sigma'\colon [n]\to\{0,\dots,3\}$ defined by*

$$\sigma'(v):=\begin{cases} l' & \textit{if } \sigma(v)=l, v>s\\ l & \textit{if } \sigma(v)=l', v>s\\ \sigma(v) & \textit{otherwise}\end{cases}$$

can be turned into a proper 4*-colouring σ'' of H by colouring all vertices $w\in[s-\beta n,s+\beta n]$ satisfying $\sigma(w)=l$ with colour 0.*

We shall say that σ'' is obtained from σ by an (l,l')*-switch at vertex s*. Note that $\sigma''(u)\neq 0$ for all vertices $u\ge s+\beta n$.

Proof. Indeed, as σ' is derived from the proper colouring σ by interchanging the colours l and l' after the vertex s, the only monochromatic edges that σ' can possibly yield are edges $\{u,v\}$ with $u\le s$ and $s<v$ and $\{\sigma(u),\sigma(v)\}=\{l,l'\}$. Since H has bandwidth at most βn, we must have that $u\in[s-\beta n,s]$ and $v\in[s+1,s+\beta n]$.

Suppose now that we construct a new colouring σ'' obtained from σ' through recolouring all the vertices of colour l in the interval $[s-\beta n,s+\beta n]$ by colour 0. Thus all previous monochromatic edges have disappeared: If $\sigma(u)=l$ and $\sigma(v)=l'$, then $\sigma''(u)=0$ and $\sigma''(v)=l$. If $\sigma(u)=l'$ and $\sigma(v)=l$, then $\sigma''(u)=l'$ and $\sigma''(v)=0$. Moreover, the newly 0-coloured vertices cannot be adjacent to each other (because they were all assigned colour l by σ). Furthermore, by assumption for vertices $u\in[s-2\beta,s-\beta n]$ we have $\sigma(u)\neq 0$ and hence $\sigma''(u)\neq 0$. Therefore, due to the bandwidth assumption on H no new monochromatic edges of colour 0 can appear in σ'' and, hence, it is a proper 4-colouring. $\square$

The next observation is based on repeated applications of the two preceding facts. Roughly speaking, it states that 3-chromatic graphs H with small bandwidth can be 4-coloured, so that one colour is "very rare" (see (21)) and the other three colours appear "equally distributed" (see (20)). For the inductive proof we consider the following somewhat technical statement.

Observation 19. *Let H be a 3-colourable graph on vertex set $[n]$ with bandwidth at most βn and suppose that the vertices are in bandwidth order. Let ξ be a constant with $\beta<\xi/6$ and assume that $1/\xi$ is an integer. For all integers $i\in[1/\xi]$ there exists a proper 4-colouring $\sigma_i\colon [n]\to\{0,\dots,3\}$ of the vertices of H with the following properties. For all $j\in[i]$*

$$\max_{l\in[3]}\left\{|\sigma_i^{-1}(l)\cap[j\xi n]|\right\}\le\min_{l\in[3]}\left\{|\sigma_i^{-1}(l)\cap[j\xi n]|\right\}+\xi n+5j\beta n. \tag{20}$$

and

$$\sigma_i^{-1}(0)\subseteq\bigcup_{j\in[i-1]}[j\xi n,j\xi n+5\beta n] \tag{21}$$

Proof. We prove this statement by induction on i. Clearly, for $i = 1$, we let σ_1 be the proper 3-colouring of H. Then (20) holds trivially and no vertices of colour 0 are needed. Now suppose that σ_i is given. We will obtain σ_{i+1} from σ_i by appropriate (l, l')-switches at $i\xi n + \beta n$ and $i\xi n + 4\beta n$.

More precisely, suppose w.l.o.g. that the smallest colour class of σ_i on the first $i\xi n$ vertices of H is that of colour 1, and the largest is that of colour 3. Since every permutation of the set [3] can be written as the composition of at most two transpositions, there must be colours l_1, l_1', l_2, l_2' such that if we obtain σ_{i+1} from σ_i by an (l_1, l_1')-switch at $i\xi n + \beta n$ followed by an (l_2, l_2')-switch at $i\xi n + 4\beta n$, then the smallest colour class of σ_{i+1} on

$$I := [i\xi n + 5\beta n, (i + 1)\xi n]$$

is that of colour 3, and the largest is that of colour 1. Clearly, the assumptions of Observation 18 are satisfied before each of the switches, since before the first switch by induction assumption $\sigma_i(u) \neq 0$ for all $u > (i - 1)\xi n + 5\beta n$ and, hence, for all $u \geq i\xi n - \beta n$, as $\xi > 6\beta$. Similarly, after the first switch the largest vertex v of colour 0 obeys $v \leq i\xi n + 2\beta n$. It follows from Observation 18 that σ_{i+1} is a proper 4-colouring of H.

It is now easy to check that σ_{i+1} satisfies the requirements of the claim. Indeed, as $\sigma_{i+1}(v) = \sigma_i(v)$ for all $v \leq i\xi n$, we already know that (20) holds for all $j \in [i]$ and thus, by induction we now have

$$\begin{aligned}|\sigma_{i+1}^{-1}(1) \cap [i\xi n]| \leq |\sigma_{i+1}^{-1}(2) \cap [i\xi n]| &\leq |\sigma_{i+1}^{-1}(3) \cap [i\xi n]| \\ &\leq |\sigma_{i+1}^{-1}(1) \cap [i\xi n]| + \xi n + 5i\beta n.\end{aligned}$$

Since, trivially,

$$\begin{aligned}|\sigma_{i+1}^{-1}(3) \cap I| \leq |\sigma_{i+1}^{-1}(2) \cap I| &\leq |\sigma_{i+1}^{-1}(1) \cap I| \\ &\leq |\sigma_{i+1}^{-1}(3) \cap I| + \xi n \leq |\sigma_{i+1}^{-1}(3) \cap I| + \xi n + 5i\beta n,\end{aligned}$$

we can now apply Observation 17 to see that

$$\begin{aligned}&\max_{l\in[3]} \left\{|\sigma_{i+1}^{-1}(l) \cap [(i + 1\xi n]|\right\} \\ &\qquad \leq \min_{l\in[3]} \left\{|\sigma_{i+1}^{-1}(l) \cap [(i + 1)\xi n]|\right\} + \xi n + 5i\beta n + \big|[i\xi n, (i + 1)\xi n] \setminus I\big|,\end{aligned}$$

which implies equation (20) for $j = i + 1$ as well. Finally, we note that (21) follows directly from the induction assumption on σ_i and the definition of σ_{i+1}. □

In the following lemma we sum up what we have achieved so far. First note that (20) and (21) imply that for all $i \in [1/\xi]$, every $j \in [i]$, and $l \in [3]$

$$\frac{j(\xi - 5\beta)n}{3} - (\xi + 5j\beta)n \leq |\sigma_i^{-1}(l) \cap [j\xi n]| \leq \frac{j(\xi + 5\beta)n}{3} + (\xi + 5j\beta)n. \qquad (22)$$

In other words, the colourings σ_i use the colours $1, 2, 3$ almost evenly, at least if we consider intervals of the form $[j\xi n]$. Moreover, colour 0 is only used in certain relatively small intervals. The following definitions try to capture these features in a form that is convenient for the proof of Lemma 12.

For $x \in \mathbb{N}$, a colouring $\sigma\colon [n] \to \{0, \dots, 3\}$ is called *x-balanced*, if for each interval $[a, b] \subseteq [n]$ and each $l \in [3]$, we have

$$\frac{b - a}{3} - x \leq \left|\sigma^{-1}(l) \cap [a, b]\right| \leq \frac{b - a}{3} + x.$$

Moreover, σ is called *x-zero free*, if for each $t \in [n]$ there exists a $t' \in [n]$ with $t - 2x \le t' \le t + 2x$ such that $\sigma(u) \neq 0$ for all $u \in [t' - x, t' + x]$. We also say that the interval $[t' - x, t' + x]$ is *zero free*.

Lemma 20 (Balancing lemma)**.** *Let H be a 3-colourable graph on vertex set $[n]$ with bandwidth at most βn and suppose that the vertices are in bandwidth order. Let ξ be a constant with $\beta < \xi^2/10$ and assume that $1/\xi$ is an integer. Then there exists a proper 4-colouring $\sigma\colon V(H) \to \{0, \dots, 3\}$ that is $5\beta n$-zero free and $5\xi n$-balanced.*

Proof. Given β, let H and ξ be as required. We set $i := 1/\xi$ and claim that the colouring $\sigma = \sigma_i$ guaranteed by Observation 19 has the desired properties.

First it is easy to check that σ is indeed $5\beta n$-zero free because $5\beta < \xi$ and we know from (21) that the vertices of colour zero all lie in intervals of the form $[j\xi n, j\xi n + 5\beta n]$ with $j \in [1/\xi]$.

Second, observe that by Observation 19, properties (20) and (21) and, consequently, (22) hold for σ. Moreover, since $\beta \le \xi^2/10 < 3\xi^2/20$, we infer from (22) that for every $j \in [1/\xi]$

$$\frac{j\xi n}{3} - 2\xi n < |\sigma_i^{-1}(l) \cap [j\xi n]| < \frac{j\xi n}{3} + 2\xi n\,. \tag{23}$$

Now for an arbitrary interval $[a, b] \subseteq [n]$, we choose j, $j' \in [i]$ such that

$$a - \xi n \le j\xi n \le a \le b \le j'\xi n \le b + \xi n\,.$$

This yields that

$$|\sigma^{-1}(l) \cap [(j+1)\xi n, (j'-1)\xi n]| \le \ \left|\sigma^{-1}(l) \cap [a, b]\right| \ \le |\sigma^{-1}(l) \cap [j\xi n, j'\xi n]|.$$

The lower bound is equal to

$$\begin{aligned}
&|\sigma^{-1}(l) \cap [(j'-1)\xi n]| - |\sigma^{-1}(l) \cap [(j+1)\xi n)| \\
&\qquad\qquad \ge \left(\frac{(j'-1)\xi n}{3} - 2\xi n\right) - \left(\frac{(j+1)\xi n}{3} + 2\xi n + 1\right) \\
&\qquad\qquad \ge \left(\frac{b - \xi n}{3} - 2\xi n\right) - \left(\frac{a + \xi n}{3} + 2\xi n\right) - 1 \ge \frac{b-a}{3} - 5\xi n.
\end{aligned}$$

Similarly, the upper bound equals

$$\begin{aligned}
&|\sigma^{-1}(l) \cap [j'\xi n]| - |\sigma^{-1}(l) \cap [j\xi n)| \\
&\qquad\qquad \le \left(\frac{j'\xi n}{3} + 2\xi n\right) - \left(\frac{j\xi n}{3} - 2\xi n - 1\right) \\
&\qquad\qquad \le \left(\frac{b + \xi n}{3} + 2\xi n\right) - \left(\frac{a - \xi n}{3} - 2\xi n\right) + 1 \le \frac{b-a}{3} + 5\xi n.
\end{aligned}$$

Thus, σ is $5\xi n$-balanced. □

After these preparations, we are ready to prove the lemma for H, Lemma 12.

Proof of Lemma 12. Given k and β let ξ, R_k and H be as required, with $V(H) = [n]$ in bandwidth order. Set $\xi' = \xi/21$, and note that $\beta \le \xi^2/10^4 \le \xi'^2/10$. Therefore, by Lemma 20 with input β, ξ', and H, there is a $5\beta n$-zero free and $5\xi' n$-balanced colouring $\sigma\colon V(H) \to \{0, \dots, 3\}$ of H.

Observe that for each triple of vertices in R_k, the common neighbourhood of these vertices is nonempty, because $\delta(R_k) > 2k/3$. It follows that for each $j \in [k/3]$

there exists a vertex $r_j \in V(R_k)$ that is adjacent in R_k to each vertex of the j-th triangle of R_k^{**}. These vertices r_j will be needed to construct the mapping f.

Given an equitriangular partition $m_1, \dots, m_k$ of n set

$$M_j := m_{3(j-1)+1} + m_{3(j-1)+2} + m_{3(j-1)+1}$$

for $j \in [k/3]$. The aim now is to cut H into intervals of length approximately $M_1, \dots, M_{k/3}$ and then define f in such a way that it maps almost all vertices of the j-th interval to the j-th triangle of R_k^{**}.

For this purpose, set $t_0 := 0$ and $t_{k/3} := n$, and for every $j = 1, \dots, k/3 - 1$ choose a vertex

$$t_j \in \left[\sum_{j'=1}^{j} M_{j'} - 10\beta n, \sum_{j'=1}^{j} M_{j'} + 10\beta n \right]$$

such that σ is zero free on $[t_j - 5\beta n, t_j + 5\beta n]$.

Such a t_j indeed exists since σ is $5\beta n$-zero free. For a vertex $u \in V(H)$, let $j(u)$ be the index in $[k/3]$ for which $u \in [t_{j(u)-1}, t_{j(u)}]$. We say, that $(t_{j-1}, t_j]$ is the j-th interval of H. The first βn vertices of such an interval are called *early*, the last βn *late*. Early and late vertices are also called *untimely*, all other vertices are *timely*. Observe that the choice of the t_j implies that

($*$) untimely vertices are not assigned colour 0 by σ and they also have no neighbours of colour 0.

Using σ, we will now construct f and X. For each $j \in [k/3]$, and each $v \in (t_{j-1}, t_j]$ in the j-th interval of H we set

$$f(v) := \begin{cases} r_j & \text{if } \sigma(v) = 0, \\ 3j + 1 & \text{if } \sigma(v) = 1 \text{ and } v \text{ is late}, \\ 3(j-2) + 3 & \text{if } \sigma(v) = 3 \text{ and } v \text{ is early}, \\ 3(j-1) + \sigma(v) & \text{otherwise.} \end{cases}$$

Let further

$$X := \{v \in V(H) : \sigma(v) = 0\} \cup \{v \in V(H) : v \text{ untimely and } \sigma(v) \in \{1, 3\}\}.$$

It remains to show that f and X satisfy properties (a)–(d) of Lemma 12.

Since σ is $5\xi' n$-balanced, $(n/3) - 5\xi' n \leq |\sigma^{-1}(l)| \leq (n/3) + 5\xi' n$ for all $l \in [3]$. Consequently at most $15\xi' n$ vertices of of H receive colour 0. It follows, that

$$|X| \leq 15\xi' n + 2\beta n \frac{k}{3} \leq 16k\xi' n \leq k\xi n,$$

which shows (a).

For (b), observe that for each $i \in [k]$ with $i = 3(j-1) + l$, f maps all timely vertices v in the j-th interval of H with $\sigma(v) = l \in [3]$ to i. Since σ is $5\xi' n$-balanced and by the choice of t_{j-1} and t_j, there are at most $(M_j + 10\beta n)/3 + 5\xi' n \leq m_i + 4\beta n + 5\xi' n$ such vertices, and at least $m_i - 4\beta n - 5\xi' n$. In addition, some late vertices of the j-th and $(j-1)$-st interval, some early vertices of the j-th and $(j+1)$-st interval and some vertices of colour 0 might be mapped to i. It follows, that

$$|W_i| \leq m_i + 4\beta n + 5\xi' n + 4\beta n + 15\xi' n \leq m_i + 21\xi' n = m_i + \xi n.$$

Similarly, $|W_i| \geq m_i - \xi n$ and this shows (b).

Now, we turn to (c) and (d). Let $\{u, v\}$ be an edge of H. Clearly, $\sigma(u) \neq \sigma(v)$. Moreover, if u and v are in different intervals of H, then one of them is late and

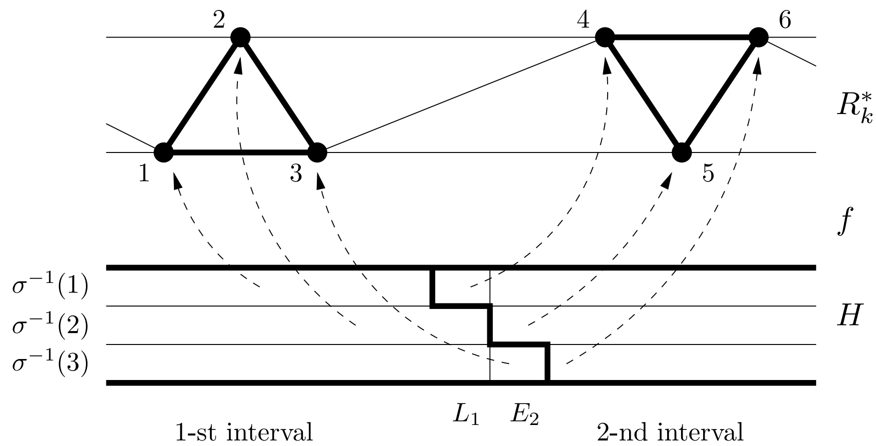

FIGURE 2. The mapping f from H to R_k^*. The late vertices of the 1-st interval are denoted by L_1 and the early vertices of the 2-nd interval by E_2.

the other one is early, because the bandwidth of H is at most βn. Since not both vertices can have colour 2, it follows that one of them is in X.

We will first consider the case where neither $u \in X$ nor $v \in X$. Consequently, u and v are in the same interval of H, i.e., $j(u) = j(v)$. Thus, for all $w \in V(H) \setminus X$ we have $f(w) = 3(j(w)-1)+\sigma(w)$ and hence $\{f(u), f(v)\} \in E(R_k^{**})$, which proves (d).

It remains to investigate the case $u \in X$. If $\sigma(u) = 0$, then $\sigma(v) \neq 0$ and, due to $(*)$ both u and v are timely. Therefore, $j(v) = j(u)$, and we have $f(v) = 3(j(v) - 1) + \sigma(v)$ and $f(u) = r_{j(v)}$. Hence $\{f(u), f(v)\} \in E(R_k)$. If, on the other hand, both $\sigma(u)$ and $\sigma(v)$ are not 0, then $u \in X$ implies u is untimely and either of colour 1 or of colour 3. If $\sigma(u) = 1$, then u is either mapped to $3(j(u) - 1) + 1$ or to $3j(u) + 1$. In the former case, u is early, and so, either $f(v) = 3(j(u) - 1) + \sigma(v)$ or $f(v) = 3(j(u) - 2) + \sigma(v)$. In both cases, $\{f(u), f(v)\} \in E(R_k)$. If u is mapped to $3j(u) + 1$, on the other hand, then u is late, and so $f(v) = 3(j(u) - 1) + \sigma(v)$ or $f(v) = 3j(u) + \sigma(v)$, which, again, implies $\{f(u), f(v)\} \in E(R_k)$. The case where $\sigma(u) = 3$ follows analogously. Therefore (c) holds for f, too. □

REFERENCES

1. S. Abbasi, *Embedding low bandwidth bipartite graphs*, unpublished. 1
2. ———, *How tight is the Bollobás-Komlós conjecture?*, Graphs Combin. **16** (2000), no. 2, 129–137. 1
3. M. Aigner and S. Brandt, *Embedding arbitrary graphs of maximum degree two*, J. London Math. Soc. (2) **48** (1993), no. 1, 39–51. 1
4. N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster, *The algorithmic aspects of the regularity lemma*, J. Algorithms **16** (1994), no. 1, 80–109. 4
5. N. Alon and E. Fischer, 2-*factors in dense graphs*, Discrete Math. **152** (1996), no. 1-3, 13–23. 1

6. N. Alon and R. Yuster, *Almost H-factors in dense graphs*, Graphs Combin. **8** (1992), no. 2, 95–102. 1
7. ———, *H-factors in dense graphs*, J. Combin. Theory Ser. B **66** (1996), no. 2, 269–282. 1
8. C. Chvatál, V. Rödl, E. Szemerédi, and W. T. Trotter, Jr., *The Ramsey number of a graph with bounded maximum degree*, J. Combin. Theory Ser. B **34** (1983), no. 3, 239–243. 2.3, 2.3, 4
9. K. Corradi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta Math. Acad. Sci. Hungar. **14** (1963), 423–439. 1
10. B. Csaba, A. Shokoufandeh, and E. Szemerédi, *Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three*, Combinatorica **23** (2003), no. 1, 35–72, Paul Erdős and his mathematics (Budapest, 1999). 1
11. A. Czygrinow and H. A. Kierstead, *2-factors in dense bipartite graphs*, Discrete Math. **257** (2002), no. 2-3, 357–369, Kleitman and combinatorics: a celebration (Cambridge, MA, 1999). 1
12. R. Diestel, *Graph theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005. 2.3
13. G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3) **2** (1952), 69–81. 1
14. P. Erdős, *Problem 9*, Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963) (M. Fiedler, ed.), Publ. House Czechoslovak Acad. Sci., Prague, 1964, p. 159. 1
15. P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar **1** (1966), 51–57. 1
16. P. Erdös and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. 1
17. G. Fan and H. A. Kierstead, *The square of paths and cycles*, J. Combin. Theory Ser. B **63** (1995), no. 1, 55–64. 1, 1
18. ———, *Hamiltonian square-paths*, J. Combin. Theory Ser. B **67** (1996), no. 2, 167–182. 1, 1
19. A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623. 1
20. H. Hàn, *Bipartite Graphen mit kleiner Bandbreite in dichten Graphen*, Master's thesis, Humboldt-Universität zu Berlin, Institut für Informatik, Fall 2006, expected. 1
21. J. Komlós, *The blow-up lemma*, Combin. Probab. Comput. **8** (1999), no. 1-2, 161–176, Recent trends in combinatorics (Mátraháza, 1995). 1, 1
22. ———, *Tiling Turán theorems*, Combinatorica **20** (2000), no. 2, 203–218. 1
23. J. Komlós, G. N. Sárközy, and E. Szemerédi, *On the square of a Hamiltonian cycle in dense graphs*, Random Structures Algorithms **9** (1996), no. 1-2, 193–211. 1
24. ———, *Blow-up lemma*, Combinatorica **17** (1997), no. 1, 109–123. 1, 2.3, 9
25. ———, *An algorithmic version of the blow-up lemma*, Random Structures Algorithms **12** (1998), no. 3, 297–312. 4
26. ———, *On the Pósa-Seymour conjecture*, J. Graph Theory **29** (1998), no. 3, 167–176. 1, 1
27. ———, *Proof of the Seymour conjecture for large graphs*, Ann. Comb. **2** (1998), no. 1, 43–60. 1
28. ———, *Proof of the Alon-Yuster conjecture*, Discrete Math. **235** (2001), no. 1-3, 255–269. 1
29. ———, *Spanning trees in dense graphs*, Combin. Probab. Comput. **10** (2001), no. 5, 397–416. 1, 3
30. J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352. 2.1
31. D. Kühn and D. Osthus, *Spanning triangulations in graphs*, J. Graph Theory **49** (2005), no. 3, 205–233. 1
32. ———, *Critical chromatic number and the complexity of perfect packings in graphs*, Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2006, Miami, Florida, USA, January 22-26, 2006, ACM Press, 2006, pp. 851–859. 1
33. D. Kühn, D. Osthus, and A. Taraz, *Large planar subgraphs in dense graphs*, J. Combin. Theory Ser. B **95** (2005), no. 2, 263–282. 1, 2.1, 2.2
34. V. Rödl and A. Ruciński, *Perfect matchings in ϵ-regular graphs and the blow-up lemma*, Combinatorica **19** (1999), no. 3, 437–452. 2.3

35. V. Rödl, A. Ruciński, and M. Wagner, *Matchings meeting quotas and their impact on the blow-up lemma*, SIAM J. Comput. **31** (2001), no. 2, 428–446. 4
36. P. Seymour, *Problem section*, Combinatorics (Proc. British Combinatorial Conf. 1973) (T. P. McDonough and V. C. Mavron, eds.), Cambridge Univ. Press, London, 1974, pp. 201–204. London Math. Soc. Lecture Note Ser., No. 13. 1
37. A. Shokoufandeh and Y. Zhao, *Proof of a tiling conjecture of Komlós*, Random Structures Algorithms **23** (2003), no. 2, 180–205. 1
38. E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Orsay, 1976), Colloques Internationaux CNRS, vol. 260, CNRS, 1978, pp. 399–401. 1, 2.1
39. P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452. 1

Zentrum Mathematik, Technische Universität München, Boltzmannstrasse 3, D-85747 Garching bei München, Germany
E-mail address: `boettche@ma.tum.de`

Institut für Informatik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany
E-mail address: `schacht@informatik.hu-berlin.de`

Zentrum Mathematik, Technische Universität München, Boltzmannstrasse 3, D-85747 Garching bei München, Germany
E-mail address: `taraz@ma.tum.de`