The bandwidth conjecture for 3-colourable graphs

Julia Böttcher

Technische Universität München

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(with Mathias Schacht & Anusch Taraz)

Outline

Let G be a graph.
We are interested in sufficient degree conditions on G for the appearance of a given graph H in G.

Small Graphs

Theorem (follows from Turán 1941)

If $\delta(G) > \frac{r-2}{r-1}n$ then $K_r \subseteq G$.

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Theorem (follows from Erdős, Stone;Simonovits 1946;1966)

For all graphs H and constants γ there is an n_0 s.t. for all graphs $|G| = n \geq n_0$ the following holds.
If $\delta(G) \geq \left(\frac{r-2}{r-1} + \gamma\right) n$ then $H \subseteq G$, where $r = \chi(H)$.

From Small Graphs to Big Graphs

- A graph H of constant size is forced in G, when

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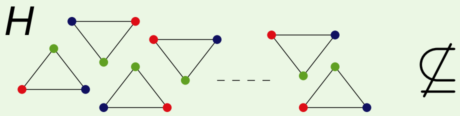
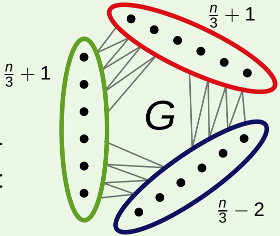
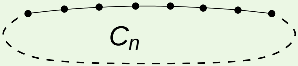
- Spanning graphs H are not, because:





Big Graphs

- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

$\subseteq G$ Dirac'52

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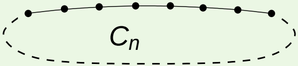
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

$\subseteq G$ — DIRAC'52
- $\delta(G) \geq \frac{r-1}{r}n \Rightarrow \frac{n}{r}$ disj. copies of $K_r \subseteq G$. — HAJNAL,SZEMERÉDI'69

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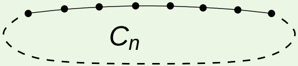
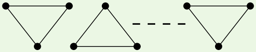
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$  $\subseteq G$ DIRAC'52

- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$ HAJNAL,SZEMERÉDI'69

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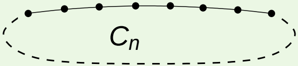
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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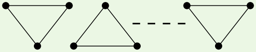
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$ — HAJNAL, SZEMERÉDI'69
- $\delta(G) \geq \frac{r-1}{r}n \Rightarrow (C_n)^r \subseteq G.$

FAN, KIERSTEAD '95

KOMLÓS, SÁRKÖZY, AND SZEMERÉDI '98

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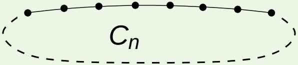
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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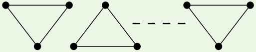
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$ 
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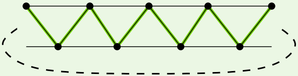
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad \mathrm{Ham}^2$ 
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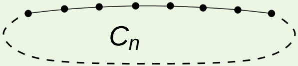
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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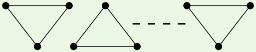
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$

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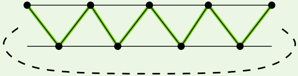
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad \mathrm{Ham}^2$  $\subseteq G$

  FAN, KIERSTEAD '95

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- other results: trees, F-factors, planar graphs . . .

  KOMLÓS, SÁRKÖZY, AND SZEMERÉDI '95

  ALON, YUSTER '96

  KÜHN, OSTHUS, TARAZ '05

A Generalization

Conjecture (naïve)

*Forall Δ and γ there is an n_0 s.t. for all $|H| = |G| = n \geq n_0$:
If $\Delta(H) \leq \Delta$ and $\delta(G) \geq \left(\frac{r-1}{r} + \gamma\right) n$ then $H \subseteq G$
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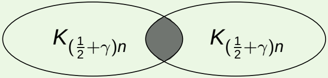
Counterexample:

- H : random bipartite graph with $\Delta(H) \leq \Delta$.
- G : two cliques of size $\left(\frac{1}{2} + \gamma\right) n$ sharing $2\gamma n$ vertices.



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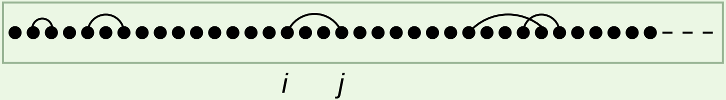
Bandwidth:

- $\mathrm{bw}(G) \leq b$ if there is a labelling of $V(G)$ by $1, \ldots, n$ s.t. for all $\{i, j\} \in E(G)$ we have $|i - j| \leq b$.



A Generalization

Conjecture (Bollobás & Komlós)

Forall Δ and γ there are β and n_0 s.t. for all $|H| = |G| = n \geq n_0$: If $\Delta(H) \leq \Delta$, $\mathrm{bw}(H) \leq \beta n$, and $\delta(G) \geq \left(\frac{r-1}{r} + \gamma\right) n$ then $H \subseteq G$ where $r = \chi(H)$.

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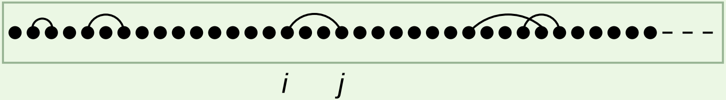

The 3-chromatic Case

Theorem

For all Δ and γ there are β and n_0 s.t. for all $|H| = |G| = n \geq n_0$: If $\Delta(H) \leq \Delta$, $\mathrm{bw}(H) \leq \beta n$, $\chi(H) = 3$, and $\delta(G) \geq \left(\frac{2}{3} + \gamma\right) n$ then $H \subseteq G$.

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- The proof is constructive.
- For the proof we use the regularity and the blow-up lemmma.

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3. Take care of edges between the different parts of H with the help of (a variant of) the embedding lemma.
4. Complete the embedding with the blow-up lemma.

The Lemma for G

Lemma (Lemma for G)

Let G be a suff. large graph with $\delta(G) \geq (\frac{2}{3}+\gamma)n$. There is a regular partition $V(G) = V'_1 \dot\cup \cdots \dot\cup V'_k$ of G with reduced graph R with the following properties:

- $\delta(R) \geq (\frac{2}{3}+\gamma/2)|V(R)|$ *and thus* $R \supseteq Ham^2 \supseteq K_3^*$,
- $V'_1 \dot\cup \cdots \dot\cup V'_k$ *is super-regular on* K_3^*,
- *The parts* V'_i *are almost of equal size.*

Moreover, if we are asked to change the sizes of the parts V'_i by ξn, we can do so. (resulting partition: $V_1 \dot\cup \cdots \dot\cup V_k$)

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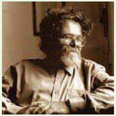

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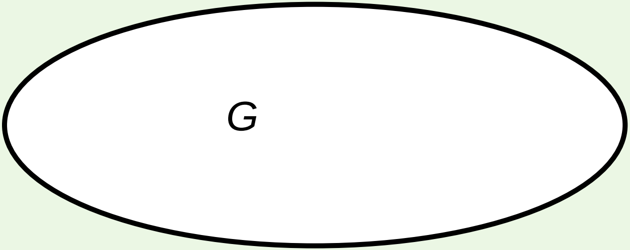

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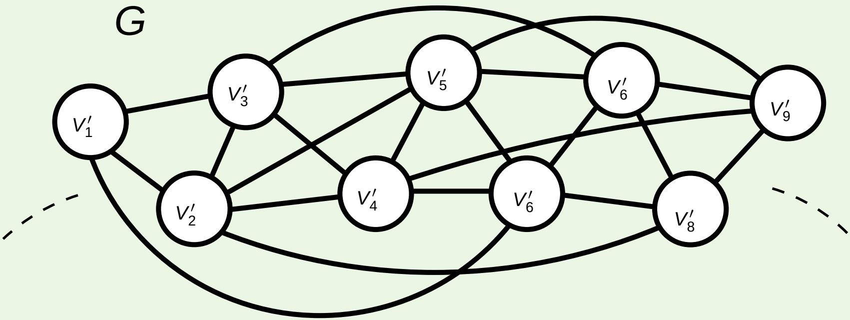
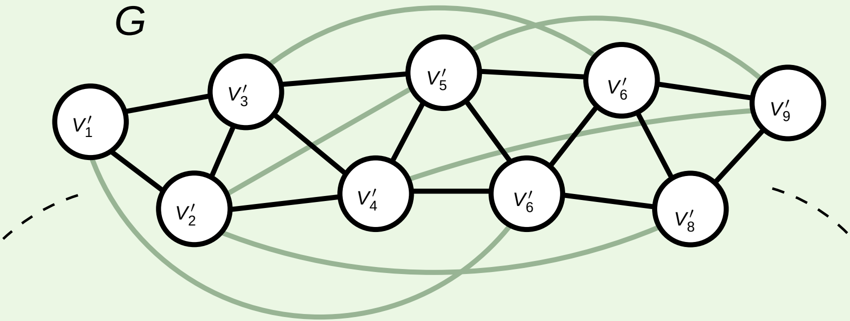


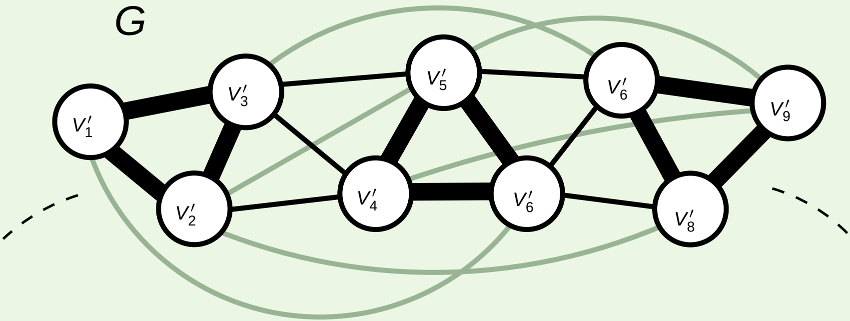
A regular partition of G with parts of almost equal size.

The Lemma for G



Since $\delta(R) \geq (\frac{2}{3} + \gamma/2)|V(R)|$ we have $R \supseteq \mathrm{Ham}^2 \supseteq K_3^*$.

The Lemma for G



We make the partition super-regular on K_3^*.

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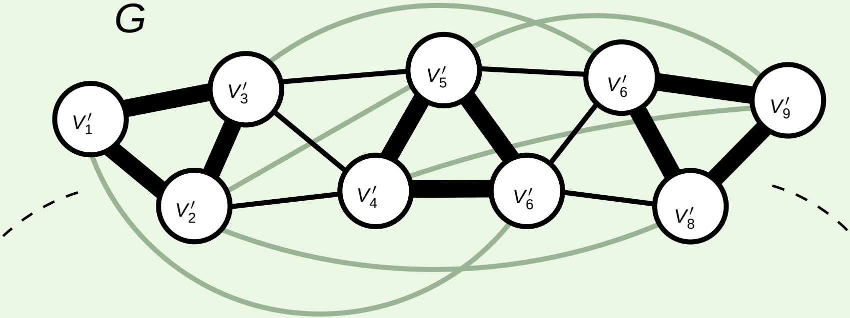

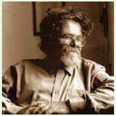

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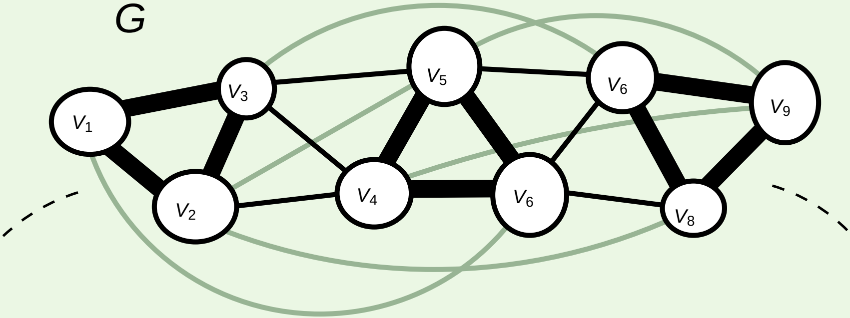


We can change the partition by moving some vertices to obtain $V_1 \dot\cup \cdots \dot\cup V_k$.

The Lemma for H

Lemma (Lemma for H)

Let H be a graph with $\Delta(H) \leq \Delta$, bandwidth $\leq \beta n$ and $\chi(H) \leq 3$, and let $R \supseteq Ham^2 \supseteq K_3^*$ be a graph with $\delta(R_k) > 2k/3$. Then there is a homomorphism f from H to R such that

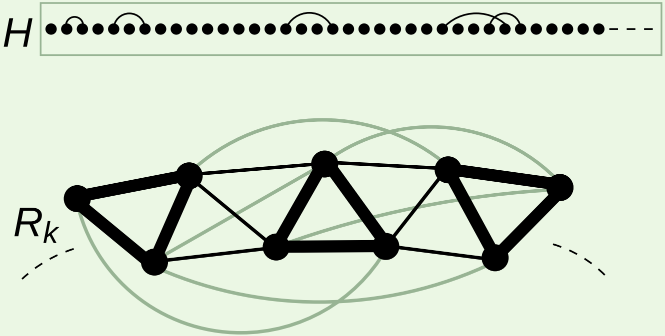
- *f sends roughly the same number of vertices to each $v \in V(R)$,*
- *Almost all edges of H are mapped to edges of K_3^*.*

Moreover, the set of special vertices

$$X := \{v \in V(H) \, : \, \text{some edge } uv \text{ is not mapped to } K_3^*\}$$

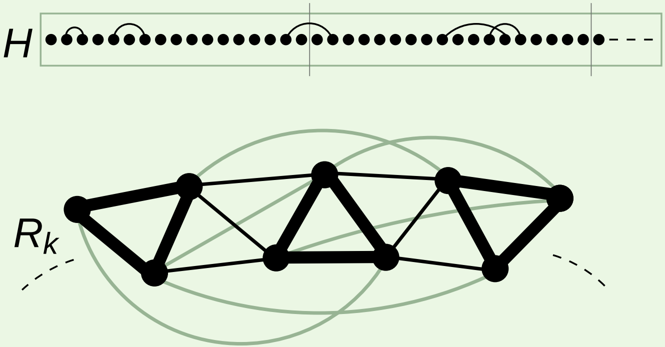
is very small.

The Lemma for H



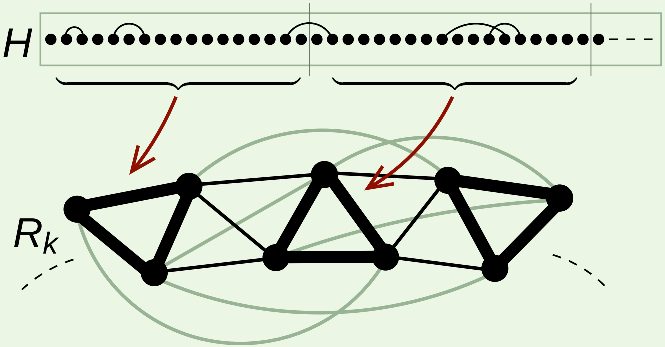
H is given in an order respecting the bandwidth bound.

The Lemma for H



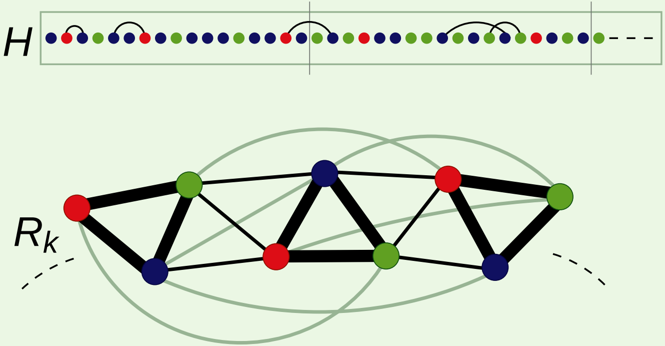
Idea: Cut H into pieces

The Lemma for H



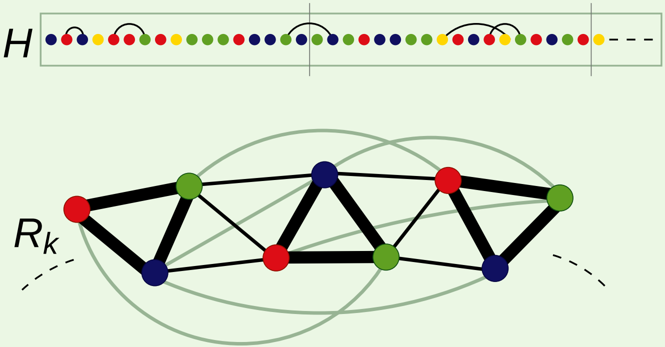
Idea: Cut H into pieces and map each piece to a triangle of K_3^*.

The Lemma for H



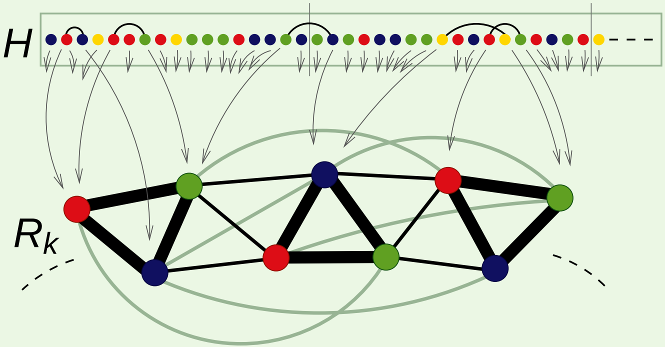
This is possible, since $\chi(H) = 3$.
But the colour classes of H may vary in size a lot.

The Lemma for H



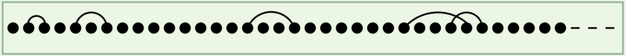
We can find a large subgraph of H with a balanced 3-colouring.

The Lemma for H



We map all edges of H to R, and most of them to K_3^*.

Proof of the Main Theorem

H 

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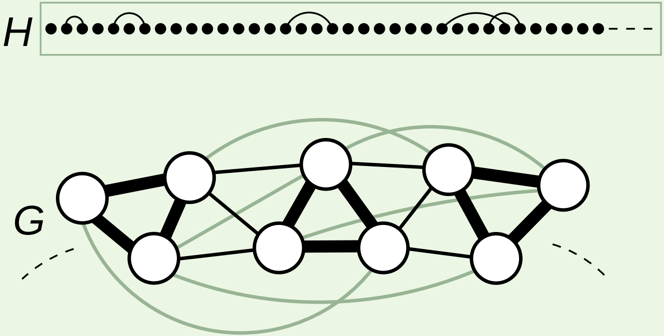
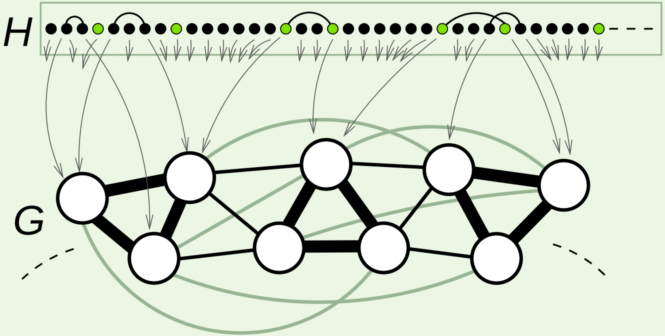


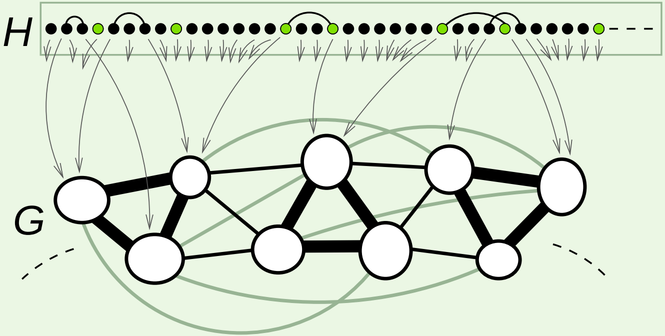
The Lemma for G constructs a regular partition $V_1' \dot\cup \cdots \dot\cup V_k'$ of G.

Proof of the Main Theorem



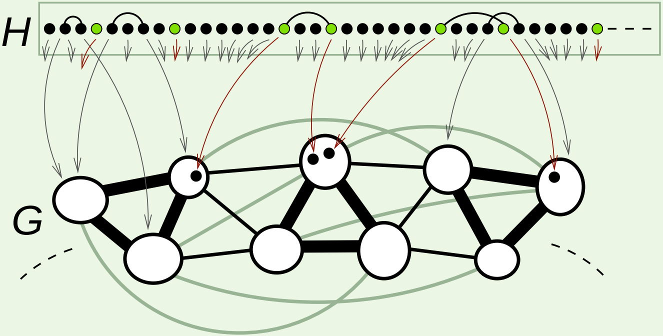
The Lemma for H constructs a homomorphism f and a set of special vertices X.

Proof of the Main Theorem



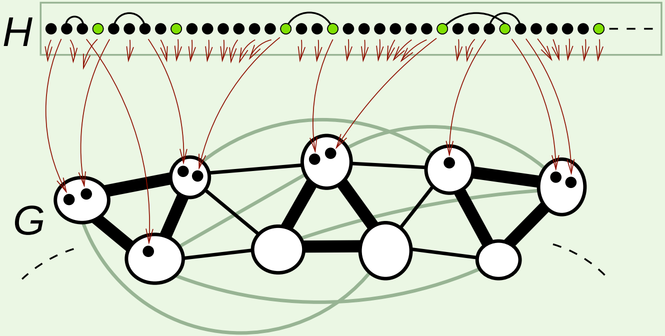
The Lemma for G adjusts the partition of G s.t. $|V_i| = f^{-1}(i)$.

Proof of the Main Theorem



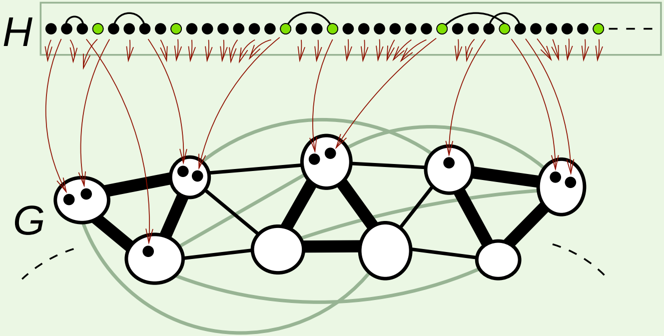
We embed the special vertices X into G using the embedding lemma.

Proof of the Main Theorem



We embed all other vertices using the blow-up lemma.

Proof of the Main Theorem



$H \subseteq G$

Summary/Outlook

- The bandwidth conjecture generalizes many known results about sufficient degree conditions for the appearance of spanning subgraphs.
- We proved the bandwidth conjecture for $\chi(H) = 3$.
- The proof does not (quite) carry over to $\chi(H) > 3$.
- What happens if we replace bandwidth by pathwidth / treewidth?