On the bandwidth conjecture for 3-colourable graphs

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(joint work with Mathias Schacht & Anusch Taraz)

Extremal graph theory

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Extremal combinatorics was invented and is loved by hungarians.

BÉLA BOLLOBÁS

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Good jazz is when the leader jumps on the piano, waves his arms, and yells.

Fine jazz is when a tenorman lifts his foot in the air.

Great jazz is when he heaves a piercing note for 32 bars and collapses on his hands and knees.

A pure genius of jazz is manifested when he and the rest of the orchestra run around the room while the rhythm section grimaces and dances around their instruments.

CHARLES MINGUS

Subgraph containment problem

Question

- Given a graph H or family $\mathcal{H} = \{H_n \colon n \in \mathbb{N}\}$. Which conditions on an n-vertex graph $G = (V, E)$ ensure $H \subseteq G$ or $H_n \subseteq G$?

Subgraph containment problem

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- Our aim: every graph $G = (V, E)$ with minimum degree $\delta(G) \geq$?? contains a given graph H.

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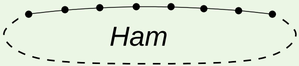
Classical example:

- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

$\subseteq G$ DIRAC'52

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H of small/fixed size

Erdős–Stone: $\delta(G) \geq \left(\frac{\chi(H)-2}{\chi(H)-1} + o(1)\right) n$.

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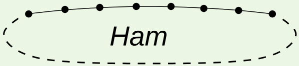
This talk

H_n is a spanning subgraph of G.

Big Graphs

- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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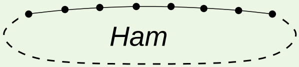
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

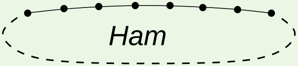
$\subseteq G$

  DIRAC'52

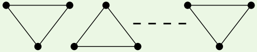
- $\delta(G) \geq \frac{r-1}{r}n \Rightarrow \frac{n}{r}$ disj. copies of $K_r \subseteq G$.

  HAJNAL,SZEMERÉDI'69

Big Graphs

- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

$\subseteq G$ — DIRAC'52

- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$ — HAJNAL,SZEMERÉDI'69

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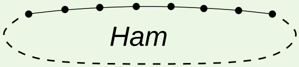
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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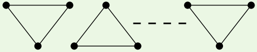
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$

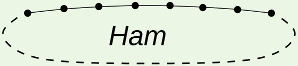
  Hajnal, Szemerédi'69

- $\delta(G) \geq \frac{r-1}{r}n \Rightarrow (Ham)^r \subseteq G.$

  Fan, Kierstead '95

  Komlós, Sárközy, and Szemerédi '98

Big Graphs

- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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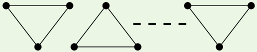
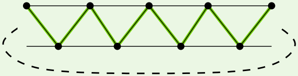
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$ — Hajnal,Szemerédi'69

- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad \mathrm{Ham}^2$  $\subseteq G$

  Pósa's conjecture

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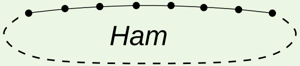
- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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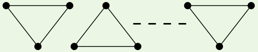
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad K_3^*$  $\subseteq G$

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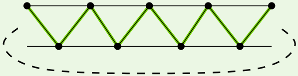
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad \mathrm{Ham}^2$  $\subseteq G$

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  FAN, KIERSTEAD '95

  KOMLÓS, SÁRKÖZY, AND SZEMERÉDI '98

- other results: trees, F-factors, planar triangulations . . .

  KOMLÓS, SÁRKÖZY, AND SZEMERÉDI '95

  ALON, YUSTER '96

  KÜHN, OSTHUS, TARAZ '05

From Small Graphs to Big Graphs

- A graph H of constant size is forced in G, when

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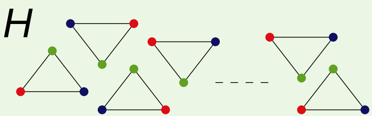
- Spanning graphs H are not, because:



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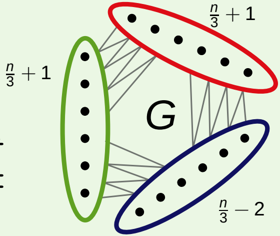

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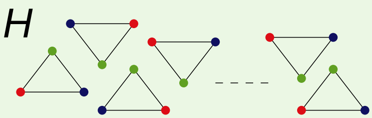

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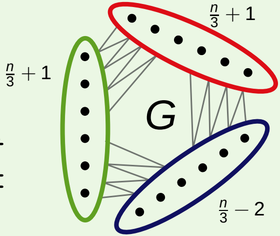

- What about $\frac{\chi(H)-1}{\chi(H)}$?

Generalizing conjecture

Naïve conjecture

For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0

H

G

Generalizing conjecture

Naïve conjecture

For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0

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- $\chi(H) = k$

G

- $\delta(G) \geq (\frac{k-1}{k} + \gamma)n$

Generalizing conjecture

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For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0

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- $\Delta(H) \leq \Delta$

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$\Longrightarrow$ G contains H.

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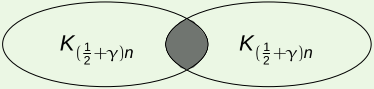
Counterexample:

- H : random bipartite graph with $\Delta(H) \leq \Delta$.
- G : two cliques of size $\left(\frac{1}{2} + \gamma\right) n$ sharing $2\gamma n$ vertices.



Generalizing conjecture

Conjecture of Bollobás and Komlós

For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0 and $\beta > 0$

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- $\mathrm{bw}(H) \leq \beta n$

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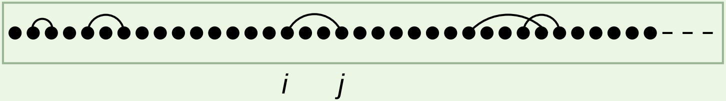
$\Longrightarrow$ G contains H.

Bandwidth:

- $\mathrm{bw}(G) \leq b$ if there is a labelling of $V(G)$ by $1, \ldots, n$ s.t. for all $\{i, j\} \in E(G)$ we have $|i - j| \leq b$.



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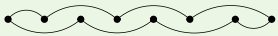
Examples for H:

- Hamiltonian cycles (bandwidth 2)



- square grid (bandwidth $\sqrt{n}$), binary trees (bandwidth $n/\log(n)$)
- graphs of constant tree width

New result

The 3-chromatic case

For all Δ and γ there are β and n_0 s.t. for all $n \geq n_0$ and all n-vertex graphs H and G the following holds.
If $\Delta(H) \leq \Delta$, $\mathrm{bw}(H) \leq \beta n$, $\chi(H) = 3$, and $\delta(G) \geq \left(\frac{2}{3} + \gamma\right) n$ then G contains H.

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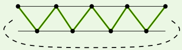
- Abbasi '98 annouced 2-chromatic case
- additional γn is necessary
- Proof uses
  - regularity lemma
  - blow-up lemma
  - affirmative solution of Pósa's conjecture

$G \supseteq$ 

- proof gives $O(n^{3.376})$ algorithm for embedding H into G

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1. partition G with the regularity lemma

Lemma for G

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2. partition H and assign parts of H to parts of G

   Lemma for H

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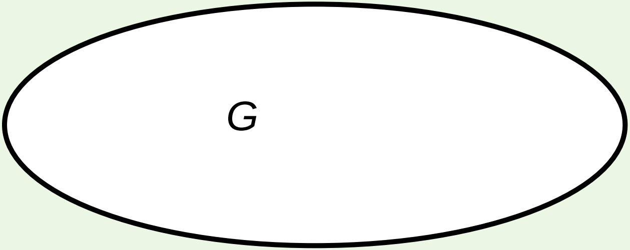
3. take care of edges of H that run between different parts — partial embedding lemma

Strategy of the proof

1. partition G with the regularity lemma — Lemma for G
2. partition H and assign parts of H to parts of G — Lemma for H
3. take care of edges of H that run between different parts — partial embedding lemma
4. embed the parts of H into the corresponding parts of G — blow-up lemma

The Lemma for G



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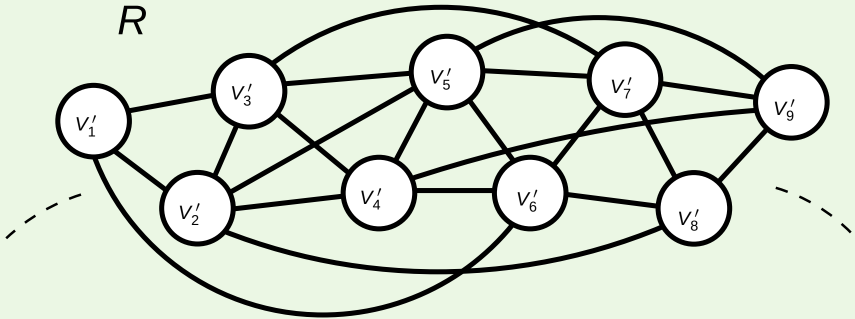
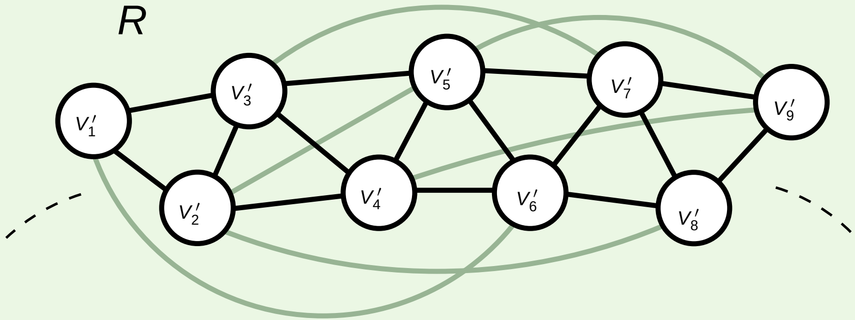


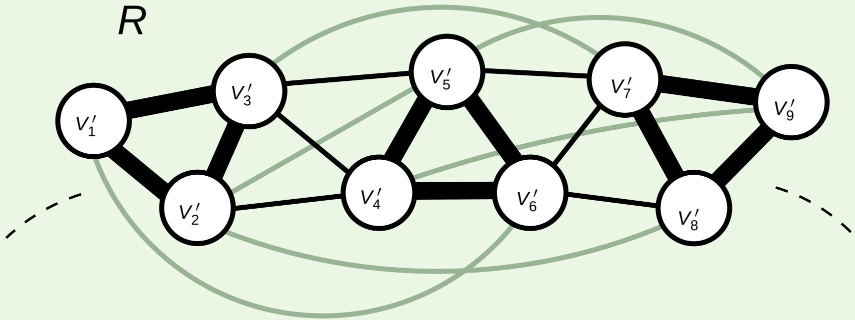
regular partition of G

The Lemma for G



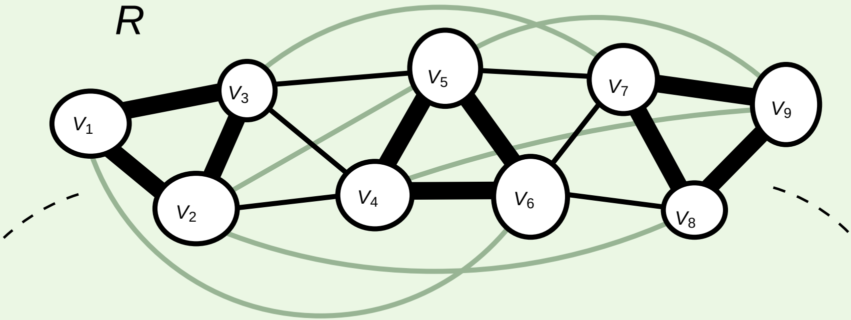
Since $\delta(R) \geq (\frac{2}{3} + \gamma/2)|V(R)|$ we have $R \supseteq \mathrm{Ham}^2 \supseteq K_3^*$

The Lemma for G



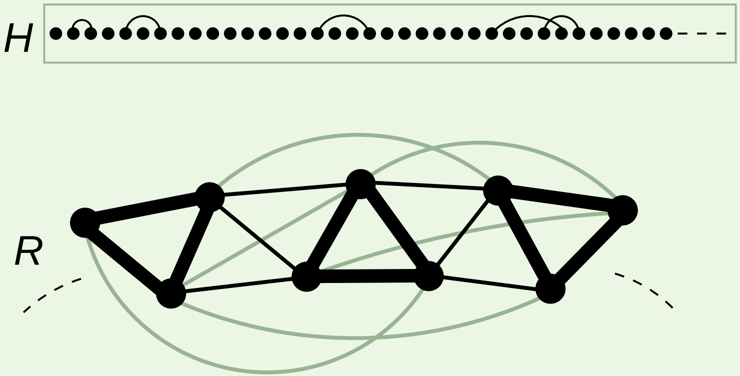
Make the partition super-regular on K_3^*.

The Lemma for G



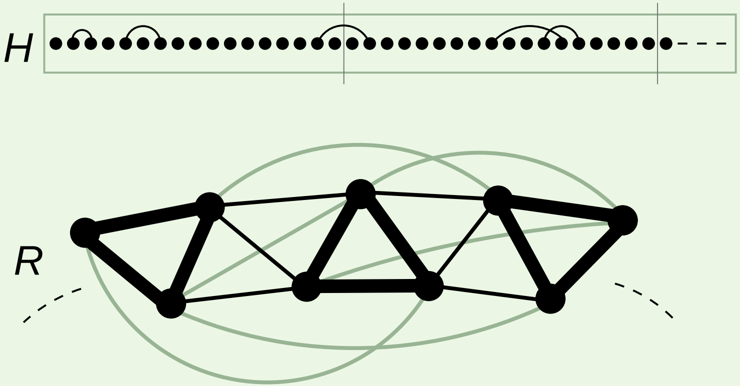
Change the partition by moving some vertices to obtain $V_1 \dot\cup \cdots \dot\cup V_k$

Lemma for H



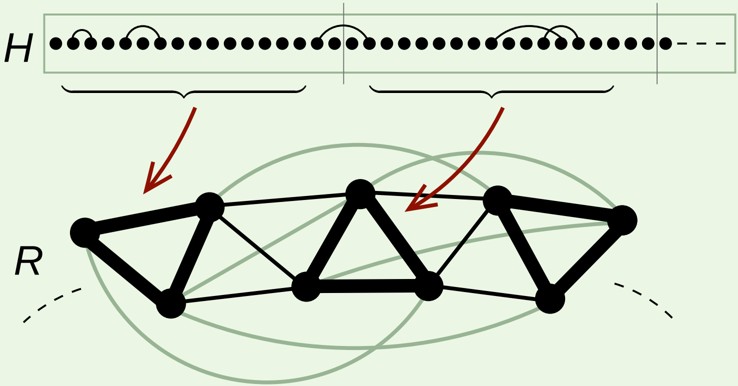
H is given in an order respecting the bandwidth bound.

Lemma for *H*



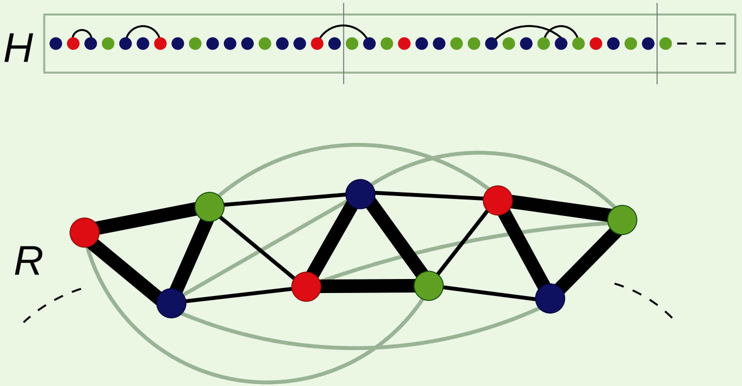
Idea: Cut *H* into pieces

Lemma for H



Idea: Cut H into pieces and map each piece to a triangle of K_3^*.

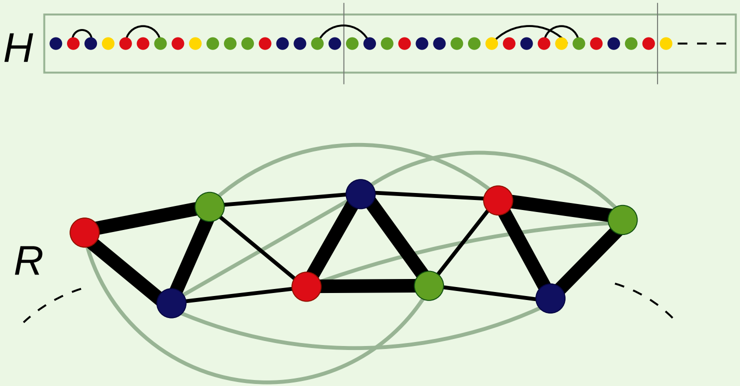
Lemma for *H*



This is possible, since $\chi(H) = 3$.

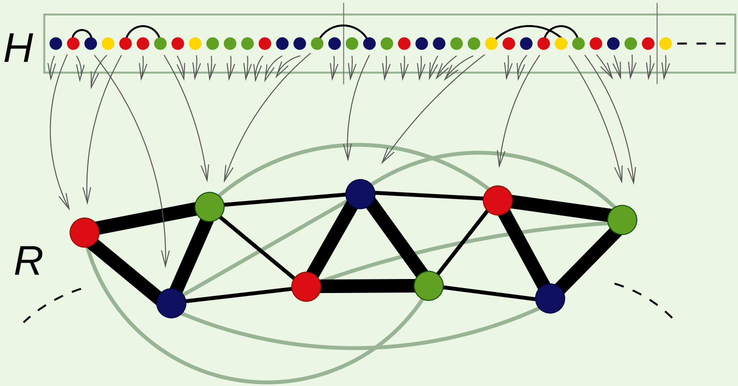
But the colour classes of H may vary in size a lot.

Lemma for *H*



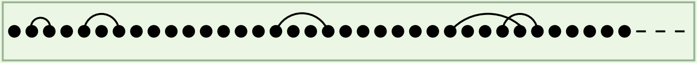
We can find a large subgraph $H \setminus X$ of H with a balanced 3-colouring.

Lemma for H



$f : V(H) \rightarrow V(R)$ maps all edges of H to edges of R, and most of them to K_3^*.

Proof of the theorem

H 

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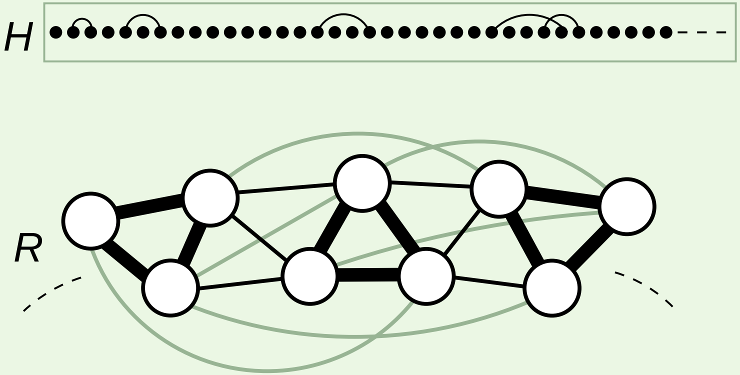
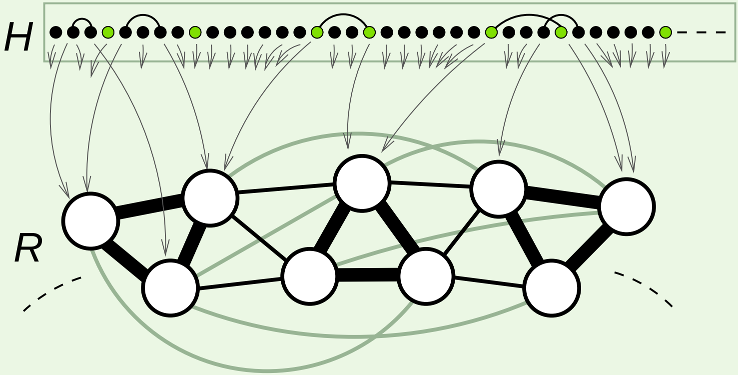


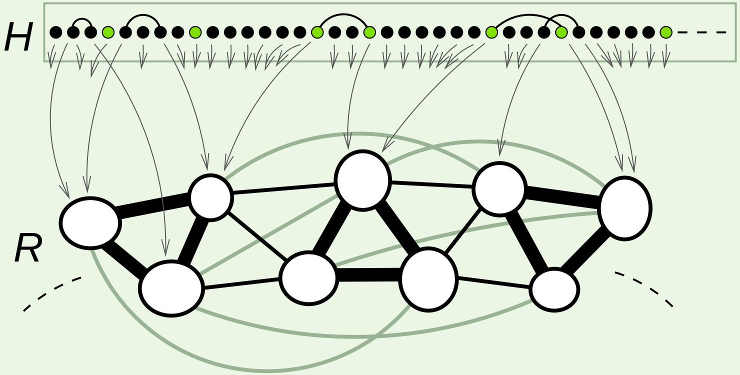
The Lemma for G constructs a regular partition $V_1' \dot\cup \cdots \dot\cup V_k'$ of G.

Proof of the theorem



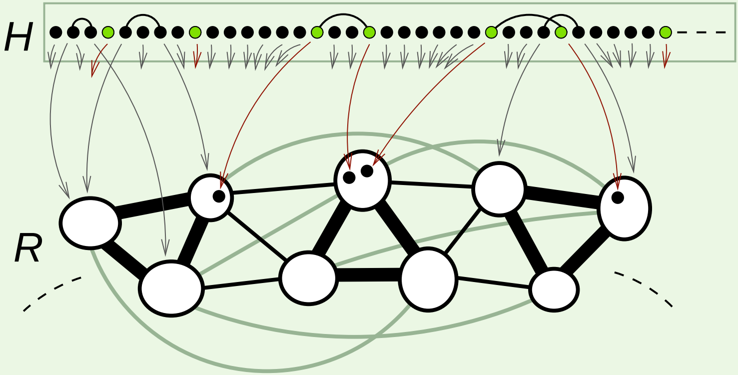
The Lemma for H constructs a homomorphism $f : V(H) \to V(R)$ and a set of special vertices X.

Proof of the theorem



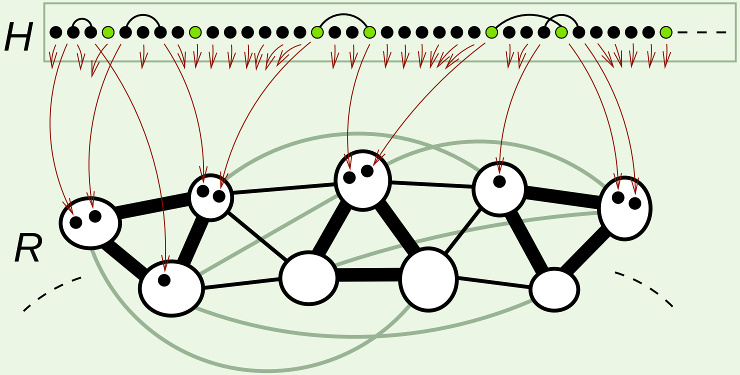
The Lemma for G adjusts the partition of G s.t. $|V_i| = f^{-1}(i)$.

Proof of the theorem



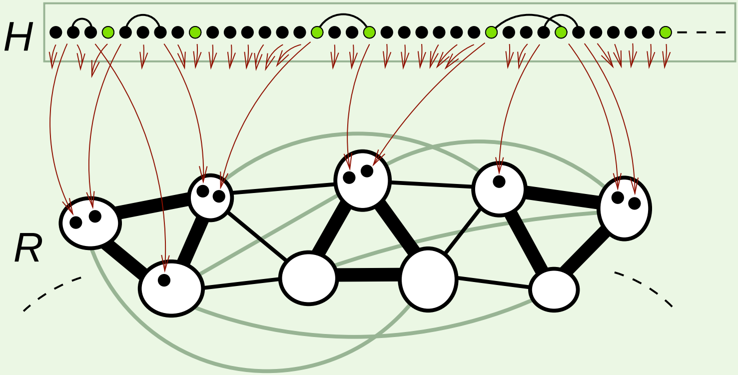
We embed the special vertices X into G using the embedding lemma.

Proof of the theorem



We embed all other vertices using the blow-up lemma.

Proof of the theorem



$H \subseteq G$

Concluding remarks

- What about k-chromatic H? work in progress
- What is the correct dependency of γ and β?
- Which graphs have bandwidth at most βn?