Almost spanning subgraphs of random graphs after adversarial edge removal

Julia Böttcher [1],

Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D–85747 Garching bei München, Germany

Yoshiharu Kohayakawa [2],

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508–090 São Paulo, Brazil

Anusch Taraz [1]

Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D–85747 Garching bei München, Germany

Abstract

Let $\Delta \geq 2$ be a fixed integer. We show that the random graph $\mathcal{G}_{n,p}$ with $p \geq c(\log n/n)^{1/\Delta}$ is robust with respect to the containment of almost spanning bipartite graphs H with maximum degree Δ and sublinear bandwidth in the following sense. If an adversary deletes arbitrary edges in $\mathcal{G}_{n,p}$ such that each vertex loses less than half of its neighbours, then asymptotically almost surely the resulting graph still contains a copy of H.

Keywords: graph theory, extremal problems, random graphs, sparse regularity

[1] Email: {boettche|taraz}@ma.tum.de

[2] Email: yoshi@ime.usp.br

1 Introduction and results

In this paper we study graphs that are robust in the following sense: even after adversarial removal of a specified proportion of their edges, they still contain copies of every graph from a certain class of graphs.

In order to make this precise, we use the notion of *resilience* (see [12]). Let $\mathcal{P}$ be a monotone increasing graph property and $G = (V, E)$ be a graph. The *global resilience* $R_g(G, \mathcal{P})$ of G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that deleting a suitable set of $r \cdot |E|$ edges from E creates a graph which is not in $\mathcal{P}$. The *local resilience* $R_\ell(G, \mathcal{P})$ of G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that deleting a suitable set of at most $r \cdot \deg_G(v)$ edges incident to v for every vertex $v \in V$ creates a graph which is not in $\mathcal{P}$.

For example, using this terminology, the classical theorems of Turán [13] and Dirac [8] can be stated as follows: the global resilience of the complete graph K_n with respect to containing a clique on r vertices is $\frac{1}{r-1} - o(1)$ and the local resilience of K_n with respect to containing a Hamilton cycle is $\frac{1}{2} - o(1)$. Here, we stay quite close to the scenario of these two examples insofar as we will also consider properties that deal with subgraph containment. However, we are interested in the resilience of graphs which are much sparser than the complete graph.

It turns out that the random graph $\mathcal{G}_{n,p}$ is well suited for this purpose ($\mathcal{G}_{n,p}$ is defined on vertex set $[n] = \{1, \ldots, n\}$ and edges exist independently of each other with probability p). Sudakov and Vu [12] showed that asymptotically almost surely (a.a.s.) the local resilience of $\mathcal{G}_{n,p}$ with respect to containing a Hamilton cycle is $\frac{1}{2} - o(1)$ if $p > \log^4 n/n$.

A result of Dellamonica et al. [6] implies that a.a.s. the local resilience of $\mathcal{G}_{n,p}$ with respect to containing cycles of length at least $(1 - \alpha)n$ is $\frac{1}{2} - o(1)$ for any $0 < \alpha < \frac{1}{2}$ and $p \gg 1/n$. We shall discuss the various lower bounds for the edge probability p occuring in these and later results at the end of Section 2.

Now we extend the scope of investigations to the containment of a much larger class of subgraphs. A graph has *bandwidth* at most b if there exists a labelling of the vertices by numbers $1, \ldots, n$, such that for every edge ij of the graph we have $|i - j| \leq b$. Let $\mathcal{H}(m, \Delta)$ denote the class of all graphs on m vertices with maximum degree at most Δ, and $\mathcal{H}_2^b(m, \Delta)$ denote the class of

The first and third author were partially supported by DFG grant TA 309/2-1. The second author was partially supported by CNPq (Proc. 308509/2007-2, 485671/2007-7 and 486124/2007-0). The cooperation of the three authors was supported by a joint CAPES-DAAD project (415/ppp-probral/po/D08/11629, Proj. no. 333/09).

all *bipartite* graphs in $\mathcal{H}(m,\Delta)$ which have bandwidth at most b. Our result [3] asserts that the local resilience of $\mathcal{G}_{n,p}$ with respect to containing any graph H from $\mathcal{H}_2^{\beta n}((1-\eta)n,\Delta)$ is $\frac{1}{2}-o(1)$ for small β and η and for $p=p(n)=o(1)$ sufficiently large.

Theorem 1.1 *For each $\eta,\gamma>0$ and $\Delta\geq 2$ there exist positive constants β and c such that the following holds a.a.s. for $p\geq c(\log n/n)^{1/\Delta}$. Every spanning subgraph $G=(V,E)$ of $\mathcal{G}_{n,p}$ with $\deg_G(v)\geq(\frac{1}{2}+\gamma)\deg_{\mathcal{G}_{n,p}}(v)$ for all $v\in V$ contains a copy of every graph H in $\mathcal{H}_2^{\beta n}((1-\eta)n,\Delta)$.*

We note that several important classes of graphs have sublinear bandwidth, and hence Theorem 1.1 does apply to them: this is the case for, e.g., the class of all bounded degree planar graphs (see [4]).

2 Background

As we saw at the end of the last section, we are looking for graphs that do not only contain one specific subgraph but a large class of graphs. A graph G is called *universal* for a class of graphs $\mathcal{H}$ if G contains every graph from $\mathcal{H}$ as a subgraph. In this section we briefly sketch some results concerning universality in general, and then come back to resilience with respect to universality.

In [7] it is shown that $\mathcal{G}_{n,p}$ a.a.s. is universal for $\mathcal{H}(n,\Delta)$ if $p=\tilde{\Omega}(n^{-\frac{1}{2\Delta}})$ (where $\tilde{\Omega}$ hides polylogarithmic factors). It is also shown in [7] that the lower bound for the edge probability p can be improved if we restrict our attention to balanced bipartite graphs: Let $\mathcal{H}_2(m,m,\Delta)$ denote the class of bipartite graphs in $\mathcal{H}(2m,\Delta)$ with two colour classes of equal size. Then $\mathcal{G}_{2n,p}$ a.a.s. is universal for $\mathcal{H}_2(n,n,\Delta)$ if $p=\tilde{\Omega}(n^{-\frac{1}{\Delta}})$. The same lower bound for p also guarantees universality for *almost spanning* graphs of arbitrary chromatic number: Alon et al. [3] prove that for every $\eta>0$ and $p=\tilde{\Omega}(n^{-\frac{1}{\Delta}})$, the random graph $\mathcal{G}_{n,p}$ a.a.s. is universal for $\mathcal{H}((1-\eta)n,\Delta)$. Alon and Capalbo [1,2] gave explicit constructions of graphs with average degree $\tilde{\Omega}(n^{-\frac{2}{\Delta}})n$ that are universal for $\mathcal{H}(n,\Delta)$.

Moving on to resilience, it is clear that an adversary can destroy any spanning subgraph by deleting the edges incident to a single vertex. Hence any graph must have trivial global resilience with respect to universality for spanning subgraphs.

However, if we focus on subgraphs of smaller order, then sparse random graphs have a global resilience arbitrarily close to 1: Alon et al. [3] show

[3] For proofs see http://www-m9.ma.tum.de/foswiki/pub/Allgemeines/JuliaBoettcher/thesis.pdf.

that for every $\gamma > 0$ there is a constant $\eta > 0$ such that for $p = \tilde{\Omega}(n^{-\frac{1}{2\Delta}})$ the random graph $\mathcal{G}_{n,p}$ a.a.s. has global resilience $1-\gamma$ with respect to universality for $\mathcal{H}_2(\eta n, \eta n, \Delta)$. In other words, $\mathcal{G}_{n,p}$ contains *many* copies of all graphs from $\mathcal{H}_2(\eta n, \eta n, \Delta)$ *everywhere*.

Finally, the concept of local resilience allows for non-trivial results concerning universality for almost spanning subgraphs. For example, a conjecture of Bollobás and Komlós proven in [5] asserts that the local resilience of the complete graph K_n with respect to universality for $\mathcal{H}_r^{\beta n}(n, \Delta)$ is $\frac{1}{r} - o(1)$. Here $\mathcal{H}_r^{\beta n}(n, \Delta)$ is the class of all r-colourable n-vertex graphs with maximum degree at most Δ and bandwidth at most βn, and one can show that the bandwidth constraint cannot be omitted. Our Theorem 1.1 replaces K_n by the much sparser graph $\mathcal{G}_{n,p}$, but it only treats almost spanning subgraphs and the case $r = 2$.

Before we conclude this section, let us briefly explain the lower bounds for the edge probability p mentioned in the results above, summarized in Table 1.

First, a straightforward counting argument shows that any graph that is universal for $\mathcal{H}(n, \Delta)$ must have at least $\Omega(n^{2-2/\Delta})$ edges. Moreover, it is easy to see that an edge probability $p = n^{-\frac{2}{\Delta}+\varepsilon}$ with $\varepsilon < \frac{1}{\Delta^2}$ is not sufficient to guarantee that $\mathcal{G}_{n,p}$ is universal for the even more restrictive class $\mathcal{H}_2(\eta n, \eta n, \Delta)$. Indeed, consider the graph $H \in \mathcal{H}_2(\eta n, \eta n, \Delta)$ consisting of $\eta n/\Delta$ copies of $K_{\Delta,\Delta}$. The expected number of copies of $K_{\Delta,\Delta}$ in $\mathcal{G}_{n,p}$ is at most

$$n^{2\Delta} p^{\Delta^2} = n^{2\Delta} (n^{-\frac{2}{\Delta}+\varepsilon})^{\Delta^2} = n^{2\Delta - 2\Delta + \varepsilon\Delta^2} \ll n,$$

and hence a.a.s. $\mathcal{G}_{n,p}$ does not contain a copy of H.

	Result	$\boldsymbol{p}$	**Ref**
Universality	$\mathcal{H}_2(n, n, \Delta) \subseteq \mathcal{G}_{2n,p}$	$p = n^{-\frac{1}{\Delta}}$	[7]
	$\mathcal{H}(n, \Delta) \subseteq \mathcal{G}_{n,p}$	$p = n^{-\frac{1}{2\Delta}}$	[7]
	$\mathcal{H}((1-\eta)n, \Delta) \subseteq \mathcal{G}_{n,p}$	$p = n^{-\frac{1}{\Delta}}$	[3]
Resilience	$R_g\big(\mathcal{G}_{n,p}, \mathcal{H}_2(\eta n, \eta n, \Delta)\big) \geq 1 - \gamma$	$p = n^{-\frac{1}{2\Delta}}$	[6]
	$R_\ell\big(\mathcal{G}_{n,p}, \mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)\big) \geq \frac{1}{2} - \gamma$	$p = n^{-\frac{1}{\Delta}}$	Thm. 1.1

Table 1
Summary of (best) known universality and resilience results (logarithmic factors for p are omitted).

3 Sparse regularity and the proof of Theorem 1.1

The proof of Theorem 1.1 uses the regularity method for sparse graphs (see [9], [10]) in a novel way, and it also resorts to several ideas and methods from, e.g., [3], [5], and [11]. Roughly speaking, the main steps of our proof can be summarized as follows.

The sparse regularity lemma. Given a subgraph G of $\mathcal{G}_{n,p}$ with the properties required by Theorem 1.1, we apply the sparse regularity lemma to G and obtain a regular partition of this graph. In the reduced graph R of this partition we find a perfect matching M and a suitable structure S connecting the edges of this matching. For finding M and S we use the fact that the reduced graph R of the regular partition is a graph with minimum degree $(\frac{1}{2}+\frac{1}{2}\eta)|V(R)|$. Accordingly we can use results about *dense* graphs for detecting the desired structure in R.

Small bandwidth. The structure S in the reduced graph R is chosen in such a way that we can construct a (graph) homomorphism h of each bipartite graph H that satisfies the requirements of Theorem 1.1 to the reduced graph R. For the remaining steps of the proof it is essential that this homomorphism should have the property that only few edges of H are not mapped to the matching M in R. More precisely, the deletion of a few vertices of H results in a graph H' such that h maps *all* edges of H' to M. For constructing this homomorphism the bandwidth bound on H in Theorem 1.1 is crucial.

The blow-up lemma. In [3] it is shown that almost spanning bounded-degree graphs which have the same number of vertices in each partition class embed into sparse regular pairs. This can be seen as a sparse bipartite analogue of the blow-up lemma for dense regular pairs. In our proof we apply this fact to the sparse regular pairs corresponding to the edges of M in order to embed the graph H', which contains almost all vertices of H, into G.

Connections. It remains to embed the vertices of H not covered by H' into G. For this purpose we use a strategy developed in [11] that allows for the embedding of bounded-degree graphs F into *systems* of regular pairs as long as F is much smaller than these regular pairs. In this step, however, we have to deal (among others) with the following problem. Most vertices of H were embedded already in the previous step. Assume that some vertex u of H was embedded to a vertex v of G in this process. Consequently, neighbours u' of u are confined to the neighbourhood $N_G(v)$ of v in their future embedding. Hence, we have to communicate certain constraints between the embedding of H' in the previous step of the proof and the embedding of the remaining vertices in this step of the proof.

References

[1] Alon, N. and M. Capalbo, *Sparse universal graphs for bounded-degree graphs*, Random Structures Algorithms **31** (2007), pp. 123–133.

[2] Alon, N. and M. Capalbo, *Optimal universal graphs with deterministic embedding*, in: *Proc. of the 19th Annual ACM-SIAM Symp. on Discrete Algorithms SODA*, 2008, pp. 373–378.

[3] Alon, N., M. Capalbo, Y. Kohayakawa, V. Rödl, A. Ruciński and E. Szemeredi, *Universality and tolerance*, in: *Proc. of the 41st Annual Symp. on Foundations of Computer Science FOCS*, 2000, pp. 14–21.

[4] Böttcher, J., K. P. Pruessmann, A. Taraz and A. Würfl, *Bandwidth, expansion, treewidth, separators, and universality for bounded degree graphs*, submitted, 2009.

[5] Böttcher, J., M. Schacht and A. Taraz, *Proof of the bandwidth conjecture of Bollobás and Komlós*, Math. Ann. **343** (2009), pp. 175–205.

[6] Dellamonica, D., Y. Kohayakawa, M. Marciniszyn and A. Steger, *On the resilience of long cycles in random graphs*, Electron. J. Combin. **15** (2008), R32.

[7] Dellamonica, D., Y. Kohayakawa, V. Rödl and A. Ruciński, *Universality of random graphs*, in: *Proc. of the 19th Annual ACM-SIAM Symp. on Discrete Algorithms SODA*, 2008, pp. 782–788.

[8] Dirac, G. A., *Some theorems on abstract graphs*, Proc. London Math. Soc. (3) **2** (1952), pp. 69–81.

[9] Kohayakawa, Y., *Szemerédi's regularity lemma for sparse graphs*, in: *Foundations of computational mathematics (Rio de Janeiro, 1997)*, Springer, 1997 pp. 216–230.

[10] Kohayakawa, Y. and V. Rödl, *Szemerédi's regularity lemma and quasi-randomness*, in: *Recent advances in algorithms and combinatorics*, CMS Books Math./Ouvrages Math. SMC. **11**, Springer, 2003 pp. 289–35.

[11] Kohayakawa, Y., M. Schacht, V. Rödl and E. Szemerédi, *Sparse partition universal graphs for graphs of bounded degree*, submitted, 2008.

[12] Sudakov, B. and V. Vu, *Local resilience of graphs*, Random Structures Algorithms **33** (2008), pp. 409–433.

[13] Turán, P., *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), pp. 436–452.