MEI LT TS Problem Set 2[1]

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[1] Available on http://personal.lse.ac.uk/carayolt/ec402.htm

Question 1

Some theory

- The two following questions are instances of applications of the LM test to situations where MLE coincides with NLS and the restrictions do now involve σ^2.
- Then the LM test has the particularly simple form given to you in the lecture notes. Let's prove it again slowly.
- Starting point: $y_t = g(x_t, \beta) + \varepsilon_t$, where ε_t is contemporaneously independent of x_t (x_t can include lagged y_t's and lagged ε_t's); $\varepsilon \sim i.i.d.N(0, \sigma^2)$.
- Denote $\psi = \begin{pmatrix} \beta & \sigma^2 \end{pmatrix}$.

Question 1

Some theory

- Under these conditions, suffices to write down the log-likelihood to see that $\hat{\beta}_{MLE}$ is the same as $\hat{\beta}_{NLS}$.
- Indeed,

$$\log L(\psi) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_t \varepsilon_t(\beta)^2$$

so that $\hat{\beta}$ from maximizing $\log L$ w.r.t. ψ is the same as from minimizing RSS w.r.t. β.

Question 1

Some theory

- Score w.r.t. β gives

$$\frac{\partial \log L}{\partial \beta}(\psi) = \frac{1}{\sigma^2}\sum_t \varepsilon_t(\beta) z_t(\beta) = \frac{1}{\sigma^2} Z(\beta)'\varepsilon(\beta)$$

  with $z_t(\beta) = -\frac{\partial \varepsilon_t}{\partial \beta}(\beta)$ and $Z(\beta) = \begin{pmatrix} z_1 & z_2 & \cdots & z_T \end{pmatrix}'$.
- FOC w.r.t σ^2 gives

$$\frac{\partial \log L}{\partial (\sigma^2)}(\psi) = -\frac{T}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_t \varepsilon_t(\beta)^2 = 0$$

  i.e. $\hat{\sigma^2} = \frac{\sum_t \varepsilon_t(\hat{\beta})^2}{T}$

Question 1

Some theory

- We want to test a (potentially nonlinear) $r \times 1$ restriction on β, say $H_0 : R(\beta) = 0$.
- Denote $\hat{\psi}_0 = \begin{pmatrix} \hat{\beta}_0 & \hat{\sigma^2}_0 \end{pmatrix}$ the constrained MLE.
- Then, taking as granted the result

$$LM = \frac{\partial \log L}{\partial \psi'}(\hat{\psi}_0) I(\hat{\psi}_0)^{-1} \frac{\partial \log L}{\partial \psi}(\hat{\psi}_0) \xrightarrow{d} \chi^2(r)$$

we will show that we can rewrite LM as related to the R^2 of an artificial regression.

Question 1

Some theory

- First note that even though ψ includes both β and σ^2, we actually only care about β in the above, because since σ^2 is not constrained the corresponding score is 0 (in other words at the constrained optimum $\hat{\psi}_0$ is such that $\frac{\partial \log L}{\partial(\sigma^2)}(\hat{\psi}_0) = 0$. (Otherwise, would be possible to find a σ^2 that achieves higher log likelihood.)
- This means that actually,

$$LM = \frac{\partial \log L}{\partial \beta'}(\hat{\psi}_0) I(\hat{\psi}_0)^{-1}_{\beta\beta} \frac{\partial \log L}{\partial \beta}(\hat{\psi}_0) \xrightarrow{d} \chi^2(r)$$

Question 1

Some theory

- Also note that

$$I(\psi)_{\beta\beta} = -E\left(\frac{\partial^2 \log L}{\partial\beta\beta'}(\psi)\right) = \frac{1}{\sigma^2}E\left(\sum_t(\varepsilon_t\frac{\partial^2\varepsilon_t}{\partial\beta\beta'} + \frac{\partial\varepsilon_t}{\partial\beta}\frac{\partial\varepsilon_t}{\partial\beta'})\right)$$
$$= \frac{1}{\sigma^2}E(Z(\beta)'Z(\beta))$$

- So a consistent estimator for $\frac{I(\hat{\psi}_0)_{\beta\beta}}{T}$ is

$$\frac{1}{T.\hat{\sigma^2}_0}Z(\hat{\beta}_0)'Z(\hat{\beta}_0)$$

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Some theory

- Hence, given the expression for the score w.r.t β found earlier, we can rewrite (omitting in the notations, but keeping in mind, that Z and ε are taken at $\hat{\beta}_0$)

$$LM = \left(\frac{Z'\varepsilon}{\hat{\sigma^2}_0}\right)' \left[\frac{1}{\hat{\sigma^2}_0} Z'Z\right]^{-1} \left(\frac{Z'\varepsilon}{\hat{\sigma^2}_0}\right) = \frac{1}{\hat{\sigma^2}_0} \varepsilon' Z(Z'Z)^{-1} Z'\varepsilon$$

- Now note that if you see something like $y'X(X'X)^{-1}X'y$, you should recognize the (uncentered) ESS of the regression of y on X. Indeed:

$$y'X(X'X)^{-1}X'y = y'P_X y = (P_X y)' P_X y = \hat{y}\hat{y}$$

Question 1

Some theory

- So $\varepsilon(Z'Z)^{-1}Z'\varepsilon$ from above is the ESS of the regression of ε on Z. $\varepsilon(\hat{\beta}_0) = y_t - g(x_t, \hat{\beta}_0)$ on $Z(\hat{\beta}_0)$
- Remember also that $\hat{\sigma^2}_0 = \frac{\varepsilon(\hat{\beta}_0)'\varepsilon(\hat{\beta}_0)}{T}$, i.e. $\frac{1}{T}$ of the TSS from the same regression.
- So $LM = T.\frac{ESS}{TSS} = TR^2$ where the R^2 is from the regression of $\varepsilon(\hat{\beta}_0)$ on $Z(\hat{\beta}_0)$.

Question 1 Section 1

Question

- 1.$y_t = \varepsilon_t + \theta\varepsilon_{t-1}$, for $t = 1, \ldots, T$ and with $\varepsilon_t \sim i.i.d.\mathbf{N}(0, \sigma^2)$; $\varepsilon_0 = 0$.
- i.e. y_t is $MA(1)$.
- Derive the Lagrange-Multiplier test of the null $\theta = 0$.

Answer

- $\epsilon_t(\theta) = y_t - \theta\epsilon_{t-1}(\theta)$, so with previous notation

$$z_t(\theta) = -\frac{\partial\varepsilon_t}{\partial\theta}(\theta) = -\left(-\epsilon_{t-1}(\theta) - \theta\frac{\partial\varepsilon_{t-1}}{\partial\theta}(\theta)\right)$$

Question 1 Section 1

Answer

- Hence at the restricted estimate $\hat{\theta}_0$,

$$\epsilon_t(\hat{\theta}_0) = y_t$$

  and

$$z_t(\hat{\theta}_0) = \epsilon_{t-1}(\hat{\theta}_0) = y_{t-1}$$

- Therefore under H_0, $LM = TR^2 \sim \chi^2(1)$, where R^2 refers to the R^2 from the regression of y_t on y_{t-1}. Compare LM to the relevant quantile of a χ^2.
- So LM test is computationally very simple here; LR would be more complicated. (Would need to actually solve the unrestricted ML problem).

Question 1 Section 2

Question

- 2. $y_t = \beta_1 x_{1t} + \beta_2 \frac{1}{2(x_{2t}-\gamma)^2} + \varepsilon_t$, for $t = 1, \ldots, T$ and with $\varepsilon_t \sim i.i.d.\mathbf{N}(0, \sigma^2)$; x_1, x_2 process independent from ε.
- Derive the Lagrange-Multiplier test of the null $\gamma = 0$.

Answer

- As opposed to 1., here we have to first solve the restricted ML problem (was trivial in 1; not so here). If $\gamma = 0$, then LM (or NLS) is the same as OLS of y on x_1 and $1/2x_t$.
- $\epsilon_t(\psi) = y_t - \beta_1 x_{1t} + \beta_2 \frac{1}{2(x_{2t}-\gamma)^2}$, so with previous notation

$$z_t(\psi) = -\frac{\partial \varepsilon_t}{\partial \psi}(\psi)$$

Question 1 Section 2

Answer

- Hence evaluating at the restricted estimate $\hat{\psi}_0$,

$$z_t(\hat{\psi}_0) = \begin{pmatrix} x_{1t} \\ \frac{1}{2x_{2t}^2} \\ -\frac{\hat{\beta}_{20}}{x_{2t}^3} \end{pmatrix}$$

- Therefore under H_0, $LM = TR^2 \sim \chi^2(1)$, where R^2 refers to the R^2 from the regression of $\hat{\epsilon_t} = \epsilon_t(\hat{\psi}_0)$ (obtained from OLS of y on x_1 and $1/2x_t$) on x_{1t}, $1/(x_{2t})^2$ and $1/(x_{2t})^3$. Compare LM to the relevant quantile of a χ^2.
- Again LM test is computationally simple here; LR would be more complicated. (Would need to actually solve the unrestricted ML problem).

Question 2

I do this one on the board.