MEI MT Problem Set 1

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Introduction

- Your class teacher:
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  - `http:\\personal.lse.ac.uk\carayolt\EC402.htm`
- Office hours: Tuesdays 11.30 to 12.30 in S684.
- 3 class groups on Fridays: 9 to 10 in K105, 3 to 4 in K105 and 5 to 6 in D211.
- My slides will be available on my website after classes.

Question 1

What is a random variable?

Question 2

- (a) and (b) basically same question, but (a) discrete and (b) continuous: given the distribution of r.v. X, we want the distribution of $Y = f(X) = X^2 + 2$.
- The way to answer is very different.

Question 2 (a)

$$X = \begin{cases} -1 \text{ with probability } \frac{1}{3} \\ 0 \text{ with probability } \frac{1}{3} \\ 1 \text{ with probability } \frac{1}{3} \end{cases}$$

We can conclude directly, as this maps, through function f, to:

$$Y = f(X) = \begin{cases} 3 \text{ with probability } \frac{2}{3} \\ 2 \text{ with probability } \frac{1}{3} \end{cases}$$

Question 2 (b)

- X uniformly distributed on $[-1, +1]$; i.e. its pdf is .5 on that interval, 0 elsewhere.
- Remember September course: we had a formula to derive the pdf of a *monotonic* function of a random variable. Here, $f(X) = X^2 + 2$ is *NOT* monotonic on $[-1, +1]$ so we can't apply this directly.
- Let us think in terms of cdf. The cdf of X is:

$$F_X(x) = \begin{cases} 0 \text{ if } x \leq -1 \\ \frac{x+1}{2} \text{ if } -1 \leq x \leq 1 \\ 1 \text{ if } 1 \leq x \end{cases}$$

Question 2 (b)

- $$F_Y(y) = P(Y \leq y) = P(X^2 + 2 < y) = P(X^2 < y - 2)$$
  $$= P(-\sqrt{y-2} < X < \sqrt{y-2})$$
  $$= F_X(\sqrt{y-2}) - F_X(-\sqrt{y-2})$$
  $$= 2F_X(\sqrt{y-2}) - 1$$
  $$= \begin{cases} 0 \text{ if } y \leq 2 \\ \sqrt{y-2} \text{ if } 2 \leq y \leq 3 \\ 1 \text{ if } 3 \leq y \end{cases} \tag{1}$$

- (From there, can differentiate to get pdf)

Question 3

- Linearity of expectations.
-

$$
\begin{aligned}
E(aX+bY) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(ax+by)f_{X,Y}(x,Y)dxdy \\
&= a\int_{-\infty}^{+\infty} x\left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy\right)dx \\
&\quad + b\int_{-\infty}^{+\infty} y\left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx\right)dy \\
&= a\int_{-\infty}^{+\infty} xf_X(x)dx + b\int_{-\infty}^{+\infty} yf_Y(y)dy
\end{aligned}
$$

Question 4

- Important question about the relationship between independence, mean-independence and uncorrelatedness. See also p.99 of your handout.
- (a) Independence stronger than mean-independence.
- (b) Mean-independence stronger than uncorrelatedness.
- Also noteworthy:
  - Mean-independence of X on Y is not the same thing as mean-independence of Y on X.
  - X mean-independent on Y and Y mean-independent on X does NOT imply X and Y independent (can find counterexample)
  - A subtle point. The statement on top of p.100 is only correct if X and Y are *jointly* normal, which is stronger than saying that they are both normal. More on this next week.

Question 4 (a)

$$
\begin{aligned}
E(X|Y) &= \int_{-\infty}^{+\infty} x f_{X|Y}(x|Y)dx = \int_{-\infty}^{+\infty} x f_X(x)dx \text{ by independence} \\
&= E(X)
\end{aligned}
$$

Question 4 (b)

$$
\begin{aligned}
cov(X, Y) &= E(XY) - E(X)E(Y) \\
&= E(E(XY|Y)) - E(X)E(Y) \\
&= E(YE(X|Y)) - E(X)E(Y) \\
&= E(YE(X)) - E(X)E(Y) \\
&= E(X)E(Y) - E(X)E(Y) = 0
\end{aligned}
$$

Question 5

Definition

$X_1, .., X_n$ are jointly normal (i.e., the vector $(X_1, X_2, .., X_n)$ is gaussian) if either of these (equivalent) propositions is satisfied:

- Any linear combination of the X_i's is normal (in particular, each X_i is normal–but that is NOT sufficient)
- The joint pdf of $X_1, .., X_n$ is $f(x_1, .., x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{det(\Sigma)}} e^{-\frac{1}{2}((x-m)'\Sigma^{-1}(x-m)}$, where m is the expected value of the gaussian vector and Σ is its variance-covariance matrix.

Question 5

- So here, by definition we know that $Z = 5 + 2Y - X$ is normally distributed, as it is a linear combination of the elements of a gaussian vector. So all we have to do is find its mean and variance.
- $E(Z) = 5 + 2E(Y) - E(X) = 5 + 2\mu_Y - \mu_X$
- $Var(Z) = Var(2Y - X) = Var(2Y) + Var(X) - 2cov(2Y, X) = 4Var(Y) + Var(X) - 4cov(X, Y) = 4\sigma_Y^2 + \sigma_X^2 - 4\sigma_{XY}$
- Hence we conclude $Z \sim \mathcal{N}(5 + 2\mu_Y - \mu_X, 4\sigma_Y^2 + \sigma_X^2 - 4\sigma_{XY})$.

Question 6 and 7

- Question 6: sample
- Question 7: tests
- Let's talk about it quickly in class.

Question 8

- X_i: uniform distribution over $[\theta, 100]$.
- Method of moments: $\hat{\theta}_{MM}$ such that the sample counterpart of the first moment (e.g., sample mean) would coincide with the population mean for $\theta = \hat{\theta}_{MM}$.
- Here: sample mean is $\bar{X}$; population mean is $\frac{100+\theta}{2}$. So $100 + \hat{\theta}_{MM} = 2\bar{X}$; hence $\hat{\theta}_{MM} = 2\bar{X} - 100 = 2 \times 50 - 100 = 0$