MEI MT Problem Set 2[1]

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[1] Available on http://personal.lse.ac.uk/carayolt/ec402.htm

Question 1

Question

A is an $n \times m$ matrix and $A'A = 0$, prove that $A = 0$.

Answer

- Denote $A = (a_{ij})$
- $A'A$ is a $m \times m$ matrix whose (i, i) element is $\sum_{k=1}^{n} a'_{ik} a_{ki} = \sum_{k=1}^{n} a_{ki} a_{ki} = \sum_{k=1}^{n} {a_{ki}}^2$.
- Hence the only way that $A'A$ be zero is if $\forall i \sum_{k=1}^{n} a_k i^2 = 0 \Leftrightarrow \forall (i, k) a_{ki} = 0 \Leftrightarrow A = 0$.

Question 2 (a)

Question

Show that A positive definite (p.d.) $\Rightarrow$ A non-singular.

Answer

- "Reductio ad absurdum": Suppose $\exists x \neq 0$ such that $Ax = 0_n$. Then for this x, $x'Ax = x'0_n = 0$, which contradicts A p.d.

Question 2 (b)

Question

Show that if X is $n \times k$ and full column rank, then $X'X$ is p.d. (and hence non-singular).

Answer

- Let z any non-zero $k \times 1$ vector $\Rightarrow Xz$ is a $n \times 1$ vector.
- $z'(X'X)z = (Xz)'(Xz)$ is the inner product of a $n \times 1$ product by itself. Therefore $z'(X'X)z > 0$ as soon as $Xz \neq 0$.
- Since X is full column rank, Xz, which is a linear combination of the columns of X, is only 0 if z is 0_n.
- Hence for any non-zero z, $z'(X'X)z > 0$, which proves that $(X'X)$ is positive definite.

Question 3

Question

Show that if A and B square, conformable and non-singular, then (a) $(AB)^{-1} = B^{-1}A^{-1}$ and (b) $(A')^{-1} = (A^{-1})'$.

Answer

- Very simple: $AB(B^{-1}A^{-1}) = AIA^{-1} = I$, therefore $B^{-1}A^{-1}$ is the inverse of AB.
- $A'A^{-1'} = (A^{-1}A)' = I$, therefore $A^{-1'}$ is the inverse of A'.

Question 4

Question

If x $(n \times 1)$ is distributed $\mathcal{N}(\mu, \Sigma)$, what is the distribution of $a'x$, where a is a vector of constants?

Answer

- Remember last week: one of the the characterizations of joint normality is that ANY linear combination of the x_i's is normally distributed. Hence, $a'x$ is normally distributed.
- $E(a'x) = a'E(x) = a'\mu$
- $Var(a'x) = a'Var(x)a = a'\Sigma a$
- So: $a'x \sim \mathcal{N}(a'\mu, a'\Sigma a)$.

Question 5 (a)

Question

A is symmetric, positive definite. Show that there exists a square non-singular matrix P such that $A = QQ'$.

Answer

- Given that A is symmetric, the spectral theorem tells us that we can find a matrix P such that $M = P\Lambda P'$, with P orthogonal ($P' = P^{-1}$) and $\Lambda = diag(\lambda_1, .., \lambda_n)$ diagonal matrix of eigenvalues of A.
- Given that A is also positive definite, its eigenvalues are strictly positive. So we can define $\sqrt{\Lambda} = diag(\sqrt{\lambda_1}, .., \sqrt{\lambda_n})$, so that $\sqrt{\Lambda}^2 = \Lambda$.
- From here: $A = P\sqrt{\Lambda}\sqrt{\Lambda}P' = \underbrace{(P\sqrt{\Lambda})}_{Q}(P\sqrt{\Lambda})'$

Question 5 (b)

Question

$x \sim \mathcal{N}(0, \Sigma)$. Show that $z = x'\Sigma^{-1}x \sim \chi^2(n)$.

Answer

- Σ, being a variance covariance matrix and full rank, is symmetric and positive definite. Σ^{-1} is therefore also symmetric and positive definite. It plays the role of A in the previous question.
- Hence we can write $\Sigma^{-1} = PP'$ with P non-singular.

Question 5 (b) (cont.)

Answer

- Therefore: $z = x'PP'x = (P'x)'P'x = y'y$ where $y = P'x$. $y = P'x$ is a random vector, each element of which is a linear combination of the elements of x; therefore any linear combination of the elements of $y = P'x$ is also a linear combination of the elements of x, which are jointly normal; hence that linear combination is normally distributed. Hence y is jointly normal.
- $E(y) = E(P'x) = P'E(x) = 0$; $Var(y) = Var(P'x) = P'Var(x)P = P'\Sigma P = P'(PP')^{-1}P = P'P'^{-1}P^{-1}P = I_n$. So $y \sim \mathcal{N}(0, I_n)$; in other words, the elements of y are independent standard normals.
- $z = y'y = \sum_i y_i^2$. z is the sum of n squared standard normals, which is the definition of a χ^2 with n degrees of freedom.

Question 6

Question

Show that if A and B square, conformable and non-singular, then (a) $tr(A') = tr(A)$; (b) $tr(A + B) = tr(A) + tr(B)$; and (c) $tr(AB) = tr(BA)$.

Answer

- (a) and (b) are trivial. (c) is not.
- Let us prove something (a bit) stronger: $\forall\ (k, n) \in \mathbb{N}^2$, if A is a $(N \times K)$ matrix and B is a $(K \times N)$ matrix, then $tr\,(AB) = tr\,(BA)$.
- $C = AB \Rightarrow c_{ij} = \sum_{k=1}^{K}(a_{ik}b_{kj})$, hence $tr\,(AB) = \sum_{n=1}^{N} c_{nn} = \sum_{n=1}^{N}\sum_{k=1}^{K}(a_{nk}b_{kn}) = \sum_{k=1}^{K}\sum_{n=1}^{N}(b_{kn}a_{nk}) = \sum_{k=1}^{K} d_{kk}$ where $D = BA$, so $tr\,(AB) = tr\,(BA)$.

Question 7

Question

Suppose M is $n \times n$, symmetric, idempotent, with rank J. Suppose x is $n \times 1$, $\sim \mathcal{N}(0_n, I_n)$. Prove that $z = x'Mx \sim \chi^2(J)$.

Answer

- Given that M is symmetric, the spectral theorem tells us that we can find a matrix P such that $M = P\Lambda P'$, with P orthogonal ($P' = P^{-1}$) and $\Lambda = diag(\lambda_1, .., \lambda_n)$ diagonal matrix of eigenvalues of M. Furthermore, as M is idempotent of rank J, its eigenvalues are J ones and $n - J$ zeros. Hence $\Lambda^2 = \Lambda$.
- Then $z = x'Mx = x'P\Lambda P'x = (x'P\Lambda)(\Lambda P'x) = (\Lambda P'x)'(\Lambda P'x)$.

Question 7 (cont.)

Answer

- $y = P'x$ is a random vector, each element of which is a linear combination of the elements of x; therefore any linear combination of the elements of $y = P'x$ is also a linear combination of the elements of x, which are jointly normal; hence that linear combination is normally distributed. Hence y is jointly normal.
- $E(y) = E(P'x) = P'E(x) = 0$; $Var(y) = Var(P'x) = P'Var(x)P = P'I_nP = P'P = I_n$ as P is orthogonal. Hence $y \sim \mathcal{N}(0_n, I_n)$, i.e. the elements of y are independent standard normals.
- $z = (\Lambda y)'(\Lambda y) = \sum_{i=1}^{n}(\lambda_i^2 y_i^2)$. Since J λ_i's are ones and $n - J$ are zeros, this is a sum of J squared standard normals–i.e., by definition of a χ^2, $z \sim \chi^2(J)$.

Question 8

Question

i_n is a $n \times 1$ vector of ones. (a) What is the effect of the transformation $A = I_n - \frac{1}{n} i_n i_n'$? (b) Show that A is symmetric and idempotent. (c) What is $A i_n$? What does it imply?

Answer

- (a) A demeans the vector that it multiplies. e.g., let $z = \begin{pmatrix} z_1 & \dots & z_i & \dots & z_n \end{pmatrix}'$. Then $$Az = z - \frac{1}{n} i_n i_n' z = z - \frac{1}{n} \begin{pmatrix} \sum_{i=1}^n z \\ \vdots \\ \sum_{i=1}^n z \end{pmatrix} = z - \bar{z} i_n$$ where $\bar{z}$ is the average of the elements of z.

Question 8 (cont.)

Answer

- (b) $A' = I_n' - \frac{1}{n}(i_n i_n')' = A$;
  $A^2 = I_n - 2I_n \frac{1}{n} i_n i_n' + \frac{1}{n^2} i_n i_n' i_n i_n' = I_n - 2\frac{1}{n} i_n i_n' + \frac{1}{n^2} i_n n i_n' = A.$
  (The deviations from the mean of the deviations from the mean are the deviations from the mean, because the mean of the deviations from the mean is zero.)
- (c) $Ai_n = 0_n$: i_n is a vector of ones, so it never deviates from its mean.