MEI MT Problem Set 3[1]

Timothee Carayol

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[1]Available on http://personal.lse.ac.uk/carayolt/ec402.htm

Question 1

Question

Suppose $A1, A2$ hold (and we use the standard notations from the MEI handout).

Answer

- (a) $y_t = \beta_1 + \beta_2 x_t + \varepsilon_t$ holds by assumption $A2$ ($A2$ further says that $E(\varepsilon) = 0$).
- (b) $y_t = \hat{\beta}_1 + \hat{\beta}_2 x_t + \hat{\varepsilon}_t$ holds by definition of $\hat{\varepsilon}_t$.
- (c) $\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 x_t$ by definition of $\hat{y}_t$

Question 1 (cont)

Answer

- (d) $y_t = \hat{\beta}_1 + \hat{\beta}_2 x_t + \varepsilon_t$ wrong (from (a)) unless we get exactly the right $\hat{\beta}$ (which of course happens with probability zero).
- (e) $y_t = \beta_1 + \beta_2 x_t$ wrong unless there is no unexplained randomness in y (i.e. $\varepsilon_t = 0$)
- (f) $y_t = \beta_1 + \beta_2 x_t + \hat{\varepsilon}_t$ wrong (from (a)) unless we get exactly the right $\hat{\beta}$ (which of course happens with probability zero).
- (g) $\sum_t \varepsilon_t = 0$ true in expected value (from $A2$), but no reason why it would hold for a given sample.
- (h) $\sum_t x_t \varepsilon_t = 0$ likewise: true in expected value (if some version of $A3$ holds) but no reason why it would be true for our sample.

Question 1 (cont)

Answer

- Note: For (g) and (h), those equalities hold if you substitute the error ε with the residual $\hat{\varepsilon}$. This is a property of OLS, a proof of which goes as follows:
  - $X'\hat{\varepsilon} = X'(y - X\hat{\beta}) = X'(y - X(X'X)^{-1}X'y) = X'y - X'X(X'X)^{-1}y = X'y - X'y = 0$
  - Hence the inner product of any column of X by ε is zero. For the first column (the constant) this gives $\sum_t \hat{\varepsilon_t} = 0$; for any other column i it gives $\sum_t x_i \hat{\varepsilon_t}$.
  - NB: If there was no intercept in our specification, the residuals would not sum to zero anymore; but $x_{it}\varepsilon_t$ would still sum to zero.
  - NB2: Those are basically the sample counterparts of $A3$, and we can find OLS by performing GMM with these moment conditions.

Question 2

Question

$y_t = \beta_1 + \beta_2 x_t + \varepsilon_t$ where $A1, A2, A3, A4$ hold. (a) $Var(\hat{\beta}_1)$ (b) $cov(\hat{\beta}_1, \hat{\beta}_2)$ (c) $E(\sum_t \hat{\varepsilon_t}^2)$.

Answer

- Start with $Var(\hat{\beta}) = \sigma^2(X'X)^{-1}$. Then the answer to (a) is the first diagonal element of this 2×2 matrix, and (b) is the off-diagonal element.
- $$X'X = \begin{pmatrix} 1'_T \\ x' \end{pmatrix} \begin{pmatrix} 1_T & x \end{pmatrix} = \begin{pmatrix} T & \sum_t x_t \\ \sum_t x_t & \sum_t x_t^2 \end{pmatrix}$$
  $$= T \begin{pmatrix} 1 & \bar{x} \\ \bar{x} & \overline{x^2} \end{pmatrix}$$
- $|X'X| = T^2(\overline{x^2} - \bar{x}^2)$

Question 2 (cont)

Answer

- $(X'X)^{-1} = \frac{T}{|X'X|} \begin{pmatrix} \bar{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$
- $Var(\hat{\beta}_1) = \sigma^2 \frac{T\bar{x^2}}{T^2(\bar{x^2}-\bar{x}^2)} = \sigma^2 \frac{1}{T} \frac{\bar{x^2}-\bar{x}^2+\bar{x}^2}{\bar{x^2}-\bar{x}^2} = \frac{\sigma^2}{T} \frac{\bar{x}^2}{\bar{x^2}-\bar{x}^2}$ which is the answer to (a).
- $cov(\hat{\beta}_1, \hat{\beta}_2) = \sigma^2 \frac{-T\bar{x}}{T^2(\bar{x^2}-\bar{x}^2)}$ is the answer to (b).
- For (c), see extra handout[2] for the general case.

[2] Available on http://personal.lse.ac.uk/carayolt/ec402.htm

Question 3 (a)

Question

$y = X\beta + \varepsilon$ with X: $T \times K$ [3], $A1, A2, A3, A4, A5$. (a) Construct $(1-\alpha)$ confidence interval for β_i based on $\beta_{i\,OLS}$.

Answer

- $\hat{\beta} = (X'X)^{-1}X'y$ with $y \sim \mathcal{N}(X\beta, \sigma^2 I_T)$. Hence $\hat{\beta} \sim \mathcal{N}(\beta, \sigma^2(X'X)^{-1})$.
- Therefore $\forall i, \hat{\beta}_i \sim \mathcal{N}(\beta_i, \sigma^2 {(X'X)^{-1}}_{ii})$.
- Therefore $Z = \frac{\beta_i - \hat{\beta}_i}{\sigma\sqrt{{(X'X)^{-1}}_{ii}}} \sim \mathcal{N}(0, 1)$, which would allow use to do inference iff we knew the true σ^2.

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[3] The question from the handout uses only $K = 2$, but let us do the more general case.

Question 3 (a) (cont)

Answer

- Typically we do not know σ^2 and instead have to estimate it (without bias) with $s^2 = \frac{1}{N-K}\sum \hat{\varepsilon_t}^2$.
- Can prove that $t = \frac{\beta_i - \hat{\beta}_i}{s\sqrt{(X'X)^{-1}{}_{ii}}} \sim t(T-K)$.
- Therefore, $(1-\alpha)$ C.I. for β_i is given by $[\hat{\beta}_i - t_{T-K,1-\alpha/2} s\sqrt{(X'X)^{-1}{}_{ii}}, \hat{\beta}_i + t_{T-K,1-\alpha/2} s\sqrt{(X'X)^{-1}{}_{ii}}]$

Question 3 (b)

Question

Show that the two-tailed test of $H_0 : \beta_i = 0$ at the level α will fail to reject iff zero lies inside the $(1 - \alpha)$ C.I. for β_i.

Answer

- Zero lies inside the $(1 - \alpha)$ C.I. for β_i

$$\Leftrightarrow \hat{\beta}_i - t_{T-K,1-\alpha/2} s \sqrt{(X'X)^{-1}{}_{ii}} \leq 0 \leq \hat{\beta}_i + t_{T-K,1-\alpha/2} s \sqrt{(X'X)^{-1}$$

$$\Leftrightarrow -t_{T-K,1-\alpha/2} s \sqrt{(X'X)^{-1}{}_{ii}} \leq -\hat{\beta}_i \leq t_{T-K,1-\alpha/2} s \sqrt{(X'X)^{-1}{}_{ii}}$$

$$\Leftrightarrow -t_{T-K,1-\alpha/2} \leq \underbrace{\frac{0 - \hat{\beta}_i}{s\sqrt{(X'X)^{-1}_{ii}}}}_{\text{Test statistic}} \leq t_{T-K,1-\alpha/2}$$

- The quantity in the middle is our test statistic for $\beta_i = 0$, distributed $t(T - K)$ under the null; hence this inequality is the condition under which we fail to reject the null.

Question 4

Question

$y = X\beta + \varepsilon$ is estimated by OLS. $\hat{y}$ and $\hat{\varepsilon}$ denote predicted value and residual, respectively. Show that $TSS = ESS + RSS$, and that the R^2 is the square of the sample correlation between y and $\hat{y}$.

Answer

- $$\begin{aligned} TSS &= (y-\bar{y})'(y-\bar{y}) = (y-\hat{y}+\hat{y}-\bar{y})'(y-\hat{y}+\hat{y}-\bar{y}) \\ &= (y-\hat{y})'(y-\hat{y}) + (\hat{y}-\bar{y})'(\hat{y}-\bar{y}) + 2(y-\hat{y})'(\hat{y}-\bar{y}) \\ &= RSS + ESS + 2\hat{\varepsilon}'(\hat{y}-\bar{y}) \qquad (1) \\ &= RSS + ESS + 2(\hat{\varepsilon}'X\beta) - 2\bar{y}\sum_t \hat{\varepsilon}_t = RSS + ESS \qquad (2) \end{aligned}$$
- (1) to (2) comes from $X'\hat{\varepsilon} = 0$, which we have seen in question 1.

Question 4 (cont)

Answer

- $$\begin{aligned} corr^2(y,\hat{y}) &= \frac{((y-\bar{y})'(\hat{y}-\bar{y}))^2}{[(y-\bar{y})'(y-\bar{y})]\,[(\hat{y}-\bar{y})'(\hat{y}-\bar{y})]} \\ &= \frac{((y-\hat{y}+\hat{y}-\bar{y})'(\hat{y}-\bar{y}))^2}{TSS.ESS} \\ &= \frac{(\hat{\varepsilon}'(\hat{y}-\bar{y})+(\hat{y}-\bar{y})'(\hat{y}-\bar{y}))^2}{TSS.ESS} \qquad (3) \\ &= \frac{(0+ESS)^2}{TSS.ESS} = \frac{ESS}{TSS} = 1-\frac{RSS}{TSS} \qquad (4) \end{aligned}$$