MEI MT Problem Set 7 Part 1[1]

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[1] Available on http://personal.lse.ac.uk/carayolt/ec402.htm

Question 1

Question

- $y_t = \alpha + x_t\beta + \varepsilon_t$
- T observations, 1 independent variable (plus the constant).
- Assumptions:
  - ε_t are *iid* with mean 0 and variance σ_ε^2.
  - x_t are random, *iid* with mean μ_x and variance σ_x^2, and $\forall t$ x_t and ε_t are independent.
  - Also assume that $x_t\varepsilon_t$ are independently distributed (not stated in the question, but otherwise the exercise is a bit more complicated, e.g. need to use Chebyshev-Markov *LLN*4 instead of Khinshine. We use that LLN at some point in question 2, so for now stick to the simpler case $x_t\varepsilon_t$ *iid*).
- Denote

$$Z = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_T \end{pmatrix} = \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_T' \end{pmatrix}$$

Question 1

Answer

- (a) $Z'Z$. $Z'Z = \begin{pmatrix} z_1 & z_2 & \cdots & z_T \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_T' \end{pmatrix} = \sum_{t=1}^{T} z_t z_t'$.

  (This is 2×2.)

- $Z'\varepsilon$. $Z'\varepsilon = \begin{pmatrix} z_1 & z_2 & \cdots & z_T \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{pmatrix} = \sum_{t=1}^{T} z_t \varepsilon_t$.

  (This is 2×1.)

Question 1

Answer

- (b) $E(z_t z_t')$.

$$E(z_t z_t') = E\left(\begin{pmatrix} 1 \\ x_t \end{pmatrix}\begin{pmatrix} 1 & x_t \end{pmatrix}\right) = \begin{pmatrix} E(1) & E(x_t) \\ E(x_t) & E(x_t^2) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \mu_x \\ \mu_x & \mu_x^2 + \sigma_x^2 \end{pmatrix} = \Sigma_{zz}$$

- (c) $plim(\frac{Z'Z}{T})$. From Khinchine LLN on $w_t = z_t z_t'$: w_t are *iid* (because z_t are) and have finite expected value, hence $plim\left(\frac{1}{T}\left(\sum_t w_t\right)\right) = E(w_t) = \Sigma_{zz}$.
- (d) $E(z_t \varepsilon_t) = E\left(\begin{pmatrix} 1 \\ x_t \end{pmatrix}\varepsilon_t\right) = \begin{pmatrix} E(\varepsilon_t) \\ E(x_t \varepsilon_t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ from our assumptions.
- (e) $Var(z_t \varepsilon_t) = E(z_t \varepsilon_t \varepsilon_t z_t') = E(\varepsilon_t^2) E(z_t z_t') = \sigma_\varepsilon^2 \Sigma_{zz}$ (from independence of z_t and ε_t).

Question 1

Answer

- (f) $plim(\frac{Z'\varepsilon}{T})$. $v_t = z_t\varepsilon_t$ are independently distributed (i.e., $\forall t \neq s$, v_s and v_t are independent), because ε_t are *iid* and we assumed $x_t\varepsilon_t$ to be independent accross t as well. So we can use $LLN1$ (Chebyshev); this, combined with (d), gives the results.
- (g) $\sqrt{T}(\frac{Z'\varepsilon}{T}) \rightarrow^d \mathcal{N}(0, \sigma_\varepsilon^2\Sigma_{zz})$. This is $CLT1$ combined with (d) and (e).
- (h) $\sqrt{T}(\frac{Z'Z}{T})^{-1}(\frac{Z'\varepsilon}{T}) \rightarrow^d \mathcal{N}(0, \sigma_\varepsilon^2\Sigma_{zz}^{-1})$. (c) implies (through Slutsky theorem) $plim\left(\frac{1}{T}(Z'Z)\right)^{-1} = \Sigma_{zz}^{-1}$. This combined with (g) gives the result (through Cramér).
- Clear from (c) and (f) that OLS is consistent. (Slutsky).

Question 1

Answer

$$
\begin{aligned}
Var(\hat{\beta}) &= Var\left([(Z'Z)^{-1}Z'\varepsilon]_2\right) \\
&= E\left([(Z'Z)^{-1}Z'\varepsilon((Z'Z)^{-1}Z'\varepsilon)']_{22}\right) \\
&= E\left([(Z'Z)^{-1}Z'\varepsilon\varepsilon'Z(Z'Z)^{-1}]_{22}\right) \\
&= E\left([(Z'Z)^{-1}Z'E(\varepsilon\varepsilon'|Z)Z(Z'Z)^{-1}]_{22}\right) \\
&= \sigma_\varepsilon^2 E\left([(Z'Z)^{-1}Z'Z(Z'Z)^{-1}]_{22}\right) \\
&= \sigma_\varepsilon^2 E\left([(Z'Z)^{-1}]_{22}\right) \\
&= \sigma_\varepsilon^2 E\left(\frac{1}{\sum_t x_t^2 - \frac{1}{T}(\sum_t x_t)^2}\right) \\
&= \sigma_\varepsilon^2 E\left(\frac{1}{\sum_t (x_t - \frac{1}{T}\sum_t x_t)^2}\right)
\end{aligned}
\tag{1}
$$

Question 1

Answer

In (1) I use $E\left((Z'Z)^{-1}Z'\varepsilon\right) = 0$ (because ε and Z are independent). Slutsky and $LLN2$ (Khinchine) allow us to conclude (also using that s^2 is consistent for σ_ε^2, which we show in the next question).

Question 2

Question

- $y = X\beta + \varepsilon$, where X is $T \times K$. ε_t are $iid(0, \sigma_\varepsilon^2)$; x_t are random, $iid(0, \Sigma_x)$; $\forall s, t$, x_s independent of ε_t: very strong assumption which we relax later.
- $s^2 = \frac{1}{T-K} \sum_{t=1}^{T} \hat{\varepsilon_t}^2$. Want to prove that s^2 is consistent for σ_ε^2.

Answer

- $$s^2 = \frac{1}{T-K} \hat{\varepsilon}' \hat{\varepsilon} = \left(\frac{T}{T-K} \right) \frac{\varepsilon' M_X \varepsilon}{T}$$
$$= \left(\frac{T}{T-K} \right) \frac{\varepsilon'\varepsilon - (X'\varepsilon)'(X'X)^{-1}(X'\varepsilon)}{T}$$
$$= \left(\frac{T}{T-K} \right) \left(\left(\frac{\varepsilon'\varepsilon}{T} \right) - \left(\frac{X'\varepsilon}{T} \right)' \left(\frac{X'X}{T} \right)^{-1} \left(\frac{X'\varepsilon}{T} \right) \right)$$

Question 2

Answer

- Hence, using the Slutsky theorem:

$$plim(s^2) = 1\left(\sigma_\varepsilon^2 - \left(0_K'\Sigma_X^{-1}0_K\right)\right) = \sigma_\varepsilon^2$$

because

  - $plim\left(\frac{T}{T-K}\right) = 1$ trivial.
  - $plim\left(\frac{\varepsilon'\varepsilon}{T}\right) = E(\varepsilon_t^2) = \sigma_\varepsilon^2$ from Khinchine $LLN2$.
  - $plim\left(\frac{X'\varepsilon}{T}\right) = E(x_t\varepsilon_t) = 0$ from Khinchine $LLN2$ (because $\forall s, t,\ v_s = x_s\varepsilon_s$ is independent from v_t from our assumptions, i.e. the v_t are *iid*).
  - And finally, $plim\left(\frac{X'X}{T}\right) = E(x_t x_t') = \Sigma_X$ from Khinchine $LLN2$.

Question 2

Answer

- Now relax our assumption that $\forall s, t$, x_s independent of ε_t.
- We have instead $y = X\beta + \varepsilon$, where X is $T \times K$. ε_t are $iid(0, \sigma_\varepsilon^2)$; x_t are random, $iid(0, \Sigma_x)$; $E(x_t \varepsilon_t) = 0$.
- Still the case that

$$s^2 = \left(\frac{T}{T-K}\right)\left(\left(\frac{\varepsilon'\varepsilon}{T}\right) - \left(\frac{X'\varepsilon}{T}\right)'\left(\frac{X'X}{T}\right)^{-1}\left(\frac{X'\varepsilon}{T}\right)\right)$$

- More tricky than previously: cannot use Chebyshev or Khinshine LLN for $x_t \varepsilon_t$ anymore. Instead we'll have to use Chebyshev-Markov ($LLN4$).

Question 2

Answer

- To use Chebyshev-Markov on $\frac{1}{T}\sum_t x_t\varepsilon_t = \frac{1}{T}\sum_t u_t$: need $Var\left(\frac{1}{T}\sum_t u_t\right) = \frac{1}{T^2}\left(\sum_t \sigma_t^2 + 2\sum_{t\neq s}\sigma_{ts}\right) \rightarrow 0$ as $T \rightarrow \infty$ with $V(u_t) = \sigma_t^2$ and $cov(u_t, u_s) = \sigma_{ts}$

Question 2

Answer

$$
\begin{aligned}
Var\left(\frac{1}{T}\sum_t u_t\right) &= \frac{1}{T^2} Var\left(\sum_t x_t \varepsilon_t\right) \\
&= \frac{1}{T^2} Var\left(X'\varepsilon\right) \\
&= \frac{1}{T^2} E\left([X'\varepsilon - E(X'\varepsilon)][X'\varepsilon - E(X'\varepsilon)]'\right) \\
&= \frac{1}{T^2} E\left(X'\varepsilon\varepsilon' X\right) \\
&= \frac{1}{T^2} E\left(X' E(\varepsilon\varepsilon'|X) X\right) \\
&= \frac{1}{T^2} E\left(X' \sigma_\varepsilon^2 X\right) \qquad (2) \\
&= \frac{1}{T}\left(\sigma_\varepsilon^2 \Sigma_X\right) \rightarrow_{T\rightarrow\infty} 0
\end{aligned}
$$

Question 2

Answer

- In (2) I use $E(\varepsilon\varepsilon'|X) = \sigma_{\varepsilon}^{2} I_T$, which is not explicitly mentioned in the question (we only know it is true unconditional on X). But we cannot conclude without making an additional assumption.